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Rational Ehrhart Theory

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Abstract. The Ehrhart quasipolynomial of a rational polytope P encodes fundamental arithmetic data of P, namely, the number of integer lattice points in positive integral dilates of P. Ehrhart quasipolynomials were introduced in the 1960s, satisfy several fundamental structural results and have applications in many areas of mathematics and beyond. The enumerative theory of lattice points in rational (equivalently, real) dilates of rational polytopes is much younger, starting with work by Linke (2011), Baldoni–Berline–Köppe–Vergne (2013), and Stapledon (2017). We introduce a generating-function *ansatz* for rational Ehrhart quasipolynomials, which unifies several known results in classical and rational Ehrhart theory. In particular, we define γ-rational Gorenstein polytopes, which extend the classical notion to the rational setting and encompass the generalized reflexive polytopes studied by Fiset–Kasprzyk (2008) and Kasprzyk–Nill (2012).

Keywords: rational polytope, Ehrhart quasipolynomial, integer lattice point, rational Ehrhart series, Gorenstein polytope

1 Introduction

This extended abstract summarizes the main results of [5]. Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a d-dimensional **lattice polytope**; that is, P is the convex hull of finitely many points in $\mathbb{Z}^d$. Ehrhart's famous theorem [10] then says that the counting function $\operatorname{ehr}(\mathsf{P}; n) := |n\mathsf{P} \cap \mathbb{Z}^d|$ is a polynomial in n, the **Ehrhart polynomial** of P. Equivalently, the corresponding **Ehrhart series** is of the form

$$\operatorname{Ehr}(\mathsf{P}; t) := 1 + \sum_{n \in \mathbb{Z}_{>0}} \operatorname{ehr}(\mathsf{P}; n)\, t^n = \frac{\mathrm{h}^*(\mathsf{P}; t)}{(1-t)^{d+1}}$$

where $\mathrm{h}^*(\mathsf{P}; t) \in \mathbb{Z}[t]$ is a polynomial of degree $\leq d$. More generally, let $\mathsf{P} \subseteq \mathbb{R}^d$ be a **rational polytope** with **denominator** k, *i.e.*, k is the smallest positive integer such

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that $k\mathsf{P}$ is a lattice polytope. Then $\operatorname{ehr}(\mathsf{P};n)$ is a **quasipolynomial**, *i.e.*, of the form $\operatorname{ehr}(\mathsf{P};n) = c_d(n)n^d + \cdots + c_1(n)n + c_0(n)$ where $c_0, c_1, \ldots, c_d \colon \mathbb{Z} \to \mathbb{R}$ are periodic functions. The least common period of $c_0(n), c_1(n), \ldots, c_d(n)$ is the **period** of $\operatorname{ehr}(\mathsf{P};n)$; this period divides the denominator k of P; again this goes back to Ehrhart [10]. Equivalently,

$$\operatorname{Ehr}(\mathsf{P};t) := 1 + \sum_{n \in \mathbb{Z}_{>0}} \operatorname{ehr}(\mathsf{P};n)\, t^n = \frac{\mathrm{h}^*(\mathsf{P};t)}{(1-t^k)^{d+1}} \tag{1.1}$$

where $\mathrm{h}^*(\mathsf{P};t) \in \mathbb{Z}[t]$ has degree $< k\,(d+1)$.

Because polytopes can be described by a system of linear equalities and inequalities, they appear in a wealth of areas; likewise Ehrhart quasipolynomials have applications in number theory, combinatorics, computational geometry, commutative algebra, representation theory, and many other areas. For general background on Ehrhart theory and connections to various mathematical fields, see, *e.g.*, [6].

Our aim is to study Ehrhart counting functions with a real dilation parameter. We define the **rational Ehrhart counting function** and the **real Ehrhart counting function**

$$\operatorname{rehr}(\mathsf{P};\lambda) := \left|\lambda\mathsf{P} \cap \mathbb{Z}^d\right|, \qquad \overline{\operatorname{r}}\operatorname{ehr}(\mathsf{P};\lambda) := \left|\lambda\mathsf{P} \cap \mathbb{Z}^d\right|,$$

where $\lambda \in \mathbb{Q}$ or $\lambda \in \mathbb{R}$ respectively. As P is a rational polytope, it suffices to compute $\operatorname{rehr}(\mathsf{P};\lambda)$ at certain rational arguments to fully understand $\overline{\operatorname{r}}\operatorname{ehr}(\mathsf{P};\lambda)$; we will (quantify and) make this statement precise shortly. To the best of our knowledge, Linke [14] initiated the study of $\overline{\operatorname{r}}\operatorname{ehr}(\mathsf{P};\lambda)$ from the Ehrhart viewpoint. She proved several fundamental results starting with the fact that $\overline{\operatorname{r}}\operatorname{ehr}(\mathsf{P};\lambda)$ is a **quasipolynomial** in the real variable λ, that is,

$$\overline{\operatorname{r}}\operatorname{ehr}(\mathsf{P};\lambda) = c_d(\lambda)\,\lambda^d + c_{d-1}(\lambda)\,\lambda^{d-1} + \cdots + c_0(\lambda)$$

where $c_0, c_1, \ldots, c_d \colon \mathbb{R} \to \mathbb{R}$ are periodic functions. As a first running example, the real Ehrhart counting function of the line segment $[1,2]$ is $\overline{\operatorname{r}}\operatorname{ehr}([1,2];\lambda) = \lfloor 2\lambda \rfloor - \lceil \lambda \rceil + 1$.

Linke views the coefficient functions as piecewise-defined polynomials, which allows her, among many other things, to establish differential equations relating the coefficient functions. Essentially concurrently, Baldoni–Berline–Köppe–Vergne [1] developed an algorithmic theory of **intermediate sums** for polyhedra, which includes $\overline{\operatorname{r}}\operatorname{ehr}(\mathsf{P};\lambda)$ as a special case.

Our goal is to add a generating-function viewpoint to [1, 14], one that is inspired by [16, 17]. Suppose the rational polytope P is given by the irredundant halfspace description

$$\mathsf{P} = \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\,\mathbf{x} \le \mathbf{b}\right\}, \tag{1.2}$$

where $\mathbf{A} \in \mathbb{Z}^{n \times d}$ and $\mathbf{b} \in \mathbb{Z}^n$ such that the greatest common divisor of b_i and the entries in the ith row of $\mathbf{A}$ equals 1, for every $i \in \{1, \ldots, n\}$.[1] We define the **codenominator** r

[1] If P is a *lattice* polytope then we do not need to include b_i in this gcd condition.

of P to be the least common multiple of the nonzero entries of $\mathbf{b}$: $r := \operatorname{lcm}(\mathbf{b})$. As we assume that P is full dimensional, the codenominator is well-defined. Our nomenclature arises from determining r using duality, as follows. Let P° denote the relative interior of P, and let $(\mathbb{R}^d)^\vee$ be the dual vector space. If $\mathsf{P} \subseteq \mathbb{R}^d$ is a rational polytope such that $\mathbf{0} \in \mathsf{P}^\circ$, the **polar dual polytope** is $\mathsf{P}^\vee := \{\mathbf{x} \in (\mathbb{R}^d)^\vee : \langle \mathbf{x}, \mathbf{y} \rangle \geq -1 \text{ for all } \mathbf{y} \in \mathsf{P}\}$, and $r = \min\{q \in \mathbb{Z}_{>0} : q\,\mathsf{P}^\vee \text{ is a lattice polytope}\}$.

We will see in Section 2 that $\overline{\mathrm{r}}\mathrm{ehr}(\mathsf{P};\lambda)$ is fully determined by evaluations at rational numbers with denominator $2r$ (see Corollary 6 below for details); if $\mathbf{0} \in \mathsf{P}$ then we actually need to know only evaluations at rational numbers with denominator r. Thus we associate two generating series to the rational Ehrhart counting function, the **rational Ehrhart series** and the **refined rational Ehrhart series**, to a full-dimensional rational polytope P with codenominator r:

$$\mathrm{REhr}(\mathsf{P};t) := 1 + \sum_{n \in \mathbb{Z}_{>0}} \mathrm{rehr}\Big(\mathsf{P}; \frac{n}{r}\Big) t^{\frac{n}{r}}, \quad \mathrm{RREhr}(\mathsf{P};t) := 1 + \sum_{n \in \mathbb{Z}_{>0}} \mathrm{rehr}\Big(\mathsf{P}; \frac{n}{2r}\Big) t^{\frac{n}{2r}}.$$

Continuing our comment above, we typically study $\mathrm{REhr}(\mathsf{P};t)$ for polytopes such that $\mathbf{0} \in \mathsf{P}$, and $\mathrm{RREhr}(\mathsf{P};t)$ for polytopes such that $\mathbf{0} \notin \mathsf{P}$.

Section 2 also contains, as a first set of main results, structural theorems about these generating functions: rationality and its consequences for the quasipolynomial $\overline{\mathrm{r}}\mathrm{ehr}(\mathsf{P};\lambda)$ (Theorem 7 and Theorem 11), nonnegativity theorems (Corollary 10), connections to the h^*-polynomial in classical Ehrhart theory (Corollary 13), and combinatorial reciprocity theorems (Corollary 16 and Corollary 17).

One can find a precursor of sorts to our generating functions $\mathrm{REhr}(\mathsf{P};t)$ and $\mathrm{RREhr}(\mathsf{P};t)$ in work by Stapledon [16, 17], and in fact this work was our initial motivation to look for and study rational Ehrhart generating functions. We explain the connection of [17] to our work in Section 3.

A $(d+1)$-dimensional, pointed, rational cone $\mathsf{C} \subseteq \mathbb{R}^{d+1}$ is called **Gorenstein** if there exists a point $(p_0, \mathbf{p}) \in \mathsf{C} \cap \mathbb{Z}^{d+1}$ such that $\mathsf{C}^\circ \cap \mathbb{Z}^{d+1} = (p_0, \mathbf{p}) + \mathsf{C} \cap \mathbb{Z}^{d+1}$ (see, *e.g.*, [3, 9, 15]). The point $(p_0, \mathbf{p})$ is called the **Gorenstein point** of the cone. We define the **homogenization** $\hom(\mathsf{P}) \subset \mathbb{R}^{d+1}$ of a rational polytope $\mathsf{P} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\,\mathbf{x} \leq \mathbf{b}\}$ as

$$\hom(\mathsf{P}) := \operatorname{cone}(\{1\} \times \mathsf{P}) := \Big\{ (x_0, \mathbf{x}) \in \mathbb{R}^{d+1} : \ \mathbf{A}\mathbf{x} \leq x_0 \mathbf{b}\,,\ x_0 \geq 0 \Big\}.$$

For a cone $\mathsf{C} \subseteq \mathbb{R}^{d+1}$, the **dual cone** $\mathsf{C}^\vee \subseteq (\mathbb{R}^{d+1})^\vee$ is

$$\mathsf{C}^\vee := \Big\{ (y_0, \mathbf{y}) \in (\mathbb{R}^{d+1})^\vee : \ \langle (y_0, \mathbf{y}), (x_0, \mathbf{x}) \rangle \geq 0 \text{ for all } (x_0, \mathbf{x}) \in \mathsf{C} \Big\}.$$

A lattice polytope $\mathsf{P} \subset \mathbb{R}^d$ is **Gorenstein** if the homogenization $\hom(\mathsf{P})$ of P is Gorenstein; in the special case where the Gorenstein point of that cone is $(1, \mathbf{q})$, for some $\mathbf{q} \in \mathbb{Z}^d$, we call P **reflexive** [2, 12]. Reflexive polytopes can alternatively be characterized as those lattice polytopes (containing the origin) whose polar duals are also lattice polytopes, *i.e.*, they have codenominator 1. This definition has a natural extension to rational polytopes [11]. Gorenstein and reflexive polytopes (and their rational versions) play an important role in Ehrhart theory, as they have palindromic h^*-polynomials. In Section 4 we give the analogous result in rational Ehrhart theory *without* reference to the polar dual (Theorem 23). We will see that there are many more *rational* Gorenstein polytopes than among lattice polytopes; *e.g.*, any rational polytope containing the origin in its interior is rational Gorenstein (Corollary 24). We mention the recent notion of an l**-reflexive polytope** P ("reflexive of higher index") [13]. A lattice point $\mathbf{x} \in \mathbb{Z}^d$ is **primitive** if the gcd of its coordinates is equal to one. The l-reflexive polytopese are precisely the lattice polytopes of the form (1.2) with $\mathbf{b} = (l, l, \ldots, l)$ and primitive vertices; note that this means P has codenominator l and $\frac{1}{l}\mathsf{P}$ has denominator l.

2 Rational Ehrhart Dilations

We assume throughout this article that all polytopes are full dimensional, and call a d-dimensional polytope in $\mathbb{R}^d$ a d**-polytope**. We note that, consequently, the leading coefficient of $\mathrm{ehr}(\mathsf{P}; n)$ is constant (namely, the volume of P), and thus the rational generating function $\mathrm{Ehr}(\mathsf{P}; t)$ has a unique pole of order $d+1$ at $t = 1$. So we could write the rational generating function $\mathrm{Ehr}(\mathsf{P}; t)$ with denominator $(1-t)(1-t^k)^d$; in other words, $\mathrm{h}^*(\mathsf{P}; t)$ always has a factor $(1 + t + \cdots + t^{k-1})$. For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ (resp. $\lceil x \rceil$) denote the largest integer $\le x$ (resp. the smallest integer $\ge x$), and $\{x\} = x - \lfloor x \rfloor$.

Example 1. We feature the following line segments as running examples. First, we compute the real Ehrhart counting function.

• $\mathsf{P}_2 := \left[0, \frac{2}{3}\right]$, codenominator $r = 2$

$$\overline{\mathrm{r}}\mathrm{ehr}(\mathsf{P}_2; \lambda) \;=\; \left\lfloor \tfrac{2}{3}\lambda \right\rfloor + 1 \;=\; \tfrac{2}{3}n + 1 \quad \text{if } n \le \lambda < n + \tfrac{3}{2}, \quad \text{for some } n \in \tfrac{3}{2}\mathbb{Z}_{>0}\,.$$

• $\mathsf{P}_3 := [1, 2]$, codenominator $r = 2$

$$\overline{\mathrm{r}}\mathrm{ehr}(\mathsf{P}_3; \lambda) = \lfloor 2\lambda \rfloor - \lceil \lambda \rceil + 1 = \begin{cases} n+1 & \text{if } \lambda = n \qquad\qquad \text{for some } n \in \mathbb{Z}_{>0}\,, \\ n & \text{if } n < \lambda < n + \frac{1}{2} \qquad \text{for some } n \in \mathbb{Z}_{>0}\,, \\ n+1 & \text{if } n + \frac{1}{2} \le \lambda < n+1 \quad \text{for some } n \in \mathbb{Z}_{>0}\,. \end{cases}$$

The real Ehrhart function $\overline{\mathrm{r}}\mathrm{ehr}(\mathsf{P}_3; \lambda)$ is not monotone. For example, $\overline{\mathrm{r}}\mathrm{ehr}(\mathsf{P}_3; 0) = 1$, $\overline{\mathrm{r}}\mathrm{ehr}(\mathsf{P}_3; \frac{1}{4}) = 0$, $\overline{\mathrm{r}}\mathrm{ehr}(\mathsf{P}_3; \frac{1}{2}) = 1$. We can see in these examples (and will prove below in general terms) that $\overline{\mathrm{r}}\mathrm{ehr}(\mathsf{P}; \lambda)$ is a quasipolynomial in the real variable λ.

Remark 2. If P is a lattice polytope, then the denominator of $\frac{1}{r}\mathsf{P}$ divides r. On the other hand, the denominator of $\frac{1}{r}\mathsf{P}$ need not equal r, *e.g.*, for $\mathsf{P}_4 := 2\mathsf{P}_3 = [2, 4]$.

Remark 3. If $\frac{1}{r}\mathsf{P}$ is a lattice polytope, its Ehrhart polynomial is invariant under lattice translations. Unfortunately, this does not clearly translate to invariance of $\operatorname{rehr}(\mathsf{P};\lambda)$. Consider the line segment $[-1,1]$ and its translation $\mathsf{P}_4 = [2,4]$. For any $\lambda \in (0,\frac{1}{4})$, $\operatorname{rehr}([-1,1],\lambda) = 1$ and $\operatorname{rehr}(\mathsf{P}_4,\lambda) = 0$. This observation raises two related questions: Is there an example of a polytope and a translate with the same codenominator? We expect not in dimension one. Given a rational polytope P, for which r and $\widetilde{\mathsf{P}}$ could $\mathsf{P} = \frac{1}{r}\widetilde{\mathsf{P}}$?

Lemma 4. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be rational d-polytope. If $\mathbf{0} \in \mathsf{P}$, then $\operatorname{rehr}(\lambda)$ is monotone for $\lambda \in \mathbb{Q}_{\geq 0}$.*

Proposition 5. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a rational d-polytope with codenominator r.*

(i) *The number of lattice points in $\lambda\mathsf{P}$ is constant for $\lambda \in (\frac{n}{r},\frac{n+1}{r})$, $n \in \mathbb{Z}_{\geq 0}$.*

(ii) *If $\mathbf{0} \in \mathsf{P}$, then the number of lattice points in $\lambda\mathsf{P}$ is constant for $\lambda \in [\frac{n}{r},\frac{n+1}{r})$, $n \in \mathbb{Z}_{\geq 0}$.*

It follows that we can compute the real Ehrhart function $\overline{\operatorname{r}}\operatorname{ehr}$ from the rational Ehrhart function:

Corollary 6. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a rational d-polytope with codenominator r. Then*

$$\overline{\operatorname{r}}\operatorname{ehr}(\mathsf{P};\lambda) \;=\; \begin{cases} \operatorname{rehr}(\mathsf{P};\lambda) & \text{if } \lambda \in \frac{1}{r}\mathbb{Z}_{\geq 0}\,, \\ \operatorname{rehr}(\mathsf{P};\lfloor\lambda\rceil) & \text{if } \lambda \notin \frac{1}{r}\mathbb{Z}_{\geq 0}\,, \end{cases}$$

where

$$\lfloor\lambda\rceil := \frac{2j+1}{2r} \quad \text{for} \quad \left|\lambda - \frac{2j+1}{2r}\right| < \frac{1}{2r} \quad \text{and} \quad j \in \mathbb{Z}\,.$$

In words, $\lfloor\lambda\rceil$ is the element in $\frac{1}{2r}\mathbb{Z}$ with odd numerator that has the smallest Euclidean distance to λ on the real line. Furthermore, if $\mathbf{0} \in \mathsf{P}$ then

$$\overline{\operatorname{r}}\operatorname{ehr}(\mathsf{P};\lambda) \;=\; \operatorname{rehr}\left(\mathsf{P};\frac{\lfloor r\lambda\rfloor}{r}\right).$$

Theorem 7. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a rational d-polytope with codenominator r, and let $m \in \mathbb{Z}_{>0}$ such that $\frac{m}{r}\mathsf{P}$ is a lattice polytope. Then*

$$\operatorname{REhr}(\mathsf{P};t) \;:=\; \sum_{n\in\mathbb{Z}_{\geq 0}} \operatorname{rehr}\left(\mathsf{P};\frac{n}{r}\right)t^{\frac{n}{r}} \;=\; \frac{\operatorname{rh}^*(\mathsf{P};t)}{\left(1-t^{\frac{m}{r}}\right)^{d+1}}$$

where $\operatorname{rh}^*(\mathsf{P};t)$ is a polynomial in $\mathbb{Z}[t^{\frac{1}{r}}]$ with nonnegative integral coefficients. Consequently, $\operatorname{rehr}(\mathsf{P};\lambda)$ and $\overline{\operatorname{r}}\operatorname{ehr}(\mathsf{P};\lambda)$ are quasipolynomials.

Our implicit definition of $\mathrm{rh}^*(\mathsf{P};t)$ depends on m. We will sometimes use the notation $\mathrm{rh}^*_m(\mathsf{P};t)$ to make this dependency explicit. Naturally, one often tries to choose m minimal, which gives a canonical definition of $\mathrm{rh}^*(\mathsf{P};t)$, but sometimes it pays to be flexible. By usual generatingfunctionology the degree of $\mathrm{rh}^*_m(\mathsf{P};t)$ is less than or equal to $m(d+1)-1$ as a polynomial in $t^{\frac{1}{r}}$.

Corollary 8. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a rational d-polytope with codenominator r, and let $m \in \mathbb{Z}_{>0}$ such that $\frac{m}{r}\mathsf{P}$ is a lattice polytope. Then the period of the quasipolynomial $\mathrm{rehr}(\mathsf{P};\lambda)$ divides $\frac{m}{r}$, i.e., this period is of the form $\frac{j}{r}$ with $j \mid m$.*

Corollary 9. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a rational d-polytope with codenominator r, and let $m \in \mathbb{Z}_{>0}$ such that $\frac{m}{r}\mathsf{P}$ is a lattice polytope. Then the period of the quasipolynomial $\mathrm{ehr}(\mathsf{P};\lambda)$ divides $\frac{m}{\gcd(m,r)}$.*

Corollary 10. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a lattice d-polytope with codenominator r. Then*

$$\mathrm{REhr}(\mathsf{P};t) \;=\; \frac{\mathrm{rh}^*_r(\mathsf{P};t)}{(1-t)^{d+1}}$$

where $\mathrm{rh}^*_r(\mathsf{P};t)$ is a polynomial in $\mathbb{Z}[t^{\frac{1}{r}}]$ with nonnegative coefficients.

For polytopes that do not contain the origin, the following variant of Theorem 7 is useful. Many of the following assertions come in two versions, one for REhr and one for the refined rational Ehrhart series RREhr defined below.

Theorem 11. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a rational d-polytope with codenominator r, and let $m \in \mathbb{Z}_{>0}$ such that $\frac{m}{2r}\mathsf{P}$ is a lattice polytope. Then*

$$\mathrm{RREhr}(\mathsf{P};t) \;:=\; 1 + \sum_{n \in \mathbb{Z}_{>0}} \mathrm{rehr}\Big(\mathsf{P};\frac{n}{2r}\Big)\, t^{\frac{n}{2r}} \;=\; \frac{\mathrm{rrh}^*(\mathsf{P};t)}{\Big(1-t^{\frac{m}{2r}}\Big)^{d+1}}$$

where $\mathrm{rrh}^*(\mathsf{P};t)$ is a polynomial in $\mathbb{Z}[t^{\frac{1}{2r}}]$ with nonnegative coefficients.

Corollary 12. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a lattice d-polytope with codenominator r. The real and rational Ehrhart functions, $\overline{\mathrm{r}}\mathrm{ehr}(\mathsf{P},\lambda)$ and $\mathrm{rehr}(\mathsf{P},\lambda)$, are given by quasipolynomials of period 1.*

Corollary 13. *If $\frac{m}{r}$ (resp. $\frac{m}{2r}$) in Theorem 7 (resp. Theorem 11) is integral we can retrieve the h^*-polynomial from the rh^*-polynomial (resp. rrh^*-polynomial) by applying the operator Int that extracts from a polynomial in $\mathbb{Z}[t^{\frac{1}{r}}]$ the terms with integer powers of t: $\mathrm{h}^*(\mathsf{P};t) = \mathrm{Int}(\mathrm{rh}^*(\mathsf{P};t))$ (resp. $\mathrm{h}^*(\mathsf{P};t) = \mathrm{Int}(\mathrm{rrh}^*(\mathsf{P};t))$).*

Example 14. Here are the (refined) rational Ehrhart series of the running examples. Recall that the rational Ehrhart series of P in the variable t can be computed as the Ehrhart series of $\frac{1}{r}\mathsf{P}$ in the variable $t^{\frac{1}{r}}$ (resp. the refined rational Ehrhart as the Ehrhart series of $\frac{1}{2r}\mathsf{P}$ in the variable $t^{\frac{1}{2r}}$).

• $\mathsf{P}_2 := [0, \frac{2}{3}]$, $r = 2$, $m = 3$

$$\mathrm{REhr}(\mathsf{P}_2; t) = \frac{1}{\left(1 - t^{\frac{1}{2}}\right)\left(1 - t^{\frac{3}{2}}\right)} = \frac{1 + t^{\frac{1}{2}} + t}{\left(1 - t^{\frac{3}{2}}\right)^2}$$

• $\mathsf{P}_3 := [1, 2]$, $r = 2$. $\frac{1}{4}\mathsf{P}_3 = [\frac{1}{4}, \frac{1}{2}]$ and $m = 4$, so $\frac{m}{2r} = 1$. See Figure 1.

$$\mathrm{RREhr}(\mathsf{P}_3; t) = \frac{1 + t^{\frac{1}{2}} + t^{\frac{3}{4}} + t^{\frac{5}{4}}}{(1 - t)^2}.$$

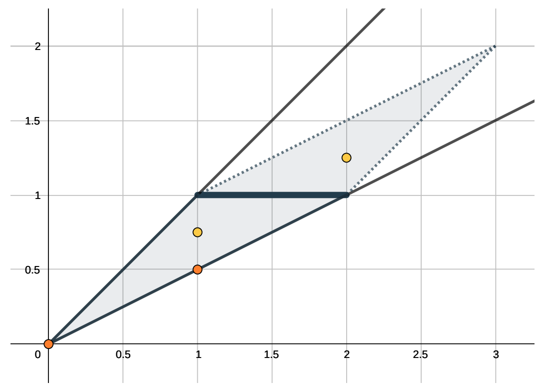

Figure 1: The cone $\hom(\mathsf{P}_3)$ over $\mathsf{P}_3 = [1, 2]$. The lattice points in the fundamental parallelepiped with respect to the lattice $\frac{1}{4}\mathbb{Z} \times \mathbb{Z}$ are $(0, 0)$, $(\frac{1}{2}, 1)$, $(\frac{3}{4}, 1)$, $(\frac{5}{4}, 2)$.

Applying the operater Int yields the Ehrhart series, $\mathrm{Ehr}(\mathsf{P}_3; t) = \frac{1}{(1-t)^2}$, as described in Corollary 13.

Remark 15. For a rational d-polytope $\mathsf{P} \subseteq \mathbb{R}^d$ with denominator k, the sum of the coefficients of the h^*-polynomial equals $d!k^{d+1}\,\mathrm{vol}(\mathsf{P})$ (see, *e.g.*, [6, Example 3.34], [7, Section 4.5]). This implies the sum of the rh^*_m-coefficients equals $d!m^{d+1}\,\mathrm{vol}(\frac{1}{r}\mathsf{P}) = d!m^{d+1}r^{-d}\,\mathrm{vol}(\mathsf{P})$.

We recover the reciprocity result for the rational Ehrhart function of rational polytopes proved by Linke [14, Corollary 1.5].

Corollary 16. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a rational d-polytope. Then $(-1)^d\,\overline{\mathrm{r}}\mathrm{ehr}(\mathsf{P}; -\lambda)$ equals the number of interior lattice points in $\lambda\mathsf{P}$, for any $\lambda > 0$.*

Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a rational d-polytope and set $\mathrm{rehr}(\mathsf{P}^\circ; \lambda) := |\lambda\mathsf{P}^\circ \cap \mathbb{Z}^d|$. We define the (refined) rational Ehrhart series of the interior of a polytope as follows:

$$\mathrm{REhr}(\mathsf{P}^\circ; t) := \sum_{\lambda \in \frac{1}{r}\mathbb{Z}_{>0}} \mathrm{rehr}(\mathsf{P}^\circ; \lambda)t^\lambda, \qquad \mathrm{RREhr}(\mathsf{P}^\circ; t) := \sum_{\lambda \in \frac{1}{2r}\mathbb{Z}_{>0}} \mathrm{rehr}(\mathsf{P}^\circ; \lambda)t^\lambda,$$

where r as usual denotes the codenominator of P.

Corollary 17. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a rational d-polytope. The (refined) rational Ehrhart series of the open polytope P° have the rational expressions*

$$\mathrm{REhr}(\mathsf{P}^\circ;t) = \frac{\mathrm{rh}^*_m(\mathsf{P}^\circ;t)}{\left(1-t^{\frac{m}{r}}\right)^{d+1}} \quad \text{and} \quad \mathrm{RREhr}(\mathsf{P}^\circ;t) = \frac{\mathrm{rrh}^*_m(\mathsf{P}^\circ;t)}{\left(1-t^{\frac{m}{2r}}\right)^{d+1}},$$

where $\mathrm{rh}^*_m(\mathsf{P}^\circ;t)$ and $\mathrm{rrh}^*_m(\mathsf{P}^\circ;t)$ are polynomials in $\mathbb{Z}[t^{\frac{1}{r}}]$ and $\mathbb{Z}[t^{\frac{1}{2r}}]$, respectively. The (refined) rational Ehrhart series fulfill the reciprocity relations

$$\mathrm{REhr}(\mathsf{P}^\circ;t) = (-1)^{d+1}\,\mathrm{REhr}\left(\mathsf{P};\frac{1}{t}\right) \quad \text{and} \quad \mathrm{RREhr}(\mathsf{P}^\circ;t) = (-1)^{d+1}\,\mathrm{RREhr}\left(\mathsf{P};\frac{1}{t}\right).$$

The rh^*- and rrh^*-polynomials of the polytope P and its interior P° are related by

$$\mathrm{rh}^*_m(\mathsf{P}^\circ;t) = \left(t^{\frac{m}{r}}\right)^{d+1}\mathrm{rh}^*_m\left(\mathsf{P};\frac{1}{t}\right) \quad \text{and} \quad \mathrm{rrh}^*_m(\mathsf{P}^\circ;t) = \left(t^{\frac{m}{2r}}\right)^{d+1}\mathrm{rrh}^*_m\left(\mathsf{P};\frac{1}{t}\right).$$

3 Stapledon

We recall the setup from [17]. Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a lattice d-polytope with codenominator r and $\mathbf{0} \in \mathsf{P}$. Let $\partial_{\neq 0}(\mathsf{P})$ denote the union of facets of P that do not contain the origin. In order to study all rational dilates of the boundary of P, Stapledon introduces the generating function

$$\mathrm{WEhr}(\mathsf{P};t) := 1 + \sum_{\lambda\in\mathbb{Q}_{>0}} \left|\partial_{\neq 0}(\lambda\mathsf{P}) \cap \mathbb{Z}^d\right| t^\lambda = \frac{\widetilde{\mathrm{h}}(\mathsf{P};t)}{(1-t)^d},$$

where $\widetilde{\mathrm{h}}(\mathsf{P};t)$ is a polynomial in $\mathbb{Z}[t^{\frac{1}{r}}]$ with fractional exponents. The generating function WEhr is closely related to the (rational) Ehrhart series: for any $\omega \in \mathbb{Q}_{>0}$, the truncated sum $1 + \sum_{\lambda\in\mathbb{Q}>0}^{\omega} |\partial_{\neq 0}(\lambda\mathsf{P}) \cap \mathbb{Z}^d|$ equals the number of lattice points in $\omega\mathsf{P}$. Proposition 5 allows us to discretize this sum:

Corollary 18. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a lattice d-polytope with codenominator r and $\mathbf{0} \in \mathsf{P}$. The number of lattice points in $\omega\mathsf{P}$ equals $1 + \sum_{\lambda\in\frac{1}{r}\mathbb{Z}_{>0},\,\lambda<\omega} |\partial_{\neq 0}(\lambda\mathsf{P}) \cap \mathbb{Z}^d|$.*

Similarly, $\widetilde{\mathrm{h}}(\mathsf{P};t)$ is related to $\mathrm{h}^*(\frac{1}{r}\mathsf{P};t^{\frac{1}{r}})$ and to $\mathrm{rh}^*_m(\mathsf{P};t)$, as we show in Lemma 19. Recall that we use $\mathrm{rh}^*_m(\mathsf{P};t)$ to keep track of the denominator of $\mathrm{REhr}(\mathsf{P};t) = \frac{\mathrm{rh}^*_m(\mathsf{P};t)}{(1-t^{\frac{m}{r}})^{d+1}}$.

Lemma 19. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a lattice d-polytope with codenominator r such that $\mathbf{0} \in \mathsf{P}$. Let k be the denominator of $\frac{1}{r}\mathsf{P}$. Then*

$$\mathrm{rh}^*_k(\mathsf{P};t) = \mathrm{h}^*\left(\frac{1}{r}\mathsf{P};t^{\frac{1}{r}}\right) = \frac{\left(1-t^{\frac{k}{r}}\right)^{d+1}}{\left(1-t^{\frac{1}{r}}\right)(1-t)^d}\,\widetilde{\mathrm{h}}(\mathsf{P};t).$$

Remark 20. Lemma 19 corrects [17, Remark 3], which was missing the factor between $\mathrm{h}^*(\frac{1}{r}\mathsf{P}; t^{\frac{1}{r}})$ and $\widetilde{\mathrm{h}}(\mathsf{P}; t)$.

Remark 21. In [16, Equation (14)] and [17, Equation (6)], Stapledon shows that we have $\mathrm{h}^*(\mathsf{P}; t) = \Psi(\widetilde{\mathrm{h}}(\mathsf{P}; t))$, where $\Psi\colon \bigcup_{r\in\mathbb{Z}_{>0}} \mathbb{R}[t^{\frac{1}{r}}] \to \mathbb{R}[t]$ is defined by $\Psi(t^\lambda) = t^{\lceil\lambda\rceil}$. In the case of a lattice polytope with $\frac{m}{r} \in \mathbb{Z}$ we give a different construction to recover the h^*-polynomial from the rrh^*- and rh^*-polynomial by applying the operator Int (see Corollary 13). Lemma 19 shows that, after a bit of computation, these two constructions are equivalent.

Remark 22. For a lattice d-polytope $\mathsf{P} \subseteq \mathbb{R}^d$ with codenominator r, $\mathbf{0} \in \mathsf{P}$, and denominator of $\frac{1}{2r}\mathsf{P}$ equal k, we can relate $\mathrm{rrh}^*(\mathsf{P}; t)$ and $\mathrm{h}^*(\frac{1}{2r}\mathsf{P}; t^{\frac{1}{2r}})$ in a similar way. We again write $\mathrm{rrh}^*_k(\mathsf{P}; t)$ to emphasize that it is the numerator of $\frac{\mathrm{rrh}^*_k(\mathsf{P};t)}{(1-t^{\frac{k}{2r}})^{d+1}}$. Then

$$\mathrm{rrh}^*_k(\mathsf{P}; t) = \mathrm{h}^*\left(\frac{1}{2r}\mathsf{P}; t^{\frac{1}{2r}}\right) = \frac{\left(1 - t^{\frac{k}{2r}}\right)^{d+1}}{\left(1 - t^{\frac{1}{2r}}\right)(1-t)^d}\,\widetilde{\mathrm{h}}(\mathsf{P}; t)\,.$$

4 Gorenstein Musings

Our main goal in this section is to extend the notion of Gorenstein polytopes to the rational case. A rational d-polytope $\mathsf{P} \subseteq \mathbb{R}^d$ is **γ-rational Gorenstein** if $\hom(\frac{1}{\gamma}\mathsf{P})$ is a Gorenstein cone. In this paper we explore this definition for parameters $\gamma = r$ and $\gamma = 2r$, other parameters are still to be investigated. The archetypal r-rational Gorenstein polytope is a rational polytope that contains the origin in its interior, see Corollary 24. The definition of γ-rational Gorenstein does not require that the origin is contained in the polytope, hence, it does not require the existence of a polar dual. A lattice polytope P is 1-rational Gorenstein if and only if it is a Gorenstein polytope in the classical sense.

Analogous to the lattice case, the following theorem shows that a polytope containing the origin is r-rational Gorenstein if and only if it has a palindromic rh^*-polynomial. Let $\mathsf{P} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\,\mathbf{x} \le \mathbf{b}\}$ be a rational d-polytope, as in (1.2). We may assume that there is an index $0 \le\le n$ such that $b_j = 0$ for $j = 1, \dots, i$ and $b_j \ne 0$ for $j = i+1, \dots, n$; thus we can write P as follows:

$$\mathsf{P} = \left\{ \begin{matrix} \mathbf{x} \in \mathbb{R}^d : \langle \mathbf{a}_j, \mathbf{x}\rangle \le 0 & \text{for } j = 1, \dots, i \\ \langle \mathbf{a}_j, \mathbf{x}\rangle \le b_j & \text{for } j = i+1, \dots, n \end{matrix} \right\}, \tag{4.1}$$

where $\mathbf{a}_j$ are the rows of $\mathbf{A}$.

Theorem 23. *Let $\mathsf{P} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\,\mathbf{x} \le \mathbf{b}\}$ be a rational d-polytope with codenominator r and $\mathbf{0} \in \mathsf{P}$, as in (1.2) and (4.1). Then the following are equivalent for $g, m \in \mathbb{Z}_{\ge 1}$ and $\frac{m}{r}\mathsf{P}$ a lattice polytope:*

(i) P *is r-rational Gorenstein with Gorenstein point* $(g,\mathbf{y}) \in \hom(\frac{1}{r}\mathsf{P})$.

(ii) *There exists a (necessarily unique) integer solution* $(g,\mathbf{y})$ *to*

$$\begin{aligned} -\langle \mathbf{a}_j, \mathbf{y}\rangle &= 1 \quad \text{for } j = 1,\ldots,i \\ b_j\, g - r\,\langle \mathbf{a}_j, \mathbf{y}\rangle &= b_j \quad \text{for } j = i+1,\ldots,n\,. \end{aligned}$$

(iii) $\mathrm{rh}^*(\mathsf{P};t)$ *is palindromic:*

$$t^{(d+1)\frac{m}{r} - \frac{g}{r}}\ \mathrm{rh}^*_m\left(\mathsf{P};\frac{1}{t}\right) \ = \ \mathrm{rh}^*_m(\mathsf{P};t)\,.$$

(iv) $(-1)^{d+1} t^{\frac{g}{r}}\,\mathrm{REhr}(\mathsf{P};t) = \mathrm{REhr}(\mathsf{P};\frac{1}{t})$.

(v) $\mathrm{rehr}(\mathsf{P};\frac{n}{r}) = \mathrm{rehr}(\mathsf{P}^\circ;\frac{n+g}{r})$ *for all* $n \in \mathbb{Z}_{\ge 0}$.

(vi) $\hom(\frac{1}{r}\mathsf{P})^\vee$ *is the cone over a lattice polytope, i.e., there exists a lattice point* $(g,\mathbf{y})$ *in* $\hom(\frac{1}{r}\mathsf{P})^\circ \cap \mathbb{Z}^{d+1}$ *such that for every primitive ray generator* $(v_0,\mathbf{v})$ *of* $\hom(\frac{1}{r}\mathsf{P})^\vee$

$$\langle (g,\mathbf{y})\,,(v_0,\mathbf{v})\rangle = 1\,.$$

The equivalence of (i) and (vi) is well known (see, *e.g.*, [4, Definition 1.8] or [8, Exercises 2.13, 2.14]). As usual there is a version of Theorem 23 for the refined rational Ehrhart series and the rrh^*-polynomial. Here, the polytopes under consideration are not required to contain the origin. Except for minor differences, the statement and proof are the same as that of Theorem 23 so we omit them, see [5, Theorem 27].

Corollary 24. *Let* $\mathsf{P} \subseteq \mathbb{R}^d$ *be a rational d-polytope with codenominator r. If* $\mathbf{0} \in \mathsf{P}^\circ$*, then* P *is r-rational Gorenstein with Gorenstein point* $(1,0,\ldots,0)$ *and* $\mathrm{rh}^*(\mathsf{P};t)$ *is palindromic.*

Example 25. We check the Gorenstein criterion for the running example $\mathsf{P}_2 := \left[0,\frac{2}{3}\right]$, where $r = 2$ and $m = 3$. Then $\mathrm{rh}^*_3(\mathsf{P}_2;t) = 1 + t^{\frac{1}{2}} + t$, which is palindromic. Therefore, P_2 is 2-rational Gorenstein with Gorenstein point $(g,\mathbf{y}) = (4,1) \in \hom(\frac{1}{2}\mathsf{P}_2)$.

Example 26. The *Haasenlieblingsdreieck* $\Delta := \mathrm{conv}\{(0,0),(2,0),(0,2)\}$ is not Gorenstein in the classic (integral) setting, but it is 2-rational Gorenstein: we compute

$$\mathrm{REhr}(\mathsf{P},t) \ = \ \frac{1}{\left(1-t^{\frac{1}{2}}\right)^3} \ = \ \frac{1+3t^{\frac{1}{2}}+3t+t^{\frac{3}{2}}}{(1-t)^3}\,.$$

Example 27. The triangle $\nabla := \mathrm{conv}\{(0,0),(0,1),(3,1)\}$ has codenominator 1. It is not 1-rational Gorenstein as $|\nabla^\circ \cap \mathbb{Z}^2| = 0$ and $|(2\nabla)^\circ \cap \mathbb{Z}^2| = 2$. One can easily check that the points $(1,1)$ and $(2,1)$, which appear simultaneously as interior lattice points in $\hom(\nabla)$, do not satisfy the first type of equations in Theorem 23 (ii). Note that these equations are independent of height in the cone.

Corollary 28. (i) *If* $\mathbf{0} \in \mathsf{P}^\circ$*, then* P *is also* $2r$*-rational Gorenstein with the same Gorenstein point* $(1, 0 \ldots, 0)$ *(see Corollary 24).*

(ii) *If* $\mathbf{0} \in \mathsf{P}$ *and* P *is* r*-rational Gorenstein, then* P *is also* $2r$*-rational Gorenstein.*

(iii) *If* P *is* $2r$*-rational Gorenstein and the first coordinate* g *of the Gorenstein point* $(g, \mathbf{y})$ *is even, then* P *is also* r*-rational Gorenstein.*

This could be generalized to ℓr-rational Gorenstein polytopes for $\ell \in \mathbb{Z}_{>0}$. However it is not clear that computationally this would provide any new insights to the (rational) Ehrhart theory of the polytopes.

Example 29. We check the Gorenstein criterion for $\mathsf{P}_3 := [1, 2]$, where $r = 2$, $m = 4$, and $\mathrm{rrh}^*_4(\mathsf{P}_3; t) = 1 + t^{\frac{2}{4}} + t^{\frac{3}{4}} + t^{\frac{5}{4}}$. The polynomial $\mathrm{rrh}^*_4(\mathsf{P}_3; t)$ is palindromic and therefore P_3 is 4-rational Gorenstein. The Gorenstein point is $(g, \mathbf{y}) = (3, 1)$ and is highlighted in orange in Figure 2. The other lattice points $\hom(\frac{1}{4}\mathsf{P}_3)^\circ \cap \mathbb{Z}^2$ are marked in black. Observe that $(3, 1) + \hom(\frac{1}{4}\mathsf{P}_3) \cap \mathbb{Z}^2 = \hom(\frac{1}{4}\mathsf{P}_3)^\circ \cap \mathbb{Z}^2$.

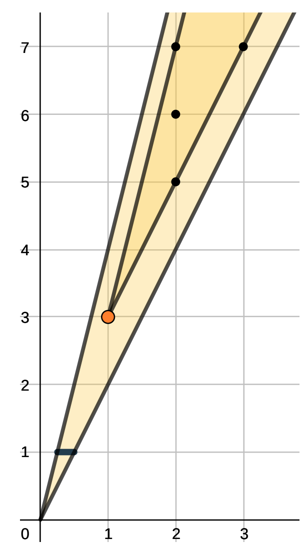

Figure 2: $\hom(\frac{1}{4}\mathsf{P}_3)$.

Example 30 (A polytope that is not $2r$-rational Gorenstein). Let $\mathsf{P}_5 = [1, 4]$. Then $r = 4$ and $2r = 8$, so $\frac{1}{2r}\mathsf{P}_5 = [\frac{1}{8}, \frac{1}{2}]$. The first lattice point in the interior of the cone $\hom(\frac{1}{8}\mathsf{P}_5)$ is $(g, \mathbf{y}) = (3, 1)$. However, $(3, 1)$ does not satisfy Condition (ii) from Theorem 23; it is at lattice distance 5 from one of the facets of $\hom(\frac{1}{8}\mathsf{P}_5)$.

Remark 31. The **codegree** of a lattice polytope is defined as $\dim(\mathsf{P}) + 1 - \deg(h^*(t))$. Analogously, in the rational case, we define the **rational codegree of** $\mathrm{rh}^*_m(\mathsf{P}; t)$ to be

$$\frac{m}{r}(\dim(\mathsf{P}) + 1) - \deg(\mathrm{rh}^*_m(\mathsf{P}; t)),$$

where the degree of $\mathrm{rh}^*_m(\mathsf{P}; t)$ is its (possibly fractional) degree as a polynomial in t. Likewise, the **rational codegree of** $\mathrm{rrh}^*_m(\mathsf{P}; t)$ is defined as $\frac{m}{2r}(\dim(\mathsf{P}) + 1) - \deg(\mathrm{rrh}^*_m(\mathsf{P}; t))$. As in the integral case, it holds that the rational codegree of $\mathrm{rh}^*(\mathsf{P}; t)$ is the smallest integral dilate of $\frac{1}{r}\mathsf{P}$ containing interior lattice points.

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References

[1] V. Baldoni, N. Berline, M. Köppe, and M. Vergne. "Intermediate sums on polyhedra: computation and real Ehrhart theory". *Mathematika* **59**.1 (2013), pp. 1–22.

[2] V. V. Batyrev. "Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties". *J. Algebraic Geom.* **3**.3 (1994), pp. 493–535.

[3] V. V. Batyrev and L. A. Borisov. "Dual cones and mirror symmetry for generalized Calabi–Yau manifolds". *Mirror Symmetry, II*. Vol. 1. AMS/IP Stud. Adv. Math. Amer. Math. Soc., Providence, RI, 1997, pp. 71–86.

[4] V. V. Batyrev and B. Nill. "Combinatorial aspects of mirror symmetry". *Integer Points in Polyhedra — Geometry, Number Theory, Representation Theory, Algebra, Optimization, Statistics*. Vol. 452. Contemp. Math. Amer. Math. Soc., Providence, RI, 2008, pp. 35–66.

[5] M. Beck, S. Elia, and S. Rehberg. "Rational Ehrhart theory". 2021. arXiv:2110.10204.

[6] M. Beck and S. Robins. *Computing the Continuous Discretely: Integer-Point Enumeration in Polyhedra*. Second Edition. Undergraduate Texts in Mathematics. Springer, New York, 2015, pp. xx+285. Link.

[7] M. Beck and R. Sanyal. *Combinatorial Reciprocity Theorems: An Invitation to Enumerative Geometric Combinatorics*. Graduate Studies in Mathematics 195. Amer. Math. Soc., Providence, RI, 2018.

[8] W. Bruns and J. Gubeladze. *Polytopes, rings, and K-theory*. Springer Monographs in Mathematics. Dordrecht: Springer, 2009, pp. xiv+461.

[9] W. Bruns and T. Römer. "h-vectors of Gorenstein polytopes". *J. Combin. Theory Ser. A* **114**.1 (2007), pp. 65–76.

[10] E. Ehrhart. "Sur les polyèdres rationnels homothétiques à n dimensions". *C. R. Acad. Sci. Paris* **254** (1962), pp. 616–618.

[11] M. H. J. Fiset and A. M. Kasprzyk. "A note on palindromic δ-vectors for certain rational polytopes". *Electron. J. Combin.* **15**.1 (2008), Note 18, 4 pp.

[12] T. Hibi. "Dual polytopes of rational convex polytopes". *Combinatorica* **12**.2 (1992), pp. 237–240.

[13] A. M. Kasprzyk and B. Nill. "Reflexive polytopes of higher index and the number 12". *Electron. J. Combin.* **19**.3 (2012), Paper 9, 18 pp.

[14] E. Linke. "Rational Ehrhart quasi-polynomials". *J. Combin. Theory Ser. A* **118**.7 (2011), pp. 1966–1978.

[15] R. P. Stanley. "Hilbert functions of graded algebras". *Adv. Math.* **28**.1 (1978), pp. 57–83.

[16] A. Stapledon. "Weighted Ehrhart theory and orbifold cohomology". *Adv. Math.* **219**.1 (2008), pp. 63–88.

[17] A. Stapledon. "Counting lattice points in free sums of polytopes". *J. Combin. Theory Ser. A* **151** (2017), pp. 51–60.