Grothendieck-to-Lascoux Expansions

Mark Shimozono[*1] and Tianyi Yu[†2]

[1]*Department of Mathematics, Virginia Tech*
[2]*Department of Mathematics, University of California, San Diego*

Abstract. We establish the conjecture of Reiner and Yong for an explicit combinatorial formula for the expansion of a Grothendieck polynomial into the basis of Lascoux polynomials. This expansion is a subtle refinement of its symmetric function version due to Buch, Kresch, Shimozono, Tamvakis, and Yong, which gives the expansion of stable Grothendieck polynomials indexed by permutations into Grassmannian stable Grothendieck polynomials. Our expansion is the *K*-theoretic analogue of a Schubert polynomial into Demazure characters, whose symmetric analogue is the expansion of a Stanley symmetric function into Schur functions. We extend our expansions to flagged Grothendieck polynomials.

Keywords: Grothendieck polynomial, Demazure character, Lascoux polynomial, set-valued tableau, Hecke insertion

1 Introduction

The Grothendieck polynomials $\mathfrak{G}_w$ of Lascoux and Schützenberger [16] are explicit polynomial representatives of the *K*-classes of structure sheaves of Schubert varieties in flag varieties. Reiner and Yong [23] conjectured an explicit combinatorial expansion of Grothendieck polynomials into the basis of Lascoux polynomials $\mathfrak{L}_\alpha$ [15]. Our first main theorem (Theorem 4.1) rediscovers a combinatorial formula for the Lascoux polynomials, which is implicit in [4]. This is used to prove our second main theorem (Theorem 7.3) which establishes the Reiner–Yong conjecture.

2 Four expansions

The Grothendieck-to-Lascoux expansion fits into a family of four related expansions. The polynomials to be expanded are the cohomological and K-theoretic Schubert bases given by the Schubert polynomials $\mathfrak{S}_w$ and the Grothendieck polynomials $\mathfrak{G}_w$ respectively, and their symmetrized or stable versions, known as the Stanley symmetric functions F_w

---

[*]mshimo@math.vt.edu
[†]tiy059@ucsd.edu

and Grothendieck symmetric functions (also known as stable Grothendieck polynomials) G_w. These are respectively expanded into type A Demazure characters (also called key polynomials) κ_α, Lascoux polynomials $\mathfrak{L}_\alpha$, Schur functions s_λ, and Grassmannian Grothendieck symmetric functions G_λ.

$$
\begin{array}{ccc}
\mathfrak{S}_w \xrightarrow{\text{symmetrize}} F_w & \qquad & \mathfrak{G}_w \xrightarrow{\text{symmetrize}} G_w \\
{\scriptstyle(a)}\downarrow{\text{expand}} \quad \text{expand}\downarrow{\scriptstyle(b)} & & {\scriptstyle(c)}\downarrow{\text{expand}} \quad \text{expand}\downarrow{\scriptstyle(d)} \\
\kappa_\alpha \xrightarrow[\text{symmetrize}]{} s_\lambda & & \mathfrak{L}_\alpha \xrightarrow[\text{symmetrize}]{} G_\lambda
\end{array}
$$

$$\text{cohomology} \qquad\qquad\qquad K\text{-theory}$$

Using the formalism of connective K-theory (equivalently, introducing a harmless grading parameter β into the Grothendieck polynomial), as we do in this article, all expansions specialize to their K-theoretic or cohomological counterparts by setting β to -1 or 0 respectively.

In chronological order, expansion (b) was established by Edelman and Greene [5] via a Schensted-type insertion algorithm for reduced words. The expansion (a) was found by Lascoux and Schützenberger and proved in [22]. Expansion (d) was established by Buch, Kresch, Shimozono, Tamvakis, and Yong [3] via Hecke insertion, which takes Hecke words as input. Expansion (c) is the topic of this article.

The expansion coefficients have geometric significance. The Stanley-to-Schur coefficients of the expansion (b) coincide with large rank affine Stanley to affine Schur coefficients [13, Proposition 9.17], which in turn coincide with Gromov–Witten invariants for the flag variety via Peterson's Quantum Equals Affine Theorem [14, 21], [11, Part 3, Section 10]. Specializing w to a Zelevinsky permutation, (a) and (b) give the expansion of cohomology classes of equioriented type A quiver loci [9, Theorem 7.14], the latter being shown by Buch and Fulton [7] to specialize to virtually all known variants of type A Schubert polynomials. Expansions (c) and (d) give analogous expansions in K-theory [2, 18].

The nonsymmetric expansions are subtle refinements of their symmetric counterparts. In the symmetric expansions there is a set of tableaux in which each tableau T in the set, gives a copy of s_λ or G_λ where λ is the shape of T. There is a corresponding term κ_α or $\mathfrak{L}_\alpha$ in the nonsymmetric expansion, but an additional datum must be supplied: a composition or extremal weight α in the symmetric group orbit of λ; see (3.10) through (3.13). Such constructions assigning a composition to a tableau go by the general name of *key*. In the crystal graph of semistandard Young tableaux of shape λ, the left and right keys of the tableau T of shape λ are given by the final and initial directions of the corresponding Littelmann path whose highest weight vector is the directed line segment from the origin to λ. The initial direction indicates the smallest Demazure crystal containing

the given tableau.

3 Grothendieck and Lascoux polynomials

The group $S_+ = \bigcup_{n \geq 1} S_n$ acts on $R = \mathbb{Z}[\beta][x_1, x_2, \dots]$ by permuting the variables: for $i \geq 1$, let s_i exchange x_i and x_{i+1}. We define the following operators on R, where an element $f \in R$ (or its fraction field) denotes the operator of left multiplication by f.

$$\partial_i := (x_i - x_{i+1})^{-1}(1 - s_i), \tag{3.1}$$

$$\pi_i := \partial_i x_i, \tag{3.2}$$

$$\partial_i^{(\beta)} := \partial_i(1 + \beta x_{i+1}), \tag{3.3}$$

$$\pi_i^{(\beta)} := \partial_i^{(\beta)} x_i. \tag{3.4}$$

All satisfy the braid relations for S_+.

Let $w_0^{(n)} \in S_n$ be the long element and $\rho^{(n)} = (n-1, n-2, \dots, 1, 0)$. For $w \in S_n$ the β-Grothendieck polynomial is defined by [16]

$$\mathfrak{G}_w^{(\beta)} := \begin{cases} x^{\rho^{(n)}} & \text{if } w = w_0^{(n)}, \\ \partial_i^{(\beta)} \mathfrak{G}_{ws_i}^{(\beta)} & \text{if } ws_i > w. \end{cases} \tag{3.5}$$

Since the $\partial_i^{(\beta)}$ satisfy the braid relations, $\mathfrak{G}_w^{(\beta)}$ is well-defined for $w \in S_n$. It is also well-defined for $w \in S_+$, that is, unchanged under the standard embedding $S_n \to S_{n+1}$ for all $n \geq 1$. The Schubert $\mathfrak{S}_w$ and Grothendieck polynomials $\mathfrak{G}_w$ are defined by

$$\mathfrak{S}_w := \mathfrak{G}_w^{(\beta)}|_{\beta=0}, \tag{3.6}$$

$$\mathfrak{G}_w := \mathfrak{G}_w^{(\beta)}|_{\beta=-1}. \tag{3.7}$$

Let $\alpha = (\alpha_1, \alpha_2, \dots)$ be a composition (sequence of nonnegative integers, almost all 0). The Lascoux polynomial $\mathfrak{L}_\alpha^{(\beta)}$ is defined by [15]

$$\mathfrak{L}_\alpha^{(\beta)} = \begin{cases} x^\alpha & \text{if } \alpha \text{ is a partition}, \\ \pi_i^{(\beta)} \mathfrak{L}_{s_i\alpha}^{(\beta)} & \text{if } \alpha_i < \alpha_{i+1}. \end{cases} \tag{3.8}$$

The Demazure character κ_α is defined by

$$\kappa_\alpha = \mathfrak{L}_\alpha^{(\beta)}|_{\beta=0}. \tag{3.9}$$

Given a composition $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$ let α^+ be the unique partition in the S_n-orbit of α. For $w \in S_n$ and $w_0 \in S_n$ the long element we have the symmetrizations

$$\pi_{w_0}^{(\beta)}(\mathfrak{L}_\alpha) = G_{\alpha^+}^{(\beta)}(x_1, \ldots, x_n), \tag{3.10}$$

$$\pi_{w_0}(\kappa_\alpha) = s_{\alpha^+}(x_1, \ldots, x_n), \tag{3.11}$$

$$\pi_{w_0}^{(\beta)}(\mathfrak{G}_w(x)) = G_w(x_1, \ldots, x_n), \tag{3.12}$$

$$\pi_{w_0}(\mathfrak{S}_w(x)) = F_w(x_1, \ldots, x_n). \tag{3.13}$$

4   Tableau formula for Lascoux polynomials

Given the definition of a certain kind of tableau which involves entries in a totally or-dered set, we say "reverse" to mean the same definition but with the total order reversed. Thus, a *reverse semistandard Young tableau* (RSSYT) is a tableau where entries weakly de-crease along rows from left to right and strictly decrease along columns from top to bottom.

For a partition λ, a *reverse set-valued tableau* (RSVT) T of shape λ is a filling of the boxes of λ by finite nonempty subsets of $\mathbb{Z}_{>0}$ satisfying the following. For the box $s \in \lambda$ let $T(s)$ be the set which occupies the box s in T.

1. $\min(T(s)) \geq \max(T(t))$ if the box t is immediately right of the box s in λ.

2. $\min(T(s)) > \max(T(t))$ if the box t is immediately below the box s in λ.

This is the reverse of Buch's set-valued tableaux [2].

Given a RSVT T, let $L(T)$ be the RSSYT obtained from T by replacing every entry $T(s)$ by its largest value $\max(T(s))$.

The *weight* $\mathrm{wt}(T)$ of a tableau T is the composition whose i-th part is the total number of times i appears in T.

A *key tableau* (or just key) is a RSSYT of partition shape such that the j-th column, viewed as a set, contains the $(j+1)$-th for all j. There is a bijection $\alpha \mapsto \mathrm{key}(\alpha)$ from compositions to keys where $\mathrm{key}(\alpha)$ is the unique RSSYT of shape α^+ and weight α. Its j-th column consists of the numbers $\{i \mid \alpha_i \geq j\}$.

The *left key* $K_-(T)$ of a RSSYT T is a key computed in the following way. For each j, let $T_{\leq j}$ be the RSSYT we get if we only keep the first j columns of T. We may anti-rectify $T_{\leq j}$ using the jeu-de-taquin (jdt). Then the leftmost column of the result becomes column j of $K_-(T)$.
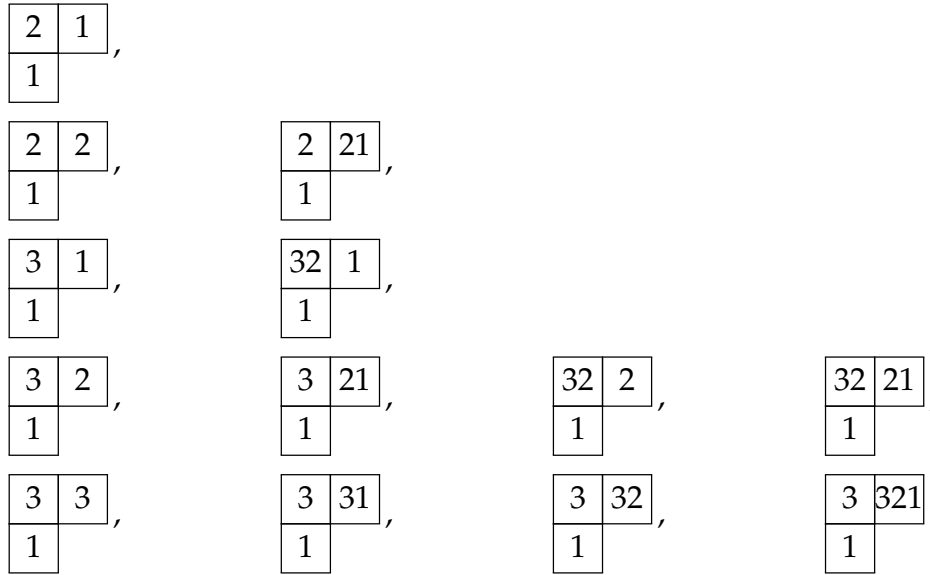
Let $|\alpha| = \sum_{i \geq 1} \alpha_i$. Let RSVT_λ be the set of reverse set-valued tableaux of shape λ. For $T \in \mathrm{RSVT}_\lambda$ let $\mathrm{ex}(T) = |\mathrm{wt}(T)| - |\lambda|$. Our first main theorem is:

Theorem 4.1. *For any composition α*

$$\mathfrak{L}_{\alpha}^{(\beta)} = \sum_{\substack{T \in \mathrm{RSVT}_{\alpha+} \\ K_-(L(T)) \leq \mathrm{key}(\alpha)}} \beta^{\mathrm{ex}(T)} x^{\mathrm{wt}(T)}. \tag{4.1}$$

Here $\leq$ indicates entrywise conparison.

Example 4.2. The following RSVTs contribute to $\mathfrak{L}_{(1,0,2)}^{(\beta)}$:

$$\begin{array}{|c|c|}\hline 2 & 1 \\\hline 1 \\\cline{1-1}\end{array},$$

$$\begin{array}{|c|c|}\hline 2 & 2 \\\hline 1 \\\cline{1-1}\end{array}, \qquad \begin{array}{|c|c|}\hline 2 & 21 \\\hline 1 \\\cline{1-1}\end{array},$$

$$\begin{array}{|c|c|}\hline 3 & 1 \\\hline 1 \\\cline{1-1}\end{array}, \qquad \begin{array}{|c|c|}\hline 32 & 1 \\\hline 1 \\\cline{1-1}\end{array},$$

$$\begin{array}{|c|c|}\hline 3 & 2 \\\hline 1 \\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 3 & 21 \\\hline 1 \\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 32 & 2 \\\hline 1 \\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 32 & 21 \\\hline 1 \\\cline{1-1}\end{array},$$

$$\begin{array}{|c|c|}\hline 3 & 3 \\\hline 1 \\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 3 & 31 \\\hline 1 \\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 3 & 32 \\\hline 1 \\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 3 & 321 \\\hline 1 \\\cline{1-1}\end{array}.$$

Thus, we may write $\mathfrak{L}_{(1,0,2)}^{(\beta)}$ as

$$x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_2 x_3 + x_1 x_3^2$$
$$+ \beta(x_1^2 x_2^2 + 2x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1^2 x_3^2 + x_1 x_2 x_3^2)$$
$$+ \beta^2(x_1^2 x_2^2 x_3 + x_1^2 x_2 x_3^2).$$

Remark 4.3. There have been a number of conjectural combinatorial formulas for Lascoux polynomials, such as the K-Kohnert move rule of Ross and Yong [24] ([8, Footnote on page 19] for the general β version), the set-valued skyline filling formula of Monical [19], and a set-valued tableau (SVT) rule of Pechenik and Scrimshaw [20]. Buciumas, Scrimshaw and Weber [4] proved the last two of these rules using solvable lattice models. In response to a previous version of this article, Travis Scrimshaw kindly informed us that Theorem 4.1 is implicit in [4]: see the proof of [4, Theorem 4.4]. We feel it is worthwhile to state these theorems in their simplest and most explicit form. We note that the naive nonreversed analogue of the RSVT formula does not yield the Lascoux polynomial.

5 Fomin–Kirillov monomial formula

Our point of departure is the explicit monomial expansion of Grothendieck polynomials due to Fomin and Kirillov [6].

The 0-*Hecke monoid* $\mathcal{H}$ is the quotient of the free monoid of words on the alphabet $\mathbb{Z}_{>0}$ by the relations

$$i(i+1)i \equiv_H (i+1)i(i+1), \tag{5.1}$$

$$ii \equiv_H i, \tag{5.2}$$

$$ij \equiv_H ji \qquad \text{for } |i-j| \geq 2. \tag{5.3}$$

$\mathcal{H}$ acts on S_+ by

$$i * w = \begin{cases} s_i w & \text{if } s_i w > w, \\ w & \text{if } s_i w < w. \end{cases}$$

Given a word $u \in \mathcal{H}$ define its *associated permutation* by $u * \mathrm{id} \in S_+$. For $w \in S_+$ let $\mathcal{H}_w$ be the words in $\mathcal{H}$ with associated permutation w. The subsets $\mathcal{H}_w \subset \mathcal{H}$ are the $\equiv_H$-equivalence classes.

Lemma 5.1. *$u \in \mathcal{H}_w$ if and only if $\mathrm{rev}(u) \in \mathcal{H}_{w^{-1}}$.*

For $a \in \mathcal{H}_w$ let $\mathrm{ex}(a) = \mathrm{length}(a) - \ell(w)$, the *excess* of the length of a above the minimum possible, the Coxeter length $\ell(w)$ of w.

The following is merely the definition in [1] but with both words reversed, which is better suited to our use of decreasing tableaux.

Definition 5.2 ([1])**.** A pair of words (a, i) is *compatible* if they satisfy

1. a, i are words of positive numbers with the same length.

2. i is weakly decreasing

3. $i_j = i_{j+1}$ implies $a_j < a_{j+1}$.

A compatible pair (a, i) is *bounded* if $i_j \leq a_j$ for all j.

Let $\mathcal{C}$ be the set of all compatible pairs, $\mathcal{C}^b$ those that are bounded, $\mathcal{C}_w$ the pairs $(a, i) \in \mathcal{C}$ such that $a \in \mathcal{H}_w$, and $\mathcal{C}_w^b = \mathcal{C}^b \cap \mathcal{C}_w$. The following monomial expansion of β-Grothendieck polynomials is due to Fomin and Kirillov [6]:

$$\mathfrak{G}_w^{(\beta)} = \sum_{(a,i) \in \mathcal{C}_{w^{-1}}^b} \beta^{\mathrm{ex}(a)} x^{\mathrm{wt}(i)}. \tag{5.4}$$

When $\beta = 0$ this is the Billey–Jockusch–Stanley formula for Schubert polynomials [1].

For $w \in S_n$ and a positive integer N let $1^N \times w$ be the permutation of S_{n+N} obtained by adding N fixed points before w. The β-Grothendieck symmetric function is defined by

$$G_w^{(\beta)} = \lim_{N \to \infty} \mathfrak{G}_{1^N \times w} \tag{5.5}$$

It lives in a completion of the ring of symmetric functions over $\mathbb{Z}[\beta]$. The Stanley and Grothendieck symmetric functions are defined by

$$F_w = G_w^{(\beta)}|_{\beta=0} \tag{5.6}$$

$$G_w = G_w^{(\beta)}|_{\beta=-1}. \tag{5.7}$$

It follows from (5.4) and the definitions that

$$G_w^{(\beta)} = \sum_{(a,i) \in \mathcal{C}_{w^{-1}}} \beta^{\mathrm{ex}(a)} x^{\mathrm{wt}(i)}. \tag{5.8}$$

6 $G_w^{(\beta)}$ to $G_\lambda^{(\beta)}$ via Hecke insertion: restriction of compatible pairs according to w

The code $c(w)$ of a permutation w is the sequence $(c_1, c_2, \dots)$ such that

$$c_i = |\{j \mid 1 \le j < w(i) \text{ and } w^{-1}(j) > i\}|.$$

For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ the *Grassmannian Grothendieck symmetric function* $G_\lambda^{(\beta)}$ is by definition equal to $G_w^{(\beta)}$ where w is the permutation with code $(\lambda_k, \dots, \lambda_1, 0, 0, \dots)$.

Buch [2] showed that the $\mathbb{Z}[\beta]$-span of the $G_w^{(\beta)}$ for $w \in S_+$, has basis given by the $G_\lambda^{(\beta)}$ and proved the increasing version of the following:

$$G_\lambda^{(\beta)} = \sum_{T \in \mathrm{RSVT}_\lambda} \beta^{\mathrm{ex}(T)} x^{\mathrm{wt}(T)} \tag{6.1}$$

For $\beta = 0$ this becomes the RSSYT formula for the Schur function s_λ.

To find the coefficients of the $G_w^{(\beta)}$ to $G_\lambda^{(\beta)}$ expansion, the *Hecke insertion* algorithm was developed in [3] in the language of *increasing tableaux*, which strictly increase along rows from left to right and strictly increase along columns from top to bottom. We use the variant of Hecke insertion for *decreasing tableaux*, which strictly decrease along rows from left to right and strictly decrease along columns from top to bottom.

It was not explicitly stated in [3] but all ingredients are there to define a Hecke Robinson–Schensted–Knuth (RSK) bijection called Insert (and its inverse bijection RevInsert)

$$\mathcal{C} \xrightleftharpoons[\text{RevInsert}]{\text{Insert}} \bigsqcup_\lambda (\text{Dec}_\lambda \times \text{RSVT}_\lambda) =: \mathcal{T}$$

$$(a,i) \rightleftharpoons (P,Q)$$

where Dec_λ is the set of decreasing tableaux of shape λ. Note that the set $\mathcal{T}$ as defined by the above diagram, consists of pairs (P,Q) of tableaux of the same partition shape with P decreasing and Q reverse set-valued.

Let $(P,Q) = \text{Insert}(a,i)$. Let $\text{word}(P)$ be the word we get if we read entries of P from left to right, and move from bottom to top within each column. Then the bijection satisfies

$$\text{rev}(a) \equiv_H \text{word}(P) \quad \text{and} \quad \text{wt}(Q) = \text{wt}(i). \tag{6.2}$$

By Lemma 5.1, the bijection Insert restricts to a bijection

$$\mathcal{C}_{w^{-1}} \leftrightarrow \bigsqcup_\lambda (\text{Dec}_\lambda^w \times \text{RSVT}_\lambda) := \mathcal{T}_w \tag{6.3}$$

where $\text{Dec}_\lambda^w = \{T \in \text{Dec}_\lambda \mid \text{word}(T) \in \mathcal{H}_w\}$. Taking the generating function of both sides we obtain

$$G_w^{(\beta)} = \sum_\lambda |\text{Dec}_\lambda^w| G_\lambda^{(\beta)}. \tag{6.4}$$

7 $\mathfrak{G}_w^{(\beta)}$ to $\mathfrak{L}_\alpha^{(\beta)}$ by Hecke insertion and keys: restriction to bounded compatible pairs

Let $\star$ denote the following right action of the monoid of words with letters in the set $\mathbb{Z}_{>0}$, on the set of subsets of $\mathbb{Z}_{>0}$.

Let $S \subseteq \mathbb{Z}_{>0}$ and let $m \in \mathbb{Z}_{>0}$. Let m' be the smallest number in S of value at least m. If m' does not exist, we let $S \star m = S \sqcup \{m\}$. Otherwise, we define $S \star m = (S - \{m'\}) \sqcup \{m\}$.

More generally, if $w = w_1 \ldots w_n$ is a word of positive integers, we define $S \star w = (S \star w_1) \star (w_2 \ldots w_n)$, and $S \star w = S$ if w is the empty word.

Example 7.1. We have:

$$\varnothing \star 3414 = \{1,4\},$$
$$\{3,4,7\} \star 3414 = \{1,4,7\}.$$

For each decreasing tableau P, we define its right key $K_+(P)$ to be the RSSYT whose j-th column is the column given by $\varnothing \star \text{word}(P_{\geq j})$ where $P_{\geq j}$ is the decreasing tableau obtained by removing the first $j - 1$ columns of P.

The *K-jeu-de-taquin* (Kjdt) of Thomas and Yong [25] may be used (following [23] for increasing tableaux) to give another definition of right key of decreasing tableau, and two definitions of right key are shown to coincide.

Let $\mathcal{T}^b$ be the subset of pairs $(P, Q) \in \mathcal{T}$ such that $K_+(P) \geq K_-(L(Q))$. With $\mathcal{T}_w$ defined as in (6.3), let $\mathcal{T}_w^b = \mathcal{T}^b \cap \mathcal{T}_w$. Then we show:

Theorem 7.2. Insert *restricts to a bijection $\mathcal{C}^b \cong \mathcal{T}^b$.*

Intersecting with the bijection (6.3), Insert restricts to a bijection

$$\mathcal{C}_{w^{-1}}^b \cong \mathcal{T}_w^b \qquad \text{for every } w \in S_+. \tag{7.1}$$

Using Theorem 4.1 we obtain our second main theorem, the Grothendieck-to-Lascoux expansion via decreasing tableaux.

Theorem 7.3.

$$\mathfrak{G}_w^{(\beta)} = \sum_\lambda \sum_{P \in \text{Dec}_\lambda^w} \mathfrak{L}_{\text{wt}(K_+(P))}^{(\beta)}. \tag{7.2}$$

8   Connecting with the Reiner–Yong conjecture

The Reiner–Yong conjecture asserts:

Theorem 8.1 ([23])**.**

$$\mathfrak{G}_w^{(\beta)} = \sum_\lambda \sum_{P \in \text{Inc}_\lambda^{w^{-1}}} \mathfrak{L}_{\text{wt}(K_-(P))'}^{(\beta)} \tag{8.1}$$

where Inc_λ^w is similar to Dec_λ^w except that the tableaux are increasing and $K_-(P)$ is the left key construction on the increasing tableau P using the Kjdt.

Proof sketch. We construct a map $\text{Dec}_\lambda \to \text{Inc}_\lambda : T \mapsto T^\sharp$. For each $T \in \text{Dec}_\lambda$, we anti-rectify it using Kjdt and then rotate the result by 180°. We show it is a bijection satisfying: $K_+(T) = K_-(T^\sharp)$ and $\text{word}(T^\sharp) \equiv_K \text{rev}(\text{word}(T))$. Using Lemma 5.1 we see that $T \in \mathcal{H}_w$ if and only if $T^\sharp \in \mathcal{H}_{w^{-1}}$. Thus the bijection restricts to a bijection $\text{Dec}_\lambda^w \cong \text{Inc}_\lambda^{w^{-1}}$ as required. $\qquad \square$

9   Flagged Grothendieck to Lascoux

In this section we extend our expansion to flagged Grothendieck polynomials.

In the literature there is a definition of flagged Grothendieck polynomial whose generality extends to the case of 321-avoiding permutations [17]; see [10] for the case of vexillary permutations. For 321-avoiding permutations there is a monomial tableau formula and a determinantal formula.

We use a divided difference definition of flagged Grothendieck polynomial from [12], which is valid for any permutation. This flagged Grothendieck polynomial has an explicit monomial expansion given in Proposition 9.1.

The main result of this subsection is a Lascoux polynomial expansion of flagged Grothendiecks.

A *flag* is a sequence of integers $f = (f_1, f_2, \cdots, f_n)$ which is weakly increasing, satisfies $f_i \geq i$ for all i, and $f_n = n$. Let $f_{\min} = (1, 2, \cdots, n)$ and $f_{\max} = (n, n, \cdots, n)$ be the minimum and maximum flags respectively. Given a flag f, define the permutation $\sigma_f \in S_n$ as follows. For the minimum flag $f_{\min}$ let $\sigma_{f_{\min}} = \mathrm{id}$. For $f \neq f_{\min}$ there is an index j such that $f_j > j$; take the minimum such. Define $\sigma_f = s_i \sigma_{f'}$ where $i + 1 = f_j$ and f' is obtained from f by replacing the $i + 1$ by i. The *flagged Grothendieck polynomial* is defined by $\mathfrak{G}_{w,f}^{(\beta)} = \pi_{\sigma_f}^{(\beta)}(\mathfrak{G}_w^{(\beta)})$.

The flagged Grothendieck polynomials have the following explicit monomial expansion.

Proposition 9.1 ([12]).

$$\mathfrak{G}_{w,f}^{(\beta)} = \sum_{\substack{(a,i) \in \mathcal{C}_{w^{-1}} \\ i_k \leq f_{a_k}}} \beta^{\mathrm{ex}(a)} x^{\mathrm{wt}(i)}. \tag{9.1}$$

Remark 9.2. Note that only the bound $i_k \leq a_k$ in (5.4) has been changed to $i_k \leq f_{a_k}$.

The flagged Grothendieck polynomials interpolate between Grothendieck polynomials and their symmetric counterparts.

Corollary 9.3. *For $w \in S_n$, let w_0 be the long element in S_n. Then*

$$\pi_{w_0}^{(\beta)}(\mathfrak{G}_w^{(\beta)}) = G_w^{(\beta)}(x_1, \ldots, x_n). \tag{9.2}$$

Proof. We have

$$\begin{aligned}
\pi_{w_0}^{(\beta)}(\mathfrak{G}_w^{(\beta)}) &= \pi_{w_0}^{(\beta)}(\mathfrak{G}_{w,f_{\min}}^{(\beta)}) \\
&= \mathfrak{G}_{w,f_{\max}}^{(\beta)} \\
&= G_w^{(\beta)}(x_1, \ldots, x_n),
\end{aligned}$$

where the last equality holds by the equality of (9.1) with (5.8) with x_i set to 0 for $i > n$. $\qquad\square$

Define the Demazure action $\circ$ of S_+ on compositions by

$$s_i \circ \alpha = \begin{cases} s_i(\alpha) & \text{if } \alpha_i > \alpha_{i+1}, \\ \alpha & \text{otherwise.} \end{cases} \tag{9.3}$$

Theorem 7.3 implies the following.

Corollary 9.4.

$$\mathfrak{G}_{w,f}^{(\beta)} = \sum_\lambda \sum_{P \in \mathrm{Dec}_\lambda^w} \mathfrak{L}_{\sigma_f \circ \mathrm{wt}(K_+(P))}^{(\beta)}. \tag{9.4}$$

Acknowledgements

References

[1]   S. C. Billey, W. Jockusch, and R. P. Stanley. "Some combinatorial properties of Schubert polynomials". *J. Algebraic Combin.* **2**.4 (1993), pp. 345–374.

[2]   A. S. Buch. "A Littlewood–Richardson rule for the K-theory of Grassmannians". *Acta Math.* **189**.1 (2002), pp. 37–78.

[3]   A. S. Buch, A. Kresch, M. Shimozono, H. Tamvakis, and A. Yong. "Stable Grothendieck polynomials and K-theoretic factor sequences". *Math. Ann.* **340**.2 (2008), pp. 359–382.

[4]   V. Buciumas, T. Scrimshaw, and K. Weber. "Colored five-vertex models and Lascoux polynomials and atoms". *J. London Math. Soc.* **102**.3 (2020), pp. 1047–1066.

[5]   P. Edelman and C. Greene. "Balanced tableaux". *Adv. Math.* **63**.1 (1987), pp. 42–99.

[6]   S. Fomin and A. N. Kirillov. "Grothendieck polynomials and the Yang–Baxter equation". *Proc. Formal Power Series and Alg. Comb.* 1994, pp. 183–190.

[7]   W. Fulton and A. S. Buch. "Chern class formulas for quiver varieties" (1999).

[8]   A. N. Kirillov et al. "Notes on Schubert, Grothendieck and key polynomials". *SIGMA. Symmetry, Integrability Geom. Methods Appl.* **12** (2016), p. 034.

[9]   A. Knutson, E. Miller, and M. Shimozono. "Four positive formulae for type A quiver polynomials". *Invent. Math,* **166**.2 (2006), pp. 229–325.

[10] A. Knutson, E. Miller, and A. Yong. "Gröbner geometry of vertex decompositions and of flagged tableaux". *J. Reine Angew. Math.* (2009), pp. 1–31.

[11] T. Lam, L. Lapointe, J. Morse, A. Schilling, M. Shimozono, and M. Zabrocki. *k-Schur Functions and Affine Schubert Calculus.* Vol. 33. Springer, 2014.

[12] T. Lam, S. J. Lee, and M. Shimozono. "Back stable K-theory Schubert calculus". 2021. arXiv:2108.10202.

[13] T. Lam, S. J. Lee, and M. Shimozono. "Back stable Schubert calculus". *Compos. Math.* **157**.5 (2021), pp. 883–962.

[14] T. Lam and M. Shimozono. "Quantum cohomology of G/P and homology of affine Grassmannian". *Acta Math.* **204**.1 (2010), pp. 49–90.

[15] A. Lascoux. "Schubert and Grothendieck: a bidecennial balance.(Schubert et Grothendieck: un bilan bidécennal.)" *Sém. Lothar. Combin.* **50** (2003), B50i–32.

[16] A. Lascoux and M.-P. Schützenberger. "Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux". *CR Acad. Sci. Paris Sér. I Math* **295**.11 (1982), pp. 629–633.

[17] T. Matsumura. "Flagged Grothendieck polynomials". *J. Algebraic Combin.* **49**.3 (2019), pp. 209–228.

[18] E. Miller. "Alternating formulas for K-theoretic quiver polynomials". *Duke Math. J.* **128**.1 (2005), pp. 1–17.

[19] C. Monical. "Set-valued skyline fillings". 2016. arXiv:1611.08777.

[20] O. Pechenik and T. Scrimshaw. "K-theoretic crystals for set-valued tableaux of rectangular shapes". 2019. arXiv:1904.09674.

[21] D. Peterson. "Quantum cohomology of G/P". *Lecture notes at MIT* (1997).

[22] V. Reiner and M. Shimozono. "Key polynomials and a flagged Littlewood—Richardson rule". *J. Combin. Theory Ser. A* **70**.1 (1995), pp. 107–143.

[23] V. Reiner and A. Yong. "The 'Grothendieck to Lascoux' conjecture". 2021. arXiv:2102.12399.

[24] C. Ross and A. Yong. "Combinatorial rules for three bases of polynomials". *Sém. Lothar. Combin.* (2015).

[25] H. Thomas and A. Yong. "A jeu de taquin theory for increasing tableaux, with applications to K-theoretic Schubert calculus". *Algebra Number Theory* **3**.2 (2009), pp. 121–148.