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Regularity of matrix Schubert varieties

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Abstract. Matrix Schubert varieties are affine varieties arising in the Schubert calculus of the complete flag variety. We give a formula for the Castelnuovo–Mumford regularity of matrix Schubert varieties, answering a question of Jenna Rajchgot. We follow her proposed strategy of studying the highest-degree homogeneous parts of Grothendieck polynomials, which we call Castelnuovo–Mumford polynomials. In addition to the regularity formula, we obtain formulas for the degrees of all Castelnuovo–Mumford polynomials and for their leading terms, as well as a complete description of when two Castelnuovo–Mumford polynomials agree up to scalar multiple. The degree of the Grothendieck polynomial is a new permutation statistic, which we call the Rajchgot index; we develop the properties of Rajchgot index and relate it to major index and to weak order.

1 Introduction

The *flag variety* Flags_n, the parameter space for complete flags of nested vector subspaces of $\mathbb{C}^n$, has a complex cell decomposition given by its *Schubert varieties*. The geometry and combinatorics of this cell decomposition are of central importance in Schubert calculus. These Schubert varieties are closely related to certain generalized determinantal varieties X_w of $n \times n$ matrices called *matrix Schubert varieties* (see [1] and Section 2 for the definition). It is natural to desire a measure of the algebraic complexity of matrix Schubert varieties. One such measure is the *Castelnuovo–Mumford regularity* of X_w, a commutative-algebraic invariant determining the extent to which the defining ideal of X_w can be resolved by low-degree polynomials.

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Jenna Rajchgot (*cf.* [7]) observed that, since matrix Schubert varieties are Cohen–Macaulay, the regularity of X_w is given by the difference between the highest-degree and lowest-degree homogeneous parts of the *K-polynomial* for X_w. These particular K-polynomials have been much studied. They were introduced by Lascoux and Schützenberger [5], under the name of *Grothendieck polynomials* $\mathfrak{G}_w(\boldsymbol{x})$, as polynomial representatives for structure sheaf classes in K-theoretic Schubert calculus of Flags_n. Grothendieck polynomials are inhomogeneous polynomials $\mathfrak{G}_w(\boldsymbol{x})$ in n variables $\boldsymbol{x} = x_1, x_2, \ldots, x_n$, indexed by permutations w in the symmetric group S_n.

The lowest-degree homogeneous part of $\mathfrak{G}_w(\boldsymbol{x})$ is the *Schubert polynomial* $\mathfrak{S}_w(\boldsymbol{x})$ [4]; Schubert polynomials are well-understood from a combinatorial perspective, and the degree of $\mathfrak{S}_w(\boldsymbol{x})$ equals the codimension of X_w or equivalently the Coxeter length $\mathrm{inv}(w)$ of the permutation w. Hence, determining the regularity of X_w reduces to answering the following question of Rajchgot: "What is the degree of a Grothendieck polynomial?"

We term the top-degree part of $(-1)^{\deg \mathfrak{G}_w(\boldsymbol{x}) - \mathrm{inv}(w)} \mathfrak{G}_w(\boldsymbol{x})$ the ***Castelnuovo–Mumford polynomial*** and write it $\mathfrak{CM}_w(\boldsymbol{x})$. (The power of -1 makes the coefficients positive.) The goal of this paper is to answer Rajchgot's question by understanding these homogeneous polynomials and in particular their degrees, thereby obtaining a formula for the Castelnuovo–Mumford regularity of X_w. In the special case of *symmetric* Grothendieck polynomials, [7] gives a formula for the degree of $\mathfrak{CM}_w(\boldsymbol{x})$. Our first main result is a degree formula for arbitrary $\mathfrak{CM}_w(\boldsymbol{x})$, answering Rajchgot's question in full generality. Since our work appeared as a preprint, additional formulas have been given for the case of *vexillary* permutations w in [2, 8].

Write a permutation $w \in S_n$ in one-line notation as $w(1)w(2)\cdots w(n)$. For each k, find an increasing subsequence of $w(k)w(k+1)\cdots w(n)$ containing $w(k)$ and of greatest length among such subsequences. Let r_k be the number of terms from $w(k)w(k+1)\cdots w(n)$ omitted from this subsequence. We call the sequence $(r_1, \ldots, r_n) = \mathrm{rajcode}(w)$ the ***Rajchgot code*** of w and its sum $\mathrm{raj}(w)$ the ***Rajchgot index*** of w.

Theorem 1.1. *For $w \in S_n$, we have $\deg \mathfrak{CM}_w(\boldsymbol{x}) = \mathrm{raj}(w)$. Moreover, for any term order satisfying $x_1 < x_2 < \cdots < x_n$, the leading term of $\mathfrak{CM}_w(\boldsymbol{x})$ is a scalar multiple of the monomial $\boldsymbol{x}^{\mathrm{rajcode}(w)} = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$. In particular, the Castelnuovo–Mumford regularity of the matrix Schubert variety X_w is $\mathrm{raj}(w) - \mathrm{inv}(w)$.*

Example 1.2. Consider $w = 293417568 \in S_9$. A longest increasing subsequence starting from 2 is $2 \bullet 34 \bullet \bullet 568$, which omits three terms, so $r_1 = 3$. In full,

$$\mathrm{rajcode}(w) = (r_1, r_2, \ldots, r_9) = (3, 7, 2, 2, 1, 2, 0, 0, 0).$$

Hence, by Theorem 1.1, the leading term of $\mathfrak{CM}_w(\boldsymbol{x})$ is a scalar multiple of the monomial $x_1^3 x_2^7 x_3^2 x_4^2 x_5 x_6^2$ and the degree of $\mathfrak{CM}_w(\boldsymbol{x})$ is $\mathrm{raj}(w) = 3+7+2+2+1+2+0+0+0 = 17$. Since $\mathrm{inv}(w) = 12$, it follows that the Castelnuovo–Mumford regularity of the matrix Schubert variety X_w is $\mathrm{raj}(w) - \mathrm{inv}(w) = 17 - 12 = 5$.

Our remaining results explore the combinatorics of Castelnuovo–Mumford polynomials and the associated permutation statistics.

While Schubert polynomials are all distinct and have distinct leading monomials, we observe that many Castelnuovo–Mumford polynomials differ only by a scalar multiple. In fact, we will show that $\mathfrak{CM}_u(\boldsymbol{x})$ and $\mathfrak{CM}_v(\boldsymbol{x})$ differ by a scalar precisely if $\mathrm{rajcode}(u) = \mathrm{rajcode}(v)$. This phenomenon is best understood in the context of *double* Castelnuovo–Mumford polynomials, as we now explain. The *double Grothendieck polynomials* are certain polynomials $\mathfrak{G}_w(x_1, \ldots, x_n; y_1, \ldots, y_n)$ in $2n$ variables, also indexed by $w \in S_n$. They represent Schubert classes in the torus-equivariant K-theory of Flags_n, and obey the relations $\mathfrak{G}_w(\boldsymbol{x}; 0) = \mathfrak{G}_w(\boldsymbol{x})$ and $\mathfrak{G}_w(\boldsymbol{x}; \boldsymbol{y}) = \mathfrak{G}_{w^{-1}}(\boldsymbol{y}; \boldsymbol{x})$. We define the ***double Castelnuovo–Mumford polynomial*** $\mathfrak{CM}_w(\boldsymbol{x}; \boldsymbol{y})$ to be the highest-degree part of $\mathfrak{G}_w(\boldsymbol{x}; \boldsymbol{y})$. We will show (Corollary 2.3) that $\mathfrak{G}_w(\boldsymbol{x}; \boldsymbol{y})$ has terms whose $\boldsymbol{x}$-degree and $\boldsymbol{y}$-degree are simultaneously maximal, so $\mathfrak{CM}_w(\boldsymbol{x}; \boldsymbol{y})$ is homogeneous in both $\boldsymbol{x}$ and $\boldsymbol{y}$.

We find that double Castelnuovo–Mumford polynomials factor as a polynomial in $\boldsymbol{x}$ times a polynomial in $\boldsymbol{y}$. We identify a special family of single Castelnuovo–Mumford polynomials, the ***Rajchgot polynomials*** $\mathfrak{R}_\pi(\boldsymbol{x})$, indexed by set partitions of $\{1, \ldots, n\}$. For each $w \in S_n$, we associate a set partition $\pi(w)$ so that the following holds.

Theorem 1.3. *Double Castelnuovo–Mumford polynomials factor into Rajchgot polynomials as*

$$\mathfrak{CM}_w(\boldsymbol{x}; \boldsymbol{y}) = \mathfrak{R}_{\pi(w)}(\boldsymbol{x}) \mathfrak{R}_{\pi(w^{-1})}(\boldsymbol{y}).$$

For any term order satisfying $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$, the leading term of $\mathfrak{CM}_w(\boldsymbol{x}; \boldsymbol{y})$ is exactly $\boldsymbol{x}^{\mathrm{rajcode}(w)} \boldsymbol{y}^{\mathrm{rajcode}(w^{-1})}$. In particular, $\mathfrak{CM}_w(\boldsymbol{x}) = \mathfrak{R}_{\pi(w^{-1})}(1, \ldots, 1) \mathfrak{R}_{\pi(w)}(\boldsymbol{x})$ has leading term $\mathfrak{R}_{\pi(w^{-1})}(1, \ldots, 1) \boldsymbol{x}^{\mathrm{rajcode}(w)}$.

In particular, Theorem 1.3 shows that, up to scalar multiple, the number of distinct Castelnuovo–Mumford polynomials for $w \in S_n$ is not $n!$, but rather the number of set partitions of n, which is also known as the nth *Bell number*.

The Rajchgot index is related to the classical *major index* statistic.

Theorem 1.4. *For all $w \in S_n$, we have*

$$\mathrm{raj}(w) = \max\{\mathrm{maj}(v) : v \leq_R w\} = \max\{\mathrm{maj}(u^{-1}) : u \leq_L w\} = \deg \mathfrak{CM}_w(\boldsymbol{x}),$$

where $\leq_L$ and $\leq_R$ denote the left and right weak orders, respectively.

To the best of our knowledge, none of the equalities in Theorem 1.4 has been observed previously. This paper is an extended abstract of [6], which contains complete proofs.

2 Background

Let $[n] := \{1, 2, \ldots, n\}$. Let S_n denote the symmetric group of permutations of $[n]$. We consider $w \in S_n$ as a map $w \colon [n] \to [n]$ and write w in ***one-line notation*** as the string $w(1)w(2)\cdots w(n)$. We will often write $w_i := w(i)$. We identify $w \in S_n$ with the permutation matrix having a 1 in each position (i, w_i) and 0s elsewhere. We write id for the identity permutation $12\cdots n$ and w_0 for the reverse permutation $n(n-1)\cdots 1$.

Let $s_i := (i\ i+1)$ denote the simple transposition that exchanges i and $i+1$, and recall that $s_1, \ldots, s_{n-1}$ generate S_n. An ***inversion*** of $w \in S_n$ is a pair $i, j \in [n]$ such that $i < j$ and $w_i > w_j$. We write $\mathrm{inv}(w)$ for the number of inversions in w and call it the ***Coxeter length*** of w. Note $\mathrm{inv}(w)$ is the length of the shortest expression for w as a product of the s_i. A factorization $w = s_{i_1} \cdots s_{i_{\mathrm{inv}(w)}}$ is called a ***reduced expression*** for w, and the sequence of subscripts $i_1 \cdots i_{\mathrm{inv}(w)}$ is called a ***reduced word*** for w.

We need three partial orders on S_n. If $w = uv$ with $\mathrm{inv}(w) = \mathrm{inv}(u) + \mathrm{inv}(v)$, then we say $v \leq_L w$ and $u \leq_R w$; the relations $\leq_L$ and $\leq_R$ are known as ***left*** and ***right weak order***, respectively. We write $\leq_{LR}$ for the partial order obtained as the transitive closure of the union of left and right weak orders and call it ***two-sided weak order***. For $u \leq_R v$, we write $[u, v]_R$ for the interval from u to v in right weak order. Similarly, we define the notations $[u, v]_L$ and $[u, v]_{LR}$. A ***descent*** of $w \in S_n$ is a value i such that $w_i > w_{i+1}$.

The ***0-Hecke monoid*** $\mathcal{H}_n$ is the free monoid on generators $\tau_1, \ldots, \tau_{n-1}$ subject to the "idempotent braid relations" $\tau_i^2 = \tau_i$, $\tau_i\tau_j = \tau_j\tau_i$, for $j \neq i \pm 1$, and $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$. There is a natural action of $\mathcal{H}_n$ on S_n induced by

$$\tau_i * w := \begin{cases} s_i w, & \text{if } \mathrm{inv}(s_i w) > \mathrm{inv}(w); \\ w, & \text{otherwise.} \end{cases}$$

For each $w \in S_n$, there is a unique element $\overline{w}$ in $\mathcal{H}_n$ with $\overline{w} * \mathrm{id} = w$. We define the ***Demazure product*** on S_n to be the binary operation $u * v$ given by $u * v = \overline{u} * \overline{v} * \mathrm{id}$.

The ***graph*** of $w \in S_n$ is obtained by plotting bullets $\bullet$ in the $n \times n$ grid in positions (i, w_i) for $i \in [n]$ (in matrix coordinates). The ***Rothe diagram*** $\mathsf{RD}(w)$ is constructed from its graph as follows. From each $\bullet$, fire a laser directly to the right and another straight down. The cells of the $n \times n$ grid that are hit by no laser are the Rothe diagram $\mathsf{RD}(w)$. The number of cells in $\mathsf{RD}(w)$ is $\mathrm{inv}(w)$. Write ℓ_i for the number of cells in row i. We call the sequence $\mathrm{invcode}(w) := (\ell_1, \ldots, \ell_n)$ the ***inv code*** of w.

The ***matrix Schubert variety*** X_w is an affine variety cut out by certain determinants. Let $Z = (z_{ij})_{1 \leq i,j \leq n}$ be a matrix of distinct indeterminates. Then X_w is a subvariety of the n^2-dimensional affine space $\operatorname{Spec} \mathbb{C}[Z]$. Consider the Rothe diagram $\mathsf{RD}(w)$. For each cell (i, j) of $\mathsf{RD}(w)$, let $r_{i,j}$ be the number of 1s appearing in the permutation matrix w northwest of the cell (i, j). Let I_w be the ideal generated by, for each such (i, j), the $(r_{i,j} + 1) \times (r_{i,j} + 1)$ minors of the matrix northwest of (i, j). The matrix Schubert variety X_w is the subvariety of $n \times n$ matrices defined by the ideal I_w. By work of Fulton [1], the ideal I is prime, so X_w is a reduced and irreducible affine variety; a $n \times n$ matrix A lies in X_w if the rank of each northwest submatrix of A is less than or equal to the rank of the same submatrix of w.

Let $R := \mathbb{C}[Z]$ be a polynomial ring and let $I \subseteq R$ be a homogeneous ideal. We write $R(-i)$ for R with all degrees shifted by i. A ***free resolution*** of R/I is a diagram of graded R-modules $0 \to \bigoplus_{i\in\mathbb{Z}} R(-i)^{b_i^k} \to \cdots \to \bigoplus_{i\in\mathbb{Z}} R(-i)^{b_i^0} \to R/I \to 0$ that is ***exact***, that is, such that the image of each map is the kernel of the next. There always exists such a free resolution with $k \leq n^2$. Up to isomorphism there is a unique free resolution simultaneously minimizing all b_i^j, the ***minimal free resolution*** of R/I. In this case, the b_i^j are invariants of R/I. The ***Castelnuovo–Mumford regularity*** $\operatorname{reg}(R/I)$ of R/I is the greatest $i - j$ such that $b_i^j \neq 0$. Conflating affine varieties with their coordinate rings, we also refer to this number as the Castelnuovo–Mumford regularity of $\operatorname{Spec} R/I$. When R/I is Cohen–Macaulay, the projective dimension of R/I equals the height of the ideal I as well as the codimension of $\operatorname{Spec} R/I$ in $\operatorname{Spec} R$.

Write $(R/I)_a$ for the degree a piece of R/I. The ***Hilbert series*** of R/I is the formal power series $H(R/I;t) = \sum_{a\in\mathbb{N}} \dim_{\mathbb{C}}(R/I)_a t^a$. If we write the Hilbert series as

$$H(R/I;t) = \frac{K(R/I;t)}{(1-t)^{n^2}}$$

the numerator $K(R/I;t)$ is the ***K-polynomial*** of R/I. The ***height*** $\operatorname{ht}(I)$ of a prime ideal I is the maximum k so that there is a nested chain of prime ideals $I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_k = I$. If I is prime, $\operatorname{ht}(I)$ is the codimension of $\operatorname{Spec} R/I$ in $\operatorname{Spec} R$.

Lemma 2.1. *Suppose R/I is Cohen–Macaulay. Then* $\operatorname{reg}(R/I) = \deg(K(R/I;t)) - \operatorname{ht}(I)$.

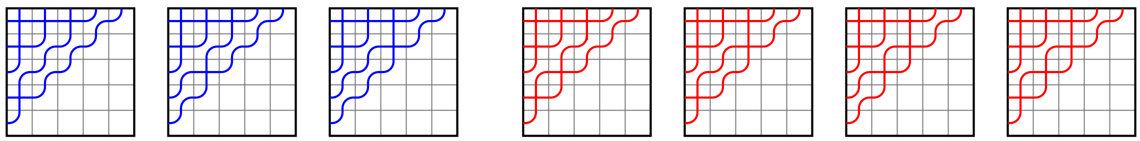
Matrix Schubert varieties are Cohen–Macaulay [1] and the codimension of X_w is the inversion statistic $\operatorname{inv}(w)$. Hence, computing the regularity of matrix Schubert varieties amounts to finding the degree of their K-polynomials.

We recall an explicit combinatorial formula for (double) Schubert polynomials and (double) Grothendieck polynomials. A ***pipe dream*** is a subset P of the cells in the strictly upper left triangular part of the $n \times n$ grid, *i.e.*, $P \subseteq \{(i,j) : 1 < i + j \leq n\}$. We represent this subset pictorially by placing a ***crossing tile*** in each cell of P and ***bumping tiles*** in the other cells. If there is a crossing tile in cell (i,j), we associate to it the simple transposition s_{i+j-1}. We then associate a reading word $\mathsf{word}(P)$ to P by reading these simple transpositions within rows from right to left, working from the top row downwards. We say P is a pipe dream for w if w is the Demazure product of $\mathsf{word}(P)$. Write $\mathsf{Pipes}(w)$ for the set of pipe dreams for w. We say $P \in \mathsf{Pipes}(w)$ is ***reduced*** if $\mathsf{word}(P)$ is a reduced word for w and write $\mathsf{Pipes}_0(w)$ for the subset of reduced pipe dreams.

Definition 2.2. For any $w \in S_n$, we have

$$\mathfrak{G}_w(\boldsymbol{x};\boldsymbol{y}) = \sum_{P\in\mathsf{Pipes}(w)} (-1)^{|P|-\operatorname{inv}(w)} \prod_{(i,j)\in P} (x_i + y_j - x_i y_j).$$

Below are the reduced (blue) and nonreduced (red) pipe dreams for $w = 42153$.



We obtain the ***(single) Grothendieck polynomials*** by specializing $\boldsymbol{y}$ to 0, that is to say, $\mathfrak{G}_w(\boldsymbol{x}) := \mathfrak{G}_w(\boldsymbol{x};\boldsymbol{0})$. The ***double Schubert polynomial*** $\mathfrak{S}_w(\boldsymbol{x},\boldsymbol{y})$ is the lowest-degree homogeneous part of the double Grothendieck polynomial $\mathfrak{G}_w(\boldsymbol{x},\boldsymbol{y})$ substituting $y_j \mapsto -y_j$, while the ***(single) Schubert polynomial*** $\mathfrak{S}_w(\boldsymbol{x})$ is the lowest-degree homogeneous part of $\mathfrak{G}_w(\boldsymbol{x})$. The degree of the Schubert polynomial $\mathfrak{S}_w(\boldsymbol{x})$ is $\operatorname{inv}(w)$.

Lemma 2.3. *Let w be a permutation and let d be the degree of $\mathfrak{G}_w(\boldsymbol{x})$. Then $\mathfrak{G}_w(\boldsymbol{x};\boldsymbol{y})$ has terms which are of bidegree (d,d) in the $\boldsymbol{x}$ and $\boldsymbol{y}$ variables, and no term in $\mathfrak{G}_w(\boldsymbol{x};\boldsymbol{y})$ has $\boldsymbol{x}$-degree or $\boldsymbol{y}$-degree higher than d.*

Define the ***Castlenuovo–Mumford polynomial*** $\mathfrak{CM}(\boldsymbol{x})$ to be $(-1)^{\deg \mathfrak{G}_w(\boldsymbol{x}) - \operatorname{inv}(w)}$ times the highest degree part of $\mathfrak{G}_w(\boldsymbol{x})$ and define the ***double Castlenuovo–Mumford polynomial***, $\mathfrak{CM}_w(\boldsymbol{x};\boldsymbol{y})$, to be $(-1)^{\operatorname{inv}(w)}$ times the highest degree part of $\mathfrak{G}_w(\boldsymbol{x};\boldsymbol{y})$. These sign factors make $\mathfrak{CM}_w(\boldsymbol{x};\boldsymbol{y})$ and $\mathfrak{CM}_w(\boldsymbol{x})$ have positive coefficients. So the degree of $\mathfrak{CM}_w(\boldsymbol{x})$, and the bidegree of $\mathfrak{CM}_w(\boldsymbol{x};\boldsymbol{y})$, are both given by the maximal number of crosses in any pipe dream for w.

If we specialize the single Grothendieck polynomial $\mathfrak{G}_w(\boldsymbol{x})$ by setting $x_i \mapsto 1-t$, we obtain the K-polynomial of the matrix Schubert variety X_w [3]. Note that this specialization does not affect the degrees of the polynomials, since all top-degree terms of $\mathfrak{G}_w(\boldsymbol{x})$ have the same sign. Moreover, $\deg \mathfrak{S}_w(\boldsymbol{x}) = \operatorname{codim} X_w = \operatorname{inv}(w)$ [1]. Thus, we can rewrite Lemma 2.1 as follows.

Corollary 2.4. *Let X_w be a matrix Schubert variety. Then* $\operatorname{reg} X_w = \deg \mathfrak{G}_w(\boldsymbol{x}) - \operatorname{inv}(w)$.

3 Simple properties of Rajchgot index

Lemma 3.1. *Let w be a permutation and let $(\ell_1(w), \ell_2(w), \ldots, \ell_n(w))$ be its inversion code. Then $r_i(w) \geq \ell_i(w)$ with equality if and only if w has no 132 pattern starting at position i. Hence,* $\operatorname{raj}(w) \geq \operatorname{inv}(w)$, *with equality if and only if w avoids the pattern 132, i.e. if and only if w is dominant.*

Say $w \in S_n$ is ***fireworks*** if the initial elements of its decreasing runs are in increasing order. Say w is ***inverse fireworks*** if w^{-1} is fireworks. For example, the permutation $41|62|853|97$ is fireworks because $4 < 6 < 8 < 9$.

Let $w \in S_n$ and let $m_i(w)$ equal the number of $j \geq i$ such that $w_j > w_{j+1}$. The ***major index*** of w is $\operatorname{maj}(w) = \sum_{i=1}^{n} m_i(w)$. Theorem 4.16 will establish a formula for raj as a maximum of many values of maj. At the moment, we have an inequality:

Lemma 3.2. *Let w be a permutation and let $m_i(w)$ equal the number of $j \geq i$ such that $w_j > w_{j+1}$. Then $r_i(w) \geq m_i(w)$. Hence, $\mathrm{raj}(w) \geq \mathrm{maj}(w)$, with equality if and only if w is fireworks.*

A ***set partition*** of $[n]$ is a collection π of pairwise-disjoint nonempty subsets of $[n]$ with union $[n]$; the subsets are called the ***blocks*** of π. We order the blocks of π by their largest elements and index them as $(\pi_t, \pi_{t+1}, \ldots, \pi_n)$, where $\max(\pi_t) < \max(\pi_{t+1}) < \cdots < \max(\pi_n)$. For brevity, we omit commas and braces from our notation, for example, writing $(1, 3, 45, 67, 8, 29)$ rather than $\{\{1\}, \{3\}, \{4, 5\}, \{6, 7\}, \{8\}, \{2,9\}\}$.

Proposition 3.3. *Fireworks permutations are enumerated by the Bell numbers.*

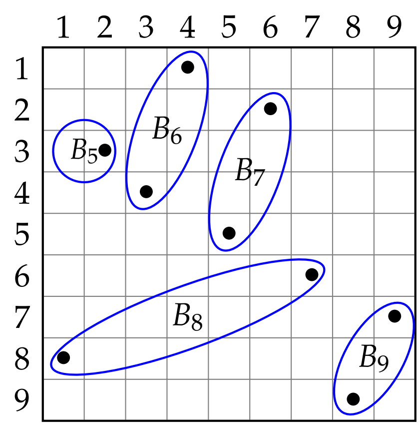
For example, the fireworks permutation 416285397 corresponds to $(14, 26, 358, 79)$.

A permutation w is ***valley*** if there is some index a with $w(1) > w(2) > \cdots > w(a) < w(a+1) < \cdots < w(n)$. There are 2^{n-1} valley permutations in S_n, because a valley permutation is uniquely determined by the subset $\{w(1), w(2), \ldots, w(a-1)\}$ of $\{2, 3, \ldots, n\}$. A permutation w is ***inverse valley*** if w^{-1} is valley.

Lemma 3.4. *A permutation w is a valley permutation if and only if w is both dominant and inverse fireworks; w is inverse valley if and only if w is both dominant and fireworks.*

4 Main results

We give a pictorial description of Rajchgot code. Draw the graph of w. Now, draw a lasso around the set of dots that are maximally southeast in the grid. Call this set of dots $B_n(w)$. Then draw another lasso around the dots that are maximally southeast among dots that are not lassoed. This next set of dots is $B_{n-1}(w)$, *etc.* We call this the ***blob diagram*** of w (see example below).



Lemma 4.1. *Let (i, w_i) be in blob B_k. Then the longest increasing subsequence starting at (i, w_i) contains $n + 1 - k$ elements.*

Corollary 4.2. *Let $(r_1(w), r_2(w), \ldots, r_n(w))$ be the Rajchgot code of w, and let (i, w_i) be in blob B_k. Then $r_i(w) = k - i$.*

Define $\pi_k(w)$ to be the set of column labels of the dots in $B_k(w)$. (By symmetry, $\pi_k(w^{-1})$ is the row labels of the dots in $B_k(w)$.) Then $\pi(w)$ is the set partition of w. Note that i and j are in the same block of the set partition exactly if $(w^{-1}(i), i)$ and $(w^{-1}(j), j)$ are in the same blob. In our example, $\pi(w)$ is the set partition $\{2, 34, 56, 17, 89\}$. (We obtain $\pi(w^{-1})$ by recording the row labels of the entries in each blob.) The ordering of the blocks of $\pi(w)$ is recoverable from $\pi(w)$, since the maximum elements of the blocks occur in increasing order. Index the blocks of $\pi(w)$ as $(\pi_t, \pi_{t+1}, \ldots, \pi_n)$, so that π_k corresponds to block B_k, and set $\alpha_k(w) = \#\pi_k(w)$. Define the composition $(\alpha_t(w), \alpha_{t+1}(w), \ldots, \alpha_n(w))$ to be the ***shape*** of w. We can express Rajchgot index in terms of shape.

Lemma 4.3. *For $w \in S_n$, we have*

$$\mathrm{raj}(w) = \sum_{k=1}^{n} k\alpha_k - \binom{n+1}{2} = \sum_{k=1}^{n} (\alpha_k + \alpha_{k+1} + \cdots + \alpha_n) - \binom{n+1}{2}.$$

Corollary 4.4. *We have* $\mathrm{raj}(w) = \mathrm{raj}(w^{-1})$.

For compositions $(\alpha_j, \alpha_{j+1}, \ldots, \alpha_n)$ and $(\beta_k, \beta_{k+1}, \ldots, \beta_n)$ of n, say α ***dominates*** β ($\alpha \succeq \beta$) if $\alpha_m + \alpha_{m+1} + \cdots + \alpha_n \geq \beta_m + \beta_{m+1} + \cdots + \beta_n$ for all m. We write $\alpha \succ \beta$ to mean $\alpha \succeq \beta$ and $\alpha \neq \beta$.

Corollary 4.5. *Let u and v be permutations of shapes α and β. If $\alpha \succeq \beta$, then $\mathrm{raj}(u) \geq \mathrm{raj}(v)$; if $\alpha \succ \beta$, then $\mathrm{raj}(u) > \mathrm{raj}(v)$.*

We now provide some lemmas without proof that are key to establishing our main results. For more details, see [6].

Lemma 4.6. *For each composition α of n, there is exactly one valley permutation, f_α of shape α, and likewise one inverse valley permutation of shape α, which is f_α^{-1}.*

Lemma 4.7. *Let u and v be permutations with $u \geq_{LR} v$. Then $\mathrm{raj}(u) \geq \mathrm{raj}(v)$. We have $\mathrm{raj}(u) = \mathrm{raj}(v)$ if and only if u and v have the same shape.*

Lemma 4.8. *If $w >_L s_i w$ and $\mathrm{raj}(w) = \mathrm{raj}(s_i w)$, then $\mathrm{rajcode}(w) = \mathrm{rajcode}(s_i w)$. Suppose that $w >_R w s_i$ and $\mathrm{raj}(w) = \mathrm{raj}(w s_i)$. Let $\mathrm{rajcode}(w) = (r_1, r_2, \ldots, r_n)$. Then $\mathrm{rajcode}(w s_i) = (r_1, r_2, \ldots, r_{i+1} + 1, r_i - 1, \ldots, r_n)$.*

Lemma 4.9. *The permutation $w \in S_n$ is fireworks if and only if the dots in each blob occupy consecutive rows of the graph of w. Likewise, w is inverse fireworks if and only if the dots in each blob occupy consecutive columns.*

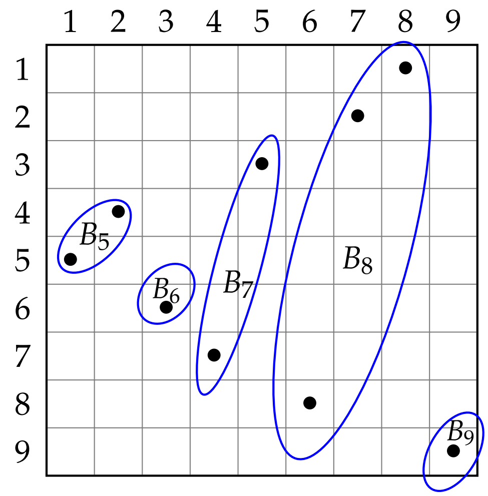
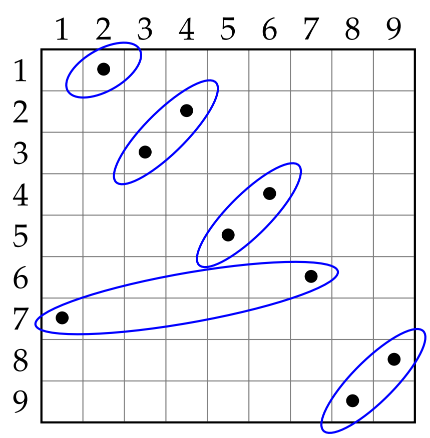


Figure 1: For $\alpha = (2,1,2,3,1)$, the proof of Lemma 4.6 gives $R = \{1 < 3 < 4 < 6 < 9\}$ and $[9]\backslash R = \{8 > 7 > 5 > 2\}$. So the corresponding valley permutation is $f = 875213469$. Here we have drawn the blob diagram of f.

We now describe a ***fireworks map*** that turns an arbitrary permutation w into a fireworks permutation $\Phi(w)$. The fireworks permutation $\Phi(w)$ corresponds to the set partition $\pi(w)$ using the bijection of Proposition 3.3. In other words, we take the dots in the graph of w and shove the dots of each blob into consecutive rows. We define $\Phi_{\text{inv}}(w) = \Phi(w^{-1})^{-1}$.

For example, let $w = 462357918$. We computed before that $\pi(w) = \{2, 34, 56, 17, 89\}$. The corresponding fireworks permutation is 243657198 with blob diagram below.



Lemma 4.10. *For any permutation w, we have $\Phi(w) \leq_R w$ and $\Phi_{\text{inv}}(w) \leq_L w$.*

Remark 4.11. Although $\Phi(w) \leq_R w$, this does not mean that Φ is order-preserving! For example, $\Phi(4312) = 1432$ and $\Phi(3412) = 3142$. Now $4312 = 3412 \cdot s_1$, so $3412 <_R 4312$, but $3142 \not\leq_R 1432$ since 1 and 3 are noninverted in 1432, but inverted in 3142.

Lemma 4.12. *The blobs B_k of $\Phi(w)$ and of w consist of dots in the same columns, the permutation w is fireworks if and only if $\Phi(w) = w$, and the permutations w and $\Phi(w)$ have the same shape. Corresponding statements hold for Φ_{inv} and being inverse fireworks.*

Corollary 4.13. *For $w \in S_n$, we have $\mathrm{raj}(w) = \mathrm{raj}(w^{-1}) = \mathrm{raj}(\Phi(w)) = \mathrm{raj}(\Phi_{\mathrm{inv}}(w))$.*

In the case of the inverse fireworks map, we can state a stronger result.

Corollary 4.14. *For any permutation w, we have $\mathrm{rajcode}(w) = \mathrm{rajcode}(\Phi_{\mathrm{inv}}(w))$.*

For a composition $\alpha = (\alpha_1, \ldots, \alpha_r)$, let S_α be the Young subgroup $S_{\alpha_1} \times S_{\alpha_2} \times \cdots \times S_{\alpha_r}$ and let e_α be its longest element. A permutation w is called ***layered*** if $w = e_\alpha$ for some α.

Lemma 4.15. *Let w be a permutation of shape α. Then $\Phi(\Phi_{\mathrm{inv}}(w)) = \Phi_{\mathrm{inv}}(\Phi(w)) = e_\alpha$.*

Theorem 4.16. *For $w \in S_n$, $\mathrm{raj}(w) = \max\{\mathrm{maj}(v) : v \leq_R w\} = \max\{\mathrm{maj}(u^{-1}) : u \leq_L w\}$.*

We now discuss interactions between the fireworks maps and weak orders.

Lemma 4.17. *If $w \in S_n$ has shape α, then w has a unique length-additive factorization $w = ue_\alpha v$, for some permutations $u, v \in S_n$. Moreover, we have $\Phi(w) = ue_\alpha$ and $\Phi_{\mathrm{inv}}(w) = e_\alpha v$.*

We can use this factorization to understand the $\leq_{LR}$ interval $[e_\alpha, w]_{LR}$.

Lemma 4.18. *Let w be a permutation of shape α, and let $w = ue_\alpha v$ be the unique length-additive factorization of w. Then the map $\mu : (u', v') \mapsto u' e_\alpha v'$ is a poset isomorphism from the product $[e_\alpha, u]_L \times [e_\alpha, v]_R$ to the two-sided weak interval $[e_\alpha, w]_{LR}$.*

Weak order gives a new perspective on fireworks permutations of a given shape.

Lemma 4.19. *Let f_α be the unique valley permutation of shape α (introduced in Lemma 4.6). Then the set of fireworks permutations of shape α is the left interval $[e_\alpha, f_\alpha^{-1}]_L$, and the set of inverse fireworks permutations of shape α is the right interval $[e_\alpha, f_\alpha]_R$.*

Lemma 4.20. *Let x be any permutation in S_n, let $\alpha = (\alpha_k, \alpha_{k+1}, \ldots, \alpha_n)$ be a composition of n, and let $y = e_\alpha * x * e_\alpha$, where $*$ denotes the Demazure product. Then $\mathrm{raj}(y) \geq \mathrm{raj}(e_\alpha)$ and we have equality if and only if $y = e_\alpha$.*

Example 4.21. We list all permutations of shape $\alpha = (1, 1, 2)$, their factorizations in the form $ue_\alpha v$, and the corresponding double Castelnuovo–Mumford polynomials:

2341		
1342		
1243	1423	4123

$s_1 s_2 e_{112}$		
$s_2 e_{112}$		
e_{112}	$e_{112} s_2$	$e_{112} s_2 s_1$

$(x_1x_2x_3)(y_1^3)$		
$(x_1x_2x_3)(y_1^2y_2+y_1y_2^2)$		
$(x_1x_2x_3)(y_1y_2y_3)$	$(x_1^2x_2+x_1x_2^2)(y_1y_2y_3)$	$(x_1^3)(y_1y_2y_3)$

The layered permutation $e_{112} = 1243$ is in the lower left, the valley permutation $f_{112} = 4123$ is in the lower right and the inverse valley permutation $f_{112}^{-1} = 2341$ is in the upper left. The permutations in the left column are fireworks; the permutations in the bottom row are inverse fireworks, and the permutations which are maximally northeast are dominant. The maps Φ and Φ_{inv} are the orthogonal projections onto the left column and bottom row, respectively.

Lemma 4.22. *If $u \leq_{LR} w$, then $\deg \mathfrak{CM}_u(\boldsymbol{x}) \leq \deg \mathfrak{CM}_w(\boldsymbol{x})$.*

Example 4.23. Note that the analogous result does not hold for the strong order. For instance, $1432 \leq 3412$ but $\deg \mathfrak{G}_{1432}(\boldsymbol{x}) = 5$ and $\deg \mathfrak{G}_{3412}(\boldsymbol{x}) = 4$.

Lemma 4.24. *If w is dominant, then $\deg \mathfrak{CM}_w(\boldsymbol{x}) = \deg \mathfrak{S}_w(\boldsymbol{x}) = \mathrm{inv}(w) = \mathrm{raj}(w)$.*

Lemma 4.25. *If $w = e_\alpha$, then $\deg \mathfrak{CM}_w(\boldsymbol{x}) = \mathrm{raj}(w)$.*

Proposition 4.26. *Let $w \in S_n$ have shape α. The permutation w is $\leq_{LR}$-maximal among permutations of shape α if and only if w is dominant.*

Theorem 4.27. *Let $w \in S_n$. Then $\deg \mathfrak{CM}_w(\boldsymbol{x}) = \mathrm{raj}(w)$.*

Definition 4.28. Let π be a set partition of $[n]$ and let w be the unique fireworks permutation with $\pi(w) = \pi$. We define the ***Rajchgot polynomial*** $\mathfrak{R}_\pi(\boldsymbol{x})$ to be $\mathfrak{CM}_w(\boldsymbol{x})$.

Theorem 4.29. *For any $w \in S_n$, we have $\mathfrak{CM}_w(\boldsymbol{x};\boldsymbol{y}) = \mathfrak{R}_{\pi(w)}(\boldsymbol{x})\mathfrak{R}_{\pi(w^{-1})}(\boldsymbol{y})$.*

Lemma 4.30. *Let $p \in [\Phi_{\mathrm{inv}}(w), w]_L$ and $q \in [\Phi(w), w]_R$. Then $q * p = w$ if and only if $p = \Phi_{\mathrm{inv}}(w)$ and $q = \Phi(w)$.*

By Lemma 4.17, each permutation $w \in S_n$ with $\alpha(w) = \alpha$ has a unique length-additive factorization $w = ue_\alpha v$, where $\Phi(w) = ue_\alpha$ and $\Phi_{\mathrm{inv}}(w) = e_\alpha v$. Therefore, the interval $[\Phi_{\mathrm{inv}}(w), w]_L$ is $\{u'e_\alpha v : \mathrm{id} \leq_L u' \leq_L u\}$ and $[\Phi(w), w]_R$ is $\{ue_\alpha v' : \mathrm{id} \leq_R v' \leq_R v\}$.

Theorem 4.31. *Let w be a permutation, let $\mathrm{rajcode}(w) = (r_1, \ldots, r_n)$ and let $\mathrm{rajcode}(w^{-1}) = (s_1, \ldots, s_n)$. For any term order satisfying $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$, every monomial of $\mathfrak{CM}_w(\boldsymbol{x};\boldsymbol{y})$ is at most $x_1^{r_1} \cdots x_n^{r_n} y_1^{s_1} \cdots y_n^{s_n}$.*

The primary statement now outstanding from Theorems 1.1 and 1.3 is that there is a pipe dream for w with $\mathrm{rajcode}_i(w)$ crossing tiles in row i and $\mathrm{rajcode}_j(w^{-1})$ crossing tiles in column j, thereby contributing the monomial $\boldsymbol{x}^{\mathrm{rajcode}(w)}\boldsymbol{y}^{\mathrm{rajcode}(w^{-1})}$ to $\mathfrak{CM}_w(\boldsymbol{x};\boldsymbol{y})$. We call such a pipe dream a ***maximal pipe dream*** for w.

Theorem 4.32. *Let w be a permutation, let $\mathrm{rajcode}(w) = (r_1, \ldots, r_n)$ and let $\mathrm{rajcode}(w^{-1}) = (s_1, \ldots, s_n)$. There is a unique pipe dream for w with r_i crosses in row i and s_j crosses in column j. Hence, the monomial $\boldsymbol{x}^{\mathrm{rajcode}(w)}\boldsymbol{y}^{\mathrm{rajcode}(w^{-1})}$ appears with coefficient 1 in $\mathfrak{CM}_w(\boldsymbol{x};\boldsymbol{y})$.*

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