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Horizontal-Strip LLT Polynomials

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Abstract. Lascoux, Leclerc, and Thibon defined a remarkable family of symmetric functions that are q-deformations of products of skew Schur functions. These LLT polynomials $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q)$ can be indexed by a tuple $\boldsymbol{\lambda}$ of skew diagrams. When each skew diagram of $\boldsymbol{\lambda}$ is a row, we define a weighted graph $\Pi(\boldsymbol{\lambda})$ associated to $\boldsymbol{\lambda}$. We show that a horizontal-strip LLT polynomial is determined by this weighted graph. When $\Pi(\boldsymbol{\lambda})$ has no triangles, we establish a combinatorial Schur expansion of $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q)$. We also explore a connection to extended chromatic symmetric functions.

Keywords: chromatic symmetric function, cocharge, Hall–Littlewood polynomial, LLT polynomial, Schur-positive, symmetric function

1 Introduction

Lascoux–Leclerc–Thibon (LLT) polynomials are symmetric functions $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q)$ indexed by a sequence $\boldsymbol{\lambda}$ of skew diagrams, and are actively studied in algebraic combinatorics and representation theory. Horizontal-strip LLT polynomials generalize Hall–Littlewood polynomials, which are the Frobenius series of cohomology rings of certain subsets of the flag variety [7]. The Shuffle Theorem [3] of Carlsson and Mellit gives a positive combinatorial formula for $\nabla(e_n)$, the Frobenius series of the space of diagonal harmonics, in terms of LLT polynomials. Haglund, Haiman, and Loehr [6] proved that Macdonald polynomials also expand positively into LLT polynomials, which implies that they are Schur-positive. LLT polynomials are also closely related to chromatic quasisymmetric functions, defined by Shareshian and Wachs [12].

Although LLT polynomials are known to be Schur-positive through Kazhdan–Lusztig theory [5, 11], it remains a predominant open problem to give a combinatorial proof.

Problem 1.1. *Find a combinatorial Schur expansion of LLT polynomials of the form*

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = \sum_{T \in S} q^{stat(T)} s_{partition(T)}. \tag{1.1}$$

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Because LLT polynomials generalize products of skew Schur functions, for which characterizing equalities is an area of active research, it is also interesting and challenging to determine when two LLT polynomials are equal.

Problem 1.2. *Characterize those sequences of skew diagrams $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ for which we have*

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = G_{\boldsymbol{\mu}}(\boldsymbol{x};q). \tag{1.2}$$

In the unicellular case, meaning that every skew diagram of $\boldsymbol{\lambda}$ consists of a single cell, we can associate a unit interval graph $\Gamma(\boldsymbol{\lambda})$ to $\boldsymbol{\lambda}$. Huh, Nam, and Yoo [8] proved a combinatorial Schur expansion of the LLT polynomial $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q)$ whenever $\Gamma(\boldsymbol{\lambda})$ is a "melting lollipop", namely

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = \sum_{T\in \mathrm{SYT}_n} q^{\mathrm{wt}_a(T)} s_{\mathrm{shape}(T)}. \tag{1.3}$$

Moreover, the unicellular LLT polynomial $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q)$ is related by a change of variables to the chromatic quasisymmetric function of $\Gamma(\boldsymbol{\lambda})$, specifically by the plethysm [3]

$$(q-1)^{-n} G_{\boldsymbol{\lambda}}[\boldsymbol{x}(q-1);q] = X_{\Gamma(\boldsymbol{\lambda})}(\boldsymbol{x};q). \tag{1.4}$$

In particular, this implies that for $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ sequences of single cells, we have

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = G_{\boldsymbol{\mu}}(\boldsymbol{x};q) \text{ if and only if } X_{\Gamma(\boldsymbol{\lambda})}(\boldsymbol{x};q) = X_{\Gamma(\boldsymbol{\mu})}(\boldsymbol{x};q). \tag{1.5}$$

More generally, we consider the horizontal-strip case, meaning that every skew diagram of $\boldsymbol{\lambda}$ is a row. We generalize the construction $\Gamma(\boldsymbol{\lambda})$ by defining a weighted interval graph $\Pi(\boldsymbol{\lambda})$ associated to $\boldsymbol{\lambda}$. We prove that a horizontal-strip LLT polynomial is determined by this weighted graph, which provides a powerful sufficient condition for equality of LLT polynomials.

Theorem 1.3 ([14, Theorem 2.7])**.** *Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be sequences of rows.*

$$\text{If } \Pi(\boldsymbol{\lambda}) \cong \Pi(\boldsymbol{\mu}), \text{ then } G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = G_{\boldsymbol{\mu}}(\boldsymbol{x};q). \tag{1.6}$$

We define a statistic on tableaux, denoted $\mathrm{cocharge}_{\Pi(\boldsymbol{\lambda})}$, and we prove the following combinatorial Schur expansion of $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q)$ whenever the weighted graph $\Pi(\boldsymbol{\lambda})$ has no triangles.

Theorem 1.4 ([13, Theorem 4.6])**.** *Let $\boldsymbol{\lambda}$ be a sequence of rows such that $\Pi(\boldsymbol{\lambda})$ is triangle-free. Then*

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = \sum_{T\in SSYT(\alpha)} q^{\mathit{cocharge}_{\Pi(\boldsymbol{\lambda})}} s_{\mathit{shape}(T)}. \tag{1.7}$$

We also show that at $q = 1$, the horizontal-strip LLT polynomial $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q)$ is related by a change of variables to the extended chromatic symmetric function $X_{\Pi(\boldsymbol{\lambda})}(\boldsymbol{x})$ defined by Crew and Spirkl [4] for a weighted graph.

Theorem 1.5. *Let $\boldsymbol{\lambda}$ be a sequence of n rows. The plethystically modified LLT polynomial $G_{\boldsymbol{\lambda}}[\boldsymbol{x}(q-1); q]$ is divisible by $(q-1)^n$, and at $q = 1$ we have*

$$\left((q-1)^{-n} G_{\boldsymbol{\lambda}}[\boldsymbol{x}(q-1); q]\right)|_{q=1} = X_{\Pi(\boldsymbol{\lambda})}(\boldsymbol{x}). \tag{1.8}$$

In particular, this implies that for $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ sequences of rows, we have that

$$\text{if } G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = G_{\boldsymbol{\mu}}(\boldsymbol{x}; q), \text{ then } X_{\Pi(\boldsymbol{\lambda})}(\boldsymbol{x}) = X_{\Pi(\boldsymbol{\mu})}(\boldsymbol{x}). \tag{1.9}$$

2 Background

A *partition* σ is a finite nonincreasing sequence of positive integers $\sigma = \sigma_1 \cdots \sigma_\ell$. By convention, we set $\sigma_i = 0$ if $i > \ell$. A *skew diagram* λ is a subset of $\mathbb{Z} \times \mathbb{Z}$ of the form

$$\lambda = \sigma/\tau = \{(i,j) : \ i \geq 1, \ \tau_i + 1 \leq j \leq \sigma_i\} \tag{2.1}$$

for some partitions σ and τ with $\sigma_i \geq \tau_i$ for every i. When τ is empty, we write σ instead of $\sigma/\emptyset$. The elements of λ are called *cells* and the *content* of a cell $u = (i,j) \in \lambda$ is the integer $c(u) = j - i$. We will focus heavily on *rows*, which are skew diagrams of the form

$$R = a/b = \{(1,j) : \ b+1 \leq j \leq a\} \tag{2.2}$$

for some $a \geq b \geq 0$. We denote by $\ell(R) = b$ and $r(R) = a - 1$ the smallest and largest contents of cells in R respectively. Note that $\ell(R)$ is the content of the leftmost cell in R, not the length of R, which is $|R| = r(R) - \ell(R) + 1$. We also denote by $R^+ = (a+1)/(b+1)$ and $R^- = (a-1)/(b-1)$ the rows obtained by shifting R right or left respectively by one cell. A *semistandard Young tableau (SSYT)* of shape λ is a function $T : \lambda \to \{1, 2, 3, \ldots\}$ that satisfies

$$T_{i,j} \leq T_{i,j+1} \text{ and } T_{i,j} < T_{i+1,j}, \tag{2.3}$$

where we write $T_{i,j}$ to mean $T((i,j))$. The *weight* of T is the sequence $w(T) = (w_1, w_2, \ldots)$, where $w_i = |T^{-1}(i)|$ is the number of times the integer i appears. We denote by SSYT_λ the set of SSYT of shape λ and by $\mathrm{SSYT}(\alpha)$ the set of SSYT of weight α. We define the *skew Schur function* of shape $\lambda = \sigma/\tau$ to be

$$s_\lambda = \sum_{T \in \mathrm{SSYT}_\lambda} \boldsymbol{x}^T, \tag{2.4}$$

where $\boldsymbol{x}^T$ is the monomial $x_1^{w_1} x_2^{w_2} \cdots$. When τ is empty, we call s_λ a *Schur function*.

A *multiskew partition* is a finite sequence of skew diagrams $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(n)})$. We say that $\boldsymbol{\lambda}$ is *unicellular* if each $\lambda^{(i)}$ is a single cell and in keeping with the terminology of Alexandersson and Sulzgruber [1], we say that $\boldsymbol{\lambda}$ is a *horizontal-strip* if each $\lambda^{(i)}$ is a row. We denote by

$$\mathrm{SSYT}_{\boldsymbol{\lambda}} = \{\boldsymbol{T} = (T^{(1)}, \ldots, T^{(n)}) : \ T^{(i)} \in \mathrm{SSYT}_{\lambda^{(i)}}\} \tag{2.5}$$

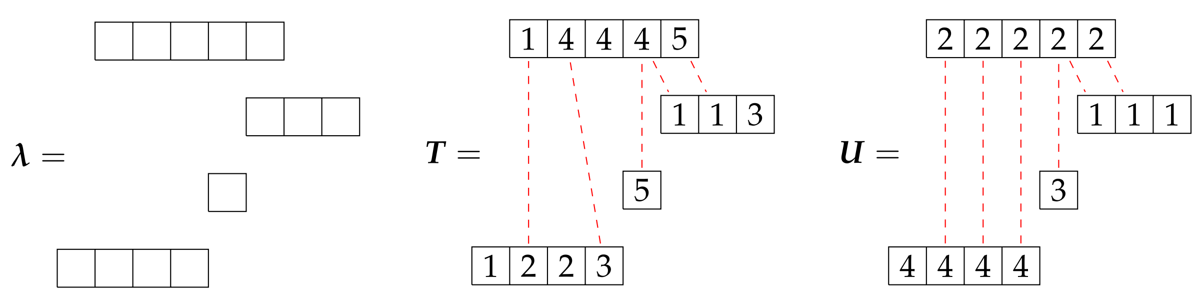
the set of *semistandard multiskew tableaux* of shape $\boldsymbol{\lambda}$. Two entries $T^{(i)}(u)$ and $T^{(j)}(v)$ with $i < j$ form an *inversion* if either

- $c(u) = c(v)$ and $T^{(i)}(u) > T^{(j)}(v)$, or
- $c(u) = c(v) + 1$ and $T^{(j)}(v) > T^{(i)}(u)$.

We denote by $\mathrm{inv}(\boldsymbol{T})$ the number of inversions of $\boldsymbol{T}$. Now we define the *LLT polynomial* [6, 10] as

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = \sum_{\boldsymbol{T} \in \mathrm{SSYT}_{\boldsymbol{\lambda}}} q^{\mathrm{inv}(\boldsymbol{T})} \boldsymbol{x}^{\boldsymbol{T}}. \tag{2.6}$$

Example 2.1. The horizontal-strip $\boldsymbol{\lambda} = (4/0, 5/4, 8/5, 6/1)$, drawn so that cells of the same content are aligned vertically, two SSYTs $\boldsymbol{S}$ and $\boldsymbol{T}$ of shape $\boldsymbol{\lambda}$ with their inversions marked by dashed lines, and the corresponding monomials of the LLT polynomial $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q)$ are given below.



$q^5 x_1^4 x_2^2 x_3^2 x_4^3 x_5^2$ $\qquad$ $q^6 x_1^3 x_2^5 x_3 x_4^4$

Some terms of the Schur function expansion of the LLT polynomial $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q)$ are

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = q^6 s_{5431}(\boldsymbol{x}) + q^6 s_{544}(\boldsymbol{x}) + \cdots + (q^6 + 2q^5) s_{733}(\boldsymbol{x}) + \cdots + 3q s_{(12)1}(\boldsymbol{x}) + s_{(13)}(\boldsymbol{x}). \tag{2.7}$$

Note that $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q)$ is *Schur-positive*, meaning that it is an $\mathbb{N}[q]$-linear combination of Schur functions. In fact, this property holds in general [5, Corollary 6.9].

3 A weighted graph description

We now define our weighted graph $\Pi(\boldsymbol{\lambda})$.

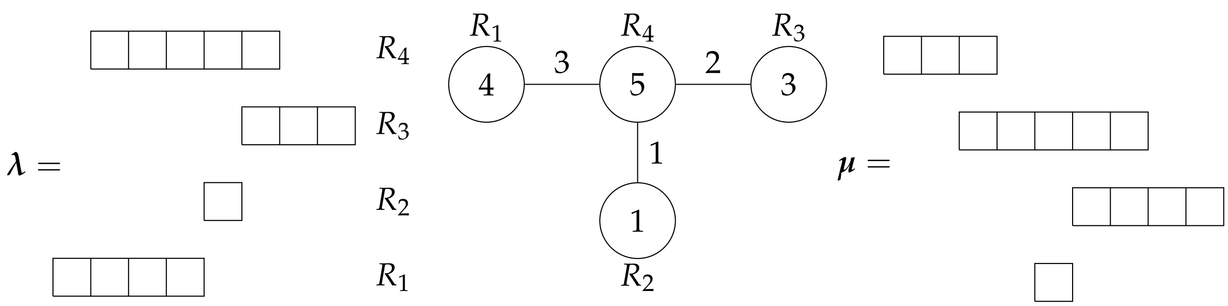
Definition 3.1 ([13, Definition 3.1])**.** Let R and R' be rows. We define the integer

$$M(R,R') = \begin{cases} |R \cap R'| & \text{if } \ell(R) \le \ell(R'), \\ |R \cap R'^{+}| & \text{if } \ell(R) > \ell(R'). \end{cases} \tag{3.1}$$

Note that $0 \le M(R,R') \le \min\{|R|,|R'|\}$. We can think of $M(R,R')$ as measuring the extent to which the rows R and R' interact. More precisely, $M(R,R')$ is the maximum number of inversions that a tableau $\boldsymbol{T} \in \mathrm{SSYT}_{\boldsymbol{\lambda}}$ can have between cells in R and R' [13, Theorem 3.5].

Definition 3.2 ([13, Definition 3.2])**.** Let $\boldsymbol{\lambda} = (R_1,\dots,R_n)$ be a horizontal-strip. We define a *weighted graph* $\Pi(\boldsymbol{\lambda})$ whose vertices are the rows of $\boldsymbol{\lambda}$. The weight of a row R_i is the number of cells $|R_i|$ and rows R_i and R_j with $i < j$ are joined by an edge of weight $M(R_i,R_j)$. By convention, we omit edges of weight zero.

Example 3.3. The horizontal-strip $\boldsymbol{\lambda} = (4/0, 5/4, 8/5, 6/1)$ and the weighted graph $\Pi(\boldsymbol{\lambda})$ are given below. We have $M(R_1,R_4) = 3$, $M(R_2,R_4) = 1$, and $M(R_3,R_4) = 2$. We have also drawn the horizontal-strip $\boldsymbol{\mu} = (5/4, 9/5, 7/2, 3/0)$, whose weighted graph $\Pi(\boldsymbol{\mu})$ is isomorphic to $\Pi(\boldsymbol{\lambda})$.



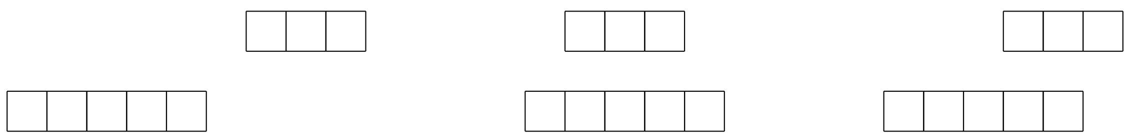
Because $\Pi(\boldsymbol{\lambda}) \cong \Pi(\boldsymbol{\mu})$, it follows from Theorem 1.3 that $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = G_{\boldsymbol{\mu}}(\boldsymbol{x};q)$. Moreover, because the weighted graph $\Pi(\boldsymbol{\lambda})$ is triangle-free, Theorem 1.4 gives a combinatorial Schur expansion of $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q)$. We will give the details in Example 3.18.

We now describe some of the tools that were used to prove Theorem 1.3 and Theorem 1.4. The following two operations, *cycling* and *commuting*, allow us to move the rows of $\boldsymbol{\lambda}$ while preserving both the weighted graph $\Pi(\boldsymbol{\lambda})$ and the LLT polynomial $G_{\boldsymbol{\lambda}}(\boldsymbol{x};q)$.

Proposition 3.4. *(Cycling) Let* $\boldsymbol{\lambda} = (R_1, R_2, \dots, R_n)$ *be a horizontal-strip and let* $\kappa(\boldsymbol{\lambda}) = (R_2,\dots,R_n,R_1^-)$. *Then* $G_{\kappa(\boldsymbol{\lambda})}(\boldsymbol{x};q) = G_{\boldsymbol{\lambda}}(\boldsymbol{x};q)$.

Definition 3.5. We say that rows R and R' *commute*, denoted $R \leftrightarrow R'$, if we have $M(R, R') = M(R', R)$, and otherwise we write $R \not\leftrightarrow R'$.

Example 3.6. The pairs of rows on the left and the middle commute and the pair of rows on the right does not. As a visual description, we have that two rows commute if and only if they are either disjoint and separated by at least one cell, or if one is contained in the other.



Lemma 3.7 (Commuting [13, Lemma 3.15])**.** *Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ be a horizontal-strip and let $\boldsymbol{\mu} = (R_1, \ldots, R_{i+1}, R_i, \ldots, R_n)$. If $R_i \leftrightarrow R_{i+1}$, then $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = G_{\boldsymbol{\mu}}(\boldsymbol{x}; q)$.*

Remark 3.8. We also prove the converse that if $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = G_{\boldsymbol{\mu}}(\boldsymbol{x}; q)$, then $R_i \leftrightarrow R_{i+1}$ and therefore $\Pi(\boldsymbol{\lambda}) \cong \Pi(\boldsymbol{\mu})$. In other words, equalities of LLT polynomials in this case are precisely characterized by the associated weighted graphs.

We now show how we can use cycling and commuting to prove Theorem 1.3 in the following very special case, in which $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q)$ is the modified Hall–Littlewood polynomial $\widetilde{H}_\lambda(\boldsymbol{x}; q)$. This argument captures the spirit of the general proof.

Lemma 3.9 ([14, Lemma 3.15])**.** *Let $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ and $\boldsymbol{\mu}$ be horizontal-strips with*

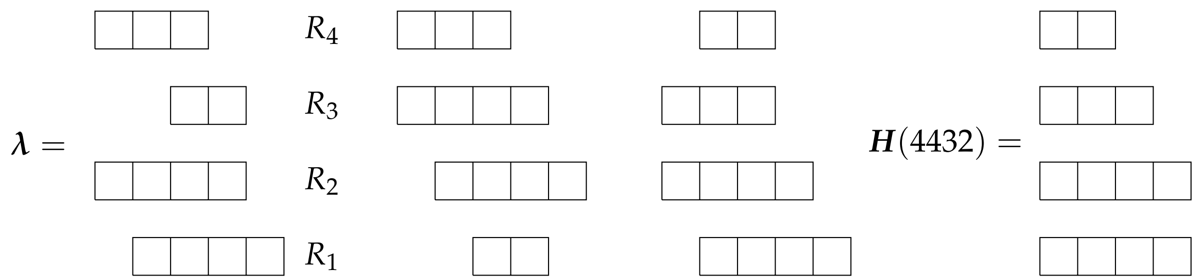
$$M(R_i, R_j) = \min\{|R_i|, |R_j|\} \tag{3.2}$$

for every $1 \le i < j \le n$. If $\Pi(\boldsymbol{\lambda}) \cong \Pi(\boldsymbol{\mu})$, then $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = G_{\boldsymbol{\mu}}(\boldsymbol{x}; q)$.

Proof sketch. Let λ denote the partition determined by the row lengths of $\boldsymbol{\lambda}$ and denote by $\boldsymbol{H}(\lambda)$ the horizontal-strip $(\lambda_1/0, \ldots, \lambda_n/0)$. We show that $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = G_{\boldsymbol{H}(\lambda)}(\boldsymbol{x}; q)$, meaning that the LLT polynomial $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q)$ only depends on λ and therefore only on the weighted graph $\Pi(\boldsymbol{\lambda})$. By translating all cells by a fixed amount, we may assume without loss of generality that $\min\{\ell(R_i) : 1 \le i \le n\} = 0$. Let us also assume that $\boldsymbol{\lambda}$ has $\sum_{i=1}^n \ell(R_i)$ minimal among such horizontal-strips with the same LLT polynomial.

We now claim that $\ell(R_i) = 0$ for every $1 \le i \le n$. If not, let j be such that $\ell(R_j) \ge 1$ is maximal. For every $1 \le i < j$, the conditions $\ell(R_i) \le \ell(R_j)$ and $M(R_i, R_j) = \min\{|R_i|, |R_j|\}$ will imply that $R_i \leftrightarrow R_j$, and therefore by commuting and cycling we have that $G_{(R_1, \ldots, R_n, R_j^-)}(\boldsymbol{x}; q) = G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q)$, contradicting minimality of $\sum_{i=1}^n \ell(R_i)$. Therefore, we have $\ell(R_i) = 0$ for every $1 \le i \le n$, and by commuting again we see that $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = G_{\boldsymbol{H}(\lambda)}(\boldsymbol{x}; q)$. This completes the proof. ∎

Example 3.10. The idea of the proof of Lemma 3.9 is illustrated below. The row R_3, which has $\ell(R_3)$ maximal, commutes with all rows below, so by commuting and cycling, we can move it down and then to the left. Continuing in this way, the horizontal-strip $\boldsymbol{\lambda}$ is shown to be similar to $\boldsymbol{H}(4432)$ on the right.



The general proof of Theorem 1.3 employs this basic technique, along with a comprehensive analysis of several cases, to arrange the rows of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ in order to apply the following recurrence relation of LLT polynomials. We can view it as a deletion-contraction relation of the corresponding weighted graphs.

Lemma 3.11 (Deletion-Contraction [13, Lemma 3.17])**.** *Let* $\boldsymbol{\lambda} = (R_1, \ldots, R_n)$ *be a horizontal-strip with* $\ell(R_{i+1}) < \ell(R_i)$ *and* $R_i \not\leftrightarrow R_{i+1}$. *Define the horizontal-strips*

$$\boldsymbol{\lambda}' = (R_1, \ldots, R_{i+1}, R_i, \ldots, R_n) \text{ and} \tag{3.3}$$
$$\boldsymbol{\lambda}'' = (R_1, \ldots, R_i \cup R_{i+1}, R_i \cap R_{i+1}, \ldots, R_n). \tag{3.4}$$

Then we have

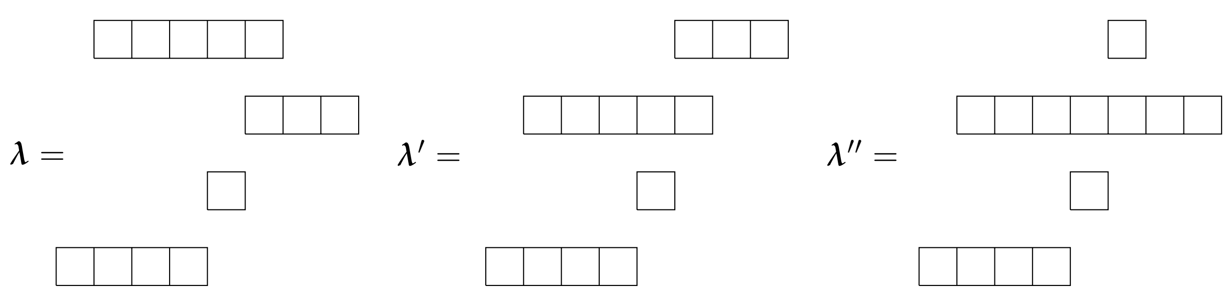
$$G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = qG_{\boldsymbol{\lambda}'}(\boldsymbol{x}; q) - (q-1)G_{\boldsymbol{\lambda}''}(\boldsymbol{x}; q). \tag{3.5}$$

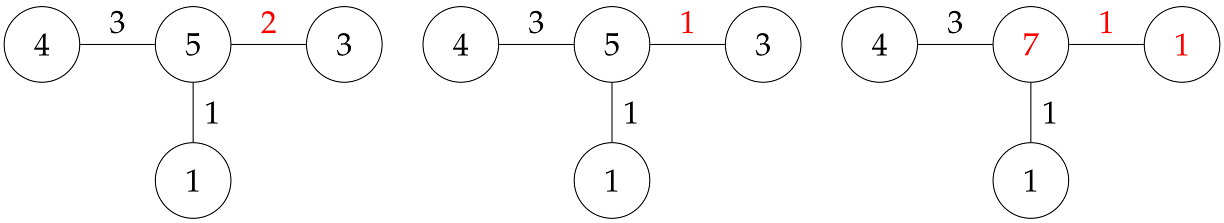
Note that the condition $R_i \not\leftrightarrow R_{i+1}$ will mean that $R_i \cup R_{i+1}$ is indeed a row.

Example 3.12. Let $\boldsymbol{\lambda} = (4/0, 5/4, 8/5, 6/1)$ and note that $\ell(R_4) < \ell(R_3)$ and $R_3 \not\leftrightarrow R_4$. Therefore, letting $\boldsymbol{\lambda}' = (4/0, 5/4, 6/1, 8/5)$ and $\boldsymbol{\lambda}'' = (4/0, 5/4, 8/1, 6/5)$, we have that

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = qG_{\boldsymbol{\lambda}'}(\boldsymbol{x}; q) - (q-1)G_{\boldsymbol{\lambda}''}(\boldsymbol{x}; q). \tag{3.6}$$

The horizontal-strips $\boldsymbol{\lambda}$, $\boldsymbol{\lambda}'$, and $\boldsymbol{\lambda}''$, and their weighted graphs $\Pi(\boldsymbol{\lambda})$, $\Pi(\boldsymbol{\lambda}')$, and $\Pi(\boldsymbol{\lambda}'')$ are given below. We can think of $\Pi(\boldsymbol{\lambda}')$ and $\Pi(\boldsymbol{\lambda}'')$ as a deletion and contraction of $\Pi(\boldsymbol{\lambda})$.





In order to prove Theorem 1.4, we found a combinatorial formula that also satisfies the deletion-contraction relation (3.5). We now describe the statistic $\mathrm{cocharge}_{\Pi}$ in our formula.

Definition 3.13. Let $T \in \mathrm{SSYT}_{\mu}(\alpha)$ be a tableau of shape μ, weight α, and with smallest entry i. We define the integer

$$f(T) = \max\{t :\ 0 \le t \le \mu_1 - \mu_2,\ t \le \alpha_i,\ T_{2,j'} > T_{1,j'+t} \text{ for all } 1 \le j' \le \mu_2\}. \tag{3.7}$$

Informally, $f(T)$ is the maximum number of i's that we can remove from T so that no entry moves down when we rectify the resulting skew tableau.

Definition 3.14. Let $T \in \mathrm{SSYT}_{\mu}(\alpha)$ and let $i < j$. We denote by $T|_{i,j}$ the rectification of the skew tableau obtained by restricting T to the entries x with $i \le x \le j$ and we define the integer

$$\mathrm{cocharge}_{i,j}(T) = \alpha_i - f(T|_{i,j}). \tag{3.8}$$

Example 3.15. Two tableaux S and T and their restrictions $S|_{2,4}$ and $T|_{2,4}$ are given below. We have $f(S|_{2,4}) = 3$ and $\mathrm{cocharge}_{2,4}(S) = 5 - 3 = 2$, and we have $f(T|_{2,4}) = 3$ because we must have $t \le 3$, so $\mathrm{cocharge}_{2,4}(T) = 3 - 3 = 0$.

$$S = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 3 & 4 \\ \hline 2 & 2 & 3 & 4 & 5 & 5 \\ \hline 1 & 1 & 1 & 2 & 2 & 2 & 3 & 4 \\ \hline \end{array} \qquad S|_{2,4} = \begin{array}{|c|c|c|c|c|c|c|}\hline 4 \\ \hline 3 & 3 & 4 \\ \hline 2 & 2 & 2 & 2 & 2 & 3 & 4 \\ \hline \end{array}$$

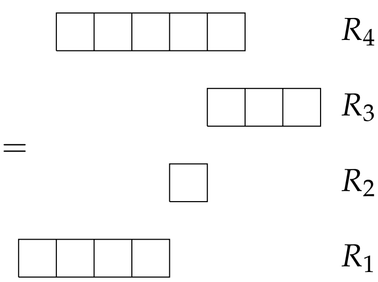
$$T = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 4 & 5 \\ \hline 2 & 2 & 4 & 4 & 5 & 5 \\ \hline 1 & 1 & 1 & 2 & 3 & 3 & 3 & 3 \\ \hline \end{array} \qquad T|_{2,4} = \begin{array}{|c|c|c|c|c|c|c|}\hline 4 & 4 & 4 \\ \hline 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ \hline \end{array}$$

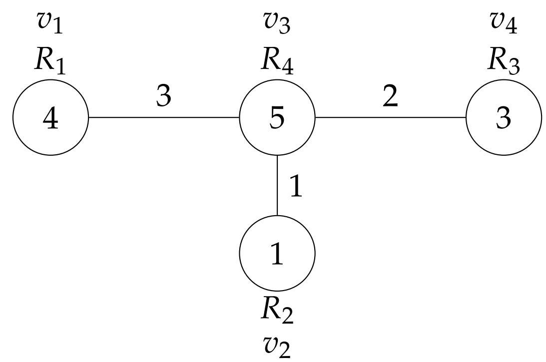
Example 3.16. In the case where $j = i + 1$, the tableau $T|_{i,j}$ has at most two rows and $\mathrm{cocharge}_{i,j}(T)$ is the number of entries on the second row of $T|_{i,j}$.

Definition 3.17. Let $\boldsymbol{\lambda}$ be a horizontal-strip such that the corrersponding weighted graph $\Pi(\boldsymbol{\lambda})$ is triangle-free. We can label the vertices of $\Pi(\boldsymbol{\lambda})$ as $v_1, \ldots, v_n$ so that if $i < j < k$ and v_i is adjacent to v_k, then the weight $M_{j,k}$ of the edge joining v_j and v_k is the weight of the vertex v_j. Let α_i be the weight of the vertex v_i and let $T \in \mathrm{SSYT}(\alpha)$. We define

$$\mathrm{cocharge}_{\Pi(\boldsymbol{\lambda})}(T) = \sum_{i<j} \min\{M_{i,j}, \mathrm{cocharge}_{i,j}(T)\}. \tag{3.9}$$

Example 3.18. The horizontal-strip $\boldsymbol{\lambda} = (4/0, 5/4, 8/5, 6/1)$ and the weighted graph $\Pi(\boldsymbol{\lambda})$ with an appropriate labelling are given below. We have $M_{1,3} = 3$, $M_{2,3} = 1$, and $M_{3,4} = 2$. Note that because v_1 and v_3 are adjacent, $M_{2,3}$ is the weight of v_2.

$\boldsymbol{\lambda} =$ 



To calculate the coefficient of s_{733}, we consider the three tableaux of weight $\alpha = 4153$ and shape 733 below.

$$T_1 = \begin{array}{|c|c|c|cccc}\hline 4&4&4\\\hline 3&3&3\\\hline 1&1&1&\multicolumn{1}{|c|}{1}&\multicolumn{1}{c|}{2}&\multicolumn{1}{c|}{3}&\multicolumn{1}{c|}{3}\\\hline\end{array} \qquad T_2 = \begin{array}{|c|c|c|cccc}\hline 4&4&4\\\hline 2&3&3\\\hline 1&1&1&\multicolumn{1}{|c|}{1}&\multicolumn{1}{c|}{3}&\multicolumn{1}{c|}{3}&\multicolumn{1}{c|}{3}\\\hline\end{array} \qquad T_3 = \begin{array}{|c|c|c|cccc}\hline 3&4&4\\\hline 2&3&3\\\hline 1&1&1&\multicolumn{1}{|c|}{1}&\multicolumn{1}{c|}{3}&\multicolumn{1}{c|}{3}&\multicolumn{1}{c|}{4}\\\hline\end{array}$$

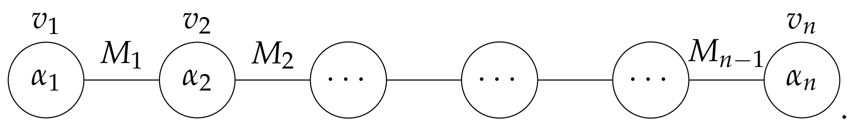
We have

$$\begin{aligned} &\mathrm{cocharge}_{\Pi(\boldsymbol{\lambda})}(T_1) \\ &\quad = \min\{3, \mathrm{cocharge}_{1,3}(T_1)\} + \min\{1, \mathrm{cocharge}_{2,3}(T_1)\} + \min\{2, \mathrm{cocharge}_{3,4}(T_1)\} \\ &\quad = \min\{3, 4-2\} + \min\{1,1\} + \min\{2,3\} = 2+1+2 = 5. \end{aligned} \tag{3.10}$$

Similarly, $\mathrm{cocharge}_{\Pi(\boldsymbol{\lambda})}(T_2) = 3+0+2 = 5$ and $\mathrm{cocharge}_{\Pi(\boldsymbol{\lambda})}(T_3) = 3+1+2 = 6$. Therefore, the coefficient of s_{733} is $(q^6 + 2q^5)$.

In the case where $\Pi(\boldsymbol{\lambda})$ is a path, our formula (1.7) takes on a more convenient form.

Corollary 3.19 ([13, Corollary 4.8])**.** *Let $\boldsymbol{\lambda}$ be a horizontal-strip whose weighted graph $\Pi(\boldsymbol{\lambda})$ is the path below.*

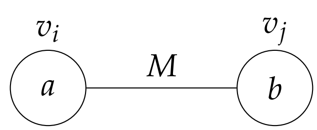


Then the LLT polynomial of $\boldsymbol{\lambda}$ is

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = \sum_{T \in SSYT(\alpha)} q^{cocharge_{\Pi(\boldsymbol{\lambda})}(T)} s_{shape(T)}, \tag{3.11}$$

where $cocharge_{\Pi(\boldsymbol{\lambda})}(T) = \sum_{i=1}^{n-1} \min\{M_i, \text{number of entries in the second row of } T|_{i,i+1}\}$.

Example 3.20. Let $\boldsymbol{\lambda}$ be a horizontal-strip with exactly two rows, so that $\Pi(\boldsymbol{\lambda})$ is



for some $a \geq b \geq M$, where (i,j) is either $(1,2)$ or $(2,1)$, so $\alpha = (a,b)$ or $\alpha = (b,a)$ respectively. In either case, for each $0 \leq k \leq b$, there is a unique tableau T_k with content α and shape $(a+b-k)k$. Therefore, by Corollary 3.19, the LLT polynomial is

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = \sum_{k=0}^{b} q^{\min\{M,k\}} s_{(a+b-k)k} = s_{(a+b)} + \cdots + q^M s_{(a+b-M)M} + \cdots + q^M s_{ab}. \tag{3.12}$$

4 Further directions

We would like to extend our results toward Problem 1.1 and Problem 1.2 to more general weighted graphs $\Pi(\boldsymbol{\lambda})$ or more general multiskew partitions $\boldsymbol{\lambda}$. Another direction is to consider expansions into k-Schur functions $s_\lambda^{(k)}$, which are Schur-positive functions introduced by Lapointe, Lascoux, and Morse [9]. They presented several conjecturally equivalent definitions based on different desired properties, and Blasiak, Morse, Pun, and Summers [2] proved that these definitions are equivalent by showing that a class of Catalan symmetric functions satisfies all of these properties.

Conjecture 4.1. *Let $\boldsymbol{\lambda}$ be a multiskew partition whose cells have contents in $\{1,\ldots,k\}$ for some k. Then there are polynomials $c_\mu(q) \in \mathbb{N}[q]$ such that*

$$\omega G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = \sum_\mu c_\mu(q) s_\mu^{(k)}. \tag{4.1}$$

Alternatively, we could consider Macdonald polynomials, which expand positively into LLT polynomials and therefore into Schur functions, but for which a combinatorial formula is open.

Problem 4.2. *Find a combinatorial Schur expansion of Macdonald polynomials of the form*

$$\widetilde{H}_\lambda(\boldsymbol{x};q,t) = \sum_{T\in S} q^{stat_1(T)} t^{stat_2(T)} s_{partition(T)}. \tag{4.2}$$

It would also be interesting to explore the connection to chromatic symmetric and quasisymmetric functions. The results (1.4) and Theorem 1.5 suggest that an extended chromatic quasisymmetric function, defined for a weighted graph and incorporating a parameter q, could generalize both results and unify all of these ideas. We pose the following problem.

Problem 4.3. *Let Π be a vertex-weighted and edge-weighted graph. Define an extended chromatic quasisymmetric function $X_\Pi(\boldsymbol{x};q)$ such that*

- *$X_\Pi(\boldsymbol{x};q)$ is manifestly quasisymmetric,*
- *if Π has all vertex weights 1 and all edge weights 0 or 1, we recover the chromatic quasisymmetric function,*
- *if $q=1$, we recover the extended chromatic symmetric function,*
- *$X_\Pi(\boldsymbol{x};q)$ satisfies a deletion-contraction relation, and*
- *if $\Pi = \Pi(\boldsymbol{\lambda})$ for a horizontal-strip $\boldsymbol{\lambda} = (R_1,\dots,R_n)$, then we have the plethystic relationship*
$$X_\Pi(\boldsymbol{x};q) = (q-1)^{-n} G_{\boldsymbol{\lambda}}[\boldsymbol{x}(q-1);q], \tag{4.3}$$
which in particular implies that for horizontal-strips $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, we have that
$$G_{\boldsymbol{\lambda}}(\boldsymbol{x};q) = G_{\boldsymbol{\mu}}(\boldsymbol{x};q) \text{ if and only if } X_{\Pi(\boldsymbol{\lambda})}(\boldsymbol{x};q) = X_{\Pi(\boldsymbol{\mu})}(\boldsymbol{x};q). \tag{4.4}$$

Note that if $\Pi = \Pi(\boldsymbol{\lambda})$ for a horizontal-strip $\boldsymbol{\lambda}$, then (4.3) can be taken as the definition of $X_\Pi(\boldsymbol{x};q)$, but it would be interesting to define $X_\Pi(\boldsymbol{x};q)$ in terms of colourings for general weighted graphs. The flexibility of a weighted graph allows us to consider a deletion-contraction relation, which exists for chromatic polynomials but not for the chromatic symmetric functions of unweighted graphs. This may be a key innovation to the main open problems of chromatic symmetric functions, namely the Stanley–Stembridge conjecture or its refinement, the Shareshian–Wachs conjecture, which we can equivalently state as follows.

Problem 4.4. *Let $\boldsymbol{\lambda}$ be a unicellular multiskew partition. Note that in this case, the chromatic quasisymmetric function $X_{\Pi(\boldsymbol{\lambda})}(\boldsymbol{x};q)$ is in fact symmetric. Find a combinatorial elementary symmetric function expansion of $X_{\Pi(\boldsymbol{\lambda})}(\boldsymbol{x};q)$ of the form*

$$X_{\Pi(\boldsymbol{\lambda})}(\boldsymbol{x};q) = \sum_{\theta \in AO(\Pi(\boldsymbol{\lambda}))} q^{asc(\theta)} e_{partition(\theta)}(\boldsymbol{x}) \tag{4.5}$$

for some statistic $partition(\theta)$ on acyclic orientations θ of the graph $\Pi(\boldsymbol{\lambda})$.

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