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Crystals and integrable systems for edge labeled tableaux

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Abstract. We define an integrable five vertex model whose partition function is the generating function E^λ of edge labeled tableau of shape λ. Using this, we prove a Cauchy-type identity. We give a crystal structure on edge labeled tableau to give a positive Schur expansion of E^λ.

Keywords: edge labeled tableau, integrable system, crystal

1 Introduction

Consider the Lie group $G = \mathrm{GL}_n(\mathbb{C})$, and fix a maximal torus T of the diagonal matrices with B being the corresponding Borel of upper triangular matrices. The (complex) Grassmannian $\mathrm{Gr}(k, n)$ is a very important object in algebraic geometry, which can be described as the set of k dimensional hyperplanes in $\mathbb{C}^n$ or as G/P, where P is the maximal parabolic of $k \times (n-k)$ block upper triangular matrices. To study its T-equivariant cohomology ring $H_T^\bullet(\mathrm{Gr}(k, n))$, the approach of Schubert calculus is to study the Schubert varieties X_λ, which are Zariski closures of the decomposition of $\mathrm{Gr}(k, n)$ into (left) B orbits. In the nonequivariant case, there is an isomorphism from $H^\bullet(\mathrm{Gr}(k, n))$ to symmetric functions modulo the ideal $(s_\lambda(\mathbf{x}) \mid \lambda \not\subseteq (n-k)^k)$, where $(n-k)^k$ denotes a $k \times (n-k)$ rectangle and $s_\lambda(\mathbf{x})$ is the Schur function and the image of the cohomology class $[X_\lambda]$ for X_λ. In $H_T^\bullet(\mathrm{Gr}(k, n))$, the factorial Schur function $s_\lambda(\mathbf{x}|\mathbf{a})$ represents $[X_\lambda]$.

The Littlewood–Richardson (LR) coefficients are the structure coefficients for Schur functions $s_\lambda(\mathbf{x}) s_\mu(\mathbf{x}) = \sum_\nu c_{\lambda\mu}^\nu s_\nu(\mathbf{x})$ (when $\mathbf{a} = 0$), which have a classical combinatorial description as certain semistandard tableaux of shape ν/λ. The problem is more subtle to compute the LR coefficients for the factorial Schurs with a manifestly positive formula. An initial solution given by Molev and Sagan [12], but it is not described in terms of skew tableaux like the usual LR rule. A skew tableau rule was given by Thomas and Yong [17] by introducing edge labeled tableaux of shape ν/λ with certain conditions.

Another natural problem is to compute the dual basis $\widehat{s}_\lambda(\mathbf{x}|\mathbf{a})$ to the factorial Schur functions under the Hall inner product, where $\langle s_\lambda(\mathbf{x}), s_\mu(\mathbf{x}) \rangle = \delta_{\lambda\mu}$. We can motivate this

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geometrically with an alternative way to construct the ring of symmetric functions using the homology $\bigoplus_{1\leq k\leq n} H_\bullet(\mathrm{Gr}(k,n))$, where the product corresponds to the induced map from the direct sum of two Grassmannians [9, Sec. 1.1]. In order to get a deformation of symmetric functions from equivariant cohomology desired by Knutson and Lederer [9], we can only use a single circle $\mathbb{S}^1$ action. It was shown in [10, Theorem 8.12] that the Schubert classes correspond to $\widehat{s}_\lambda(\mathbf{x};t)$, which equals $\widehat{s}_\lambda(\mathbf{x}|\mathbf{a})$ with $a_i = 0$ for $i \leq 0$ and $a_i = t$ for $i > 0$, by utilizing back stable Schubert calculus. We remark that $\widehat{s}_\lambda(\mathbf{x}|\mathbf{a})$ was first studied by Molev [11] and shown to be the generating function of certain tableaux with the weights being rational functions.

Dual bases must satisfy the Cauchy identity [16, Lemma 7.9.2]. Our first main result is that the dual basis $\widehat{s}_\lambda$ in finitely many variables, up to a simple overall factor of $\prod_{j=1}^m \prod_{k=1}^{m+\lambda_1} (1+a_k y_j)^{-1}$, is given by the generating function $E^\lambda(\mathbf{x}|\mathbf{a})$ of edge labeled tableaux, which we coin the *edge Schur functions*.

Theorem 1.1. *Denote $\mathbf{a}^n := (a_{i-n-1})_{i\in\mathbb{Z}_{>0}}$. For $N \geq m+\lambda_1$, we have*

$$\sum_{\substack{\lambda \\ \ell(\lambda)\leq\min(n,m)}} s_\lambda(\mathbf{x}_n|-\mathbf{a}) \prod_{\substack{1\leq k\leq N \\ 1\leq j\leq m}} (1+a_k y_j)^{-1} E^\lambda(\mathbf{y}_m|\mathbf{a}^n) = \prod_{\substack{1\leq i\leq n \\ 1\leq j\leq m}} \frac{1}{1-x_i y_j}. \tag{1.1}$$

Note that the weight is different than in [17] and is reminiscent of the refined weights for refined symmetric Grothendieck polynomials [4] from the K-theory of $\mathrm{Gr}(k,n)$. For our proof, we introduce an integrable lattice model such that the commutation of transfer matrices with the model for factorial Schur functions yields the Cauchy identity in finitely many variables. A consequence of this is $E^\lambda(\mathbf{x}|\mathbf{a})$ is a symmetric function.

The next natural question in studying $E^\lambda(\mathbf{x}|\mathbf{a})$ is to determine how they expand in terms of Schur functions. We compute this by utilizing our second main result (Theorem 4.1), there exists a $U_q(\mathfrak{sl}_n)$-crystal structure on edge labeled tableaux by breaking the tableau into diagonals (as opposed to rows or columns as detailed in [6, 13]). An immediate consequence is a *positive* Schur expansion of $E^{\lambda/\mu}(\mathbf{x}|\mathbf{a})$ by counting highest weight elements. We also provide an uncrowding algorithm and conclude this is a crystal isomorphism by properties of RSK following [2, 6, 13, 14, 15].

This extended abstract is organized as follows. In Section 2, we give the background on the requisite tableaux and generating functions. In Section 3, we give the lattice model proving Theorem 1.1. In Section 4, we describe the crystal structure on edge labeled tableaux and the corresponding uncrowding algorithm.

After this was submitted, we learned that our lattice model for edge Schur functions (3.1) and the corresponding R-matrix (Proposition 3.2) had previously appeared in the preprint of Gorbounov and Korff [5]. We thank the referee for noting this.

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \cdots \xrightarrow{n-1} \boxed{n}$$

Figure 1: Crystal $B(\Lambda_1)$ of the natural representation for $U_q(\mathfrak{sl}_n)$.

2 Tableaux generating functions

Fix a positive integer n, and let $[n] := \{1, \ldots, n\}$. A *partition* λ is a weakly decreasing finite sequence of positive integers, and we draw the Young diagram of λ using English convention. We identify partitions λ with 01-sequences by a 1 for every vertical step and 0 for every horizontal step, read from bottom-left to top-right. Boldface letters will denote an countably infinite sequence of indeterminates unless otherwise stated and a subscript indicates finitely many variables; *e.g.*, $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{x}_n = (x_1, \ldots, x_n, 0, \ldots)$. We make one main exception of $\mathbf{a} := (\ldots, a_{-1}, a_0, a_1, \ldots)$.

Following [18], an *edge labeled tableau* is a filling of the Young diagram by $\mathbb{Z}_{>0}$ and finite subsets of $\mathbb{Z}_{>0}$ on horizontal edges such that rows weakly increase (not including the edges) and vertical edges strictly increase, where for any set A on an edge, an entry above is less than $\min A$ and an entry below is greater than $\max A$. We consider the top of the partition to extend infinitely far to the right, which can also hold edge labels. For skew shapes, we do not allow edge labels only on the *top* row of boxes that have been skewed out. The set of *semistandard tableaux* of (skew) shape λ/μ, denoted $\mathrm{SSYT}(\lambda/\mu)$, is the set of edge labeled tableaux of shape λ/μ such that no entry appears on any edge.

The *factorial Schur functions* and *edge Schur functions* are defined as

$$s_{\lambda/\mu}(\mathbf{x}|\mathbf{a}) = \sum_{T \in \mathrm{SSYT}(\lambda/\mu)} \prod_{\alpha \in T} (x_\alpha - a_{\alpha+j-i}), \qquad E^{\lambda/\mu}(\mathbf{x}|\mathbf{a}) = \sum_{T \in \mathrm{ELT}(\lambda/\mu)} \prod_{\alpha \in T} x_\alpha \prod_{\ell \in ET} x_\ell a_{j-i},$$

where the product over $\alpha \in T$ (resp. $\ell \in ET$) is all boxes (resp. edge labels in the upper edge of boxes) in T, where the box is in the i-th row and j-th column. When $\mathbf{a} = 0$, then $s_{\lambda/\mu}(\mathbf{x}) = E^{\lambda/\mu}(\mathbf{x}; 0)$ are the usual skew Schur functions.

We recall the crystal structure for $\mathfrak{sl}_n$, the special linear Lie algebra of traceless $n \times n$ matrices (over $\mathbb{C}$). We use the standard identification of partitions such that $\ell(\lambda) \leq n$ with elements in the dominant weight lattice $P^+ = \mathbb{Z}^n_{\geq 0}$ by $\lambda \leftrightarrow \sum_{i=1}^n \lambda_i \epsilon_i$, where $\{\epsilon_i \mid i \in [n]\}$ is the standard basis of $\mathbb{Z}^n$. A *crystal graph* will be an edge-colored by $[n-1]$ colors, weighted, directed graph. A *highest weight crystal* will be a (weakly) connected component of crystal graph for a tensor power of the crystal $B(\Lambda_1)$ given in Figure 1. We define $\mathcal{B} = B(\Lambda_1)^{\otimes k}$ as the crystal graph with vertices $B(\Lambda_1)^k$ and an edge $b_k \otimes \cdots \otimes b_1 \xrightarrow{i} b'_k \otimes \cdots \otimes b'_1$ by the *signature rule*: Replace each i and $i+1$ with $-$ and $+$ respectively, and successively deleting any $(+-)$-pairs (in that order) until obtaining a sequence $- \cdots - + \cdots +$. Let j_- be the index for the rightmost $-$ remaining, and set

$$b'_k \otimes \cdots \otimes b'_{j_-} \otimes \cdots \otimes b'_1 = b_k \otimes \cdots \otimes \boxed{i+1} \otimes \cdots \otimes b_1,$$

where there is no edge if there is no such $-$. Define the *weight* of an element $b \in \mathcal{B}$ as $\prod_{i=1}^{n} x_i^{a_i}$, where a_i is the number of entries equal to i. Our tensor product convention follows [3], which is opposite of [7, 8]. See [3] for additional information on crystals.

A *highest weight element* is a source of the crystal graph. For a highest weight crystal, there exists a unique $\lambda \in P^+$ such that $\mathrm{wt}(b) = \mathbf{x}^\lambda$, which is the crystal basis of a quantum group $U_q(\mathfrak{sl}_n)$ module $V(\lambda)$ [8]. Moreover, the *character* of a crystal $\mathcal{B}$ is

$$\operatorname{ch} \mathcal{B} := \sum_{b \in \mathcal{B}} \mathrm{wt}(b).$$

It is a classical fact that $\operatorname{ch} B(\lambda) = s_\lambda(\mathbf{x}_n)$. Hence, we can identify elements of $B(\lambda)$ with $\mathrm{SSYT}(\lambda)$ with max entry n under *admissible reading words* [7, Theorem 7.3.6], where for any fixed box b, we read every box to its northeast after b. We will use a nonstandard reading word by reading along diagonals from bottom-to-top, where along each diagonal we read from bottom-to-top. This gives us an injection $\mathrm{rwd}\colon B(\lambda/\mu) \to B(\Lambda_1)^{\otimes|\lambda/\mu|}$.

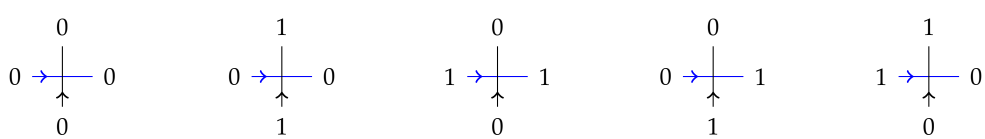
Example 2.1. Under the reading word described above, we have

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline 10 \\ \cline{1-1} \end{array} \longmapsto \boxed{10} \otimes \boxed{7} \otimes \boxed{8} \otimes \boxed{4} \otimes \boxed{9} \otimes \boxed{5} \otimes \boxed{1} \otimes \boxed{6} \otimes \boxed{2} \otimes \boxed{3}.$$

3 Lattice model

A *state* in a *five vertex model* is a (potentially infinite) square grid with vertices a subset of $\mathbb{Z}^2$ and labels $\{0,1\}$ on each edge such that around each vertex satisfies one of five possible configurations. We can assign a *Boltzmann weight* to each possible vertex, and the Boltzmann weight of a state is the product of the Boltzmann weights of each vertex. The *partition function* of a vertex model is the sum of the Boltzmann weights of all possible states. This assignment of Boltzmann weights to vertices is called an *L-matrix* (we can consider the Boltzmann weight of the other local configurations to be 0). We can realize L-matrix at position (i,j) as a linear map in $\mathrm{End}(H_i \otimes V_j)$, where $H_i \cong \mathbb{C}^2$ is the i-th quantum space and $V_j \cong \mathbb{C}^2$ is the j-th physical space. For more information, we refer the reader to [1].

We will now give a lattice model on $[n] \times \mathbb{Z}$ whose partition function is $E^{\lambda/\mu}(\mathbf{x}_n; \mathbf{a})$. The L-matrix L_{ij} at position (i,j) is defined as

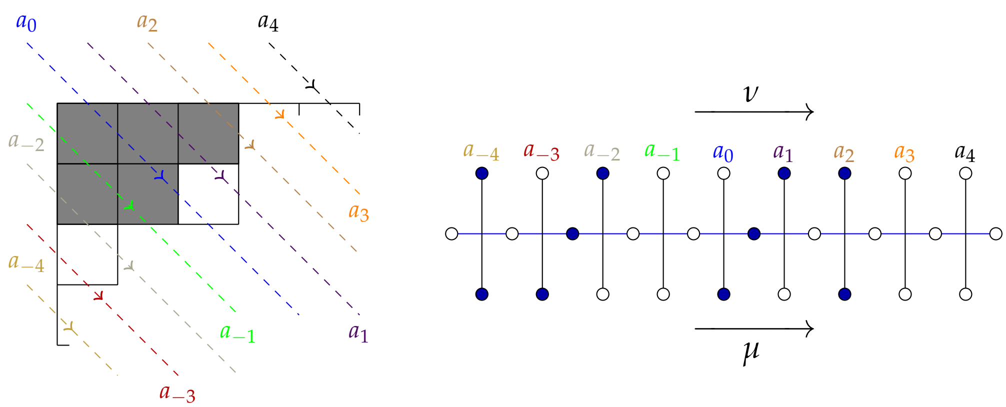


(3.1)

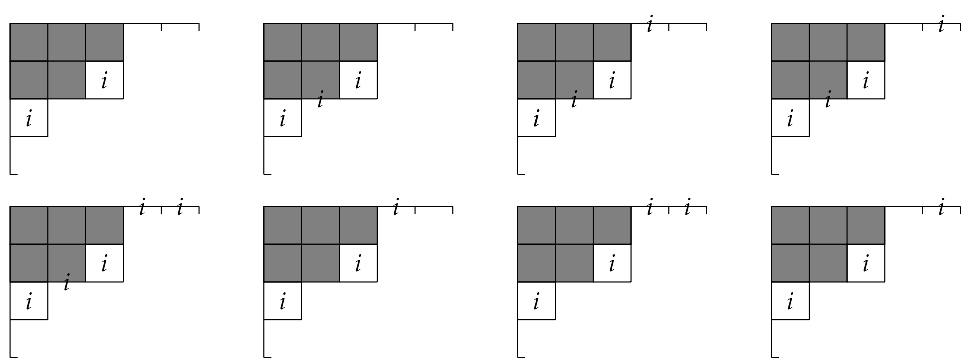
$1 + a_j x_i$ $\quad$ 1 $\quad$ x_i $\quad$ 1 $\quad$ x_i

Next, we describe the *boundary conditions*, where the top and bottom edges are the 01-sequence for the partitions λ and μ, respectively. To see that the partition function is $E^{\lambda/\mu}(\mathbf{x}_n; \mathbf{a})$, it is sufficient to consider a single row, which is equivalent to restricting to at single letter, say $\mathbf{x}_1$. We identify diagonals of the tableau with the vertical lines in the lattice model. We note that there is a unique state in this model, every vertex outside of the $[-\ell(\lambda), \lambda_1 - 1]$ vertical lines are fixed, and the placement of edge labels correspond to the choice of monomial in $(1 + a_j x_i)$.

Example 3.1. We restrict to the finite lattice $[1] \times [-4, 4]$, which means we set $a_i = 0$ for all $i > 4$. For the partitions $\lambda = (3, 3, 1)$ and $\mu = (3, 2)$, the only possible state is



The Boltzmann weight of this state is $(1 + a_{-1}x_i)(1 + a_3 x_i)(1 + a_4 x_i)x_i^2$. To translate this to edge labeled tableaux, note that we can add an i to the edges along the diagonals with index -1, 3 and 4, which are also the upper edges of boxes with those contents.



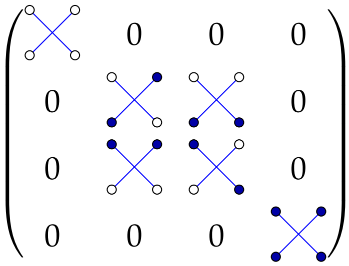
Note that we have the same partition function with the lattice $[1] \times [-k, 4]$ for any $k \geq 3$.

The lattice model perspective also makes it clear how to derive the notion of edge labeled tableau for skew shapes so that we have the branching rule

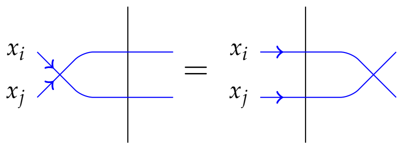
$$E^{\lambda/\mu}(\mathbf{x}, \mathbf{y}|\mathbf{a}) = \sum_{\mu \subseteq \nu \subseteq \lambda} E^{\lambda/\nu}(\mathbf{y}|\mathbf{a}) E^{\nu/\mu}(\mathbf{x}|\mathbf{a}).$$

This model is *integrable*, which means the following holds.

Proposition 3.2 ([5]). *There exists an R-matrix given by*

$R_{i,j}(x_i, x_j) =$  $_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{x_i}{x_j} & 0 \\ 0 & 1 & 1-\frac{x_i}{x_j} & 0 \\ 0 & 0 & 0 & \frac{x_i}{x_j} \end{pmatrix}_{ij} \in \operatorname{End}(H_i \otimes H_j).$

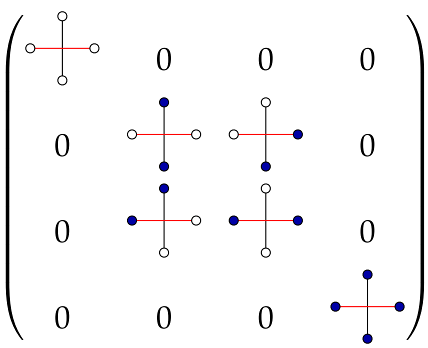
satisfying the Yang–Baxter equation: for any fixed boundary, the partition functions are equal:



$$\big(R_{ij}(x_i, x_j)L_{i,n}(x_i)L_{j,n}(x_j) = L_{j,n}(x_j)L_{i,n}(x_i)R_{ij}(x_i, x_j)\big).$$

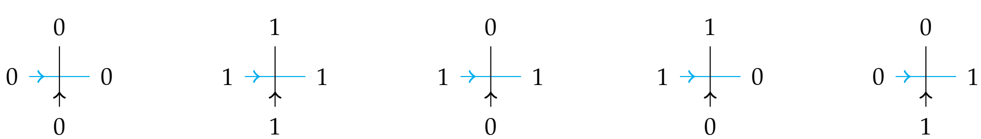
A consequence of Proposition 3.2 is $E^\lambda(\mathbf{x}|\mathbf{a})$ is a symmetric function by repeatedly using the Yang–Baxter equation (this is known as the train argument).

To prove Theorem 1.1, we need a model whose partition function is the factorial Schur function $s_\lambda(\mathbf{x}|-\mathbf{a})$. We will use the model from [19], which is also integrable, with the l-matrix

$l_{i,j} =$  $_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & x_i + a_j & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{ij} \in \operatorname{End}(H_i \otimes V_j).$

In this model, we use $\mathbb{Z}_{\geq 0} \times [n]$ with the left boundary all being 1.

The goal is to attach the vertex model for $s_\lambda(\mathbf{x}|-\mathbf{a})$ with a vertex model for $E^\lambda(\mathbf{x}|\mathbf{a})$, pass them through each other by the train argument, and then be able to easily compute the resulting partition function. However, we cannot use the "natural" model for $E^\lambda(\mathbf{x}|\mathbf{a})$. Instead we use a "dual" model, whose L^*-matrix is formed by rotating the vertices 180 degrees and interchanging $0 \leftrightarrow 1$ along the quantum space (horizontal edges):



x_i	1	$1 + a_j x_i$	1	x_i

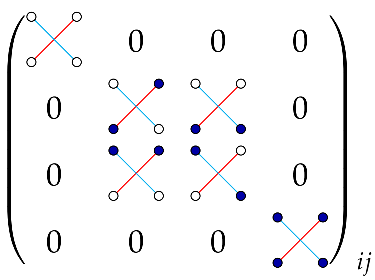
For this model, we note that when we restrict to any sufficiently large $[-N, N] \times [n]$, the left and right boundary conditions are all 1.

We call a single row of a vertex model a *transfer matrix* (with no boundary conditions), and denote the transfer matrix using the L^*-matrix (resp. ℓ-matrix) by $\mathfrak{T}^*$ (resp. $\mathfrak{t}$). Unlike more classical cases of vertex models, our model for $E^{\lambda/\mu}(\mathbf{x}|\mathbf{a})$ depends on the number of $a_i \neq 0$ for $i > 0$. We also will make a technical assumption that $|x_j| < 1$ so that no infinite products can occur. This yields the following key relation.

Proposition 3.3. *Let $a_i = 0$ for all $i \gg 1$ and $|x| < 1$. Then $\mathfrak{T}^*(y)\mathfrak{t}(x) = (1 - xy)\mathfrak{t}(x)\mathfrak{T}^*(y)$.*

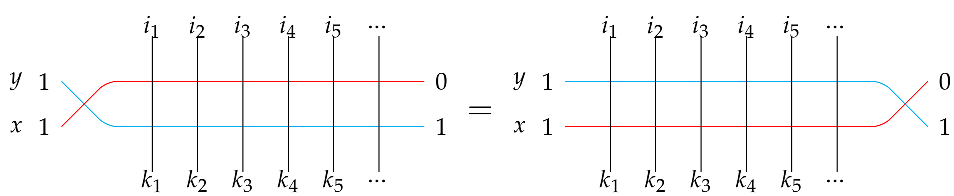
The proof is essentially the existence of a solution of the Yang–Baxter equation with

$\mathfrak{R}(x, y) =$

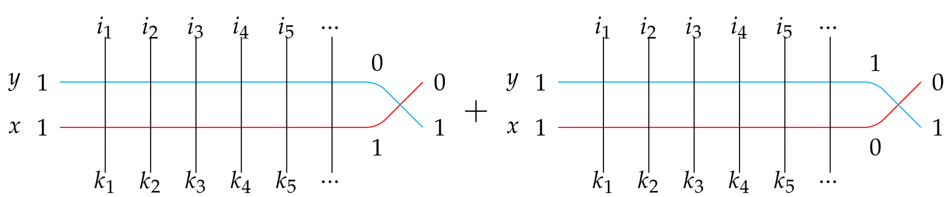


$$= \begin{pmatrix} y & 0 & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 1 & 1 - xy & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{ij} \in \operatorname{End}(H \otimes H^*).$$

Multiplying by the $\mathfrak{R}$-matrix, the train argument yields equal partition functions for

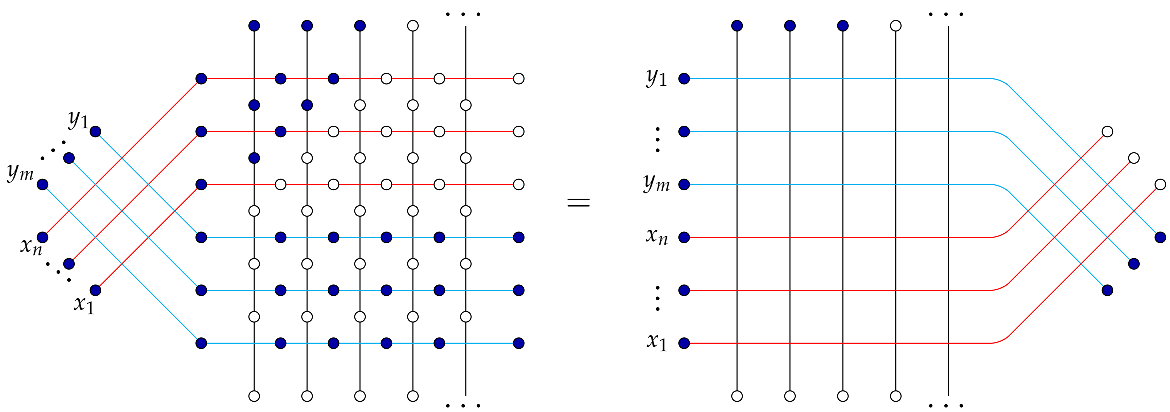


On the left side, as the weight of the $\mathfrak{R}$-matrix is 1, we have $\mathfrak{T}^*(y)\mathfrak{t}(x)$. The right side has potentially two possible configurations:



However, from our assumptions, we note that the Boltzmann weight of for any state of the left configuration must be 0. The claim follows from the weight of the $\mathfrak{R}$-matrix.

To finish the proof of Theorem 1.1 is repeatedly applying Proposition 3.3 and noting there is a unique state on one side that contributes a factor of $\prod_{1\leq k\leq N}\prod_{1\leq j\leq m}(1+a_k y_j)$:



4 Crystal structure

We define our crystal structure on edge labeled tableaux by extending the reading for a box b with an entry b and a set $A = \{a_1 < \cdots < a_k\}$ on the edge below b as

$$\underset{A}{\boxed{b}} \iff \boxed{a_k} \otimes \cdots \otimes \boxed{a_1} \otimes \boxed{b} \iff {\begin{array}{|c|c|c|c|}\hline b & a_1 & \cdots & a_k \\ \hline\end{array}}^{T}.$$

We read the tableau following the reading word rwd using this description for each box.

We define the crystal structure by using the signature rule with this reading word. This generally gives a valid edge labeled tableau with the following exception:

$$\begin{array}{|c|c|}\hline i & \underset{p}{i} \\ \hline\end{array} \xrightarrow{\;i\;} \begin{array}{|c|c|}\hline \overset{i}{p} & p \\ \hline\end{array},$$

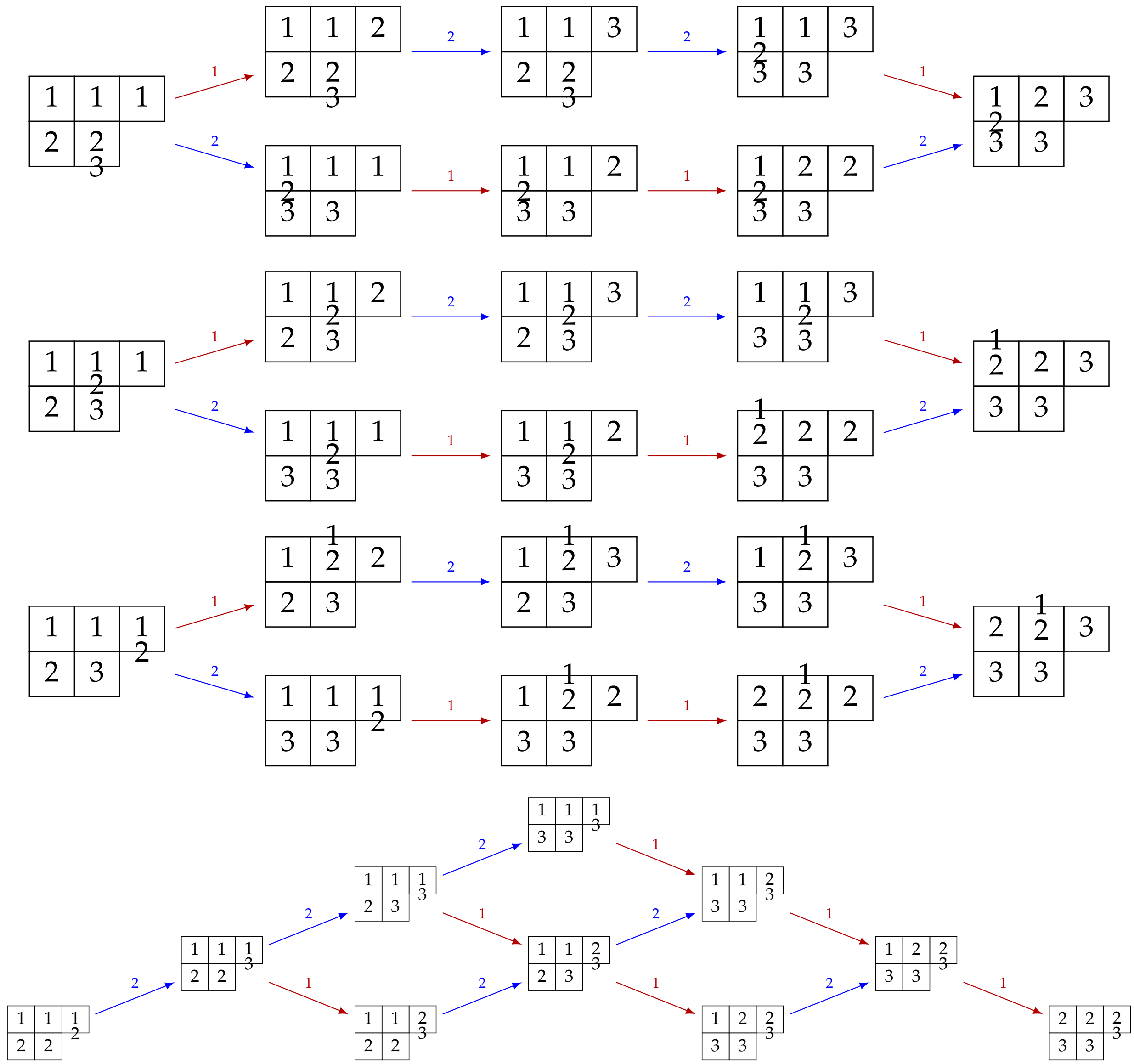
where $p = i+1$. Note that normally we would change the left i to and $i+1$, which would not be an edge-labeled tableau. It is straightforward to see that this is the same operation after taking the reading word, which yields the following.

Theorem 4.1. *The set of edge labeled tableau of shape λ/μ is a highest weight $U_q(\mathfrak{sl}_n)$-crystal. Moreover, the function $E^{\lambda/\mu}(\mathbf{x}_n|\mathbf{a})$ is Schur positive.*

Example 4.2. Let $\lambda = (3,2)$. For any $n \geq 3$, we have

$$E^{32}(\mathbf{x}_n|\mathbf{a}) = s_{32}(\mathbf{x}_n) + (a_{-2} + a_{-1} + a_0 + a_1)s_{321}(\mathbf{x}_n) + a_1 s_{33}(\mathbf{x}_n) + \sum_{i>1} a_i s_{42} + HOT.$$

We have the following crystals for the coefficients a_{-1}, a_0, and a_1 for $n = 3$:



Next, we construct an analog of the uncrowding bijection in analogy to [2, 6]. In this case, given our reading word, we will perform the uncrowding along diagonals, which requires a little more care. For simplicity, we index the diagonals so the first diagonal has content $1 - \ell(\lambda)$.

Definition 4.3 (Uncrowding algorithm)**.** We proceed along diagonals starting from the lower-left box. Start with $(P_0, Q_0) = (\emptyset, \emptyset)$. For the i-th diagonal D_i, let $P_i = P_{i-1} \xleftarrow{RSK} \mathrm{rwd}(D_i)$ denote the RSK insertion (see, *e.g.*, [16, Ch. 7]). We construct the recording tableau Q_i as the skew shape $\mu^{(i)}/\nu^{(i)}$, where $\mu^{(i)}$ is the shape of P_i and $\nu^{(i)}$ are the boxes of λ up to the i-th diagonal (counted from the lower-left box) and slid up into a straight shape. The entries of Q_i are those of Q_{i-1} shifted appropriately and then any remaining empty cells are filled with an i.

Example 4.4. Consider the edge labeled tableau

$$T = \begin{array}{|c|c|c|}\hline 1 & 1 & \underset{45}{2} \\ \hline 2 & 2 & 6 \\ \hline 3 & \underset{4}{3} \\ \cline{1-2} \underset{5}{4} & 5 \\ \cline{1-2}\end{array}$$

Under the uncrowding algorithm, we have

$$(P_0, Q_0) = \emptyset, \emptyset \xleftarrow{RSK} 54 = \begin{array}{|c|}\hline 4 \\ \hline 5 \\ \hline\end{array}, \begin{array}{|c|}\hline \cdot \\ \hline 1 \\ \hline\end{array} \xleftarrow{RSK} 53 = \begin{array}{|c|c|}\hline 3 & 5 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline \cdot & \cdot \\ \hline \cdot \\ \cline{1-1} 1 \\ \cline{1-1}\end{array} \xleftarrow{RSK} 432$$

$$= \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & 4 \\ \hline 4 & 5 \\ \hline 5 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline \cdot & \cdot \\ \hline \cdot & \cdot \\ \hline \cdot & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \xleftarrow{RSK} 21 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline 3 & 4 \\ \hline 4 & 5 \\ \hline 5 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline \cdot & \cdot \\ \hline \cdot & \cdot \\ \hline \cdot & \cdot \\ \hline \cdot & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \xleftarrow{RSK} 61$$

$$= \begin{array}{|c|c|c|}\hline 1 & 1 & 6 \\ \hline 2 & 2 \\ \cline{1-2} 3 & 3 \\ \cline{1-2} 4 & 4 \\ \cline{1-2} 5 & 5 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot \\ \cline{1-2} \cdot & \cdot \\ \cline{1-2} \cdot & \cdot \\ \cline{1-2} 1 & 3 \\ \cline{1-2}\end{array} \xleftarrow{RSK} 542 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 4 \\ \hline 2 & 2 & 5 \\ \cline{1-3} 3 & 3 & 6 \\ \cline{1-3} 4 & 4 \\ \cline{1-2} 5 & 5 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|c|}\hline \cdot & \cdot & \cdot & 6 \\ \hline \cdot & \cdot & \cdot \\ \cline{1-3} \cdot & \cdot & 6 \\ \cline{1-3} \cdot & \cdot \\ \cline{1-2} 1 & 3 \\ \cline{1-2}\end{array} = (P_6, Q_6).$$

Now we need to describe the inverse algorithm; in particular, we need to describe which which cells to remove as we will use inverse RSK at each step. We proceed by removing the diagonals in reverse order but starting with the cell at the bottom of the corresponding column. We also remove any cell labeled by i if we are at the i-th diagonal from the bottom, the result of which becomes an edge label and can be placed in a unique way such that the result is an edge labeled tableau.

We let $\mathfrak{E}(\lambda/\mu)$ denote the set of recording tableaux obtained by applying the uncrowding algorithm. We do not currently have a characterization of these tableaux other than they will have shape λ/μ. We leave this as an open question.

Theorem 4.5. *Uncrowding* $\Upsilon\colon \operatorname{ELT}(\lambda) \to \bigsqcup_{\mu\subseteq\lambda} \operatorname{SSYT}(\mu) \times \mathfrak{E}(\lambda/\mu)$ *is a crystal isomorphism, where the crystal operators on the image act only on* $\operatorname{SSYT}(\mu)$.

This gives an alternative proof of Theorem 4.1. It would be good to describe this using a formulation analogous to the alternative descriptions for uncrowding given in [14, 15].

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