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Parabolic Tamari Lattices in Linear Type B

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Abstract. We study parabolic aligned elements associated with the type-B Coxeter group and the so-called linear Coxeter element. These elements were introduced algebraically in (Mühle and Williams, 2019) for parabolic quotients of finite Coxeter groups and were characterized by a certain forcing condition on inversions. We focus on the type-B case and give a combinatorial model for these elements in terms of pattern avoidance. Moreover, we describe an equivalence relation on parabolic quotients of the type-B Coxeter group whose equivalence classes are indexed by the aligned elements. We prove that this equivalence relation extends to a congruence relation for the weak order. The resulting quotient lattice is the type-B analogue of the parabolic Tamari lattice introduced for type A in (Mühle and Williams, 2019). These lattices have not appeared in the literature before.

Résumé. Nous étudions les éléments paraboliques alignés associés aux groupes de Coxeter de type B et à l'élément de Coxeter linéaire. Ces éléments ont été introduits de façon algébrique en (Mühle et Williams, 2019) pour les quotients paraboliques des groupes de Coxeter finis, et ils ont été caractérisés par une certaine condition de forçage sur les inversions. Le cas du type B est considéré, et nous proposons un modèle combinatoire de ces éléments en termes de motifs exclus. De plus, nous décrivons une relation d'équivalence dans un quotient parabolique du groupe de Coxeter de type B, dont les classes d'équivalence sont indicées par les éléments alignés. Nous montrons que cette relation d'équivalence s'étend à une relation de congruence sur l'ordre faible. Le quotient ainsi obtenu est un analogue en type B du treillis de Tamari parabolique introduit par (Mühle et Williams, 2019) pour le type A. Ces treillis ne sont jamais apparus dans la littérature.

Keywords: sign-symmetric permutation, hyperoctahedral group, parabolic quotient, Tamari lattice, Coxeter–Catalan combinatorics

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1 Introduction

Knuth introduced in [7, Section 2.2.1] the family of *stack-sortable permutations*. These are permutations that can be converted into the identity permutation by passing through a stack. Combinatorially, stack-sortable permutations are characterized by avoiding the pattern 312. Björner and Wachs have interpreted this pattern-avoidance by a certain *forcing* of inversions [3, Section 9]. More precisely, a permutation π of $\{1, 2, \ldots, n\}$ avoids the pattern 312 if and only if for every $i < j < k$, when $\pi(i) > \pi(k)$, we have $\pi(j) > \pi(k)$.

Using a root-theoretic approach, Reading generalized this condition to all finite irreducible Coxeter groups W and all Coxeter elements c. The Coxeter group W generalizes the symmetric group, and the Coxeter element c generalizes the linear order $1 < 2 < \cdots < n$ and the reverse lexicographic order on the transpositions used to describe the above "forcing" condition. These *c-aligned elements* of W play an important role in the very active stream of Coxeter–Catalan combinatorics. First, the number of c-aligned elements of W is the W-Catalan number (for any choice of Coxeter element c). Second, the c-aligned elements provide a bijective bridge between c-noncrossing partitions and c-clusters associated with W [12].

On top of that, the c-aligned elements behave nicely from a lattice-theoretic point of view. Inside the weak order, they form the *c-Cambrian lattice*, which is semidistributive and trim. This family of lattices generalizes the famous *Tamari lattice* [11]. Another remarkable feature is that these lattices arise from a certain orientation of the 1-skeleton of the *W-associahedron*, a polytope associated with the Coxeter group W. In fact, the W-associahedron is the dual polytope of the c-cluster complex associated with W.

In [10], c-aligned elements, c-noncrossing partitions and c-clusters were generalized to *parabolic quotients* of finite irreducible Coxeter groups. In this generalization, some properties of these objects were conjectured to remain, such as the lattice property of parabolic c-aligned elements under weak order ([10, Conjecture 35]), and when W is of *coincidental type*, the equinumerosity of parabolic c-aligned elements, parabolic c-noncrossing partitions and parabolic c-clusters ([10, Conjecture 41]). The case of the symmetric group with c the increasing long cycle was settled in the same article. Further research has exhibited remarkable connections between parabolic Coxeter–Catalan objects associated with the symmetric group, certain Hopf algebras, and the theory of diagonal harmonics [4]. For more properties of these objects, readers are referred to [5, 8, 9].

In this extended abstract, we present a first study of parabolic Coxeter–Catalan objects associated with the hyperoctahedral group, i.e., the Coxeter group of type B. We start by setting up a combinatorial model of the elements of the parabolic quotient $\mathfrak{H}_\alpha$ of the hyperoctahedral group with respect to some type-B composition α using colored sign-symmetric permutations, and then describe in Section 3 a particular order on the relevant inversions we use to describe the forcing conditions that determine the parabolic aligned elements, and further characterize them using pattern-avoidance. We then prove in Section 4 that these parabolic aligned elements in type B form a quotient lattice of the weak order on the whole parabolic quotient. This quotient lattice is called the *type-B parabolic Tamari lattice*, denoted by $\mathsf{Tam}_B(\alpha)$, and has the same lattice-theoretic properties as the c-Cambrian lattice. We state this as the main result of this extended abstract.

Theorem 1.1. *For every type-B composition α, the type-B parabolic Tamari lattice $\mathsf{Tam}_B(\alpha)$ is a lattice. Moreover, it is a trim and semidistributive quotient lattice of the weak order on the parabolic quotient $\mathfrak{H}_\alpha$.*

Most of the proofs are omitted here due to space limitations. These proofs, along with more details and background, can be found in the full version [6].

2 Basics

2.1 The hyperoctahedral group

For $n > 0$, we define $[n] \overset{\text{def}}{=} \{1, 2, \ldots, n\}$ and $\pm[n] \overset{\text{def}}{=} \{-n, -n+1, \ldots, -2, -1, 1, 2, \ldots, n\}$. A permutation π of $\pm[n]$ is *sign-symmetric* if $\pi(-i) = -\pi(i)$ for all $i \in [n]$. The group of all sign-symmetric permutations of $\pm[n]$ is the *hyperoctahedral group* of degree n, denoted by $\mathfrak{H}_n$. It is easily checked that $\left|\mathfrak{H}_n\right| = 2^n n!$. We usually represent sign-symmetric permutations via their *long one-line notation*, i.e., for $\pi \in \mathfrak{H}_n$ we write down

$$\pi(-n), \pi(-n+1), \ldots, \pi(-1), \pi(1), \ldots, \pi(n-1), \pi(n).$$

By definition, the left half of this word is determined by the right half. We still want to keep it because it will be easier to spot the relevant type-B 231-patterns that we introduce in Section 2.2. For stylistic reasons, we represent negative values by an overbar instead of a minus sign, and we add a vertical bar between $\pi(-1)$ and $\pi(1)$ to emphasize the symmetry. In case of potential ambiguities, we use commas to separate the entries.

Let $[\![i]\!]$ denote the sign-symmetric permutation that exchanges i and $-i$, and let $(\!(i\ j)\!)$ denote the one that exchanges i and j (and simultaneously $-i$ and $-j$). We take

$$T = \left\{[\![i]\!] \mid 1 \leq i \leq n\right\} \cup \left\{(\!(i\ j)\!) \mid 1 \leq i < j \leq n\right\} \cup \left\{(\!(-j\ i)\!) \mid 1 \leq i < j \leq n\right\}.$$

It is well known that $\mathfrak{H}_n$ admits the following presentation, where $S = \{s_0, s_1, \ldots, s_{n-1}\}$ is the set of generators, and e is the identity:

$$\mathfrak{H}_n \overset{\text{def}}{=} \left\langle S \mid s_i^2 = \mathsf{e}, (s_0 s_1)^4 = \mathsf{e}, (s_i, s_{i+1})^3 = \mathsf{e} \text{ for } i > 0, (s_i s_j)^2 = \mathsf{e} \text{ for } j > i+1\right\rangle.$$

This can be realized by taking $s_0 \mapsto [\![1]\!]$ and $s_i \mapsto (\!(i\ i{+}1)\!)$ for $i \in [n-1]$. Thus, $\mathfrak{H}_n$ has the structure of a Coxeter group. Moreover, every $\pi \in \mathfrak{H}_n$ can be written as a product of elements of S. An *S-reduced word* for π is such a product with minimum length, and we denote its length by $\ell_S(\pi)$.

A *(right) inversion* of π is some $t \in T$ such that $\ell_S(\pi) > \ell_S(\pi t)$. A *cover inversion* of π is an inversion t of π for which there exists $s \in S$ such that $s\pi = \pi t$. We write $\mathsf{Inv}(\pi)$ (resp. $\mathsf{Cov}(\pi)$) for the sets of inversions (resp. cover inversions) of π. The *(left) weak order* on $\mathfrak{H}_n$, denoted by $\leq_{\mathsf{weak}}$, is the containment order of inversion sets.

2.2 Aligned elements

We fix the long cycle $\vec{c} \overset{\mathsf{def}}{=} s_0 s_1 \cdots s_{n-1}$, whose long one-line notation is

$$1, \overline{n}, \ldots, \overline{3}, \overline{2} \mid 2, 3, \ldots, n, \overline{1}.$$

In [12], N. Reading defined a family of sign-symmetric permutations associated with $\vec{c}$, called the *$\vec{c}$-aligned elements*, characterized in two equivalent ways. Algebraically, a sign-symmetric permutation $\pi \in \mathfrak{H}_n$ is $\vec{c}$-aligned if, for every $1 \leq i < k \leq n$, we have:

- if $[\![i]\!] \in \mathsf{Cov}(\pi)$, then for all $1 \leq j < i$, we have $[\![j]\!] \in \mathsf{Inv}(\pi)$;
- if $(\!(i\ k)\!) \in \mathsf{Cov}(\pi)$, then for all $i < j < k$, we have $(\!(i\ j)\!) \in \mathsf{Inv}(\pi)$;
- if $(\!(-k\ i)\!) \in \mathsf{Cov}(\pi)$, then
  - $[\![i]\!] \in \mathsf{Inv}(\pi)$,
  - $(\!(-j\ i)\!) \in \mathsf{Inv}(\pi)$ for all $1 \leq j < k$, $j \neq i$,
  - $(\!(-k\ j)\!) \in \mathsf{Inv}(\pi)$ for all $1 \leq j < i$.

We refer to these implications among inversions and cover inversions as *forcing relations*.

Combinatorially, $\pi \in \mathfrak{H}_n$ has a *type-B 231-pattern* if there exist indices $-n \leq i < j < k \leq n$ such that $j, k > 0$, $\pi(j) > \pi(i)$ and either $\pi(i) = \pi(k) + 1$ or $\pi(i) = 1 = -\pi(k)$. The following result is the starting point of our work.

Proposition 2.1 ([12, Lemma 4.9])**.** *A sign-symmetric permutation $\pi \in \mathfrak{H}_n$ is $\vec{c}$-aligned if and only if it avoids type-B 231-patterns.*

The main purpose of this extended abstract is to generalize this result to parabolic quotients of $\mathfrak{H}_n$ together with the appropriate definitions.

2.3 Parabolic Quotients

We are interested in minimal-length representatives of the left cosets of $\mathfrak{H}_n$ by the subgroup generated by some $J \subseteq S$. This *parabolic quotient* is equivalently defined by

$$\mathfrak{H}_n^J \overset{\mathsf{def}}{=} \{\pi \in \mathfrak{H}_n \mid \ell_S(\pi s) > \ell_S(\pi) \text{ for all } s \in J\}.$$

Combinatorially, we can describe the parabolic quotient neatly as follows. Let $n > 0$. Consider a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ of n, i.e., $\alpha_1 + \alpha_2 + \cdots + \alpha_r = n$. We define $p_0 \overset{\mathsf{def}}{=} 0$ and $p_i \overset{\mathsf{def}}{=} \alpha_1 + \alpha_2 + \cdots + \alpha_i$. There are 2^{n-1} compositions of n, as we may map α to $\{p_1, p_2, \ldots, p_{r-1}\} \subseteq [n-1]$. Conversely, the set $\{k_1, k_2, \ldots, k_{r-1}\} \subseteq [n-1]$ is associated with the composition $(k_1, k_2 - k_1, k_3 - k_2, \ldots, k_{r-1} - k_{r-2}, n - k_{r-1})$.

A *type-B composition* of $n > 0$ is a composition of n with a possible *zero-component* $\alpha_0 \overset{\mathsf{def}}{=} 0$ at the beginning. A type-B composition is *split* if it has a zero-component, and *join* otherwise. There are 2^n type-B compositions of n, because each type-B composition α is associated with a unique subset $J_\alpha \subseteq S$, with S the generating set of $\mathfrak{H}_n$. The non-zero components of α determine a subset of $\{s_1, s_2, \ldots, s_{n-1}\}$ as before, and we add s_0 to this set if and only if α is split. Through this bijection, we associate with each type-B composition α the parabolic quotient $\mathfrak{H}_n^{S \setminus J_\alpha}$ of $\mathfrak{H}_n$ determined by the *complement* $S \setminus J_\alpha$, also denoted by $\mathfrak{H}_\alpha$ hereinafter for simplicity. Clearly, if $\alpha = (0, 1, 1, \ldots, 1)$, then $\mathfrak{H}_\alpha = \mathfrak{H}_n$.

Let α be a type-B composition with r non-zero components $\alpha_1, \alpha_2, \ldots, \alpha_r$ in that order. As before, we define $p_i = \alpha_1 + \alpha_2 + \cdots + \alpha_i$, and we set $\overline{p}_i \overset{\mathsf{def}}{=} -p_i$. For $i < j$, we use the notation $[i, j] \overset{\mathsf{def}}{=} \{i, i{+}1, \ldots, j\}$. We define $\mathsf{Part}(\alpha)$ as the following partition of $\pm[n]$:

$$\begin{aligned}
&\big\{[\overline{n}, \overline{p}_{r-1}{-}1], \ldots, [\overline{p}_2, \overline{p}_1{-}1], [\overline{p}_1, \overline{1}], [1, p_1], [p_1{+}1, p_2], \ldots, [p_{r-1}{+}1, n]\big\} && (\alpha \text{ split}),\\
&\big\{[\overline{n}, \overline{p}_{r-1}{-}1], \ldots, [\overline{p}_2, \overline{p}_1{-}1], [\overline{p}_1, p_1], [p_1{+}1, p_2], \ldots, [p_{r-1}{+}1, n]\big\} && (\alpha \text{ join}).
\end{aligned}$$

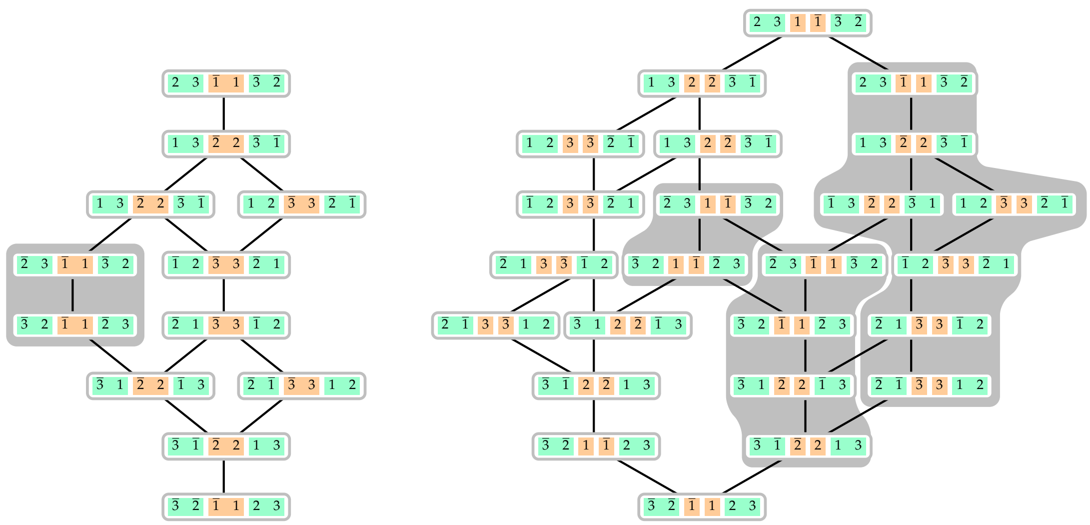
This means, if α is split, that we partition $\pm[n]$ into $2r$ symmetrically placed blocks, and if α is join, we partition it into $2r - 1$ blocks, where the middle block contains positive and negative integers. This partition also explains the terminology “join” and “split” for the type-B compositions. The partition given by a join composition has a block joining positive and negative integers, while the one given by a split composition does not.

For $a \in [n]$, we say that a is *in the* i^{th} *α-region* if $p_{i-1} \leq a < p_i$, where $p_0 = 0$. We sometimes write $\varrho_\alpha(a) = i$ in this situation.

Lemma 2.2. *For α a type-B composition, a sign-symmetric permutation $\pi \in \mathfrak{H}_n$ is in $\mathfrak{H}_\alpha$ if and only if its long one-line notation, with positions partitioned by* $\mathsf{Part}(\alpha)$*, is increasing in each part.*

We use the compositions $\alpha_{(j)} = (1, 2)$ (join) and $\alpha_{(s)} = (0, 1, 2)$ (split) as our running examples hereinafter. The partition $\mathsf{Part}(\alpha_{(j)})$ has three parts $\{-3, -2\}, \{-1, 1\}, \{2, 3\}$, while $\mathsf{Part}(\alpha_{(s)})$ has four parts $\{-3, -2\}, \{-1\}, \{1\}, \{2, 3\}$. We highlight the parts in the long one-line notation, for instance $\boxed{\overline{1}\ 3}\,\boxed{2}\,\boxed{\overline{2}}\,\boxed{\overline{3}\ 1}$ is a member of $\mathfrak{H}_{\alpha_{(s)}}$, while $\boxed{\overline{1}\ 3}\,\boxed{2\ \overline{2}}\,\boxed{\overline{3}\ 1}$ is not in $\mathfrak{H}_{\alpha_{(j)}}$, because the elements in the central block are decreasing.

The *weak order* on $\mathfrak{H}_\alpha$ is simply the restriction of the weak order on $\mathfrak{H}_n$ restricted to the subset $\mathfrak{H}_\alpha$. Figure 1a shows the weak order on $\mathfrak{H}_{\alpha_{(j)}}$ and Figure 1b shows the weak order on $\mathfrak{H}_{\alpha_{(s)}}$.



(a) The weak order on $\mathfrak{H}_{(1,2)}$. (b) The weak order on $\mathfrak{H}_{(0,1,2)}$.

Figure 1: The parabolic quotients with respect to certain type-B compositions of 3 in the weak order. The grey regions indicate congruence classes with respect to the congruence relation Θ_α defined in Section 4.

3 Aligned Elements in $\mathfrak{H}_\alpha$

The forcing relations from Section 2.2 that determine the $\vec{c}$-aligned elements of $\mathfrak{H}_n$ come from a certain total order on T together with a decomposition of the *roots* of $\mathfrak{H}_n$, which are vectors associated with the elements of T that are important in the reflection representation of the group $\mathfrak{H}_n$. For lack of space, we do not go into detail here. Instead, we present a uniform construction that works for all type-B compositions of n.

3.1 The $\vec{c}$-sorting word of the longest element

By [2, Theorem 4.1], the set $\mathfrak{H}_\alpha$ has a unique element of maximum length, which is denoted by $\pi_{\circ;\alpha}$. For our constructions, we are interested in a particular S-reduced word of $\pi_{\circ;\alpha}$, namely the *$\vec{c}$-sorting word* of $\pi_{\circ;\alpha}$. This is the S-reduced word of $\pi_{\circ;\alpha}$ that appears as far right as possible as a subword in the half-infinite word

$$^{\infty}\vec{c} \overset{\text{def}}{=} \ldots \mid s_{n-1}\cdots s_1 s_0 \mid s_{n-1}\cdots s_1 s_0.$$

Given integers $\lambda_1 \geq \lambda_2 \geq \cdots$, a *Ferrers diagram* of shape $(\lambda_1, \lambda_2, \ldots)$ is a left-aligned arrangement of unit boxes, where the i^{th} row consists of i boxes. For two Ferrers diagrams λ, μ, the associated *skew diagram* λ/μ consists of all boxes of λ that are not boxes of μ. For a type-B composition α of n with non-zero components $\alpha_1, \alpha_2, \ldots, \alpha_r$, we define

(a) The skew shape $\mathsf{skew}\big((1,2)\big)$.　(b) The skew shape $\mathsf{skew}\big((0,1,2)\big)$.

Figure 2: Skew shapes associated with certain type-B compositions of 3. The colorful triangles on the left indicate the composition.

the *α-shape* to be the skew shape $\mathsf{skew}(\alpha) \overset{\mathsf{def}}{=} \lambda_{(\alpha)}/\mu_{(\alpha)}$, where the k^{th} entry of $\mu_{(\alpha)}$ is $\mu_k \overset{\mathsf{def}}{=} \alpha_{\varrho_\alpha(k)} + \alpha_{\varrho_\alpha(k)+1} + \cdots + \alpha_r$ and the k^{th} entry of $\lambda_{(\alpha)}$ is

$$\lambda_k \overset{\mathsf{def}}{=} \begin{cases} 2n - \alpha_1, & \text{if } \alpha \text{ is join and } k \leq \alpha_1, \\ 2n+1-k, & \text{otherwise.} \end{cases}$$

We now fill $\mathsf{skew}(\alpha)$ with elements of S as follows. Let $k \in [n]$. If α is split or $k > \alpha_1$, then we fill the k^{th} row of $\mathsf{skew}(\alpha)$ with the elements $s_0, s_1, \ldots$ *from right to left* until all boxes in that row are filled. If α is join and $k \leq \alpha_1$, then we fill the k^{th} row of $\mathsf{skew}(\alpha)$ in the same way with $s_{\alpha_1+1-k}, s_{\alpha_1+2-k}, \ldots$ instead. We denote by $\mathbf{w}_\alpha$ the word obtained from reading the filling of $\mathsf{skew}(\alpha)$ from bottom to top, left to right.

Proposition 3.1. *For every type-B composition α, the word $\mathbf{w}_\alpha$ is the $\vec{c}$-sorting word of $\pi_{\circ;\alpha}$.*

Corollary 3.2. *If α is a type-B composition of n with non-zero components $\alpha_1, \alpha_2, \ldots, \alpha_r$, then*

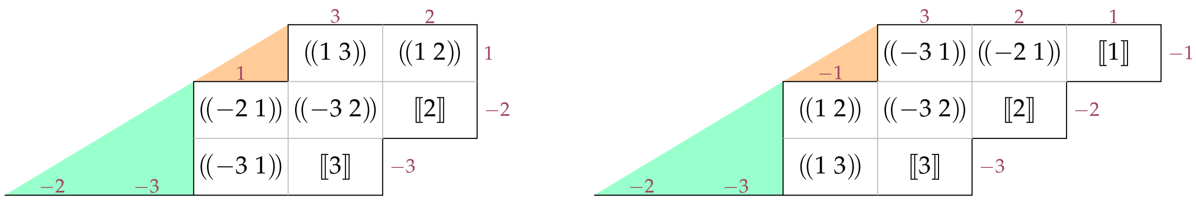
$$\ell_S(\pi_{\circ;\alpha}) = n^2 - \sum_{i=1}^{r} \binom{\alpha_i}{2} - \begin{cases} \binom{\alpha_1+1}{2}, & \textit{if } \alpha \textit{ is join,} \\ 0, & \textit{if } \alpha \textit{ is split.} \end{cases}$$

Figure 2a shows the skew shape $\mathsf{skew}(\alpha_{(j)})$ and Figure 2b shows the skew shape $\mathsf{skew}(\alpha_{(s)})$, both together with their fillings. It is easy to check that

$$\begin{aligned} \pi_{\circ;\alpha_{(j)}} &= \; 2\;3\;\bar{1}\;1\;\bar{3}\;\bar{2} \; = s_1 s_0 s_2 s_1 s_0 s_2 s_1 = \mathbf{w}_{\alpha_{(j)}}, \\ \pi_{\circ;\alpha_{(s)}} &= \; 2\;3\;1\;\bar{1}\;\bar{3}\;\bar{2} \; = s_1 s_0 s_2 s_1 s_0 s_2 s_1 s_0 = \mathbf{w}_{\alpha_{(s)}}. \end{aligned}$$

3.2 The inversion order of $\pi_{\circ;\alpha}$ with respect to $\vec{c}$

Let $w = a_1 a_2 \cdots a_k$ be an S-reduced word of some sign-symmetric permutation $\pi \in \mathfrak{H}_n$. For $i \in [k]$, we define $t_i = a_k a_{k-1} \cdots a_{k-i+1} \cdots a_{k-1} a_k$. By [1, Section 1.3], the inversion



(a) The inversion order with respect to the composition $(1,2)$.

(b) The inversion order with respect to the composition $(0,1,2)$.

Figure 3: The inversion orders associated with certain type-B compositions of 3.

set of π is then $\mathsf{Inv}(\pi) = \{t_1, t_2, \ldots, t_k\}$. The total order of $\mathsf{Inv}(\pi)$ induced by w is the *inversion order* of π. The inversion order associated with the $\vec{c}$-sorting word $\omega_{\mathsf{o};\alpha}$ of $\pi_{\mathsf{o};\alpha}$ can be obtained from $\mathsf{skew}(\alpha)$ as follows.

(i) The first n columns are labeled by $\pi_{\mathsf{o};\alpha}(n)$, $\pi_{\mathsf{o};\alpha}(n-1)$, $\ldots$, $\pi_{\mathsf{o};\alpha}(1)$ from left to right. The next $n - \alpha_1$ columns are labeled from n to $\alpha_1 + 1$. If α is split, we label the remaining α_1 columns by $\alpha_1, \alpha_1 - 1, \ldots, 1$. If α is join, there are no columns left.

(ii) If α is split, then we label the rows by -1, -2, $\ldots$, $-n$ from top to bottom. If α is join, then we label the first α_1 rows by $\alpha_1, \alpha_1 - 1, \ldots, 1$, and the remaining $n - \alpha_1$ rows by $-\alpha_1 - 1, -\alpha_1 - 2, \ldots, -n$.

Now we fill the cells of $\mathsf{skew}(\alpha)$. Consider a cell with row label r and column label c.

(i) If $r > 0$ and $c > 0$, then we fill this cell by $(\!(r\ c)\!)$. This only happens when α is join, in which case we necessarily have $r < c$.

(ii) If $r < 0$ and $c = -r$, then we fill this cell by $[\![r]\!]$.

(iii) If $r < 0$ and $c > 0$ with $c > -r$, then we fill this cell by $(\!(-c\ {-r})\!)$.

(iv) If $r < 0$ and $c > 0$ with $c < -r$, then we fill this cell by $(\!(r\ c)\!)$.

(v) If $r < 0$ and $c < 0$, then we fill this cell by $(\!(-c\ {-r})\!)$. By construction, we are in the first n columns, thus $r < c$.

Proposition 3.3. *The filling of* $\mathsf{skew}(\alpha)$*, read from top to bottom and right to left, yields the inversion order of* $\pi_{\mathsf{o};\alpha}$ *with respect to its* $\vec{c}$*-sorting word.*

Figure 3a shows the filling of the skew shape $\mathsf{skew}(\alpha_{(j)})$ with inversions, and Figure 3b shows that of $\mathsf{skew}(\alpha_{(s)})$. The corresponding inversion order of $\alpha_{(j)}$ is

$$(\!(1\ 2)\!) \prec (\!(1\ 3)\!) \prec [\![2]\!] \prec (\!(-3\ 2)\!) \prec (\!(-2\ 1)\!) \prec [\![3]\!] \prec (\!(-3\ 1)\!),$$

while that of $\alpha_{(s)}$ is

$$[\![1]\!] \prec ((-2\ 1)) \prec ((-3\ 1)) \prec [\![2]\!] \prec ((-3\ 2)) \prec ((1\ 2)) \prec [\![3]\!] \prec ((1\ 3)).$$

This can also be checked directly using the $\vec{c}$-sorting words $\mathbf{w}_{\alpha_{(j)}}$ and $\mathbf{w}_{\alpha_{(s)}}$.

3.3 Aligned elements of $\mathfrak{H}_\alpha$

We now put everything together and define the $\vec{c}$-aligned elements of $\mathfrak{H}_\alpha$. A sign-symmetric permutation $\pi \in \mathfrak{H}_\alpha$ is *$\vec{c}$-aligned* if, for all $1 \leq i < k \leq n$, we have

(1) if $[\![i]\!] \in \mathsf{Cov}(\pi)$, then $[\![j]\!] \in \mathsf{Inv}(\pi)$ for all $1 \leq j < i$ with $\varrho_\alpha(j) < \varrho_\alpha(i)$;

(2) if $((i\ k)) \in \mathsf{Cov}(\pi)$, then $((i\ j)) \in \mathsf{Inv}(\pi)$ for all $i < j < k$ with $\varrho_\alpha(i) < \varrho_\alpha(j) < \varrho_\alpha(k)$;

(3) if $((-k\ i)) \in \mathsf{Cov}(\pi)$, then

- (3a) $[\![i]\!] \in \mathsf{Inv}(\pi)$ when $i > \alpha_1$ or α is split,
- (3b) $((-j\ i)) \in \mathsf{Inv}(\pi)$ for $1 \leq j < k$ with $\varrho_\alpha(j) < \varrho_\alpha(k)$ when α is split or $j > \alpha_1$,
- (3c) $((j\ k)) \in \mathsf{Inv}(\pi)$ when $j \leq \alpha_1$, $j \neq i$ and α is join,
- (3d) $((-k\ j)) \in \mathsf{Inv}(\pi)$ for $1 \leq j < i$ with $\varrho_\alpha(j) < \varrho_\alpha(i)$ when α is split or $j > \alpha_1$,
- (3e) $((j\ i)) \in \mathsf{Inv}(\pi)$ when $i > j > \alpha_1$ and α is join.

The alignment condition can be expressed combinatorially as pattern avoidance. For $k \in \pm[n]$, we define

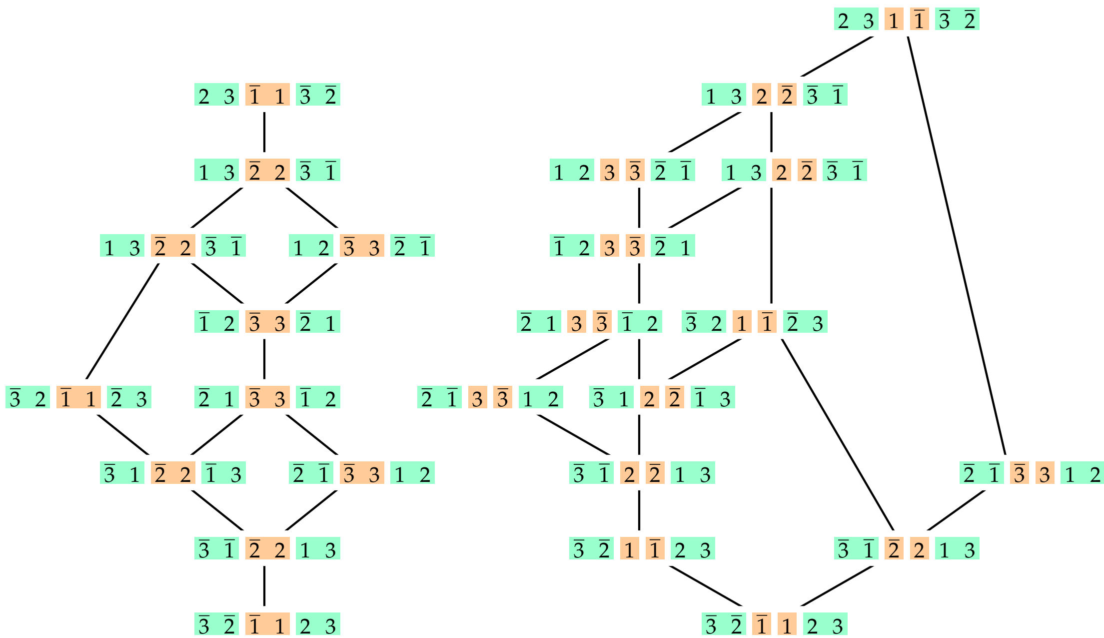
$$k^+ \stackrel{\mathsf{def}}{=} \begin{cases} k+1, & \text{if } k \neq -1, \\ 1, & \text{if } k = -1. \end{cases}$$

Let $\pi \in \mathfrak{H}_\alpha$. A *type-B $(\alpha, 231)$-pattern* is a triple (i, j, k) with $-n \leq i < j < k \leq n$ such that i, j, k are in different α-regions, $j > 0$, $\pi(i) = \pi(k)^+$ and

$$\begin{cases} \pi(i) < \pi(j), & \text{when } \alpha \text{ is split or } j > \alpha_1, \\ \pi(j) < \pi(k), & \text{when } \alpha \text{ is join and } j \leq \alpha_1. \end{cases}$$

Proposition 3.4. *For any type-B composition α of n, it holds that $\pi \in \mathfrak{H}_\alpha$ is $\vec{c}$-aligned if and only if it does not have a type-B $(\alpha, 231)$-pattern.*

Consider $\pi = \boxed{\bar{2}\ 3}\ \boxed{\bar{1}\ 1}\ \boxed{\bar{3}\ 2} \in \mathfrak{H}_{(1,2)}$. We can check easily that $((-3\ 2)) \in \mathsf{Cov}(\pi)$, but $((1\ 3)) \notin \mathsf{Inv}(\pi)$. This violates Condition (3c) above, and π is therefore not $\vec{c}$-aligned. This is, equivalently, witnessed by the $(\alpha_{(j)}, 231)$-pattern in positions $(-2, 1, 3)$.



(a) The lattice $\mathsf{Tam}_B\big((1,2)\big)$. (b) The lattice $\mathsf{Tam}_B\big((0,1,2)\big)$.

Figure 4: The type-B parabolic Tamari lattices associated with certain type-B compositions of 3.

Now take $\pi' = 1\;2\;\bar{3}\;3\;\bar{2}\;\bar{1} \in \mathfrak{H}_{(0,1,2)}$. There, we see that $[\![3]\!] \in \mathsf{Cov}(\pi')$, but $[\![1]\!] \notin \mathsf{Inv}(\pi')$. This violates Condition (1) above, and π' is thus not $\vec{c}$-aligned, which is again manifested in the $(\alpha_{(s)}, 231)$-pattern in positions $(-3, 1, 3)$.

We denote by $\mathfrak{H}_\alpha(231)$ the set of elements of $\mathfrak{H}_\alpha$ avoiding type-B $(\alpha, 231)$-patterns. By Proposition 3.4, this is also the set of $\vec{c}$-aligned elements of $\mathfrak{H}_\alpha$. The *type-B parabolic Tamari lattice* is the partially ordered set $\mathsf{Tam}_B(\alpha) \overset{\text{def}}{=} \big(\mathfrak{H}_\alpha(231), \leq_{\mathsf{weak}}\big)$. This name is justified in the next section. Figure 4a shows $\mathsf{Tam}_B\big((1,2)\big)$ and Figure 4b shows $\mathsf{Tam}_B\big((0,1,2)\big)$.

4 The parabolic Tamari lattice as a quotient lattice

Let $n > 0$. By [2, Theorem 4.1], the partially ordered set $\big(\mathfrak{H}_\alpha, \leq_{\mathsf{weak}}\big)$ is a lattice for every type-B composition α of n. We now construct a congruence relation on $\big(\mathfrak{H}_\alpha, \leq_{\mathsf{weak}}\big)$ that exhibits $\mathsf{Tam}_B(\alpha)$ as a quotient lattice. Key to this construction is the following lemma.

Lemma 4.1. *For every $\pi \in \mathfrak{H}_\alpha$ there exists a unique element $\pi_\downarrow \in \mathfrak{H}_\alpha(231)$ with $\mathsf{Inv}(\pi_\downarrow) \subseteq \mathsf{Inv}(\pi)$ such that $\mathsf{Inv}(\pi_\downarrow)$ is maximal by inclusion for any such element.*

We may therefore consider the assignment $\pi \mapsto \pi_{\downarrow}$ as a map from $\mathfrak{H}_\alpha$ to $\mathfrak{H}_\alpha(231)$. By construction, the elements of $\mathfrak{H}_\alpha(231)$ are the fixed points of this map. We also define an equivalence relation Θ_α on $\mathfrak{H}_\alpha$ by taking $(\sigma, \pi) \in \Theta_\alpha$ if and only if $\sigma_{\downarrow} = \pi_{\downarrow}$.

Sketch of proof of Theorem 1.1. The lattice property of $\mathsf{Tam}_B(\alpha)$ is a direct consequence of Lemma 4.1. Indeed, one consequence of Lemma 4.1 is that the map $\pi \mapsto \pi_{\downarrow}$ is order-preserving. As a consequence, $\mathfrak{H}_\alpha(231)$ is closed under taking joins. Since the identity belongs to $\mathfrak{H}_\alpha(231)$, the poset $\mathsf{Tam}_B(\alpha)$ has a unique smallest element. This is enough to guarantee that it is a lattice.

Moreover, it follows that the equivalence classes of Θ_α form intervals in $(\mathfrak{H}_\alpha, \leq_{\mathsf{weak}})$. The bottom elements of these intervals are precisely the members of $\mathfrak{H}_\alpha(231)$. Finally, we prove that the map that sends $\pi \in \mathfrak{H}_\alpha$ to the top element of the interval induced by elements equivalent to π in Θ_α is also order-preserving. In view of [11, Section 3], this is enough to conclude that Θ_α is a congruence relation on $(\mathfrak{H}_\alpha, \leq_{\mathsf{weak}})$. The corresponding quotient lattice is isomorphic to $\mathsf{Tam}_B(\alpha)$.

Thanks to the quotient property, the semidistributivity of $\mathsf{Tam}_B(\alpha)$ is inherited from that of the weak order on $\mathfrak{H}_\alpha$. For the trimness of $\mathsf{Tam}_B(\alpha)$, given the semidistributivity, by [13, Theorem 1.4], we only need to show that $\mathsf{Tam}_B(\alpha)$ is extremal in the sense that the number of join-irreducible elements equals the maximum length of a maximal chain. Extremality is established by showing that each suffix of the $\vec{c}$-sorting word of $\pi_{\mathsf{o};\alpha}$ is in $\mathfrak{H}_\alpha(231)$ and that for each inversion $t \in \mathsf{Inv}(\pi_{\mathsf{o};\alpha})$ there exists a unique $\pi \in \mathfrak{H}_\alpha(231)$ whose only cover inversion is t. $\square$

In Figures 1a and 1b we highlight the equivalence classes with grey boxes. The reader is invited to check that the weak order on representatives of these equivalence classes is isomorphic to the lattices shown in Figures 4a and 4b respectively.

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