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Shuffle Lattices and Bubble Lattices

Thomas McConville*[1] and Henri Mühle†[2]

[1]*Kennesaw State University, 30144 Kennesaw (GA), USA*
[2]*Technische Universität Dresden, 01062 Dresden, Germany*

Abstract. C. Greene introduced the shuffle lattice as an idealized model for DNA mutation and discovered remarkable combinatorial and enumerative properties of these structures. In this article we attempt an explanation of these properties from a lattice-theoretic point of view. To that end, we introduce and study an order extension of the shuffle lattice, the *bubble lattice*. Intriguingly, most of the combinatorics of the bubble lattice can be encoded by means of two simplicial complexes, the *noncrossing matching complex* and the *noncrossing bipartite complex*. We present an intriguing relationship between the f-vectors of these complexes and relate it to the rank-generating function of the shuffle lattice.

Keywords: shuffle words, insertion, deletion, transposition, noncrossing graphs

1 Introduction

Motivated by an idealized model for mutations in DNA sequences, C. Greene introduced the *shuffle lattice*. The ground set of this lattice is the set $\mathsf{Shuf}(m,n)$ of shuffles of order-preserving repetition-free words whose letters are taken from two disjoint, linearly ordered alphabets $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. The *shuffle order* is determined by inserting letters of Y or deleting letters of X from any given word. In [6], Greene studied this poset extensively and discovered several surprising enumerative relationships among its characteristic polynomial, its zeta polynomial and its rank-generating function. Namely, each of these invariants occurs as a specialization of the same Jacobi polynomial. Greene's enumerative results were recovered in [11] using algebraic methods, but the presence of the Jacobi polynomials remained mysterious.

In this abstract we seek to explain the enumerative relationships among the combinatorial invariants of the shuffle lattice using combinatorial lattice theory. We introduce an alternate partial ordering on the set $\mathsf{Shuf}(m,n)$, which we call the *bubble order*. This is the extension of the shuffle order, where we also allow the exchange of adjacent letters from X and Y. The resulting *bubble lattice* is indeed a lattice, and its combinatorial structure can be explained using two simplicial complexes: the *noncrossing matching complex*

*tmcconvi@kennesaw.edu
†henri.muehle@tu-dresden.de. Received funding from the European Research Council (Grant Agreement no. 681988, CSP-Infinity.)

and the *noncrossing bipartite complex*. In this abstract we state that the f-vectors of these simplicial complexes are related in an intriguing fashion and can be used to recover the rank-generating polynomial of the shuffle lattice.

This abstract is organized as follows: we start in Section 2 with the recapitulation of the basic definitions as well as the formal introduction of the bubble lattice. In Section 3, we characterize the cover relation in the bubble lattice and introduce the noncrossing matching complex. In Section 4, we sketch the proof that the bubble lattice is in fact a lattice and define the noncrossing bipartite complex. In Section 5, we conduct a refined face-enumeration in the noncrossing matching and noncrossing bipartite complex, and relate the resulting formulas to the rank-generating function of the shuffle lattice. Due to the space limitations, we have most often omitted proofs of our statements, and refer the interested reader to the full version of this abstract [7, 8].

2 Basics

2.1 Shuffle Words

For nonnegative integers m and n, we consider two disjoint sets of letters:

$$X = \{x_1, x_2, \ldots, x_m\} \quad \text{and} \quad Y = \{y_1, y_2, \ldots, y_n\}.$$

A word over the disjoint union $X \uplus Y$ is *simple* if it does not contain duplicate letters. The *support* of a word is the set of letters it contains. The empty word is denoted by ϵ and we usually (at least in the examples) write the letters of X in red and the letters of Y in blue.

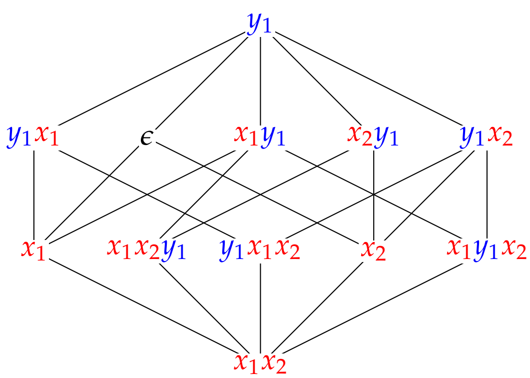
A *subword* of a simple word $\mathbf{w} = w_1 w_2 \cdots w_k$ is any word of the form $w_{i_1} w_{i_2} \cdots w_{i_\ell}$ with $1 \leq i_1 < i_2 < \cdots < i_\ell \leq k$. For $i \in [k] \overset{\text{def}}{=} \{1, 2, \ldots, k\}$ we write $\mathbf{w}_{\hat{\imath}}$ for the subword of $\mathbf{w}$ obtained by deleting the letter w_i.

If $\mathbf{u}, \mathbf{v}$ are simple words, then the *restriction* of $\mathbf{u}$ to $\mathbf{v}$, denoted by $\mathbf{u}_{\mathbf{v}}$, is the subword of $\mathbf{u}$ formed by the common letters of $\mathbf{u}$ and $\mathbf{v}$. For instance, if $\mathbf{u} = x_1 y_1 x_2 x_3 y_3$ and $\mathbf{v} = x_3 y_1 x_4$, then the restriction of $\mathbf{u}$ to $\mathbf{v}$ is $\mathbf{u}_{\mathbf{v}} = y_1 x_3$.

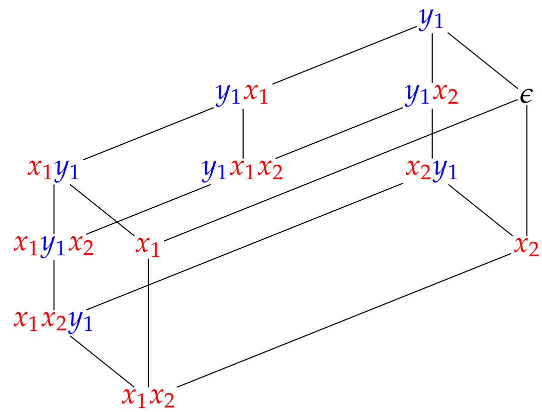
Our main interest lies in *order-preserving* simple words. That means, if we define

$$\mathbf{x} \overset{\text{def}}{=} x_1 x_2 \cdots x_m \quad \text{and} \quad \mathbf{y} \overset{\text{def}}{=} y_1 y_2 \cdots y_n,$$

then we consider the set of simple words $\mathbf{w}$ with the property that $\mathbf{w}_{\mathbf{x}}$ is a subword of $\mathbf{x}$ and $\mathbf{w}_{\mathbf{y}}$ is a subword of $\mathbf{y}$. We call such words *shuffle words* of $\mathbf{x}$ and $\mathbf{y}$, and we write $\mathsf{Shuf}(m, n)$ for the set of all shuffle words. It is easy to see that the number of shuffle words depends only on the cardinalities of X and Y, and not on the concrete elements of X and Y.



(a) The shuffle poset $\mathbf{Shuf}(2,1)$.



(b) The bubble poset $\mathbf{Bub}(2,1)$.

Figure 1: Two posets of shuffle words.

2.2 Operations on Shuffle Words

Let $\mathbf{u} = u_1u_2\cdots u_k \in \mathsf{Shuf}(m,n)$. Lemma 4.6 in [6] states that $\mathbf{u}$ is uniquely determined by its *interface*, *i.e.* the set of letters $x \in X$, $y \in Y$ for which there exists $i \in [k-1]$ such that $u_i = y$ and $u_{i+1} = x$, and its *residue*, *i.e.* the letters of $\mathbf{u}$ which are not in the interface. This motivates the following two operations on $\mathsf{Shuf}(m,n)$.

An *indel* is a relation $\mathbf{u} \to \mathbf{u}_{\hat{\imath}}$ if $u_i \in X$ or $\mathbf{u}_{\hat{\imath}} \to \mathbf{u}$ if $u_i \in Y$. In other words, an indel of $\mathbf{u}$ is a shuffle word obtained from $\mathbf{u}$ by either <u>in</u>serting a letter of $\mathbf{y}$ or <u>del</u>eting a letter of $\mathbf{x}$. A *(forward) transposition* swaps two letters u_i and u_{i+1} if $u_i \in X$ and $u_{i+1} \in Y$. In this situation, we write $\mathbf{u} \Rightarrow \mathbf{u}'$, where $\mathbf{u}' = u_1u_2\cdots u_{i-1}u_{i+1}u_iu_{i+2}\cdots u_k$. Drawing inspiration from the analogous situation for permutations, we define the *inversion set* of $\mathbf{u}$ by

$$\mathsf{Inv}(\mathbf{u}) \overset{\text{def}}{=} \{(x_s, y_t) \mid \text{ there exist } i < j \text{ such that } u_i = y_t \text{ and } u_j = x_s\}.$$

Now we use indels and transpositions to define two partial orders on $\mathsf{Shuf}(m,n)$. The *shuffle order*, denoted by $\leq_{\mathsf{shuf}}$, is the reflexive and transitive closure of indels, and the *bubble order*, denoted by $\leq_{\mathsf{bub}}$, is the reflexive and transitive closure of indels and transpositions[1]. We write $\mathbf{Shuf}(m,n) \overset{\text{def}}{=} (\mathsf{Shuf}(m,n), \leq_{\mathsf{shuf}})$ and $\mathbf{Bub}(m,n) \overset{\text{def}}{=} (\mathsf{Shuf}(m,n), \leq_{\mathsf{bub}})$ for the corresponding partially ordered sets (*posets*). Figure 1a shows $\mathbf{Shuf}(2,1)$ and Figure 1b shows $\mathbf{Bub}(2,1)$. The poset $\mathbf{Shuf}(m,n)$ was intensively studied in [6], while the poset $\mathbf{Bub}(m,n)$ is new. The main purpose of this abstract is to exhibit several remarkable structural and enumerative correspondences between these two posets and introduce some new combinatorial structures arising in the context of shuffle words.

[1] The name "bubble order" is to emphasize that this is an order extension of the shuffle order in which we have to bubble sort the words before we can perform an indel.

3 Covering Pairs and the Noncrossing Matching Complex

We first describe the "local structure" of $\mathbf{Bub}(m,n)$. To that end, we study the *covering pairs* in $\mathbf{Bub}(m,n)$, *i.e.* the relations $\mathbf{u} <_{\mathsf{bub}} \mathbf{v}$ such that there exists no $\mathbf{w}$ with $\mathbf{u} <_{\mathsf{bub}} \mathbf{w} <_{\mathsf{bub}} \mathbf{v}$. We write $\mathbf{u} \lessdot_{\mathsf{bub}} \mathbf{v}$ in that event.

While it is easily checked that every transposition corresponds to a covering pair in $\mathbf{Bub}(m,n)$, the same is not necessarily true for indels. In fact, let $\mathbf{u} = x_1x_2y_1$ and $\mathbf{v} = x_1y_1$. Then, $\mathbf{u} \to \mathbf{v}$, because we delete the letter x_2. However, this is not a covering pair, because the word $\mathbf{w} = x_1y_1x_2$ lies strictly between $\mathbf{u}$ and $\mathbf{v}$, see Figure 1b. In fact, we have $\mathbf{u} \Rightarrow \mathbf{w} \to \mathbf{v}$. In some sense, transpositions are prioritized over deletions.

Let $\mathbf{u} = u_1u_2\cdots u_k$. Recall that for $i \in [k]$ we denote by $\mathbf{u}_{\widehat{i}}$ the word obtained by deleting the letter u_i. Then, $\mathbf{u}_{\widehat{i}} \to \mathbf{u}$ if u_i is in $\mathbf{y}$ and $\mathbf{u} \to \mathbf{u}_{\widehat{i}}$ if u_i is in $\mathbf{x}$. We define a new relation $\hookrightarrow$ on $\mathsf{Shuf}(m,n)$ by setting

$$\mathbf{v} \hookrightarrow \mathbf{v}' \quad \text{if and only if} \quad \begin{cases} \mathbf{v} = \mathbf{u} \text{ and } \mathbf{v}' = \mathbf{u}_{\widehat{i}} & \text{if } u_i, u_{i+1} \in X, \\ \mathbf{v} = \mathbf{u}_{\widehat{i}} \text{ and } \mathbf{v}' = \mathbf{u} & \text{if } u_i, u_{i+1} \in Y. \end{cases}$$

In both cases, if $i = k$, then we just have to check the condition for u_k. We call $\hookrightarrow$ a *right indel*, because the inserted (resp. deleted) letter needs to be as far right as possible. Clearly, $\mathbf{u} \hookrightarrow \mathbf{v}$ implies $\mathbf{u} \to \mathbf{v}$, but the converse is not true. For instance, $x_1y_2x_2y_4 \to x_1y_2y_4 \to y_1x_1y_2y_4$ are not right indels, because in the first indel, the deleted letter x_2 is neither at the end of the word nor followed by another letter from X. In the second indel, the inserted letter y_1 ends up before a letter from X.

Lemma 3.1. *For $\mathbf{u}, \mathbf{v} \in \mathsf{Shuf}(m,n)$ we have $\mathbf{u} \lessdot_{\mathsf{bub}} \mathbf{v}$ if and only if either $\mathbf{u} \Rightarrow \mathbf{v}$ or $\mathbf{u} \hookrightarrow \mathbf{v}$.*

The definition of $\hookrightarrow$ and $\Rightarrow$ implies that there are essentially three different types of covering pairs. We use this to define the following labeling of the covering pairs of $\mathbf{Bub}(m,n)$:

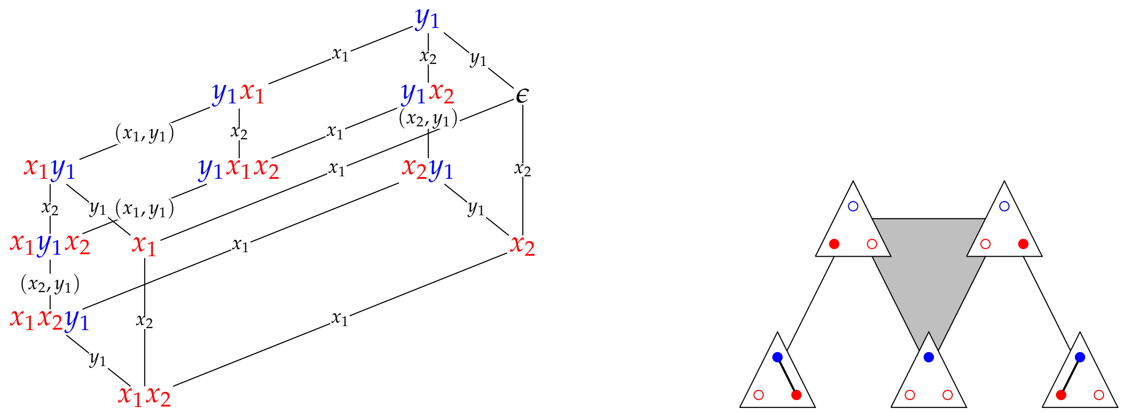
$$\lambda(\mathbf{u},\mathbf{v}) \overset{\mathsf{def}}{=} \begin{cases} x & \text{if } \mathbf{u} \hookrightarrow \mathbf{v}, \mathbf{u_y} = \mathbf{v_y}, \mathbf{u_x} \setminus \mathbf{v_x} = \{x\}, \\ y & \text{if } \mathbf{u} \hookrightarrow \mathbf{v}, \mathbf{u_x} = \mathbf{v_x}, \mathbf{v_y} \setminus \mathbf{u_y} = \{y\}, \\ (x,y) & \text{if } \mathbf{u} \Rightarrow \mathbf{v}, \mathsf{Inv}(\mathbf{v}) \setminus \mathsf{Inv}(\mathbf{u}) = \big\{(x,y)\big\}. \end{cases} \tag{3.1}$$

For $\mathbf{v} \in \mathsf{Shuf}(m,n)$, let

$$\lambda_{\downarrow}(\mathbf{v}) \overset{\mathsf{def}}{=} \big\{\lambda(\mathbf{u},\mathbf{v}) \mid \mathbf{u} \hookrightarrow \mathbf{v} \text{ or } \mathbf{u} \Rightarrow \mathbf{v}\big\}.$$

Figure 2a shows $\mathbf{Bub}(2,1)$ with this labeling.

We now construct a simplicial complex whose faces are in bijection with the shuffle words in $\mathsf{Shuf}(m,n)$. Recall that $X = \{x_1,\ldots,x_m\}$ and $Y = \{y_1,\ldots,y_n\}$, and define $\mathcal{T} \overset{\mathsf{def}}{=} X \uplus Y \uplus (X \times Y)$. We call the elements of $X \uplus Y$ *loops* and elements of $X \times Y$ *edges*. Then a loop z and an edge $\{x,y\}$ are *crossing* if $z = x$ or $z = y$. Two edges (x_{s_1}, y_{t_1}) and



(a) The poset $\mathbf{Bub}(2,1)$ labeled by λ.

(b) The noncrossing matching complex $\Gamma(2,1)$. The gray triangle indicates a two-dimensional face. In each node, the vertex at the top represents y_1, and the two vertices at the bottom represent x_1 and x_2.

Figure 2: The noncrossing matching complex of a bubble lattice.

(x_{s_2}, y_{t_2}) are *crossing* if $s_1 < s_2$ and $t_1 > t_2$. Two elements of $\mathcal{T}$ are *noncrossing* if they do not cross in one of the mentioned ways. The *noncrossing matching complex* $\Gamma \stackrel{\text{def}}{=} \Gamma(m,n)$ is the abstract simplicial complex on $\mathcal{T}$ whose faces are collections of pairwise noncrossing elements of $\mathcal{T}$. The term "crossing" is motivated by the following graphical representation of Γ. We draw vertices labeled by $x_1, x_2, \ldots, x_m$ on a horizontal line and vertices labeled by $y_1, y_2, \ldots, y_n$ on another horizontal line such that the line containing the y's is above the line containing the x's. An edge (x_s, y_t) is then illustrated by connecting the vertices x_s and y_t. Then, two edges are crossing if and only if their corresponding lines intersect.

The noncrossing matching complex recovers essential information on the bubble lattice.

Proposition 3.2. *The map $\lambda_{\downarrow} \colon \mathsf{Shuf}(m,n) \to \Gamma(m,n)$ is a bijection.*

Proof sketch. It is straightforward to verify that for every $\mathbf{v} \in \mathsf{Shuf}(m,n)$, the set of labels $\lambda_{\downarrow}(\mathbf{v})$ consists of mutually noncrossing elements of $\mathcal{T}$.

Since any $\mathbf{v} \in \mathsf{Shuf}(m,n)$ is uniquely determined by its interface and its residue, it is clear that the assignment $\mathbf{v} \mapsto \lambda_{\downarrow}(\mathbf{v})$ is a bijection from $\mathsf{Shuf}(m,n)$ to (the set of faces of) $\Gamma(m,n)$. □

We now prove that $\Gamma(m,n)$ has another intriguing topological property: it is (non-pure) vertex decomposable in the sense of [1, 10]. Let us briefly recall the necessary definitions and fix a simplicial complex Δ with vertex set M. For $F \in \Delta$, the *link* of F in

Δ is

$$\mathsf{lk}_\Delta(F) \stackrel{\text{def}}{=} \{G \in \Delta \mid F \cap G = \emptyset \text{ and } F \cup G \in \Delta\}.$$

The *deletion* of F in Δ is

$$\mathsf{del}_\Delta(F) \stackrel{\text{def}}{=} \{G \in \Delta \mid F \nsubseteq G\}.$$

A vertex $v \in M$ is a *shedding vertex* if both the link $\mathsf{lk}_\Delta(v)$ and the deletion $\mathsf{del}_\Delta(v)$ are vertex decomposable and the complexes $\mathsf{lk}_\Delta(v)$ and $\mathsf{del}_\Delta(v)$ do not share facets. A simplicial complex Δ is *vertex decomposable* if it is a simplex or contains a shedding vertex. A simplicial complex Δ is *shellable* if there exists a total order $\prec$ on the facets of Δ such that for every two facets F, G with $F \prec G$ there exists a facet $H \prec G$ such that $F \cap G \subseteq H \cap G$ and $\dim H \cap G = \dim \Delta - 1$. Such a total order (if it exists) is a *shelling order* of Δ. By [1, Theorem 11.3], every vertex-decomposable complex is shellable.

Proposition 3.3. *The noncrossing matching complex $\Gamma(m, n)$ is vertex decomposable, and therefore shellable.*

Proof sketch. The proof proceeds by induction on $m + n$. The induction base is trivial for $n = 0$, since $\Gamma(m, 0)$ is a simplex. For $n = 1$, the claim can be deduced from the results of [9], where it was shown that a simplicial complex isomorphic to $\Gamma(m, 1)$ is vertex decomposable. For $n > 1$, we consider the vertex $v = (x_s, y_t)$ for $s \in [m]$ and $t \in [n]$ and show that it is a shedding vertex of $\Gamma(m, n)$. Indeed, the link $\mathsf{lk}_{\Gamma(m,n)}(v)$ is isomorphic to the join $\Gamma(m_1, n_1) * \Gamma(m_2, n_2)$, for appropriate choices of m_1, n_1, m_2, n_2. By successively deleting vertices $(x_s, y_{t'})$ for $t' \in [n]$ we may show that v satisfies the remaining properties. $\square$

4 Lattice Structure and the Noncrossing Bipartite Complex

Now we move to the "global structure" of $\mathsf{Bub}(m, n)$. Recall that a *lattice* is a poset in which every two elements have a least upper bound and a greatest lower bound.

Proposition 4.1. *For $m, n \geq 0$, the poset $\mathsf{Bub}(m, n)$ is a lattice.*

Proof sketch. It is straightforward to verify that $\mathsf{Bub}(m, n)$ is isomorphic to the dual poset of $\mathsf{Bub}(n, m)$. Therefore, it remains to establish the existence of joins in $\mathsf{Bub}(m, n)$. In order to explicitly construct the join of $\mathbf{u}, \mathbf{v} \in \mathsf{Shuf}(m, n)$, we first determine the letters of X contained in both $\mathbf{u}$ and $\mathbf{v}$, and then we determine the letters of Y which are contained in $\mathbf{u}$ or $\mathbf{v}$. These letters form the support of several shuffle words, and we choose the minimal shuffle word from this set which satisfies the "disorder" introduced by both $\mathbf{u}$ and $\mathbf{v}$, *i.e.* which letters from Y must necessarily appear before which letters of X. $\square$

Example 4.2. Let $m = n = 5$ and consider $\mathbf{u} = x_2x_4y_1y_4x_5y_5$ and $\mathbf{v} = x_3y_1y_3x_4x_5$. Then, we consider all words in $\mathsf{Shuf}(m, n)$ using the letters $\{x_4, x_5\}$ and $\{y_1, y_3, y_4, y_5\}$. The minimal element with these letters is clearly $\widetilde{\mathbf{w}} = x_4x_5y_1y_3y_4y_5$. We observe that x_5 appears after y_1 and y_4 in $\mathbf{u}$ and after y_1 and y_3 in $\mathbf{v}$. Moreover, x_4 also appears after y_1 and y_3 in $\mathbf{v}$. Therefore, the join of $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{w} = y_1y_3x_4y_4x_5y_5$.

In the remainder of this section, we present a geometric interpretation of $\mathsf{Bub}(m,n)$. More precisely, we construct a certain simplicial complex with the property that the 1-skeleton of its dual polytope can be oriented in such a way that it agrees with the Hasse diagram of $\mathsf{Bub}(m,n)$.

Let $r \stackrel{\text{def}}{=} m+n$. We introduce two new letters x_0 and y_0 and define $\widetilde{\mathbf{x}} \stackrel{\text{def}}{=} x_0x_1\cdots x_m$ and $\widetilde{\mathbf{y}} \stackrel{\text{def}}{=} y_0y_1\cdots y_n$. Let $\widetilde{X} \stackrel{\text{def}}{=} X \uplus \{x_0\}$ and $\widetilde{Y} \stackrel{\text{def}}{=} Y \uplus \{y_0\}$ be the supports of $\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{y}}$, respectively. Consider a multigraph $\widetilde{G} = \widetilde{G}(m,n)$ with vertex set $\widetilde{X} \uplus \widetilde{Y}$ and edge set $\mathcal{U} = \mathcal{L} \uplus \mathcal{E}$ where

$$\mathcal{L} \stackrel{\text{def}}{=} \big\{\{z\} \mid z \in X \uplus Y\big\}, \quad \text{and} \quad \mathcal{E} \stackrel{\text{def}}{=} \big\{\{x,y\} \mid x \in \widetilde{X},\ y \in \widetilde{Y}\big\} \setminus \big\{\{x_0,y_0\}\big\}.$$

In other words, $\widetilde{G}$ is formed by deleting the edge $\{x_0,y_0\}$ from the complete bipartite graph with independent sets $\widetilde{X}$ and $\widetilde{Y}$, then adding a loop at each vertex except x_0 and y_0.

We say that edges $\{x_{s_1},y_{t_1}\}$ and $\{x_{s_2},y_{t_2}\}$ are *crossing* if $s_1 < s_2$ and $t_1 > t_2$. A loop $\{z\}$ and an edge $\{x,y\}$ are *crossing* if $z = x$ or $z = y$. Any two elements of $\mathcal{U}$ that are not crossing in one of these two ways are said to be *noncrossing*.

The *noncrossing bipartite complex* $\Delta \stackrel{\text{def}}{=} \Delta(m,n)$ is the abstract simplicial complex with ground set $\mathcal{U}$ whose faces are collections of pairwise noncrossing elements. See Figure 3a for an illustration of $\Delta(2,1)$. By construction, the noncrossing matching complex $\Gamma(m,n)$ is a subcomplex of $\Delta(m,n)$.

Let $\mathbf{w} = w_1w_2\cdots w_k \in \mathsf{Shuf}(m,n)$. We let $\widetilde{\mathbf{w}} = w_{-1}w_0w_1\cdots w_k$ where $w_{-1} = x_0$ and $w_0 = y_0$, and define $\phi(\mathbf{w})$ to be the following r-element subset of $\mathcal{U}$:

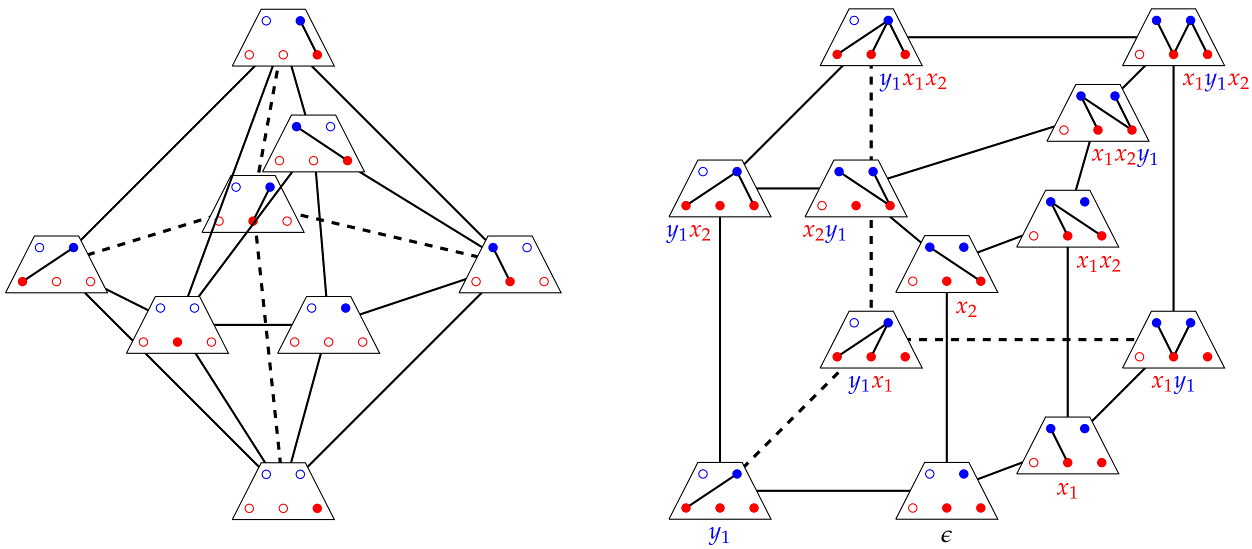
- if $z \in X \uplus Y$ is not in the support of $\mathbf{w}$, then $\{z\} \in \phi(\mathbf{w})$;
- if $w_j \in X$ and $i = \max\{i' < j \mid w_{i'} \in \widetilde{Y}\}$, then $\{w_j, w_i\} \in \phi(\mathbf{w})$;
- if $w_j \in Y$ and $i = \max\{i' < j \mid w_{i'} \in \widetilde{X}\}$, then $\{w_i, w_j\} \in \phi(\mathbf{w})$.

For example, if $m = n = 3$, then

$$\phi(x_2y_1y_2x_3) = \big\{\{x_1\},\ \{y_3\},\ \{x_2,y_0\},\ \{x_2,y_1\},\ \{x_2,y_2\},\ \{x_3,y_2\}\big\}.$$

Proposition 4.3. *The map ϕ is a bijection from $\mathsf{Shuf}(m,n)$ to the set of facets of $\Delta(m,n)$.*

By construction, $\phi(\mathbf{w})$ contains $r = m+n$ elements. Thus, Proposition 4.3 implies that $\Delta(m,n)$ is a pure simplicial complex of dimension $r-1$. The next two lemmas establish that the 1-skeleton of the dual of $\Delta(m,n)$ can be oriented in such a way that one obtains the Hasse diagram of $\mathsf{Bub}(m,n)$. This is illustrated in Figure 3b, which



(a) The noncrossing bipartite complex $\Delta(2,1)$.

(b) The dual polytope of $\Delta(2,1)$.

Figure 3: The noncrossing bipartite complex and its dual polytope. These are two-dimensional polytopes realized in a three-dimensional space. We have drawn a projection where hidden edges are dashed.

shows the dual of $\Delta(m,n)$. The vertices of this dual polytope (which are the facets of $\Delta(m,n)$) are labeled by their corresponding shuffle words under the map ϕ^{-1}. It is left to the reader to recover the orientation of the 1-skeleton of this polytope that recovers the bubble lattice $\mathsf{Bub}(m,n)$; see also Figure 1b.

Lemma 4.4. *For any $\sigma \in \Delta(m,n)$, the set of shuffle words $\mathbf{w} \in \mathsf{Shuf}(m,n)$ with $\sigma \subseteq \varphi(\mathbf{w})$ is a closed interval of $\mathsf{Bub}(m,n)$.*

Lemma 4.5. *For shuffle words $\mathbf{u}, \mathbf{v}$, we have $\dim \phi(\mathbf{u}) \cap \phi(\mathbf{v}) = r - 2$ if and only if either $\mathbf{u} \lessdot_{\mathsf{bub}} \mathbf{v}$ or $\mathbf{v} \lessdot_{\mathsf{bub}} \mathbf{u}$.*

We conclude this section with the fact that $\Delta(m,n)$ is shellable.

Proposition 4.6. *Any linear extension of $\mathsf{Bub}(m,n)$ induces a shelling order on the facets of $\Delta(m,n)$. Consequently, $\Delta(m,n)$ is shellable.*

Proof sketch. Fix a linear extension $\prec$ of $\leq_{\mathsf{bub}}$ and pick $\mathbf{u}, \mathbf{v} \in \mathsf{Shuf}(m,n)$ with $\mathbf{u} \prec \mathbf{v}$. By abuse of notation, we use the same symbol $\prec$ for the total order on the facets of $\Delta(m,n)$ induced via the map ϕ. We thus have $\phi(\mathbf{u}) \prec \phi(\mathbf{v})$. Now, $\mathbf{u} <_{\mathsf{bub}} \mathbf{v}$ and we consider $\sigma = \phi(\mathbf{u}) \cap \phi(\mathbf{v})$. By Lemma 4.4, any $\mathbf{w} \in \mathsf{Shuf}(m,n)$ with $\sigma \subseteq \phi(\mathbf{w})$ must belong to the closed interval $[\mathbf{u}, \mathbf{v}]$ in $\mathsf{Bub}(m,n)$. In particular, we may choose $\mathbf{w}$ such that

$\mathbf{u} \leq_{\mathsf{bub}} \mathbf{w} \lessdot_{\mathsf{bub}} \mathbf{v}$, which then implies $\phi(\mathbf{w}) \prec \phi(\mathbf{v})$. But then we have $\sigma \subseteq \phi(\mathbf{w}) \cap \phi(\mathbf{v})$ and with Lemma 4.5 we get $\dim \phi(\mathbf{w}) \cap \phi(\mathbf{v}) = r - 2 = \dim \Delta(m,n) - 1$. Thus $\prec$ is a shelling order of $\Delta(m,n)$. $\square$

5 Enumerative Properties

Let us now turn to some enumerative aspects of the simplicial complexes $\Gamma(m,n)$ and $\Delta(m,n)$. In general, the *f-vector* of a simplicial complex Δ is the sequence

$$f_\Delta \stackrel{\mathsf{def}}{=} (f_{-1}, f_0, f_1, \ldots),$$

where f_i counts the i-dimensional faces of Δ.

We start by computing a refined variant of the f-vector of the noncrossing matching complex $\Gamma(m,n)$. We define the *in-degree* of $\mathbf{u} \in \mathsf{Shuf}(m,n)$ by

$$\mathsf{in}(\mathbf{u}) \stackrel{\mathsf{def}}{=} \bigl|\bigl\{\mathbf{u}' \in \mathsf{Shuf}(m,n) \mid \mathbf{u}' \lessdot_{\mathsf{bub}} \mathbf{u}\bigr\}\bigr|.$$

By Proposition 3.2, $\mathsf{in}(\mathbf{u})$ describes the size of the corresponding face $\lambda_\downarrow(\mathbf{u})$ of $\Gamma(m,n)$. By Lemma 3.1, a covering pair in $\mathbf{Bub}(m,n)$ can either correspond to an indel or to a transposition. We thus define the *indel-degree* of $\mathbf{u}$ by

$$\mathsf{in}_{\hookrightarrow}(\mathbf{u}) \stackrel{\mathsf{def}}{=} \bigl|\bigl\{\mathbf{u}' \in \mathsf{Shuf}(m,n) \mid \mathbf{u}' \hookrightarrow \mathbf{u}\bigr\}\bigr|,$$

and the *transpose-degree* of $\mathbf{u}$ by

$$\mathsf{in}_{\Rightarrow}(\mathbf{u}) \stackrel{\mathsf{def}}{=} \bigl|\bigl\{\mathbf{u}' \in \mathsf{Shuf}(m,n) \mid \mathbf{u}' \Rightarrow \mathbf{u}\bigr\}\bigr|.$$

Evidently, $\mathsf{in}(\mathbf{u}) = \mathsf{in}_{\hookrightarrow}(\mathbf{u}) + \mathsf{in}_{\Rightarrow}(\mathbf{u})$.

Lemma 5.1. *The number of elements $\mathbf{u} \in \mathsf{Shuf}(m,n)$ with $\mathsf{in}_{\Rightarrow}(\mathbf{u}) = a$ and $\mathsf{in}_{\hookrightarrow}(\mathbf{u}) = b$ is $\binom{m}{a}\binom{n}{a}\binom{m+n-2a}{b}$.*

Proof sketch. Any $\mathbf{u} \in \mathsf{Shuf}(m,n)$ is determined by its interface, *i.e.* pairs (x_s, y_t) where y_t immediately precedes x_s. Choosing an interface consisting of a such pairs can be done in $\binom{m}{a}\binom{n}{a}$ ways and each such pair is the label of a covering pair corresponding to a transposition. Any letter of X not in the support of $\mathbf{u}$ is the label of a covering pair corresponding to an indel and any letter of Y in the support of $\mathbf{u}$, but not part of a pair in the interface, is the label of a covering pair corresponding to an indel. Clearly, these letters can be obtained in $\binom{m+n-2a}{b}$ ways. $\square$

We may thus compute the *H-triangle* of $\mathbf{Bub}(m,n)$, defined by

$$H_{m,n}(p,q) \stackrel{\mathsf{def}}{=} \sum_{\mathbf{u} \in \mathsf{Shuf}(m,n)} p^{\mathsf{in}(\mathbf{u})} q^{\mathsf{in}_{\hookrightarrow}(\mathbf{u})}.$$

Proposition 5.2. *For $m, n \geq 0$, we have*

$$H_{m,n}(p,q) = \sum_{a\geq 0} \binom{m}{a}\binom{n}{a} p^a (1+pq)^{m+n-2a}.$$

The f-vector of $\Gamma(m,n)$ is then obtained as the coefficient sequence of $H_{m,n}(p,1)$.

Remark 5.3. Somewhat surprisingly, the polynomial $H_{m,n}(p,1)$ appears already in [6, Corollary 4.8] and was used to establish the rank symmetry of $\mathsf{Shuf}(m,n)$ and for showing its decomposition into symmetrically placed Boolean lattices.

Next, we wish to describe the f-polynomial of the noncrossing bipartite complex $\Delta(m,n)$. To that end, we define the *F-triangle* by

$$F_{m,n}(p,q) \overset{\mathsf{def}}{=} \sum_{\sigma\in\Delta(m,n)} p^{|\sigma\cap\mathcal{E}|} q^{|\sigma\cap\mathcal{L}|}.$$

More precisely, the variable p keeps track of the number of edges per face and the variable q keeps track of the number of loops per face. The f-vector of $\Delta(m,n)$ is obtained from the coefficient sequence of $F_{m,n}(p,p)$. It is not immediately obvious, but follows for instance from the next result that the h-vector of $\Delta(m,n)$ is obtained from the coefficient sequence of $H_{m,n}(p,1)$.

Our main result relates the F- and the H-triangle associated with $\mathsf{Bub}(m,n)$ by an explicit variable substitution. This establishes a remarkable connection between the f-vectors of the noncrossing matching and the noncrossing bipartite complex.

Theorem 5.4. *For $m, n \geq 0$, we have*

$$H_{m,n}(p,q) = (p-1)^{m+n} F_{m,n}\left(\frac{1}{p-1}, \frac{1+p(q-1)}{p-1}\right).$$

Equivalently, we have

$$F_{m,n}(p,q) = p^{m+n} H_{m,n}\left(\frac{p+1}{p}, \frac{q+1}{p+1}\right).$$

Proof sketch. Using the shellability of $\Delta(m,n)$ and the fact that $\mathsf{Bub}(m,n)$ is obtainable from an orientation of the 1-skeleton of the dual of $\Delta(m,n)$ we may relate the reverse h-vector of $\Delta(m,n)$ to the f-vector of $\Gamma(m,n)$. Some explicit computation then yields the result. □

Example 5.5. By inspection of Figure 2a, we compute

$$H_{2,1}(p,q) = p^3q^3 + 3p^2q^2 + 2p^2q + 3pq + 2p + 1 = (1+pq)^3 + 2p(1+pq)$$

in accordance with Proposition 5.2. The evaluation $H_{2,1}(p,1)$ has the coefficient sequence $(1,5,5,1)$ which agrees with the f-vector of $\Gamma(2,1)$, see Figure 2b. By Theorem 5.4, we get

$$F_{2,1}(p,q) = 3p^3 + 5p^2q + 3pq^2 + q^3 + 7p^2 + 8pq + 3q^2 + 5p + 3q + 1.$$

The evaluation $F_{2,1}(p,p)$ has the coefficient sequence $(12,18,8,1)$, which is the (reverse) f-vector of $\Delta_{2,1}$, see Figure 3a.

Lastly, we state a relation of the F- and H-triangle associated with $\mathsf{Bub}(m,n)$ with a natural polynomial associated with the shuffle lattice $\mathsf{Shuf}(m,n)$. Recall that the *Möbius function* of a finite poset $\mathbf{P} = (P, \leq)$ is defined for all $x, y \in P$ by

$$\mu_{\mathbf{P}}(x,y) \stackrel{\mathsf{def}}{=} \begin{cases} 1 & \text{if } x = y, \\ -\sum_{x<z\leq y} \mu_{\mathbf{P}}(x,z) & \text{if } x < y, \\ 0 & \text{otherwise.} \end{cases}$$

It was shown in [6] that $\mathsf{Shuf}(m,n)$ is a graded poset. If we denote by rk the rank function and by μ the Möbius function of the *shuffle* poset $\mathsf{Shuf}(m,n)$, then the *M-triangle* of $\mathsf{Shuf}(m,n)$ is

$$M_{m,n}(p,q) \stackrel{\mathsf{def}}{=} \sum_{\mathbf{u},\mathbf{v}\in\mathsf{Shuf}(m,n)} \mu(\mathbf{u},\mathbf{v}) p^{\mathsf{rk}(\mathbf{u})} q^{\mathsf{rk}(\mathbf{v})}.$$

The second main outcome of this abstract is the following conjectural relation of the M-triangle with the F- and H-triangle.

Conjecture 5.6. *For $m, n \geq 0$, we have*

$$\begin{aligned} M_{m,n}(p,q) &= (1-q)^{m+n} H_{m,n}\left(\frac{q(p-1)}{1-q}, \frac{p}{p-1}\right) \\ &= (pq-1)^{m+n} F_{m,n}\left(\frac{1-q}{pq-1}, \frac{1}{pq-1}\right). \end{aligned}$$

Example 5.7. It can be verified directly in Figure 1a that

$$M_{2,1}(p,q) = p^3q^3 - 5p^2q^3 + 5p^2q^2 + 7pq^3 - 12pq^2 - 3q^3 + 5pq + 7q^2 - 5q + 1,$$

which confirms Conjecture 5.6 in this case.

We wish to end this abstract with the following observation. The structural and enumerative relationships between the bubble lattice and the shuffle lattice as well as the noncrossing matching and the noncrossing bipartite complex that we have outlined here is completely analogous to the relationship between remarkable, important structures associated with a finite irreducible Coxeter group W and a Coxeter element $c \in W$. More precisely, the analogous structures in this setting are the *c-Cambrian lattice* and the *c-noncrossing partition lattice*, as well as the *canonical join complex* of the c-Cambrian lattice and the *c-cluster complex*. F-, H- and M-triangles associated with these structures were defined in [3, 4] and they exhibit *the same* relations as the ones stated in Theorem 5.4 and Conjecture 5.6.

Other structures exhibiting a similar behavior were studied for instance in [2, 5]. This indicates that there should exist a more general setting of which all these structures are concrete examples.

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