Séminaire Lotharingien de Combinatoire **86B** (2022)
Article #64, 12 pp.

Proceedings of the 34th Conference on Formal Power Series and Algebraic Combinatorics (Bangalore)

Multiplication Theorems for Self-Conjugate Partitions

David Wahiche*[1]

[1] *Univ. Lyon, Université Claude Bernard Lyon 1, UMR 5208, Institut Camille Jordan, France*

Abstract. In 2011, Han and Ji proved, by using fine properties of the Littlewood decomposition, multiplication theorems for integer partitions. From the latter, they derived modular analogues of many classical identities involving hook-length. In the present paper, we prove multiplication theorems for the subset of self-conjugate partitions. Although difficulties arise due to parity questions, we are almost always able to include the BG-rank introduced by Berkovich and Garvan.

Keywords: integer partitions, hook-length, Littlewood decomposition, q-series

1 Introduction and results

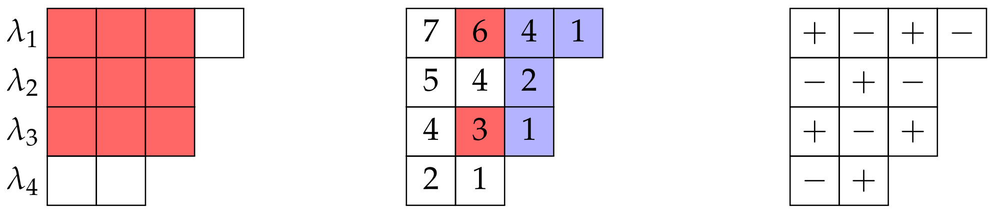
Formulas involving hook-length of integer partitions abound in combinatorics and representation theory. One illustrative example is the hook-length formula discovered in 1954 by Frame, Robinson and Thrall [5]. It states the equality between the number f^λ of standard Young tableaux of shape λ and size n, and the number of permutations of $\{1, \ldots, n\}$ divided by the product of the elements of the hook-lengths multiset $\mathcal{H}(\lambda)$ of λ. A much more recent identity is the Nekrasov–Okounkov formula. It was discovered independently by Nekrasov and Okounkov in their work on random partitions and Seiberg–Witten theory [11] and by Westbury in his work on universal characters for $\mathfrak{sl}_n$ [16]. This formula is commonly stated as follows:

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z}{h^2}\right) = \prod_{k \geq 1} \left(1 - q^k\right)^{z-1}, \tag{1.1}$$

where z is a fixed complex number. This identity was later obtained independently by Han [8], using combinatorial tools and the Macdonald identities for affine root systems of type $\widetilde{A}_t$ [10]. The well-known generating series for the set of partitions $\mathcal{P}$ can also be obtained from (1.1) by setting $z = 0$:

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} = \prod_{j \geq 1} \frac{1}{1 - q^j}. \tag{1.2}$$

*wahiche@math.univ-lyon1.fr



(a) Durfee square (b) hook-lengths (c) BG-rank

Figure 1: Ferrers diagram and some partition statistics.

Each partition can be represented by its Ferrers diagram, which is a finite collection of boxes arranged in left-justified rows, with the row lengths in non-increasing order. The *Durfee square* of λ is the maximal square fitting in the Ferrers diagram. We call its diagonal the *main diagonal* of λ. Its size will be denoted $d = d(\lambda) := \max(s|\lambda_s \geq s)$. As an example, in Figure 1a, the Durfee square of $\lambda = (4,3,3,2)$, which is a partition of 12 of length 4, is coloured in red.

For each box v in the Ferrers diagram of a partition λ (for short we will say for each box v in λ), one defines the *arm-length* (respectively *leg-length*) as the number of boxes u in the same row (respectively in the same column) as v strictly to the right of (respectively strictly below) the box v. One defines the *hook-length* of v, denoted by $h_v(\lambda)$ or h_v, the number of boxes u such that either $u = v$, or u lies strictly below (respectively to the right) of v in the same column (respectively row). The *hook-length multiset* of λ, denoted by $\mathcal{H}(\lambda)$, is the multiset of all hook-lengths of λ. For any positive integer t, the multiset of all hook-lengths that are congruent to $0 \pmod t$ is denoted by $\mathcal{H}_t(\lambda)$. Notice that $\mathcal{H}(\lambda) = \mathcal{H}_1(\lambda)$. A partition ω is a *t-core* if $\mathcal{H}_t(\omega) = \emptyset$. In Figure 1b, the Ferrers diagram is filled with the hook-lengths of all boxes and the boxes are shaded in red if they are part of $\mathcal{H}_3(\lambda) = \{3,6\}$. The boxes shaded in blue are part of the rim-hook ribbon of length 4.

A *rim hook* (or border strip, or ribbon) is a connected skew shape containing no 2×2 square. The length of a rim hook is the number of boxes in it, and its height is one less than its number of rows. By convention, the height of an empty rim hook is zero.

Recall from the work of Berkovich and Garvan [1] that the BG-*rank* of the partition λ, denoted by $\mathrm{BG}(\lambda)$, is defined as follows. First fill each box in the Ferrers diagram of λ with alternating ± 1's along rows and columns beginning with a "$+1$" in the $(1,1)$ position (see Figure 1c). Then sum their values over all the boxes. Note that all boxes belonging to the diagonal of a Ferrers diagram are filled with a "$+1$".

Let a and q be complex numbers such that $|q| < 1$. Recall that the *q-Pochhammer symbol* is defined as $(a;q)_0 = 1$ and for any integer $n \geq 1$:

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}) \qquad \text{and} \qquad (a;q)_\infty = \prod_{j\geq 0}(1-aq^j).$$

A classical bijection in partition theory is the Littlewood decomposition (see for instance [9, Section 2]). Roughly speaking, for any positive integer t, it transforms $\lambda \in \mathcal{P}$ into two components, namely the t-core ω and the t-quotient $\underline{\nu}$ (see Section 2 for precise definitions and properties):

$$\lambda \in \mathcal{P} \mapsto (\omega, \underline{\nu}) \in \mathcal{P}_{(t)} \times \mathcal{P}^t.$$

In [9], Han and Ji underline some important properties of the Littlewood decomposition, which enable them to prove the following multiplication theorem.

Theorem 1 ([9, Theorem 1.5]). *Let t be a positive integer and let ρ_1 be a function defined on $\mathbb{N}$. Let f_t be the following formal power series:*

$$f_t(q) := \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \rho_1(th).$$

Then we have

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} \prod_{h \in \mathcal{H}_t(\lambda)} \rho_1(h) = t \frac{(q^t;q^t)_\infty^t}{(q;q)_\infty} \left(f_t(xq^t)\right)^t.$$

Note that Walsh and Warnaar in [15] prove as well multiplication theorems giving rise to hook-length formulas. They also get interesting extensions regarding leg-length.

Theorem 1 gives modular analogues of many classical formulas. For instance, setting $\rho_1(h) = 1 - z/h^2$ for any complex number z, it provides the modular analogue of the Nekrasov–Okounkov formula (1.1) originally proved in [8, Theorem 1.2]:

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} \prod_{h \in \mathcal{H}_t(\lambda)} \left(1 - \frac{z}{h^2}\right) = \frac{(q^t;q^t)_\infty^t}{(xq^t;xq^t)_\infty^{t-z/t} (q;q)_\infty}. \tag{1.3}$$

In the present work, we extend Theorem 1 to an important subset of $\mathcal{P}$, namely the self-conjugate partitions, and derive several applications regarding these. A partition λ is *self-conjugate* if its Ferrers diagram is symmetric along the main diagonal. We denote the set of self-conjugate partitions by $\mathcal{SC}$. This subset of partitions has been of particular interest within the works of Pétréolle [12, 13] where two Nekrasov–Okounkov type formulas for affine root systems of $\widetilde{C}$ and $\widetilde{C}\check{}$ are derived. In [12, 13], he proved the following $\mathcal{SC}$ Nekrasov–Okounkov type formula similar to (1.1), which stands for any complex number z:

$$\sum_{\lambda \in \mathcal{SC}} (-q)^{|\lambda|} \prod_{\substack{u \in \lambda \\ h_u \in \mathcal{H}(\lambda)}} \left(1 - \frac{2z}{h_u \varepsilon_u}\right) = \left(\frac{(q^2;q^2)_\infty^{z+1}}{(q;q)_\infty}\right)^{2z-1}.$$

Here for a box u of λ of coordinates (i,j), ε_u is defined as -1 if u is strictly below the main diagonal of the Ferrers diagram and as 1 otherwise. The Littlewood decomposition, when restricted to $\mathcal{SC}$, also has interesting properties and can be stated as follows (see for instance [6, 13]):

$$\lambda \in \mathcal{SC} \mapsto \begin{cases} (\omega, \underline{\widetilde{\nu}}) \in \mathcal{SC}_{(t)} \times \mathcal{P}^{t/2} & \text{if } t \text{ is even,} \\ (\omega, \underline{\widetilde{\nu}}, \mu) \in \mathcal{SC}_{(t)} \times \mathcal{P}^{(t-1)/2} \times \mathcal{SC} & \text{if } t \text{ is odd.} \end{cases} \tag{1.4}$$

As can be seen above, if one aims to provide an analogue of Theorem 1 for self-conjugate partitions, the t even case is simpler to handle. We therefore first restrict ourselves to this setting. Nevertheless, it yields a result where the BG-rank can be incorporated.

Theorem 2. *Set t an even integer and let $\widetilde{\rho_1}$ be a function defined on $\mathbb{Z} \times \{-1, 1\}$. Set also $f_t(q)$ the formal power series defined by:*

$$f_t(q) := \sum_{\nu \in \mathcal{P}} q^{|\nu|} \prod_{h \in \mathcal{H}(\nu)} \widetilde{\rho_1}(th, 1)\widetilde{\rho_1}(th, -1).$$

Then we have

$$\sum_{\lambda \in \mathcal{SC}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\mathrm{BG}(\lambda)} \prod_{\substack{u \in \lambda \\ h_u \in \mathcal{H}_t(\lambda)}} \widetilde{\rho_1}(h_u, \varepsilon_u)$$
$$= f_t(x^2 q^{2t})^{t/2} (q^{2t}; q^{2t})_\infty^{t/2} (-bq; q^4)_\infty (-q^3/b; q^4)_\infty.$$

As a direct consequence, by choosing $\widetilde{\rho_1}(a, \varepsilon) = \rho_1(a)$, we obtain the following unsigned version of Theorem 2.

Corollary 3. *Let t be a positive even integer and let ρ_1 be a function defined on $\mathbb{N}$. Let f_t be the formal power series defined as:*

$$f_t(q) := \sum_{\nu \in \mathcal{P}} q^{|\nu|} \prod_{h \in \mathcal{H}(\nu)} \rho_1(th)^2.$$

Then we have

$$\sum_{\lambda \in \mathcal{SC}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\mathrm{BG}(\lambda)} \prod_{h \in \mathcal{H}_t(\lambda)} \rho_1(h) = f_t(x^2 q^{2t})^{t/2} (q^{2t}; q^{2t})_\infty^{t/2} (-bq; q^4)_\infty (-q^3/b; q^4)_\infty.$$

By setting $\widetilde{\rho_1}(a, \varepsilon) = 1$, we obtain the following new trivariate generating series:

$$\sum_{\lambda \in \mathcal{SC}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\mathrm{BG}(\lambda)} = \frac{(q^{2t}; q^{2t})_\infty^{t/2}}{(x^2 q^{2t}; x^2 q^{2t})_\infty^{t/2}} \left(-bq; q^4\right)_\infty \left(-q^3/b; q^4\right)_\infty.$$

We also derive the modular self-conjugate versions of (1.3) by setting $\widetilde{\rho_1}(a, \varepsilon) = 1 - z/(a\varepsilon)$ in Theorem 2 and $\rho_1(h) = \left(1 - z/h^2\right)^{1/2}$ in Corollary 3.

Corollary 4. *For any complex number z and any even positive integer t, we have:*

$$\sum_{\lambda\in\mathcal{SC}} q^{|\lambda|}x^{|\mathcal{H}_t(\lambda)|}b^{\mathrm{BG}(\lambda)} \prod_{h\in\mathcal{H}_t(\lambda)} \left(1-\frac{z}{h^2}\right)^{1/2} = \sum_{\lambda\in\mathcal{SC}} q^{|\lambda|}x^{|\mathcal{H}_t(\lambda)|}b^{\mathrm{BG}(\lambda)} \prod_{\substack{u\in\lambda\\ h_u\in\mathcal{H}_t(\lambda)}} \left(1-\frac{z}{h_u\varepsilon_u}\right)$$
$$= \left(x^2q^{2t};x^2q^{2t}\right)_\infty^{(z/t-t)/2}\left(q^{2t};q^{2t}\right)_\infty^{t/2}\left(-bq;q^4\right)_\infty\left(-q^3/b;q^4\right)_\infty.$$

It is also possible to prove a result similar to Theorem 2 when t is odd; nevertheless more difficulties arise due to the additional $\mu\in\mathcal{SC}$ appearing in the Littlewood decomposition. However, as will be seen later, the subset $\mathcal{BG}^t$ of $\mathcal{SC}$ for which μ is empty in (1.4), can be handled almost similarly as for Theorem 2. The interesting thing here is that $\mathcal{BG}^t$ actually corresponds to partitions called $\mathcal{BG}_t$ in [2], which are algebraically involved in representation theory of the symmetric group over a field of characteristic t when t is an odd prime number.

Theorem 5. *Let t be a positive odd integer and set $\widetilde{\rho}_1$ a function defined on $\mathbb{Z}\times\{-1,1\}$. Let f_t be the formal power series defined in Theorem 2. Then we have*

$$\sum_{\lambda\in\mathcal{BG}^t} q^{|\lambda|}x^{|\mathcal{H}_t(\lambda)|} \prod_{\substack{u\in\lambda\\ h_u\in\mathcal{H}_t(\lambda)}} \widetilde{\rho}_1(h_u,\varepsilon_u) = f_t(x^2q^{2t})^{(t-1)/2}\frac{(q^{2t};q^{2t})_\infty^{(t-1)/2}(-q;q^2)_\infty}{(-q^t;q^{2t})_\infty}.$$

Note that by taking $\widetilde{\rho}_1(a,\varepsilon)=\rho_1(a)$ in the above result, we get an analogue of Corollary 3 for t odd and $b=1$, restricted to the set $\mathcal{BG}^t=\mathcal{BG}_t$.

This paper is organized as follows. In Section 2, we provide the necessary background and properties regarding the Littlewood decomposition for self-conjugate partitions. Section 3 is devoted to the proof of Theorem 2. Finally in Section 4 we study the odd case. We point out that more general addition-multiplication theorems and their consequences are presented in [14].

2 Littlewood decomposition for self-conjugate partitions

In this section, we use the formalism of Han and Ji in [9]. Recall that a partition μ is a *t-core* if it has no hook that is a multiple of t. For any $A\subset\mathcal{P}$, we denote by $A_{(t)}$ the subset of elements of A that are t-cores. For example, the only 2-cores are the "staircase" partitions $(k,k-1,\dots,1)$, for any positive integer k. These are also the only $\mathcal{SC}$ 2-cores.

Let $\partial\lambda$ be the border of the Ferrers diagram of λ. Each step on $\partial\lambda$ is either horizontal or vertical. Encode the walk along the border from the South-West to the North-East as depicted in Figure 2: take "0" for a vertical step and "1" for a horizontal step. This yields a $0/1$ sequence denoted $s(\lambda)$. The resulting word $s(\lambda)$ over the $\{0,1\}$ alphabet contains infinitely many "0"'s (respectively "1"'s) at the beginning (respectively the end). This word is indexed by $\mathbb{Z}$, and is written $(c_i)_{i\in\mathbb{Z}}$.

This writing as a sequence is not unique since for any k, sequences $(c_{k+i})_{i\in\mathbb{Z}}$ encode the same partition. Hence it is necessary for that encoding to be bijective to set the index 0 uniquely. To tackle that issue, we set the index 0 when the number of "0"'s to the right of that index is equal to the number of "1"'s to the left. In other words, the number of horizontal steps along $\partial\lambda$ corresponding to a "1" of negative index in $(c_i)_{i\in\mathbb{Z}}$ must be equal to the number of vertical steps corresponding to "0"'s of nonnegative index in $(c_i)_{i\in\mathbb{Z}}$ along $\partial\lambda$. The delimitation between the letter of index -1 and the one of index 0 is called the *median* of the word, marked by a $|$ symbol. The size of the Durfee square is then equal to the number of "1"'s of negative index. Hence a partition is bijectively associated by the application s to the word $s(\lambda) = (c_i)_{i\in\mathbb{Z}} = (\ldots c_{-2}c_{-1}|c_0c_1c_2\ldots)$, where $c_i \in \{0,1\}$ for any $i\in\mathbb{Z}$, and such that $\#\{i \leq -1, c_i = 1\} = \#\{i \geq 0, c_i = 0\}$.

Moreover, this application maps bijectively a box u of hook-length h_u of the Ferrers diagram of λ to a pair of indices $(i_u, j_u) \in \mathbb{Z}^2$ of the word $s(\lambda)$ such that $i_u < j_u$, $c_{i_u} = 1$, $c_{j_u} = 0$, $j_u - i_u = h_u$. Now we recall the following classical map, often called the Littlewood decomposition (see for instance [6, 9]).

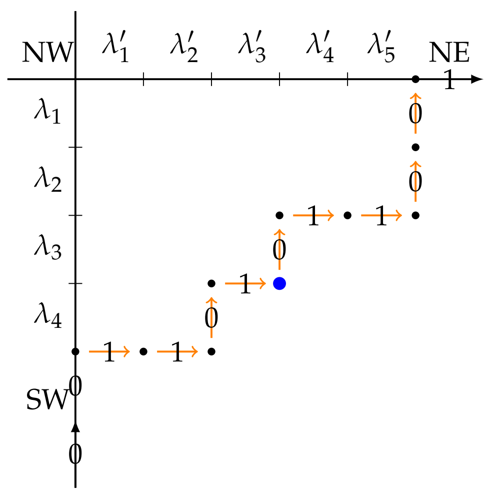


Figure 2: $\partial\lambda$ and its binary correspondence for $\lambda = (5,5,3,2)$.

Definition 6. *Let $t \geq 2$ be an integer and consider:*

$$\Phi_t : \left| \begin{array}{ccc} \mathcal{P} & \to & \mathcal{P}_{(t)} \times \mathcal{P}^t \\ \lambda & \mapsto & (\omega, \nu^{(0)}, \ldots, \nu^{(t-1)}), \end{array} \right.$$

where ω is the partition obtained from removing all ribbons of length t from λ's Ferrers diagram. ω is the t-core of λ denoted by $core_t(\lambda)$ and the tuple $\underline{\nu} = \left(\nu^{(0)}, \ldots, \nu^{(t-1)}\right)$ is called the t-

quotient of λ and is denoted by $quot_t(\lambda)$. The latter is defined as follows: if we set $s(\lambda) = (c_i)_{i\in\mathbb{Z}}$, then for all $k \in \{0, \dots, t-1\}$, one has $\nu^{(k)} := s^{-1}\left((c_{ti+k})_{i\in\mathbb{Z}}\right)$.

Note that obtaining the t-quotient is straightforward from $s(\lambda) = (c_i)_{i\in\mathbb{Z}}$ (see for instance [14]): we just look at subwords with indices congruent to the same values modulo t. The sequence 10 within these subwords are replaced iteratively by 01 until the subwords are all the infinite sequence of "0"'s before the infinite sequence of "1"'s. In fact, this consists in removing all rim hooks in λ of length congruent to $0 \pmod t$. Then ω is the partition corresponding to the word which has the subwords $\pmod t$ obtained after the removal of the 10 sequences.

Many interesting properties of the Littlewood decomposition are given in [9, Theorem 2.1]. Now we recall analogues of this properties for $\mathcal{SC}$ partitions. Let t be a positive integer, take $\lambda \in \mathcal{SC}$, and set $s(\lambda) = (c_i)_{i\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ and $(\omega, \underline{\nu}) = (core_t(\lambda), quot_t(\lambda))$. Then we have (see for instance [6, 13]):

$$\begin{aligned}\lambda \in \mathcal{SC} &\text{ if and only if for all } i_0 \in \{0, \dots, t-1\}, \text{ for all } j \in \mathbb{N}, c_{i_0+jt} = 1 - c_{t-(i_0+1)-t(j-1)}\\ &\text{ if and only if for all } i_0 \in \{0, \dots, t-1\}, \nu^{(i_0)} = \left(\nu^{(t-i_0-1)}\right)' \text{ and } \omega \in \mathcal{SC}_{(t)}.\end{aligned}$$

Therefore λ is uniquely defined if its t-core is known as well as the $\lfloor t/2 \rfloor$ first elements of its t-quotient, which are partitions without any constraints. It implies that if t is even, there is a one-to-one correspondence between a self-conjugate partition and a pair made of one $\mathcal{SC}$ t-core and $t/2$ generic partitions. If t is odd, the Littlewood decomposition is a one-to-one correspondence between a self-conjugate partition and a triple made of one $\mathcal{SC}$ t-core, $(t-1)/2$ generic partitions and a self-conjugate partition $\mu = \nu^{((t-1)/2)}$. Hence the analogues of the above theorems when applied to self-conjugate partitions are as follows.

Proposition 7 ([13, Lemma 4.7])**.** *Let t be a positive integer. The Littlewood decomposition Φ_t maps a self-conjugate partition λ to $\left(\omega, \nu^{(0)}, \dots, \nu^{(t-1)}\right) = (\omega, \underline{\nu})$ such that:*

- $(SC1)$ *the first component ω is a $\mathcal{SC}$ t-core and $\nu^{(0)}, \dots, \nu^{(t-1)}$ are partitions,*
- $(SC2)$ *for all $j \in \{0, \dots, \lfloor t/2 \rfloor - 1\}, \nu^{(j)} = \left(\nu^{(t-1-j)}\right)'$,*
- $(SC'2)$ *if t is odd, $\nu^{((t-1)/2)} = \left(\nu^{((t-1)/2)}\right)' =: \mu$,*
- $(SC3)$ $$|\lambda| = \begin{cases} |\omega| + 2t \displaystyle\sum_{t=0}^{t/2-1} |\nu^{(i)}| & \text{if } t \text{ is even},\\ |\omega| + 2t \displaystyle\sum_{t=0}^{(t-1)/2-1} |\nu^{(i)}| + t|\mu| & \text{if } t \text{ is odd},\end{cases}$$
- $(SC4)$ $\mathcal{H}_t(\lambda) = t\mathcal{H}(\underline{\nu})$.

The set $D(\lambda) = \{h_{(i,i)}(\lambda), i = 1, 2, \dots\}$ is called the *set of main diagonal hook-lengths of* λ. For short, we will denote $h_{(i,i)}$ by δ_i. It is clear that if $\lambda \in \mathcal{SC}$, then $D(\lambda)$ determines λ, and elements of $D(\lambda)$ are all distinct and odd. Hence, as observed in [4], for a self-conjugate partition λ, the set $D(\lambda)$ can be divided into the following two disjoint subsets:

$$D_1(\lambda) := \{\delta_i \in D(\lambda) : \delta_i \equiv 1 \pmod 4\},$$
$$D_2(\lambda) := \{\delta_i \in D(\lambda) : \delta_i \equiv 3 \pmod 4\}.$$

We have the following result whose proof can be found in the long version [14].

Lemma 8. *For a self-conjugate partition λ, set $r := |D_1(\lambda)|$ and $s := |D_3(\lambda)|$. Then*

$$\mathrm{BG}(\lambda) = r - s.$$

In the case $t = 2$, we can combine Lemma 8 and [9, Theorem 2.2] to derive the following additional result.

Proposition 9. *The Littlewood decomposition Φ_2 has the further property:*

$$(SC5) \quad \mathrm{BG}(\lambda) = r - s = \begin{cases} \frac{\ell(\omega)+1}{2} & \text{if } \mathrm{BG}(\lambda) > 0, \\ -\frac{\ell(\omega)}{2} & \text{if } \mathrm{BG}(\lambda) \leq 0. \end{cases}$$

Our goal in section 3 is to prove a multiplication theorem similar to Theorem 2 including the ε sign statistic defined in Section 1. To do so, we will need a refinement of Proposition 7 $(SC4)$, which is an immediate consequence of the Littlewood decomposition: for $\lambda \in \mathcal{P}$ and any box $u \in \lambda$ with hook-length $h_u \in \mathcal{H}_t(\lambda)$ (here t is any positive integer), there exists a unique $k \in \{0, \dots, t-1\}$ and a unique box $u_k \in \nu^{(k)}$ such that $h_u = t h_{u_k}$, where h_{u_k} is the hook-length of u_k in the partition $\nu^{(k)}$. We will say that the box u_k is *associated to* the box u. We have the following lemma for self-conjugate partitions whose proof can be found in [14].

Lemma 10. *Set $\lambda \in \mathcal{SC}$, let t be a positive even integer. Set $u \in \lambda$ such that $h_u \in \mathcal{H}_t(\lambda)$. Then the following properties hold true:*

1. *The box u does not belong to the main diagonal of λ.*
2. *The application $u \mapsto u'$, where u' is the reflection of u with respect to the main diagonal of λ, is well-defined on λ, bijective and satisfies $h_{u'} = h_u \in \mathcal{H}_t(\lambda)$ and $\varepsilon_u = -\varepsilon_{u'}$.*
3. *If u_k and u_l are the boxes associated to u and u' respectively, then $l = t - 1 - k$.*

3 Proof of Theorem 2

First we will compute the term

$$\sum_{\substack{\lambda\in\mathcal{SC}\\ core_t(\lambda)=\omega}} q^{|\lambda|}x^{|\mathcal{H}_t(\lambda)|}b^{\mathrm{BG}(\lambda)}\prod_{u\in\underline{\nu}}\widetilde{\rho}_1(th_u,\varepsilon_u), \tag{3.1}$$

where $\omega\in\mathcal{SC}_{(t)}$ is fixed. Let us remark that for $\lambda\in\mathcal{SC}$ and $\omega = core_t(\lambda)$, one has $\mathrm{BG}(\lambda)=\mathrm{BG}(\omega)$. Indeed ω is obtained by removing from λ ribbons of even length t and these have BG-rank 0. Hence (3.1) can be rewritten as follows

$$b^{\mathrm{BG}(\omega)}q^{|\omega|}\sum_{\substack{\lambda\in\mathcal{SC}\\ core_t(\lambda)=\omega}} q^{|\lambda|-|\omega|}x^{|\mathcal{H}_t(\lambda)|}\prod_{u\in\underline{\nu}}\widetilde{\rho}_1(th_u,\varepsilon_u).$$

Hence using properties $(SC3)$ and $(SC4)$ from Proposition 7, this is equal to

$$b^{\mathrm{BG}(\omega)}q^{|\omega|}\sum_{\underline{\nu}\in\mathcal{P}^t} q^{t|\underline{\nu}|}x^{|\underline{\nu}|}\prod_{u\in\underline{\nu}}\widetilde{\rho}_1(th_u,\varepsilon_u), \tag{3.2}$$

where ω is in $\mathcal{SC}_{(t)}$ and $|\underline{\nu}| := \sum_{i=0}^{t-1}|\nu^{(i)}|$. The product part $q^{t|\underline{\nu}|}x^{|\underline{\nu}|}\prod_{u\in\underline{\nu}}\widetilde{\rho}_1(th_u,\varepsilon_u)$ inside the sum over $\underline{\nu}$ can be rewritten as follows

$$\prod_{i=0}^{t/2-1} q^{t\left(|\nu^{(i)}|+|\nu^{(t-1-i)}|\right)}x^{|\nu^{(i)}|+|\nu^{(t-1-i)}|}\prod_{h\in\mathcal{H}(\nu^{(i)})}\widetilde{\rho}_1(th,1)\widetilde{\rho}_1(th,-1).$$

Indeed, by Lemma 10, each box $u\in\nu^{(i)}$, with $0\le i\le t-1$, is bijectively paired with a box $u'\in\nu^{(t-1-i)}$ satisfying $\widetilde{\rho}_1(th_{u'},\varepsilon_{u'})=\widetilde{\rho}_1(th_u,-\varepsilon_u)$. Therefore (3.2), and thus (3.1), become

$$2b^{\mathrm{BG}(\omega)}q^{|\omega|}\prod_{j=0}^{t/2-1}\left(\sum_{\nu^{(i)}\in\mathcal{P}} q^{2t|\nu^{(i)}|}x^{2|\nu^{(i)}|}\prod_{h\in\mathcal{H}(\nu^{(i)})}\widetilde{\rho}_1(th,1)\widetilde{\rho}_1(th,-1)\right).$$

Hence we get:

$$\sum_{\substack{\lambda\in\mathcal{SC}\\ core_t(\lambda)=\omega}} q^{|\lambda|}x^{|\mathcal{H}_t(\lambda)|}b^{\mathrm{BG}(\lambda)}\prod_{h\in\mathcal{H}_t(\lambda)}\rho_1(h) = b^{\mathrm{BG}(\omega)}q^{|\omega|}f_t\left(x^2q^{2t}\right)^{t/2}.$$

To finish the proof, it remains to show that

$$\sum_{\omega\in\mathcal{SC}_{(t)}} q^{|\omega|}b^{\mathrm{BG}(\omega)} = \left(q^{2t};q^{2t}\right)_\infty^{t/2}\left(-bq;q^4\right)_\infty\left(-q^3/b;q^4\right)_\infty. \tag{3.3}$$

For an integer k, let $c_{t/2}(k)$ be the number of $t/2$-core partitions of k. Following [4], define for a nonnegative integer m:

$$\mathcal{SC}^{(m)}(n) := \left\{\lambda \in \mathcal{SC}(n) : |D_1(\lambda)| - |D_3(\lambda)| = (-1)^{m+1}\lceil m/2 \rceil\right\}.$$

Setting $p = 1$ in [4, Proposition 4.7], we get that for any integer $m \geq 0$, the number of self-conjugate t-core partitions ω such that $|D_1(\omega)| - |D_3(\omega)| = (-1)^{m+1}\lceil m/2 \rceil$ is

$$sc_{(t)}^{(m)}(n) = \begin{cases} c_{t/2}(k) & \text{if } n = 4k + \frac{m(m+1)}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

To prove this, the authors define a bijection $\phi^{(m)}$ in [4, Corollary 4.6] between $\omega \in \mathcal{SC}_{(t)}^{(m)}$ and $\kappa \in \mathcal{P}_{(t/2)}$ with $|\omega| = 4|\kappa| + m(m+1)/2$ and κ independent of m.

Recall from Lemma 8 that $\mathrm{BG}(\lambda) = r - s = |D_1(\lambda)| - |D_3(\lambda)|$. Therefore

$$m = \begin{cases} 2\,\mathrm{BG}(\lambda) - 1 & \text{if } \mathrm{BG}(\lambda) > 0, \\ -2\,\mathrm{BG}(\lambda) & \text{if } \mathrm{BG}(\lambda) \leq 0. \end{cases}$$

Hence the bijection $\phi^{(m)}$ maps a t-core self-conjugate partition ω with BG-rank j to a $t/2$-core partition independent of j. Then Property $(SC5)$ from Proposition 9 implies that $|\omega| = j(2j-1) + 4|\kappa|$ with κ independent of j. Therefore we deduce

$$\sum_{\omega \in \mathcal{SC}_{(t)}} q^{|\omega|} b^{\mathrm{BG}(\omega)} = \sum_{j=-\infty}^{\infty} b^j q^{j(2j-1)} \times \sum_{\kappa \in \mathcal{P}_{(t/2)}} q^{4|\kappa|}. \tag{3.4}$$

Now we compute the sum over j. The Jacobi triple product [7, Appendix, (II.28)] is

$$\sum_{j=-\infty}^{+\infty} (-1)^j z^j q^{j(j-1)/2} = (z;q)_\infty \,(q/z;q)_\infty \,(q;q)_\infty\,.$$

Therefore, setting $z = -bq$ and then replacing q by q^4 in the above identity yields

$$\sum_{j=-\infty}^{+\infty} b^j q^{j(2j-1)} = \left(-bq;q^4\right)_\infty \left(-q^3/b;q^4\right)_\infty \left(q^4;q^4\right)_\infty. \tag{3.5}$$

Finally, to complete the proof of Theorem 2, it remains to compute the generating function of $t/2$-core partitions which is well-known (see [6, 8, 14]). However we briefly recall its computation. By direct application of the Littlewood decomposition, using $(SC3)$ and the generating series (1.2) for $\mathcal{P}$ where q is replaced by $q^{t/2}$, we derive:

$$\sum_{\omega \in \mathcal{P}_{(t/2)}} q^{|\omega|} = \frac{\left(q^{t/2};q^{t/2}\right)_\infty^{t/2}}{(q;q)_\infty}. \tag{3.6}$$

Replacing q by q^4 in (3.6), and using (3.4) and (3.5), this proves (3.3) and the theorem.

4 The odd case

Recall that [6, Formula (3.4)] gives a connection between the BG-rank of a partition, and its t-quotient and its t-core when t is odd. However the formula implies a dependence between t-core and t-quotient, which is not convenient for multiplication theorems. This is why we formulate in Theorem 5 such a result without the BG-rank.

Moreover, because of the partition $\mu \in \mathcal{SC}$ appearing in (1.4), more difficulties arise which make a general result less elegant than in the even case. For this reason, we focus here on a subset of self-conjugate partitions for which μ is empty, which, as will be explained, is algebraically interesting.

For a fixed positive odd integer t and Φ_t from Definition 6, let us define

$$\mathcal{BG}^t := \{\lambda \in \mathcal{SC}, \Phi_t(\lambda) = (\omega, \underline{\nu}) \in \mathcal{SC}_{(t)} \times \mathcal{P}^t \text{ with } \nu^{((t-1)/2)} = \emptyset\}.$$

Note that λ is in $\mathcal{BG}^t$ if and only if the partition μ is empty in (1.4). Following [2], we also define for an odd prime number $p > 0$, the set of self-conjugate partitions with no diagonal hook-length divisible by p:

$$\mathcal{BG}_p := \{\lambda \in \mathcal{SC} \mid \text{for all } i \in \{1, \ldots, d\}, p \nmid h_{(i,i)}\}.$$

Algebraically, this set yields interesting properties in representation theory of the symmetric group over a field of characteristic $p > 0$, see for instance [2, 3]. Combinatorially, it is natural to extend this definition to a set $\mathcal{BG}_t$ for any positive odd number t.

The following result explains the connection between the two above sets and is proved in [3, Lemma 3.4] for any prime number p. Nevertheless, it can be generalized to any positive odd integer t, which is proved in [14].

Lemma 11. *For any positive odd integer t, we have that $\mathcal{BG}^t = \mathcal{BG}_t$.*

Combining this argument and following the same path as the proof of Theorem 2 yields Theorem 5. For details and applications, we refer to [14].

References

[1] A. Berkovich and F. G. Garvan. “On the Andrews-Stanley refinement of Ramanujan’s partition congruence modulo 5 and generalizations”. *Trans. Amer. Math. Soc.* **358**.2 (2006), pp. 703–726. Link.

[2] A. Bernal. “On self-Mullineux and self-conjugate partitions”. *Electron. J. Combin.* **28**.1 (2021), Paper No. 1.57, 33. Link.

[3] O. Brunat and J.-B. Gramain. “A basic set for the alternating group”. *J. Reine Angew. Math.* **641** (2010), pp. 177–202. Link.

[4] H. Cho, J. Huh, and J. Sohn. “A bijection between ordinary partitions and self-conjugate partitions with same disparity”. *Sém. Lothar. Combin.* **82B** (2020), Art. 30, 12.

[5] J. S. Frame, G. d. B. Robinson, and R. M. Thrall. “The hook graphs of the symmetric groups”. *Canad. J. Math.* **6** (1954), pp. 316–324. Link.

[6] F. Garvan, D. Kim, and D. Stanton. “Cranks and t-cores”. *Invent. Math.* **101**.1 (1990), pp. 1–17. Link.

[7] G. Gasper and M. Rahman. *Basic Hypergeometric Series*. Vol. 96. Second Edition, Encyclopedia of Mathematics And Its Applications. Cambridge University Press, Cambridge, 2004.

[8] G.-N. Han. “The Nekrasov–Okounkov hook length formula: refinement, elementary proof, extension and applications”. *Ann. Inst. Fourier (Grenoble)* **60**.1 (2010), pp. 1–29. Link.

[9] G.-N. Han and K. Q. Ji. “Combining hook length formulas and BG-ranks for partitions via the Littlewood decomposition”. *Trans. Amer. Math. Soc.* **363**.2 (2011), pp. 1041–1060. Link.

[10] I. G. Macdonald. “Affine root systems and Dedekind’s η-function”. *Invent. Math.* **15** (1972), pp. 91–143. Link.

[11] N. A. Nekrasov and A. Okounkov. “Seiberg–Witten theory and random partitions”. *The Unity of Mathematics*. Vol. 244. Progr. Math. Birkhäuser Boston, Boston, MA, 2006, pp. 525–596. Link.

[12] M. Pétréolle. “A Nekrasov–Okounkov type formula for $\widetilde{C}$”. *Proceedings of FPSAC 2015*. Discrete Math. Theor. Comput. Sci. Proc. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2015, pp. 535–546.

[13] M. Pétréolle. “Quelques développements combinatoires autour des groupes de Coxeter et des partitions d’entiers”. Theses. Université Claude Bernard - Lyon I, Nov. 2015. Link.

[14] D. Wahiche. “Multiplication theorems for self-conjugate partitions”. 2021. arXiv:2107.06793.

[15] A. Walsh and S. O. Warnaar. “Modular Nekrasov–Okounkov formulas”. *Sém. Lothar. Combin.* **81** (2020), Art. B81c, 28.

[16] B. W. Westbury. “Universal characters from the Macdonald identities”. *Adv. Math.* **202**.1 (2006), pp. 50–63. Link.