

Weak Faces and a Formula for Weights of Highest Weight Modules Via Parabolic Partial Sum Property for Roots

G. Krishna Teja^{*1}

¹*Department of Mathematics, Indian Institute of Science, Bangalore — 560012, India*

Abstract. Let \mathfrak{g} be a finite or an affine type Lie algebra over \mathbb{C} with root system Δ . We show a parabolic generalization of the partial sum property for Δ , which we term the parabolic partial sum property. It allows any root β involving (any) fixed subset S of simple roots, to be written as an ordered sum of roots, each involving exactly one simple root from S , with each partial sum also being a root. We show three applications of this property to weights of highest weight \mathfrak{g} -modules: (1) We provide a minimal description for the weights of all non-integrable simple highest weight \mathfrak{g} -modules, refining the weight formulas shown by Khare [*J. Algebra* 2016] and Dhillon–Khare [*Adv. Math.* 2017]. (2) We provide a Minkowski difference formula for the weights of an arbitrary highest weight \mathfrak{g} -module. (3) We completely classify and show the equivalence of two combinatorial subsets — weak faces and 212-closed subsets — of the weights of all highest weight \mathfrak{g} -modules. These two subsets were introduced and studied by Chari–Greenstein [*Adv. Math.* 2009], with applications to Lie theory including character formulas. We also show (3') a similar equivalence for root systems.

Keywords: Root systems, highest weight modules, 212-closed subsets, weak faces

1 Introduction

Let $\mathbb{Z} \subset \mathbb{R} \subset \mathbb{C}$ be the set of integers, real and complex numbers, respectively. Throughout, $\mathfrak{g} = \mathfrak{g}(A)$ stands for a complex finite-dimensional simple or an affine Kac–Moody Lie algebra — accordingly, in short we say \mathfrak{g} is of finite or affine type — corresponding to a Cartan matrix A . We fix for \mathfrak{g} : a Cartan subalgebra \mathfrak{h} , triangular decomposition $\mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, root system $\Delta \subset \mathfrak{h}^*$, set of simple roots $\Pi = \{\alpha_i \mid i \in \mathcal{I}\}$ in Δ and simple co-roots $\Pi^\vee = \{\alpha_i^\vee \mid i \in \mathcal{I}\} \subset \mathfrak{h}$. Here \mathcal{I} is the (fixed) common indexing set for the simple roots, simple co-roots, rows/columns of A , and the set of nodes in the Dynkin diagram of \mathfrak{g} . We say Δ is of finite or affine type depending on the type of \mathfrak{g} .

This note showcases in finite and affine types, some of the main results in [10, 11], which study the root systems and weights of arbitrary highest weight modules (see

^{*}tejag@iisc.ac.in

Section 3 for the definition) over \mathfrak{g} . The results stated below were shown more generally over all Kac–Moody algebras in the above two papers. These are inspired by and build on the works of Chari, Khare, and their co-authors, *e.g.*, [2, 3, 4, 7, 8].

1.1 Parabolic partial sum property and its applications

Begin by recalling the well-known property of root systems Δ , the *partial sum property (PSP)*: every root in Δ is an ordered sum of simple roots such that all partial sums are also roots. In this paper, we give a novel “parabolic-generalization”, termed the *parabolic partial sum property* — or *parabolic PSP* — which we state formally in the next section. It has many applications to representation theory and algebraic combinatorics; this note records three of them. Informally, the parabolic PSP says that given a nonempty set $S \subseteq \Pi$ of simple roots, every positive root β involving some simple roots from S is an ordered sum of positive roots, such that each root involves exactly one simple root from S , and all partial sums are also roots. Observe, the case $S = \Pi$ is the “usual” PSP.

The parabolic PSP was originally a question of Khare, whose motivation was to obtain a “minimal description” for the weights of *all* non-integrable simple highest weight \mathfrak{g} -modules, and more generally, all parabolic Verma modules — thus, the label “parabolic”.

The simple highest weight modules over \mathfrak{g} are crucial in representation theory, combinatorics, and physics among other areas. These modules form the building blocks of, and thereby pave ways for the study of, modules central to Lie theory. Their weights and characters, particularly those of *integrable* (simple) highest weight \mathfrak{g} -modules, have rich combinatorial properties and numerous applications. For instance, when \mathfrak{g} is of finite type A , these characters are precisely the Schur polynomials.

Recall, the simple highest weight \mathfrak{g} -modules are indexed by their highest weights $\lambda \in \mathfrak{h}^*$; correspondingly, let $L(\lambda)$ denote the simple module. For a weight module V — meaning, V is a direct sum of its weight spaces/the simultaneous eigenspaces of V for the action of \mathfrak{h} — let $\text{wt}V$ denote the set of weights of V .

The weights and characters of the integrable simple modules $L(\lambda)$ — for all *dominant integral* $\lambda \in \mathfrak{h}^*$ — were well understood decades ago. However, for non-integrable simple \mathfrak{g} -modules $L(\lambda)$, even their sets of weights seem to have not been known until 2016 [7]. In this paper, Khare showed three formulas for the weights of a large class of highest weight \mathfrak{g} -modules, including all modules $L(\lambda)$. Dhillon–Khare [4] extended these formulas to hold over all Kac–Moody \mathfrak{g} . Here we focus on one of these three formulas, the Minkowski difference formula (3.1). This uses the $\mathbb{Z}_{\geq 0}$ -cone of the positive roots lying outside a subroot system Δ_{J_λ} . Finding minimal generating sets for these cones would yield a minimal description for the weights of simple highest weight \mathfrak{g} -modules. In this note we prove the parabolic PSP, thereby solving the minimal description problem.

As a second application, the analysis and understanding of the above minimal generating sets and the parabolic PSP, were fruitful in showing several results about weights

of arbitrary highest weight \mathfrak{g} -modules in [10, 11]. Namely, in [10] we obtain a Minkowski difference formula for the set of weights of *every* highest weight \mathfrak{g} -module, which looks similar to those shown by Dhillon–Khare for simple modules. This note discusses that result, as well as the ones from [11] explained in the next subsection.

1.2 Weak faces of root systems and weights

The parabolic partial sum property and its two applications (above) occupy the first half of this note. The second half is a third application: classifying and determining two combinatorial subsets of $\text{wt}V$ and its convex hull, for arbitrary highest weight \mathfrak{g} -modules V . These are the *weak faces* and the *212-closed subsets* (see [7] for the choice of these names), and they arise from the combinatorics of root systems Δ , and of subsets that maximize linear functionals over Δ . Such subsets were studied by Chari and her co-authors in [2, 3] and used to show several interesting results in representation theory. These include constructing Koszul algebras, obtaining character formulas for the specialization at $q = 1$ of Kirillov–Reshetikhin modules over untwisted quantum affine algebras $U_q(\widehat{\mathfrak{g}})$, and constructing irreducible ad-nilpotent ideals in parabolic subalgebras of \mathfrak{g} . For a detailed overview of these motivations, see [7, 11].

Khare and Ridenour [8] extended the results of [2] from maximizer subsets of Δ to the weights falling on the faces of *Weyl polytopes*, *i.e.*, the shapes $\text{conv}(\text{wt}L(\lambda))$ for all dominant integral $\lambda \in \mathfrak{h}^*$. Even more generally, Khare [7] considered $\text{conv}(\text{wt}V)$ for highest weight modules V with arbitrary highest weight $\lambda \in \mathfrak{h}^*$, and introduced:

Definition 1.1. Let \mathbb{A} be a fixed non-trivial additive subgroup of $(\mathbb{R}, +)$, and $\emptyset \neq Y \subseteq X$ be two subsets of a real vector space. Define $\mathbb{A}_{\geq 0} := \mathbb{A} \cap [0, \infty)$.

1. Y is said to be a *weak- \mathbb{A} -face of X* if

$$\sum_{i=1}^n r_i y_i = \sum_{j=1}^m t_j x_j \text{ and } \sum_{i=1}^n r_i = \sum_{j=1}^m t_j > 0 \quad \text{for } m, n \in \mathbb{N}, \text{ where}$$

$$y_i \in Y, x_j \in X, r_i, t_j \in \mathbb{A}_{\geq 0} \text{ for all } 1 \leq i \leq n, 1 \leq j \leq m,$$

implies $x_j \in Y$ for all $t_j \neq 0$. By *weak faces* of X we mean the collection of all weak- \mathbb{A} -faces of X for all additive subgroups $\{0\} \subsetneq \mathbb{A} \subseteq (\mathbb{R}, +)$.

2. Y is said to be a *212-closed subset of (or 212-closed in) X* if

$$(y_1) + (y_2) = (x_1) + (x_2) \text{ for some } y_1, y_2 \in Y \text{ and } x_1, x_2 \in X \text{ implies } x_1, x_2 \in Y.$$

Remark 1.2. For any pair of subsets $Y \subseteq X$ of a real vector space, and all non-trivial $\mathbb{A} \subseteq (\mathbb{R}, +)$, it can be checked by the definitions that each part below implies the next:

- (1) Y maximizes a linear functional ψ on X , *i.e.*, $\psi(x) \leq \psi(y)$ for all $x \in X$ and $y \in Y$.
- (2) Y is a weak- \mathbb{R} -face of X .
- (3) Y is a weak- \mathbb{A} -face of X .
- (4) Y is 212-closed in X .

For a subset $X \neq \emptyset$ of a real vector space, weak faces of X generalize the “classical” faces of the convex hull $\text{conv}(X)$. Namely, when $\text{conv}(X)$ is a polyhedron, [8] shows that weak faces of X are precisely the elements of X that fall on faces of $\text{conv}(X)$ — *i.e.*, conditions (1)–(3) in Remark 1.2 are equivalent. Our goal is to show that the same holds for the even weaker notion (a priori) of (4) 212-closed subsets, when X is a distinguished set in Lie theory and algebraic combinatorics, *i.e.*, of the form Δ , $\Delta \sqcup \{0\}$, $\text{wt}V$, $\text{conv}(\text{wt}V)$.

Remark 1.3. There is a natural combinatorial interpretation for “212-closed subsets $Y \subseteq X$ ” of a real vector space \mathbb{V} when X is the set of lattice points in a lattice polytope — *i.e.*, X is the intersection of $\text{conv}(X)$ with a lattice in \mathbb{V} that contains the vertices of $\text{conv}(X)$. (We explain the connection to $X = \text{wt}L(\lambda)$ in Remark 1.4.) Suppose Y denotes a subset of “colored” (or in a contemporary spirit, “infected”) lattice points in X , with the property that if $y \in Y$ is the average of two points in X , then the “color” or “infection” spreads to both points from y . (More precisely, if two pairs of — not necessarily distinct — points have the same average, and one pair is colored, then the color spreads to the other pair.) We would like to understand the extent to which the spread happens. A “continuous” variant works with the entire convex hull itself, rather than the lattice points in X .

Khare [7] classified the weak faces and 212-closed subsets for $X = \text{wt}V$, for V from a large class of highest weight \mathfrak{g} -modules including all simple highest weight modules. He showed that these two classes of subsets are equal, and they coincide with the sets of weights falling on the faces of the convex hull of weights. The interesting part here is:

Remark 1.4. $\text{conv}(\text{wt}L(\lambda))$ is a convex polytope, and moreover, a lattice polytope as in Remark 1.3; see (3.2), which explains the latter part. Khare’s (partial) results and ours below show that for *all* such lattice polytopes $\text{conv}(\text{wt}L(\lambda))$, the weak faces and 212-closed subsets of X are the same, and these two classes of subsets are equivalent to the faces of $\text{conv}(X)$. This equivalence is striking in view of the “minimality” in the definition of 212-closed subsets, particularly in contrast to the definition of (weak) faces. Furthermore, it naturally raises the question of exploring for general lattice polytopes, the extent to which these results hold; particularly, the equivalence of these three notions.

Recently in [11], we have shown all of the (partial) results of Khare [7], for important sets in Lie theory: $X = \Delta$ or $\Delta \sqcup \{0\} = \text{wt}\mathfrak{g}$, or $\text{wt}V$ or $\text{conv}(\text{wt}V)$ (the \mathbb{R} -convex hull of $\text{wt}V$) for any highest weight module V , over any Kac–Moody \mathfrak{g} . More precisely, for $X = \text{wt}V$ (or $\text{conv}(\text{wt}V)$), we completely classify these two classes of subsets of X over any Kac–Moody algebra \mathfrak{g} . More strongly, we show their equivalence with the sets of weights on the faces (respectively, with the faces) of $\text{conv}(\text{wt}V)$. For $X = \Delta \sqcup \{0\}$ or $X = \Delta$, we show the analogous results, and that these two classes of subsets of X are equal, except in two interesting cases where $X = \Delta$ is of type either A_2 or \widehat{A}_2 . In this note, we discuss these results of [11] for \mathfrak{g} of finite and affine types.

2 Parabolic partial sum property

Throughout, Δ is the root system of \mathfrak{g} which is either finite-dimensional simple or an affine Kac–Moody Lie algebra; accordingly, we say Δ is of finite type or affine type, respectively. Δ^+ denotes the set of positive roots in Δ . Recall \mathfrak{g} has the root space decomposition $\mathfrak{h} \oplus \bigoplus_{\beta \in \Delta} \mathfrak{g}_\beta$, where $\mathfrak{g}_\beta := \{x \in \mathfrak{g} \mid hx = \langle \beta, h \rangle x \text{ for all } h \in \mathfrak{h}\}$ is the root space corresponding to β ; here $\langle \beta, h \rangle$ denotes the evaluation of $\beta \in \mathfrak{h}^*$ at $h \in \mathfrak{h}$. We begin by developing some notation needed to state the parabolic partial sum property. Let \mathcal{I} index the simple roots in $\Delta^+ \subset \Delta$. For $\emptyset \neq I \subseteq \mathcal{I}$ we define a special (for this entire paper) subset $\Delta_{I,1}$ of positive roots, and for a vector $x = \sum_{i \in \mathcal{I}} c_i \alpha_i$ for $c_i \in \mathbb{C}$ we define two important height functions, as follows:

$$\text{ht}(x) := \sum_{i \in \mathcal{I}} c_i, \quad \text{and} \quad \text{ht}_I(x) := \sum_{i \in I} c_i. \quad \Delta_{I,1} := \{\beta \in \Delta \mid \text{ht}_I(\beta) = 1\} \subseteq \Delta^+. \quad (2.1)$$

The subsets $\Delta_{I,1}$ form the minimal generating sets for the cones in weights of simple highest weight modules mentioned in the introduction:

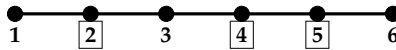
Theorem 2.1 (Parabolic Partial Sum Property). *Let Δ be a finite or an affine root system, and fix $\emptyset \neq I \subseteq \mathcal{I}$. Suppose β is a positive root with $m = \text{ht}_I(\beta) > 0$. Then*

there exist roots $\gamma_1, \dots, \gamma_m \in \Delta_{I,1}$ such that $\beta = \sum_{j=1}^m \gamma_j$ and $\sum_{j=1}^i \gamma_j \in \Delta^+$ for all $1 \leq i \leq m$.

In other words, every root with positive I -height is an ordered sum of roots, each with unit I -height, such that each partial sum of that ordered sum is also a root.

When $I = \mathcal{I}$, this is precisely the usual PSP. We next present another example:

Example 2.2. Let \mathfrak{g} be of type A_6 (7×7 trace zero matrices over \mathbb{C}), $\mathcal{I} = \{1, 2, 3, 4, 5, 6\}$, and fix $I = \{2, 4, 5\}$. The Dynkin diagram for \mathfrak{g} (with nodes from I boxed) is:



Recall, the roots in Δ are precisely $\sum_{i \in T} \alpha_i$, where $T \subset \mathcal{I}$ has consecutive indices. Now,

$$\Delta_{I,1} = \{\alpha_2, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_4, \alpha_3 + \alpha_4, \alpha_5, \alpha_5 + \alpha_6\}.$$

Let $\beta = \sum_{i=1}^6 \alpha_i$ denote the highest root in Δ . Check that $\text{ht}(\beta) = 6$ and $\text{ht}_I(\beta) = 3$. Set $\gamma_1 = \alpha_1 + \alpha_2$, $\gamma_2 = \alpha_3 + \alpha_4$ and $\gamma_3 = \alpha_5 + \alpha_6$. Observe that $\gamma_1 + \gamma_2 + \gamma_3 = \beta$, and moreover, both γ_1 and $\gamma_1 + \gamma_2$ are roots.

Proof sketch for parabolic PSP. Fix $\emptyset \neq I \subseteq \mathcal{I}$ and a root β with $m = \text{ht}_I(\beta) > 0$. The parabolic PSP trivially holds for β if $\text{ht}_I(\beta) = 1$; hence, we now assume $\text{ht}_I(\beta) > 1$. The parabolic PSP is shown by applying the following stronger, structural result:

Theorem 2.3. *Let \mathfrak{g} be of finite or affine type. Fix $\emptyset \neq I \subseteq \mathcal{I}$, and $\beta \in \Delta^+$ with $m := \text{ht}_I(\beta) > 0$. Then the root space \mathfrak{g}_β is spanned by the right normed Lie words of the form:*

$$[e_{\gamma_m}, [\dots, [e_{\gamma_2}, e_{\gamma_1}] \dots]] \text{ where } \gamma_t \in \Delta_{I,1}, e_{\gamma_t} \in \mathfrak{g}_{\gamma_t}, \text{ for all } 1 \leq t \leq m, \text{ and } \sum_{t=1}^m \gamma_t = \beta. \quad (2.2)$$

In fact, a stronger result (in [10]) shows the parabolic PSP to the best possible extent, and moreover at the level of *Lie words*, via proving Theorem 2.3 for *any* general Lie algebra (over any field) graded over *any* free abelian semigroup. Indeed, by (the stronger analogue of) Theorem 2.3, let the Lie word in (2.2) be non-zero. Then all its right normed Lie sub-words — namely, $[e_{\gamma_i}, [\dots, [e_{\gamma_2}, e_{\gamma_1}] \dots]]$ for all $1 \leq i \leq m$ — are non-zero. This immediately implies that each partial sum of $\sum_{t=1}^m \gamma_t$ is a root, proving the parabolic PSP.

Finally, the proof for Theorem 2.3 involves structure theory ideas, and goes as follows. Fix a *right normed Lie word* $0 \neq x = [e_{i_n}, [\dots, [e_{i_2}, e_{i_1}] \dots]]$ in \mathfrak{g}_β , where e_{i_t} is a (simple) root vector in the simple root space $\mathfrak{g}_{\alpha_{i_t}}$ for all $1 \leq t \leq n$ and $\sum_{t=1}^n \alpha_{i_t} = \beta$. Recall, every root space of \mathfrak{g} is spanned by such right normed Lie words. Inducting on n , we can show that x can be written as a linear combination of the Lie words as in (2.2). More precisely, we assume that $[e_{i_{n-1}}, [\dots, [e_{i_2}, e_{i_1}] \dots]]$ is in the span of Lie words as in (2.2), and then use the Jacobi identity to take e_{i_n} inside these new Lie words if $\text{ht}_I(\alpha_{i_n}) = 0$. \square

Alternately for \mathfrak{g} of finite type, the parabolic PSP is implied by:

Lemma 2.4. *Assume that \mathfrak{g} is of finite type, and $\emptyset \neq I \subseteq \mathcal{I}$. Suppose $\beta \in \Delta^+$ is such that $\text{ht}_I(\beta) > 1$. Then there exists a root $\gamma \in \Delta_{I,1}$ such that $\beta - \gamma \in \Delta^+$.*

Proof. We reach a contradiction, working with β of least height such that the lemma fails for β . Recall, β is a sum of simple roots, and the Killing form on \mathfrak{h} , hence on \mathfrak{h}^* , is positive definite, i.e. $(x, x) > 0$ for all $x \in \mathfrak{h}^*$. Thus, fix a simple root α such that $(\beta, \alpha) > 0$. Now [5, Lemma 9.4] implies $\beta - \alpha$ is a root. The assumption on β forces $\text{ht}_I(\alpha) = 0$, so $\text{ht}_I(\beta - \alpha) = \text{ht}_I(\beta) > 1$. Since $\text{ht}(\beta - \alpha) < \text{ht}(\beta)$, we have a root $\eta \in \Delta_{I,1}$ such that $\beta - \alpha - \eta \in \Delta^+$. Now applying [3, Lemma 1.1(iii)] for $\beta - \alpha - \eta, \eta, \alpha$ (in place of α, β, γ , respectively), we have either: (1) $(\beta - \alpha - \eta) + (\alpha) = \beta - \eta$ is a root, or (2) $\alpha + \eta$ is root (in which case $\alpha + \eta \in \Delta_{I,1}$). Both of these cases contradict the choice of β . \square

3 Minkowski difference formulas for weights

In this section, we state and discuss our minimal description result for $\text{wt}L(\lambda)$ for all weights $\lambda \in \mathfrak{h}^*$, and a Minkowski difference formula for the set of weights of an arbitrary highest weight \mathfrak{g} -module, using the parabolic PSP. We need the following notation.

For any subset $S \neq \emptyset$ of a real vector space, let $\mathbb{Z}S$ (or $\mathbb{Z}_{\geq 0}S$) comprise the set of \mathbb{Z} -linear (or $\mathbb{Z}_{\geq 0}$ -linear) combinations of elements of S . Let $\text{conv}(S)$ denote the \mathbb{R} -convex

hull of S . Recall, \mathcal{I} is the set of nodes in the Dynkin diagram of \mathfrak{g} . Set $I^c := \mathcal{I} \setminus I$ for $I \subseteq \mathcal{I}$. Now for $i \in \mathcal{I}$, let s_i denote the simple reflection about the hyperplane perpendicular to the simple root α_i . These generate the Weyl group W of \mathfrak{g} . Let e_i, f_i, α_i^\vee for all $i \in \mathcal{I}$ be the Chevalley generators of \mathfrak{g} .

We now fix $\emptyset \neq J \subseteq \mathcal{I}$ and define the parabolic analogues. Define \mathfrak{g}_J to be the Lie subalgebra of \mathfrak{g} generated by e_j, f_j, α_j^\vee for all $j \in J$. Since \mathfrak{g} is of finite or affine type, recall that \mathfrak{g}_J is always semisimple for $J \subsetneq \mathcal{I}$. The subalgebra \mathfrak{g}_J corresponds to the Cartan matrix $A_{J \times J}$. We denote the Cartan subalgebra, root system, the (fixed) simple roots and simple co-roots of \mathfrak{g}_J by $\mathfrak{h}_J, \Delta_J, \Pi_J := \{\alpha_j\}_{j \in J}$ and Π_J^\vee , respectively; note that $\Delta_J = \Delta \cap \mathbb{Z}\Pi_J$. The parabolic subgroup of W generated by the simple reflections $\{s_j\}_{j \in J}$ corresponding to J , is the Weyl group of \mathfrak{g}_J ; it is denoted by W_J .

Fix $\lambda \in \mathfrak{h}^*$ for this section. Recall, λ is *integral*, respectively *dominant*, if $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}$, respectively $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{R}_{\geq 0}$, for all $i \in \mathcal{I}$; λ is *dominant integral* if it is both. For a \mathfrak{g} -module V , recall the definitions of the λ -weight space and the set of weights of V :

$$V_\lambda := \{v \in M \mid h \cdot v = \langle \lambda, h \rangle v \text{ for all } h \in \mathfrak{h}\} \quad \text{and} \quad \text{wt}V := \{\mu \in \mathfrak{h}^* \mid V_\mu \neq \{0\}\}.$$

V is said to be a highest weight module of highest weight λ , if there exists a vector $0 \neq v \in V$ such that (1) v generates V over \mathfrak{g} , (2) $hv = \langle \lambda, h \rangle v$ for all $h \in \mathfrak{h}$ (so $v \in V_\lambda \neq \{0\}$), and (3) $e_i v = 0$ for all $i \in \mathcal{I}$. We call such a v to be a highest weight vector of V and it is unique up to scalars. We denote the simple highest weight module over \mathfrak{g}_J with highest weight λ (or rather λ restricted to \mathfrak{h}_J) by $L_J(\lambda)$. In this section, we deal with an important subset $J_\lambda := \{j \in \mathcal{I} \mid \langle \lambda, \alpha_j^\vee \rangle \in \mathbb{Z}_{\geq 0}\}$ of \mathcal{I} – the *integrability* of λ (or of $L(\lambda)$).

Remark 3.1. $L_{J_\lambda}(\lambda)$ is the integrable highest weight \mathfrak{g}_{J_λ} -module corresponding to λ ; i.e., f_j acts nilpotently on each vector in $L_{J_\lambda}(\lambda)$ (nilpotently on all of $L_{J_\lambda}(\lambda)$ when \mathfrak{g} is of finite type) for all $j \in J_\lambda$. So, we know well the set $\text{wt}L_{J_\lambda}(\lambda)$. It (or equivalently, the set of nodes J_λ) determines $\text{wt}L(\lambda)$, as shown by Khare [7] and Dhillon–Khare [4]:

$$\text{wt}L(\lambda) = \text{wt}L_{J_\lambda}(\lambda) - \mathbb{Z}_{\geq 0} \left(\Delta^+ \setminus \Delta_{J_\lambda}^+ \right). \quad (3.1)$$

$$\text{wt}L(\lambda) = \text{conv}(\text{wt}L(\lambda)) \cap (\lambda - \mathbb{Z}_{\geq 0}\Pi). \quad (3.2)$$

Observe by formula (3.1) that $\text{conv}(\text{wt}L(\lambda))$ equals the Minkowski difference between $\text{conv}(\text{wt}L_{J_\lambda}(\lambda))$ and the real cone $\mathbb{R}_{\geq 0}(\Delta^+ \setminus \Delta_{J_\lambda}^+)$. Formula (3.2) affirmatively answers a question of Daniel Bump on recovering weights from their convex hulls for these simple \mathfrak{g} -modules. Formula (3.1) is the Minkowski difference formula mentioned in the introduction, expressing $\text{wt}L(\lambda)$ in terms of two well-understood subsets. The parabolic partial sum property was posed in order to determine the minimal generators for the (non-negative integer) cone in (3.1), which is generated by all the positive roots outside $\Delta_{J_\lambda}^+$; these are found in the next theorem. Recall the definition of $\Delta_{J_\lambda^c, 1}$ from (2.1).

Theorem 3.2. *The cone $\mathbb{Z}_{\geq 0}(\Delta^+ \setminus \Delta_{J_\lambda}^+)$ is minimally generated (over $\mathbb{Z}_{\geq 0}$) by $\Delta_{J_\lambda^c, 1}$, which is always finite if Δ is of finite or affine type. Therefore, we have the following minimal description:*

$$\text{wt}L(\lambda) = \text{wt}L_{J_\lambda}(\lambda) - \mathbb{Z}_{\geq 0}\Delta_{J_\lambda^c, 1}, \quad \text{for all } \lambda \in \mathfrak{h}^*. \quad (3.3)$$

Proof sketch. Notice that $\Delta_{J_\lambda^c, 1} \subseteq \Delta^+ \setminus \Delta_{J_\lambda}^+$. Conversely, if $\beta \in \Delta^+ \setminus \Delta_{J_\lambda}^+$, then $\text{ht}_{J_\lambda^c}(\beta) > 0$. Now the partial sum property implies $\beta \in \mathbb{Z}_{\geq 0}\Delta_{J_\lambda^c, 1}$. Therefore, $\mathbb{Z}_{\geq 0}(\Delta^+ \setminus \Delta_{J_\lambda}^+) = \mathbb{Z}_{\geq 0}\Delta_{J_\lambda^c, 1}$. It remains to show the minimality — or irredundancy — of $\Delta_{J_\lambda^c, 1}$. Suppose there is a root $\gamma \in \Delta_{J_\lambda^c, 1}$ such that $\gamma = \sum_{i=1}^n \gamma_i$ for some roots $\gamma_1, \dots, \gamma_n \in \Delta^+ \setminus \Delta_{J_\lambda}^+$. Comparing the J_λ^c -heights on both sides, it follows that $n = 1$ and $\gamma_1 = \gamma$. \square

Next, we give our formula for the weights of an arbitrary highest weight \mathfrak{g} -module V of highest weight $\lambda \in \mathfrak{h}^*$. We define for convenience $\text{wt}_{J_\lambda} V := (\lambda - \mathbb{Z}_{\geq 0}\Pi_{J_\lambda}) \cap \text{wt}V$, which is the set of weights of the \mathfrak{g}_{J_λ} -module generated by a highest weight vector in V .

Theorem 3.3. *Let $\lambda \in \mathfrak{h}^*$, and V be a highest weight \mathfrak{g} -module with highest weight λ . Then*

$$\text{wt}V = \text{wt}_{J_\lambda} V - \mathbb{Z}_{\geq 0}\Delta_{J_\lambda^c, 1} = \text{wt}_{J_\lambda} V - \mathbb{Z}_{\geq 0}(\Delta^+ \setminus \Delta_{J_\lambda}^+). \quad (3.4)$$

Remark 3.4. When $V = L(\lambda)$, (3.4) is the same as the formula (3.1) by Dhillon and Khare, as $\text{wt}_{J_\lambda} L(\lambda) = \text{wt}L_{J_\lambda}(\lambda)$. The significance of the formula (3.4) is as follows:

The cone in it is well understood, and the unknown part is $\text{wt}_{J_\lambda} V$. In view of this, if we find the sets of weights of highest weight \mathfrak{g} -modules with dominant integral highest weights, then we have the formulas for weights of all highest weight \mathfrak{g} -modules.

Proof sketch of Theorem 3.3. The inclusion $\text{wt}V \subseteq \text{wt}_{J_\lambda} V - \mathbb{Z}_{\geq 0}\Delta_{J_\lambda^c, 1}$ follows by [10, Corollary 3.2(b)], which involves the Poincaré–Birkhoff–Witt theorem and the parabolic PSP. To show the reverse inclusion in the previous sentence, recall that [10, Lemma 4.2] says $\text{wt}_{J_\lambda} V - \mathbb{Z}_{\geq 0}\Pi_{J_\lambda^c} \subseteq \text{wt}V$. Using this inclusion as the base step, for any $\gamma_1, \dots, \gamma_n \in \Delta_{J_\lambda^c, 1}$, we induct on $\text{ht}_{J_\lambda^c}(\sum_{i=1}^n \gamma_i)$ to prove: $\text{wt}_{J_\lambda} V - \mathbb{Z}_{\geq 0}\sum_{i=1}^n \gamma_i \subseteq \text{wt}V$, which finishes the proof. Showing this crucial step is tremendously simplified if one works with the generating set $\Delta_{J_\lambda^c, 1}$ for $\mathbb{Z}_{\geq 0}(\Delta^+ \setminus \Delta_{J_\lambda}^+)$ instead of $\Delta^+ \setminus \Delta_{J_\lambda}^+$. \square

4 Weak faces and 212-closed subsets

Recall the definitions of weak faces and 212-closed subsets from Definition 1.1. Throughout, \mathfrak{g} (or equivalently, Δ) is of finite or affine type, $\mathbb{A} \neq \{0\}$ is a fixed arbitrary additive subgroup of \mathbb{R} , and V is an arbitrary highest weight \mathfrak{g} -module of highest weight $\lambda \in \mathfrak{h}^*$. The symbol Y always denotes a weak face or a 212-closed subset of X , where $X = \Delta$ or $\Delta \sqcup \{0\}$ or $\text{wt}V$ or $\text{conv}(\text{wt}V)$. The main results of this section are as follows. Theorem 4.7 completely determines as well as discusses the equivalence of the 212-closed subsets

and weak faces both for $X = \Delta$ and $X = \Delta \sqcup \{0\}$. It turns out that $X = \Delta$ of types A_2 or (affine) \widehat{A}_2 are the only two exceptions for which these two classes of subsets are not equal. In Theorem 4.8: 1) we show for $X = \text{wt}V$ that these two classes of subsets are the same as the sets of weights falling on the faces of $\text{conv}(\text{wt}V)$, and 2) for $X = \text{conv}(\text{wt}V)$ these two notions are the same as the usual faces. Let us first understand a geometric interpretation of 212-closedness.

Remark 4.1. In the definition of a 212-closed set, consider the equality $(y_1) + (y_2) = (x_1) + (x_2)$ for $y_1, y_2 \in Y$ and $x_1, x_2 \in X$ implying $x_1, x_2 \in Y$. This equality arises notably when: (1) $y_1 = y_2$ is the midpoint between x_1 and x_2 . (2) $x_1 = x_2$ is the midpoint between y_1 and y_2 . (3) y_1, y_2, x_1, x_2 form vertices of a parallelogram with y_1, y_2 (similarly, x_1, x_2) on a diagonal.

Using this remark, we now look at the 212-closed subsets of Δ for the example below:

Example 4.2. Let Δ be of type A_2 , with the simple roots α_1 and α_2 . $\Delta = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2)\}$ is the set of vertices of the hexagon in Figure (4.1a) below, where $\alpha_1 = (1, 0)$ and $\alpha_2 = (-1/2, \sqrt{3}/2)$. The figure in (4.1b) is the \mathbb{R} -convex hull of Δ .

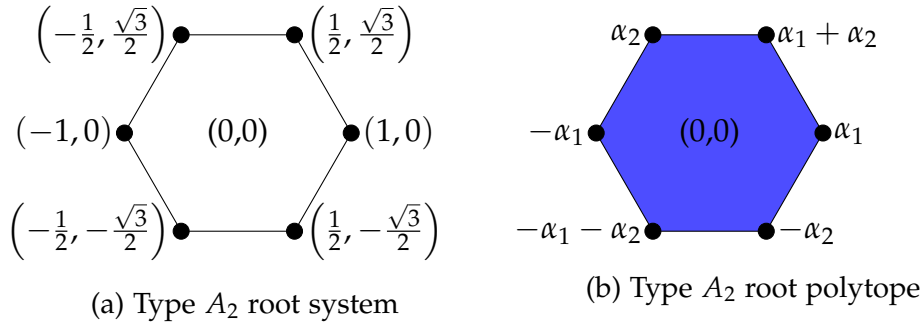


Figure 4.1: The A_2 case

Δ is 212-closed in itself trivially. By Remark 4.1(1), since no root lies between two others, all singletons in Δ are 212-closed. By Remark 4.1 the following are all the rest of the 212-closed subsets in Δ : (i) non-antipodal pairs of points; (ii) any subset of 3 vertices, such that in their induced subgraph none of them are isolated, or all of them are isolated. Next about the 212-closed subsets of $\Delta \sqcup \{0\}$, once again by Remark 4.1 points (1) and (3), if Y 212-closed in $\Delta \sqcup \{0\}$ is such that Y contains 0 or a pair of non-adjacent vertices, then $Y = \Delta \sqcup \{0\}$. Theorem 4.8(b), and the formulas given for the faces and for weights falling on the faces (4.2), can be verified/understood with the aid of the convex set in Figure (4.1b), which equals $\text{conv}(\text{wt}L(\alpha_1 + \alpha_2))$.

We now generalize below some of the notable observations in this example.

Note. For $X = \Delta$ or $\Delta \sqcup \{0\}$, throughout, every weak face and every 212-closed subset Y of X will be assumed to be a proper subset of X . This is to overcome the obstruction to

our uniform comparisons of these notions: Δ is 212-closed (also a weak- \mathbb{A} -face for any \mathbb{A}) in Δ , but not in $\Delta \sqcup \{0\}$, since if $Y = \Delta$ is 212-closed in $\Delta \sqcup \{0\}$, then

$$\text{for any } \zeta \in Y = \Delta, \quad (\zeta) + (-\zeta) = 2(0) \quad \text{implies } 0 \in \Delta \Rightarrow \Leftarrow. \quad (4.1)$$

Remark 4.3. Let Y be 212-closed in $X = \Delta$ or $\Delta \sqcup \{0\}$. Since $Y \subsetneq X$, observe via the reversed equation before the implication in (4.1) that $0 \notin Y$. This has two further implications: (1) For any root ζ , both ζ and $-\zeta$ cannot belong to Y . (2) Every 212-closed subset of $\Delta \sqcup \{0\}$ is contained in Δ , and therefore (by the definition) it is 212-closed in Δ . These assertions hold equally well for weak faces, since they are 212-closed.

Let \mathfrak{g} be of finite type, with highest long root θ . As \mathfrak{g} and $L(\theta)$ are isomorphic, $\text{wt}\mathfrak{g} = \Delta \sqcup \{0\} = \text{wt}L(\theta)$. So [7, Theorem C], which finds and shows the equivalence of all the weak faces and 212-closed subsets of $X = \text{wt}L(\lambda)$ for all $\lambda \in \mathfrak{h}^*$, applied for $\lambda = \theta$ yields:

Proposition 4.4. *For \mathfrak{g} of finite type, the following classes of subsets are all the same: 1) 212-closed subsets of $\Delta \sqcup \{0\}$, 2) weak faces of $\Delta \sqcup \{0\}$, 3) the sets of weights falling on the faces of $\text{conv}(\Delta \sqcup \{0\})$, 4) maximizer subsets for linear functionals on $\Delta \sqcup \{0\}$. These are precisely:*

$$w [(\theta - \mathbb{Z}_{\geq 0}\Pi_I) \cap \text{wt}L(\theta)] \quad \text{for all } w \in W \text{ and } I \subsetneq \mathcal{I}. \quad (4.2)$$

Remark 4.5. Recall, the convex hulls of the subsets in (4.2) are all the faces of the root polytope $\text{conv}(\Delta)$ from Borel–Tits [1], Satake [9], Vinberg [12]; for instance see [7, Theorem 2.17] which quotes this result. Let ω_i for all $i \in \mathcal{I}$ be the fundamental dominant integral weights in \mathfrak{h}^* . The set in (4.2) maximizes the linear functional $(w \sum_{j \in I^c} \omega_j, -)$. So by Remark 1.2, it is a weak- \mathbb{A} -face (also 212-closed) in $\Delta \sqcup \{0\}$ for every $\emptyset \neq \mathbb{A} \subseteq (\mathbb{R}, +)$.

Now assume that Δ is of affine type and $\mathcal{I} = \{0, 1, \dots, \ell\}$, where ℓ is the rank of A or Δ . A root β is real if $(\beta, \beta) > 0$, and imaginary if $(\beta, \beta) = 0$; every root is either real or imaginary. $W\Pi$ (the W orbit of all the simple roots) are all the real roots. The imaginary roots are $\{n\delta \mid n \in \mathbb{Z} \setminus \{0\}\}$, where δ is the smallest positive imaginary root. In a 212-closed subset of Δ , every root is real by the remark below.

Remark 4.6. Let Y be 212-closed in Δ of affine type and $\eta \in \Delta$ imaginary. Then $\eta \notin Y$:

$$\text{If } \eta \in Y, \quad \text{then } 2(\eta) = (3\eta) + (-\eta) \quad \text{implies } \pm \eta \in Y \Rightarrow \Leftarrow \text{ (by Remark 4.3)}. \quad (4.3)$$

Recall from Kac’s book [6, Tables Aff 1–Aff 3, Subsection 6.3]: (1) the subroot system $\mathring{\Delta}$ generated by $\{\alpha_1, \dots, \alpha_\ell\}$ is of finite type; (2) the roots in Δ are explicitly describable in terms of the roots in $\mathring{\Delta}$. Recall, there are at most two lengths in a finite type root system. Let $\mathring{\Delta}_s$ and $\mathring{\Delta}_l$ denote the set of roots in $\mathring{\Delta}$ of the shortest and longest lengths, respectively. Note, $\mathring{\Delta}_s = \mathring{\Delta}_l$ when $\mathring{\Delta}$ is of (simply laced) types A_n, D_n, E_6, E_7, E_8 . With the above preliminaries and observations, we are ready to state our next main theorem, with one last notation:

For Δ of type $X_\ell^{(r)}$, $r \in \{1, 2, 3\}$ — see [6, Tables Aff 1–Aff 3] — and $Y \subseteq \Delta \sqcup \{0\}$, define

$$Y_s := \begin{cases} (Y \cap \mathring{\Delta}_s) + \mathbb{Z}\delta & \text{if } Y \cap \mathring{\Delta}_s \neq \emptyset, \\ \emptyset & \text{if } Y \cap \mathring{\Delta}_s = \emptyset, \end{cases} \quad Y_l := \begin{cases} (Y \cap \mathring{\Delta}_l) + r\mathbb{Z}\delta & \text{if } Y \cap \mathring{\Delta}_l \neq \emptyset, \\ \emptyset & \text{if } Y \cap \mathring{\Delta}_l = \emptyset. \end{cases} \quad (4.4)$$

Theorem 4.7.

- (a) For $\mathfrak{g} \neq A_2$ of finite type, the following classes of subsets are all the same: 212-closed subsets of Δ , 212-closed subsets of $\Delta \sqcup \{0\}$, weak faces of Δ , weak faces of $\Delta \sqcup \{0\}$, and the subsets in (4.2).
- (b) For \mathfrak{g} of type A_2 , the following classes of subsets are all the same: 212-closed subsets of $\Delta \sqcup \{0\}$, weak faces of Δ , weak faces of $\Delta \sqcup \{0\}$, and the subsets in (4.2). All of these subsets and the following additional ones, are all the 212-closed subsets of Δ .

$$\text{W-conjugates of: } \Pi = \{\alpha_1, \alpha_2\}, \Delta^+ = \{\alpha_1, \alpha_2, \alpha_2 + \alpha_1\}, \{\alpha_1, \alpha_2, -\alpha_2 - \alpha_1\}. \quad (4.5)$$

- (c) Assume that \mathfrak{g} is of affine type. Then the following two statements 1) and 2) are equivalent:

- 1) Y is a 212-closed subset of Δ (respectively, of $\Delta \sqcup \{0\}$).
- 2) $Y = Z_s \cup Z_l$ for some 212-closed subset Z of $\mathring{\Delta}$ (respectively, of $\mathring{\Delta} \sqcup \{0\}$).

For \mathfrak{g} not of type \widehat{A}_2 (so $\mathring{\Delta}$ is not of type A_2) — similar to part (a) — we have the equality of the four classes: 212-closed subsets and weak faces of both Δ and $\Delta \sqcup \{0\}$. When \mathfrak{g} is of type \widehat{A}_2 ($\mathring{\Delta}$ is of type A_2) — as in part (b) — these classes other than the 212-closed subsets of Δ are the same. The additional 212-closed subsets of Δ correspond to those of $\mathring{\Delta}$ in (4.5).

Ideas in proof. In [11], each part of the theorem is shown in cases. Part (a) for types A_1 , B_2 and G_2 , as well as part (b), can be verified alternately via looking at their pictures/plots and the observations similar to those in Example 4.2. In particular, in this manner, for Δ of type A_2 , the disjointedness of the two lists in (4.2) and (4.5) can be verified. If Y belongs to the list in (4.5), then Y is not a weak face of Δ of type A_2 , because

$$\text{for any } \mathbb{A} \subseteq (\mathbb{R}, +) \text{ and } 0 < a \in \mathbb{A}, \quad a(\alpha_1) + a(\alpha_2) = a(\alpha_1 + \alpha_2) + (0) \text{ implies } 0 \in Y,$$

which is a contradiction. The proof in [11] of part (c), where Δ is of affine type, runs over four steps. In all of them, we heavily use the description of the roots given by the well-known result [6, Proposition 6.3]. This description and the definition of 212-closed sets immediately give the equivalence of points 1) and 2) in part (c). This close relation between the 212-closed subsets of Δ and $\mathring{\Delta}$ leads to the equivalences of the various classes of sets in part (c). \square

We conclude with our final main theorem for the set of weights and their convex hull, for an arbitrary highest weight \mathfrak{g} -module V . When $V = L(\lambda)$, part (a) of our theorem below recovers [7, Theorem C] of Khare. We define the *integrability* of V to be $I_V := \{i \in \mathcal{I} \mid f_i \text{ acts nilpotently on each vector in } V \text{ or equivalently on } V_\lambda\}$. Recall, V is an integrable \mathfrak{g}_{I_V} -module, and so $\text{wt}V$ and $\text{conv}(\text{wt}V)$ are W_{I_V} -invariant. I_V determines $\text{wt}V$ to a significant extent, and $\text{conv}(\text{wt}V)$ completely, by [7, 4].

Theorem 4.8. *Let V be an arbitrary highest weight \mathfrak{g} -module of highest weight $\lambda \in \mathfrak{h}^*$. Then:*

- (a) *The following classes of subsets are equal: (1) Weak faces of $\text{wt}V$. (2) 212-closed subsets of $\text{wt}V$. (3) The subsets*

$$w[(\lambda - \mathbb{Z}_{\geq 0}\Pi_I) \cap \text{wt}V] \quad \text{for all } w \in W_{I_V} \text{ and } I \subseteq \mathcal{I}. \quad (4.6)$$

- (b) *The following classes of subsets of $X = \text{conv}_{\mathbb{R}}(\text{wt}V)$ are equal: (1) Exposed faces, i.e., the maximizer subsets of X with respect to linear functionals. (2) Weak faces of X . (3) 212-closed subsets of X . (4) The convex hulls of the subsets in (4.6).*

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