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The Elliptic Hall Algebra Element $\mathbf{Q}^k_{m,n}(1)$

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Abstract. Suppose M and N are positive integers and let $k = \gcd(M, N)$, $m = M/k$, and $n = N/k$. We define a symmetric function $L_{M,N}$ as a weighted sum over certain tuples of lattice paths and conjecture that this function is equal (up to a constant) to the elliptic Hall algebra operator $\mathbf{Q}_{m,n}$ composed k times and applied to 1. We show that $L_{M,N}$ satisfies a generalization of Mellit and Hogancamp's recursion for the triply-graded Khovanov–Rozansky homology of the M, N-torus link. As a corollary, we obtain the triply-graded Khovanov–Rozansky homology of the M, N-torus link as a specialization of $L_{M,N}$.

Keywords: lattice paths, link homology, torus links, elliptic Hall algebra

1 Introduction

For coprime positive integers m and n, much has been discovered in recent years about the relationship between m, n-torus knots, the elliptic Hall algebra, and m, n-Dyck paths (lattice paths from $(0, 0)$ to (m, n) staying above the line $my = nx$). More precisely, the relevant objects are

(1) the triply-graded Khovanov–Rozansky homology of the m, n-torus knot,

(2) a certain elliptic Hall algebra operator $\mathbf{Q}_{m,n}$ (defined in Section 2) applied to 1, and

(3) a generating function over m, n-Dyck paths, weighted by variables q and t as well as monomials in variables $x_1, x_2, x_3 \ldots$.

Gorsky and Negut conjectured that (2) and (3) are equal up to a sign [6]. This conjecture was proved by Mellit [8]. An earlier result of Gorsky implies that (1) appears as a certain specialization of (2) and (3) [4].

Somewhat less explored, but still fairly well understood, is the case where $m = n$, *i.e.* m and n are "minimally coprime". In this case, the objects are

(I) the triply-graded Khovanov–Rozansky homology of the n, n-torus link (no longer a knot),

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(II) the symmetric function ∇p_{1^n}, where ∇ is the Macdonald eigenoperator, and

(III) a generating function over an arbitrary number of boxes placed into n columns [9].

(II) and (III) are conjectured to be equal [9]. Although this conjecture is still open, it is possible that recent developments in the area could make progress on this problem. The same specialization as before allows one to move from (II) or (III) to (I), making use of a recursion of Elias and Hogancamp [3].

The goal of this abstract is to generalize both the $\gcd(m, n) = 1$ and $m = n$ cases to any positive integers M and N. We let $k = \gcd(M, N)$, $m = M/k$, and $n = N/k$. The objects are now

(A) the triply-graded Khovanov–Rozansky homology of the M, N-torus link,

(B) the elliptic Hall algebra operator $\mathbf{Q}_{m,n}$ applied iteratively k times to 1, and

(C) a generating function over k-tuples of (variations of) m, n-Dyck paths.

The correct analog of Elias and Hogancamp's recursion for (A) is known due to Hogancamp and Mellit [7]. This recursion guided much of our progress. We develop (B) and (C) in Section 2 and Section 3, respectively. In Conjecture 1, which appears at the end of Section 3, we posit a precise relationship between (B) and (C). In Section 4, we explain what we can prove about connections between these objects and what is still unknown. Our main result in this section (Theorem 4) is a lift of Hogancamp and Mellit's recursion to the level of symmetric functions using the path tuples from (C). We also make explicit connections between our work and past work on torus link homology and the Rational Shuffle Theorem.

2 The elliptic Hall algebra

Suppose throughout that M and N are positive integers and let $k = \gcd(M, N)$, $m = M/k$, and $n = N/k$. We let Λ denote the ring of symmetric functions in variables $x_1, x_2, x_3, \ldots$ over the ground field $\mathbb{Q}(q, t, a, z)$. We use *plethystic substitution*, *i.e.* for any formal sum of monomials $A = a_1 + a_2 + a_3 + \cdots$ from the x_i's or the ground field, we define

$$p_k[A] = a_1^k + a_2^k + a_3^k + \cdots$$

for any power sum polynomial p_k. We extend plethystic substitution to all of Λ by viewing the p_i's as algebraically independent generators for Λ. We also let

$$X = x_1 + x_2 + x_3 + \cdots .$$

Our first definition is an important operator on Λ in the study of Macdonald polynomials.

Definition 1. For any nonnegative integer k and any symmetric function f,

$$\mathbf{D}_k f[X] = f[X + (1-q)(1-t)/z] \sum_{i\geq 0} (-z)^i e_i[X] \Big|_{z^k},$$

where $|_{z^k}$ extracts the coefficient of z^k from the series to its left.

Next, we define the fundamental operators for the elliptic Hall algebra.

Definition 2. For nonnegative integers m and n and $f \in \Lambda$, we define $\mathbf{Q}_{m,n} f$ by first setting

$$\mathbf{Q}_{0,n} f = \frac{qt}{qt-1} h_n[(1-qt)X/(qt)] \cdot f$$
$$\mathbf{Q}_{1,n} f = \mathbf{D}_n f.$$

Otherwise, $m \geq 2$. We assume m and n are coprime, so there are unique integers $1 \leq a < m$ and $1 \leq b < n$ such that $na - mb = 1$. Then we let

$$\mathbf{Q}_{m,n} f = \frac{1}{(1-q)(1-t)} [\mathbf{Q}_{m-a,n-b}, \mathbf{Q}_{a,b}] f.$$

$\mathbf{Q}_{m,n}$ is also defined when m and n are not coprime, but we will not need this level of generality [1]. $\mathbf{Q}_{m,n}$ applied to 1 is the symmetric function piece of the Rational Shuffle Theorem. Our next goal is to develop the combinatorial objects which we will use to propose a formula for $\mathbf{Q}^k_{m,n}(1)$.

3 Combinatorics

Again, for positive integers M and N we let $k = \gcd(M, N)$, $m = M/k$, and $n = N/k$. We will define a generating function as a sum over certain k-tuples of lattice paths. These lattice paths will depend on a pair of sequences v and w of lengths M and N, respectively, in the alphabet $\{0, 1, \bullet\}$. Furthermore, we let

$$|v| = \text{the number of 1's in } v$$

and insist that $|v| = |w|$. We will denote the resulting generating function by $L(v, w)$. When $v = 0^M$ and $w = 0^N$, we conjecture that $L(v, w)$ provides a combinatorial model for $\mathbf{Q}^k_{m,n}(1)$.

3.1 Path tuples

Definition 3. An *m,n-path*, or just a *path*, is a sequence of n unit-length north and m unit-length east steps from $(a, 0)$ to $(a+m, n)$ for some integer a that

- begins with a north step, and
- stays weakly above the line $my = nx$.

We can imagine repeating the sequence of steps in a path to get an "infinite" path in the plane. Any height-n "band" (region between $y = b$ and $y = b + n$ for some integer b) of the infinite path determines the original path. We will most often work with the original $b = 0$ band. Next, we define a labeling of the unit squares in each of k "sheets" of $\mathbb{Z}^2$.

Definition 4. We consider $(\mathbb{Z}^2)^k$ as k *sheets* of $\mathbb{Z}^2$, where the sheets are indexed from 0 to $k-1$. The *content* of a lattice square (or "cell") in the i^{th} sheet, where $0 \leq i < k$, is

$$i + My - Nx,$$

where (x, y) are the coordinates of the lower right lattice point of the square.

We depict the contents of some cells in Figure 1. Content provides a bijective correspondence between the cells in any band of $(\mathbb{Z}^2)^k$ and $\mathbb{Z}$. As a a result, given a fixed band, we can refer to a cell by its content. In the sequel, we often use the phrase "cell c" to refer to the unique cell in the current band with content equal to c.

Definition 5. For a fixed k-tuple of paths $\boldsymbol{P}$ in sheets $0, 1, \ldots, k-1$ of $\mathbb{Z}^2$, fix a cell c and consider the path P in $\boldsymbol{P}$ that is in c's sheet. We say that cell c

- *has a north step in* $\boldsymbol{P}$ if there is a north step from P on c's left boundary,
- *is left of* $\boldsymbol{P}$ if the cell in c's row that has a north step in P is weakly to the right of c, and
- *is right of* $\boldsymbol{P}$ if the cell in c's row that has a north step in P is strictly to the left of c.

Analogously, cell c

- *has an east step in* $\boldsymbol{P}$ if there is an east step from P on c's upper boundary,
- *is above* $\boldsymbol{P}$ if the cell in c's column that has an east step from P is weakly below c, and
- *is below* $\boldsymbol{P}$ if the cell in c's column that has an east step from P is strictly above c.

Figure 1: We have drawn the only two possible path tuples for $v = 000110$ and $w = 0110$, one in blue and one in red. The areas of the path tuples are 0 and 1, respectively. Both tuples have 4 path diagonal inversions: $(3,6),(4,6),(4,9),(6,9)$.

Now we use sequences v and w in the alphabet $\{0, 1, \bullet\}$ of lengths M and N, respectively, such that $|v| = |w|$ to restrict the set of paths we consider. These sequences v and w govern the relative location of the paths and the cells $0, 1, \ldots, M-1$ and $0, 1, \ldots, N-1$.

Definition 6. A *v,w-path tuple* (or just a *path tuple*) is a k-tuple of m,n-paths $\boldsymbol{P} = (P^{(0)}, P^{(1)}, \ldots, P^{(k-1)})$, one in each sheet, such that, for $v = v_0 \ldots v_{M-1}$,

(i) $v_i = 0$ if and only if cell i is right of $\boldsymbol{P}$,

(ii) $v_i = 1$ if and only if cell i has a north step in $\boldsymbol{P}$, and

(iii) $v_i = \bullet$ if and only if cell i is left of $\boldsymbol{P}$.

and, for $w = w_0 \ldots w_{N-1}$,

(i') $w_i = 0$ if and only if cell i is below $\boldsymbol{P}$,

(ii') $w_i = 1$ if and only if cell i has an east step in $\boldsymbol{P}$, and

(iii') $w_i = \bullet$ if and only if cell i is above $\boldsymbol{P}$.

The north (respectively east) steps insisted upon above are called *basement* steps and denoted with dashed lines in our figures.

As a sanity check, we note that, for any M and N, the restriction that paths are weakly above the line $my = nx$ implies that there is exactly one path tuple if $v = \bullet^M$ and $w = \bullet^N$. On the other hand, there are many pairs of sequences for which there are infinitely many path tuples, *e.g.* $v = 0^M$ and $w = 0^N$. We address this issue in Subsection 3.3. Figure 1 contains two examples of path tuples. Next, we define statistics for these objects.

Definition 7. The *area* of a path tuple $\boldsymbol{P}$ is the number of cells $c \geq M + N$ that are weakly right of $\boldsymbol{P}$, *i.e.* they are either right of $\boldsymbol{P}$ or they have north steps in $\boldsymbol{P}$.

Next, we define a notion of diagonal inversion for a path.

Definition 8. A *path diagonal inversion* in a path tuple $\boldsymbol{P}$ is a pair of cells c and d with $c \geq 0$ and $M \leq d < c + M$ such that

- c has a north step in $\boldsymbol{P}$ and
- d is weakly to the right of $\boldsymbol{P}$.

c and d may be in different sheets and the north steps mentioned above can be basement steps. We let $\operatorname{pdinv}(\boldsymbol{P})$ denote the number of path diagonal inversions in a path tuple $\boldsymbol{P}$.

3.2 Carlsson–Mellit operators and characteristic functions

Next, we use certain operators defined by Carlsson and Mellit to assign a characteristic function to each path tuple.

Definition 9. For any integer $\ell \geq 0$, we let

$$V_\ell = \mathbb{Q}[y_1, y_2, \ldots, y_\ell] \otimes \Lambda.$$

Following Carlsson and Mellit [2], we define operators

$$\begin{aligned} d_+ &\colon V_\ell \to V_{\ell+1} \qquad (\ell \geq 0), \\ d_- &\colon V_\ell \to V_{\ell-1} \qquad (\ell \geq 1), \end{aligned}$$

by

$$\begin{aligned} d_+ f &= T_1 T_2 \ldots T_\ell f[X + (t-1) y_{\ell+1}], \\ d_- f &= -y_\ell f[X - (t-1) y_\ell] \sum_{i \geq 0} h_i[-X/y_\ell] \Big|_{y_\ell^0}, \end{aligned}$$

where

$$T_i f = \frac{(t-1) y_i f + (y_{i+1} - t y_i) s_i f}{y_{i+1} - y_i}$$

and s_i swaps y_i and y_{i+1}. We define a third operator,

$$d_= \colon V_\ell \to V_\ell \qquad (\ell \geq 1)$$

which also appears in Carlsson and Mellit's work but not by this name. It acts by

$$d_= f = \frac{1}{t-1} \left(d_- d_+ f - d_+ d_- f \right).$$

Given a path tuple $\boldsymbol{P}$, we describe how to obtain a sequence of d_+, d_-, and $d_=$ operators that, when applied to 1, allow us to define $L(v, w)$.

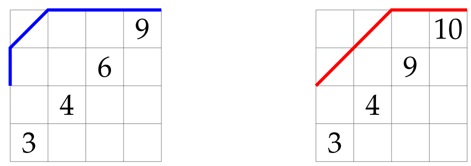

Figure 2: This figure contains the partial Schröder paths of the two path tuples from Figure 1. Their corresponding characteristic functions are $d_- d_= d_+ d_+ d_+(1)$ and $d_= d_= d_+ d_+(1)$, both of which are in V_2.

Definition 10. A *partial Schröder path* is a lattice path from $(0, \ell)$ to (N, N) consisting of steps $(1, 0)$, $(0, 1)$ and $(1, 1)$ that remains weakly above the line $y = x$ and does not contain any diagonal steps on the line $y = x$.

Definition 11. Given a path tuple $\boldsymbol{P}$, suppose $c_1 < c_2 < \ldots < c_N$ are the cells containing north steps in $\boldsymbol{P}$ and that $c_1, \ldots, c_\ell$ are the basement steps. Place the values $c_1, c_2, \ldots, c_N$ in the first N cells on the line $y = x$ from bottom left to top right. Let S be the (unique) partial Schröder path from $(0, \ell)$ to (N, N) such that

(i) if $c_i < c_j < c_i + M$, then the entire (unique) square above c_i and to the left of c_j is under S,

(ii) if $c_i + M = c_j$, then the square above c_i and to the left of c_j contains a diagonal step,

(iii) otherwise, the square above c_i and to the left of c_j is above S.

We call S the *partial Schröder path* of $\boldsymbol{P}$.

Definition 12. Given the partial Schröder path S of a path tuple $\boldsymbol{P}$, we begin with 1 and, reading S from right to left, iteratively apply

- d_+ for any horizontal step,
- d_- for any vertical step, and
- $d_=$ for any diagonal step.

The resulting element of V_ℓ is the *characteristic function* of $\boldsymbol{P}$, written $\chi(\boldsymbol{P})$.

The partial Schröder paths and characteristic functions for the path tuples in Figure 1 are depicted in Figure 2. We close this subsection by defining our main objects of study.

Definition 13. Given sequences v and w of lengths M and N, respectively, and $|v| = |w|$, we let

$$L(v, w) = \sum_{\boldsymbol{P}} t^{-\operatorname{pdinv}(\boldsymbol{P})} q^{\operatorname{area}(\boldsymbol{P})} \chi(\boldsymbol{P})$$

where the sum is over all v, w-path tuples $\boldsymbol{P}$.

Example 1. To complete this subsection, we compute $L(v,w)$ in its entirety for $v = 000110$ and $w = 0110$. There are two path tuples in this case, which appear in Figure 1. As mentioned in Figure 1, the areas of the path tuples are 0 and 1, respectively, and both tuples have 4 path diagonal inversions. Figure 2 depicts the partial Schröder paths for these path tuples. Assembling this information, and evaluating the relevant sequences of Carlsson–Mellit operators, we get

$$L(000110, 0110) = t^{-4}d_-d_=d_+d_+d_+(1) + t^{-4}qd_=d_=d_+d_+(1) = -t^{-2}y_1s_1 + t^{-3}qy_1y_2 \in V_2.$$

3.3 Compressed path tuples

By modifying the underlying objects, we can write $L(v,w)$ as a sum over a finite collection of path tuples instead of a potentially infinite collection.

Definition 14. Given sequences v and w, a path tuple $\boldsymbol{P} = (P^{(0)}, P^{(1)}, \dots, P^{(k-1)})$ is *compressed* if any $c > 0$ such that

- all of the cells $c-1, c-2, \dots, c-M$ are below the path, and
- none of the cells $c-1, c-2, \dots, c-M$ have a north step

has a north step. We call any north step in such a cell c a *forced* north step and denote the number of forced north steps in $\boldsymbol{P}$ by $\operatorname{force}(\boldsymbol{P})$.

Then we have the following finite formula for $L(v,w)$.

Theorem 1.

$$L(v,w) = \sum_{\boldsymbol{P}} t^{-\operatorname{pdinv}(\boldsymbol{P})} q^{\operatorname{area}(\boldsymbol{P})} (1-q)^{-\operatorname{force}(\boldsymbol{P})} \chi(\boldsymbol{P}),$$

where the sum is over all compressed path tuples $\boldsymbol{P}$*.*

Figure 3 shows all compressed paths for $v = 0000$ and $w = 000$.

3.4 Path labelings when $v = 0^M$ and $w = 0^N$

We can simplify the previous definitions for $L(v,w)$ when $v = 0^M$ and $w = 0^N$. As a result, we get an expression for $L(v,w)$ that does not explicitly use Carlsson–Mellit operators. We can assign labels to north steps in our paths in a manner reminiscent of parking functions.

Definition 15. A *labeled path tuple* is a k-tuple of paths $\boldsymbol{P}$ along with a function f from the north steps of $\boldsymbol{P}$ to the positive integers such that, if $c + M = d$, then $f(c) < f(d)$.

We define a notion of diagonal inversions for labeled path tuples.

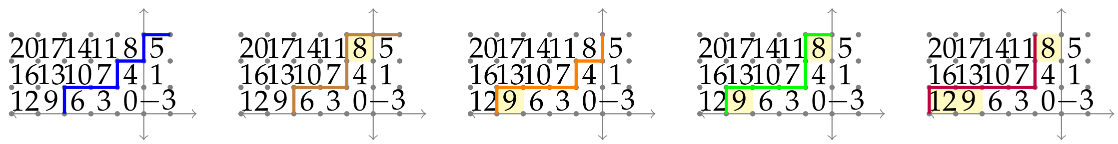

Figure 3: Above, we have drawn the 5 compressed path tuples for $v = 0000$ and $w = 000$. Since $\gcd(M, N) = 1$, each has only one sheet. Cells that contribute to the area are shaded yellow. Each path tuple has 3 path diagonal inversions and 1 forced north step (at cell 4). The usual 5 Dyck paths for this case can be recovered by starting at the lattice point $(-1, 1)$ and following the infinite repetition of each path for a total of 3 north steps and 3 east steps.

Definition 16. Given a labeled path tuple $(\boldsymbol{P}, f)$, a *labeled diagonal inversion* is a pair of cells c and d with north steps such that $c < d < c + M$ and $f(c) < f(d)$. (Again, c and d may be in different sheets.) We let $\operatorname{ldinv}(\boldsymbol{P}, f)$ be the number of labeled diagonal inversions in the labeled path tuple $(\boldsymbol{P}, f)$.

Definition 17. We let

$$L_{M,N} = \sum_{(\boldsymbol{P},f)} t^{\operatorname{ldinv}(\boldsymbol{P},f)-\operatorname{pdinv}(\boldsymbol{P})} q^{\operatorname{area}(\boldsymbol{P})} (1-q)^{-\operatorname{force}(\boldsymbol{P})} \prod_{i>0} x_i^{f^{-1}(i)},$$

where the sum is over all labeled path tuples $(\boldsymbol{P}, f)$ for M and N where $\boldsymbol{P}$ is compressed.

As an example, we show all compressed path tuples that contribute to $L_{4,3}$ in Figure 3. The following theorem comes directly from the work of Carlsson and Mellit [2].

Theorem 2. *For positive integers M and N,*

$$L_{M,N} = L(0^M, 0^N).$$

While it is theoretically possible to unwind the definitions of the Carlsson–Mellit operators to give a label-based expression for $L(v, w)$ for any v and w, the resulting expression is quite technical and not helpful in what we aim to achieve here. Understanding this expression may be a worthwhile endeavor in the future. We close this section by stating our main conjecture.

Conjecture 1. *For positive integers M and N, let $k = \gcd(M, N)$, $m = M/k$, $n = N/k$. Then*

$$\mathbf{Q}^k_{m,n}(1) = (-1)^{k(n+1)} (1-q)^k t^C L_{M,N},$$

where C is the maximum of $\operatorname{pdinv}(\boldsymbol{P})$ *over all M, N-path tuples $\boldsymbol{P}$.*

We currently do not know of a simple way to compute the value C in Conjecture 1.

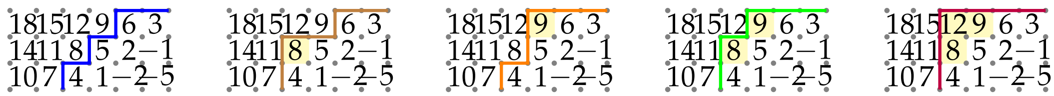

Figure 4: We have shifted the 5 compressed path tuples for $v = 0000$ and $w = 000$ up to obtain rational Dyck paths.

4 Connections

4.1 The Rational Shuffle Theorem

When M and N are coprime (so $M = m$, $N = n$, and $k = 1$), the Rational Shuffle Theorem gives a combinatorial expression for $\mathbf{Q}_{m,n}(1)$. We check that our formulas agree. Since $k = 1$, there is only one sheet in each path tuple. We obtain the usual depiction of rational parking functions by viewing each of our path tuples in the horizontal band $1 < y \leq n + 1$. We can do this same operation for compressed path tuples. For comparison, we shift each of the path tuples in Figure 3 and depict the resulting rational Dyck paths in Figure 4. Our statistics for $L_{m,n}$ match up with the statistics that appear on the combinatorial side of the Rational Shuffle Theorem.

4.2 Torus link homology

For positive integers M and N, Hogancamp and Mellit derived a recursion for computing the triply graded Khovanov–Rozansky homology of the M,N-torus link [7]. We follow Gorsky, Mazin, and Vazirani's approach by allowing $\bullet$'s to appear in v and w [5]; Hogancamp and Mellit's original recursion can be obtained by removing all $\bullet$'s. We state this theorem below for comparison with our generalization (Theorem 4).

Theorem 3 ([5, 7])**.** *For nonnegative integers M and N, the triply graded Khovanov–Rozansky homology of the M,N-torus link is free over $\mathbb{Z}$ of graded rank $p(0^M, 0^N)$, which is an element of $\mathbb{N}[q, t^{\pm 1}, a, (1-q)^{-1}]$ computed by the following recursion:*

0. $p(\bullet^M, \bullet^N) = 1$.
1. $p(\bullet v, \bullet w) = p(v\bullet, w\bullet)$.
2. $p(0v, 0w) = (1-q)^{-1} p(v1, w1)$ *if* $|v| = |w| = 0$.
3. $p(0v, 0w) = t^{-\ell} p(v1, w1) + q t^{-\ell} p(v0, w0)$ *if* $\ell = |v| = |w| > 0$.
4. $p(1v, 0w) = p(v1, w\bullet)$.

5. $p(0v, 1w) = p(v\bullet, w1)$.

6. $p(1v, 1w) = (t^{|v|} + a)p(v\bullet, w\bullet)$.

An analogous recursion holds for $L(v, w)$. In fact, the search for such a recursion motivated the discovery of $L(v, w)$.

Theorem 4. *Let v and w be sequences in the alphabet $\{0, 1, \bullet\}$ with $|v| = |w|$. We can compute $L(v, w)$ via the following recursion:*

0. $L(\bullet^M, \bullet^N) = 1$.

1. $L(\bullet v, \bullet w) = L(v\bullet, w\bullet)$.

2. $L(0v, 0w) = (1 - q)^{-1}d_- L(v1, w1)$ *if* $|v| = |w| = 0$.

3. $L(0v, 0w) = t^{-\ell}d_- L(v1, w1) + qt^{-\ell}L(v0, w0)$ *if* $\ell = |v| = |w| > 0$.

4. $L(1v, 0w) = t^{-\ell}d_= L(v1, w\bullet)$ *if* $\ell = |v| = |w|$.

5. $L(0v, 1w) = L(v\bullet, w1)$.

6. $L(1v, 1w) = d_+ L(v\bullet, w\bullet)$

The proof of Theorem 4 (which we omit in this extended abstract) is obtained by considering the different ways that cells 0, M, and N can appear with respect to a path tuple. Figure 5 contains a visual guide to these different configurations and how they correspond to cases in the recursion.

One can view Theorem 4 as a lift of Theorem 3 to the level of symmetric functions (or, more precisely, symmetric functions tensored with polynomials in variables $y_1, y_2, \ldots, y_\ell$). A reasonable approach to proving Conjecture 1 would be to mimic this recursion to define and study extensions of the operators $\mathbf{Q}_{m,n}$ to sequences v and w. A similar approach was used by Mellit to prove the Rational Shuffle Theorem [8].

It would also be interesting to investigate whether $L(v, w)$ has a precise geometric or topological meaning in particular cases. For example, Hogancamp and Mellit show that $p(1^k0^{m(k-1)}, 1^k0^{n(k-1)})$ is related to colored homology of certain torus links [7]. We close this abstract by describing how to recover any $p(v, w)$ from the corresponding $L(v, w)$.

Corollary 1. *Let ψ be the operator[1] on Λ defined by $\psi(e_i) = 1 + a$ for each (algebraically independent) e_i. Then, for any sequences v and w in the alphabet $\{0, 1, \bullet\}$ with $|v| = |w| = \ell$,*

$$\psi\left(d_-^\ell L(v, w)\right) = p(v, w).$$

[1]To experts in this area, applying ψ equivalent to taking the "Schröder inner product."

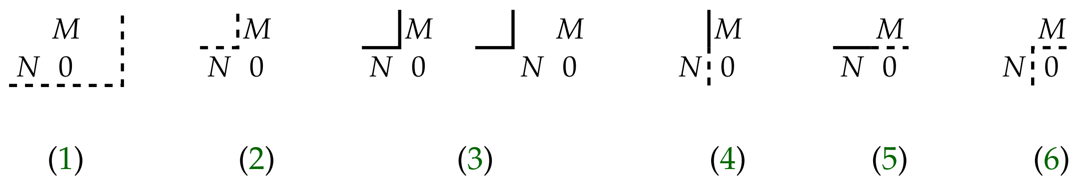

Figure 5: We sketch the different possible configurations of steps near cells 0, M, and N in a path tuple $\boldsymbol{P}$ which correspond to the different recursive cases in Theorem 4.

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