Séminaire Lotharingien de Combinatoire **86B** (2022)
Article #76, 12 pp.
Proceedings of the 34th Conference on Formal Power Series and Algebraic Combinatorics (Bangalore)

Lattice Paths and Negatively Indexed Weight-Dependent Binomial Coefficients

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Abstract. In 1992, Loeb considered a natural extension of the binomial coefficients to negative entries and gave a combinatorial interpretation in terms of hybrid sets. He showed that many of the fundamental properties of binomial coefficients continue to hold in this extended setting. Recently, Formichella and Straub showed that these results can be extended to the q-binomial coefficients with arbitrary integer values and extended the work of Loeb further by examining arithmetic properties of the q-binomial coefficients. In this paper, we give an alternative combinatorial interpretation in terms of lattice paths and consider an extension of the more general weight-dependent binomial coefficients, first defined by the second author, to arbitrary integer values. Remarkably, many of the results of Loeb, Formichella and Straub continue to hold in the general weighted setting. We also examine important special cases of the weight-dependent binomial coefficients, including ordinary, q- and elliptic binomial coefficients as well as elementary and complete homogeneous symmetric functions (with application of these cases to Stirling numbers).

Keywords: binomial theorem, commutation relations, symmetric functions, elliptic-commuting variables, elliptic binomial coefficient, elliptic hypergeometric series

1 Introduction

Loeb [5] studied a generalization of the binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, where n and k are allowed to be negative integers. He defined them for integers n and k in terms of

$$\binom{n}{k} := \lim_{\epsilon \mapsto 0} \frac{\Gamma(n+1+\epsilon)}{\Gamma(k+1+\epsilon)\Gamma(n-k+1+\epsilon)}.$$

*josef.kuestner@univie.ac.at. Supported by FWF Austrian Science Fund grant P32305.
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From this definition it follows that the binomial coefficients with integer values satisfy the recursion

$$\binom{n}{0} = \binom{n}{n} = 1 \qquad \text{for } n \in \mathbb{Z}\,, \tag{1.1a}$$

and for $n, k \in \mathbb{Z}$, provided that $(n+1, k) \neq (0, 0)$,

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}, \tag{1.1b}$$

and they can be fully characterized by this recursion. Loeb introduced a combinatorial model for these numbers and showed that many properties of binomial coefficients, as the recursion formula (1.1) and the binomial theorem, remain to be true for negative arguments. Formichella and Straub [2] generalized these results to q-binomial coefficients. In [8], the second author introduced weight-dependent binomial coefficients for nonnegative integers n, k which naturally generalize q-binomial coefficients.

In this extended abstract, we show how the weight-dependent binomial coefficients can be extended to negative integer values, analogous to the work of Loeb [5] and Formichella and Straub [2]. We study reflection formulae inspired by an involution which recently appeared in [9]. We give a combinatorial interpretation of the weight-dependent binomial coefficients in terms of lattice paths and prove a corresponding noncommutative weight-dependent generalization of the binomial theorem. As a corollary of the binomial theorem, we obtain convolution formulae analogous to the Chu–Vandermonde convolution formula. Finally, we study some important special cases of the weight-dependent binomial coefficients, such as elementary and complete homogeneous symmetric functions (with application of these cases to Stirling numbers), and elliptic binomial coefficients.

2 Weight-dependent commutation relations

2.1 A noncommutative algebra

Let $(w(s,t))_{s,t\in\mathbb{Z}}$ be a sequence of invertible variables. We start by defining the noncommutative algebra $\mathbb{C}_w[x, x^{-1}, y, y^{-1}]$ which is fundamental to the content of this extended abstract.

Definition 2.1. For a doubly-indexed sequence of invertible variables $(w(s,t))_{s,t\in\mathbb{Z}}$, let $\mathbb{C}_w[x, x^{-1}, y, y^{-1}]$ be the associative unital algebra over $\mathbb{C}$ generated by x, x^{-1}, y and y^{-1} and the sequence of invertible variables $(w(s,t)^{\pm 1})_{s,t\in\mathbb{Z}}$ satisfying the following relations:

$$x^{-1}x = xx^{-1} = 1 \tag{2.1a}$$
$$y^{-1}y = yy^{-1} = 1 \tag{2.1b}$$
$$yx = w(1,1)xy, \tag{2.1c}$$
$$xw(s,t) = w(s+1,t)x, \tag{2.1d}$$
$$yw(s,t) = w(s,t+1)y, \tag{2.1e}$$

for all $s,t \in \mathbb{Z}$.

In the following, we restrict the just defined algebra $\mathbb{C}_w[x,x^{-1},y,y^{-1}]$ to $\mathbb{C}_w[x,y,y^{-1}]$ (where we omit the generator x^{-1} and relation (2.1a)) or to $\mathbb{C}_w[x,x^{-1},y]$ (where we omit the generator y^{-1} and relation (2.1b)), respectively. In addition we find it convenient to work in the extensions of these algebras to the algebras of formal power series $\mathbb{C}_w[[x,x^{-1},y,y^{-1}]]$, $\mathbb{C}_w[[x,y,y^{-1}]]$ and $\mathbb{C}_w[[x,x^{-1},y]]$ (keeping the same relations). Note that expressions of the form $(x+y)^n$ with $n<0$ do not have a unique power series expansion in $\mathbb{C}_w[[x,x^{-1},y,y^{-1}]]$. For $n<0$ we have to decide how to expand $(x+y)^n$, say as $(x+y)^n = ((1+xy^{-1})y)^n$ as an element in $\mathbb{C}_w[[x,y,y^{-1}]]$, or rather as $(x+y)^n = (x(1+x^{-1}y))^n$ as an element in $\mathbb{C}_w[[x,x^{-1},y]]$, respectively. By default, we shall always choose the algebra of formal power series $\mathbb{C}_w[[x,y,y^{-1}]]$ and only resort to $\mathbb{C}_w[[x,x^{-1},y]]$ if we explicitly mention that.

For $l,m \in \mathbb{Z}\cup\{\pm\infty\}$ we define products of (possibly noncommutative) invertible variables A_j as follows:

$$\prod_{j=l}^{m} A_j = \begin{cases} A_l A_{l+1}\cdots A_m & \text{if } m > l-1 \\ 1 & m = l-1 \\ A_{l-1}^{-1}A_{l-2}^{-1}\cdots A_{m+1}^{-1} & \text{if } m < l-1. \end{cases} \tag{2.2}$$

Especially for the reflection formulae it will be necessary to define $\operatorname{sgn}(n)$ for $n \in \mathbb{Z}$, following [2], as

$$\operatorname{sgn}(n) = \begin{cases} 1 & \text{if } n \geq 0, \\ -1 & \text{if } n < 0. \end{cases} \tag{2.3}$$

For $s,t \in \mathbb{Z}$ and the sequence of invertible weights $(w(s,t))_{s,t\in\mathbb{Z}}$ we define

$$W(s,t) := \prod_{j=1}^{t} w(s,j). \tag{2.4}$$

We refer to the $w(s,t)$ as *small weights*, whereas to the $W(s,t)$ as *big weights*.

Lemma 2.2. *Let $(w(s,t))_{s,t\in\mathbb{Z}}$ be a doubly-indexed sequence of invertible variables, and x and y invertible variables together forming the associative algebra $A_{x,y}=\mathbb{C}_{w_{x,y}}[x,x^{-1},y,y^{-1}]$ where $w_{x,y}(s,t)=w(s,t)$. Then the following six homomorphisms are involutive algebra isomorphisms.*

$$\phi_{y,x}\colon \quad A_{x,y}\to A_{y,x} \quad \text{with} \quad w_{y,x}(s,t)=w(t,s)^{-1}, \tag{2.5a}$$
$$\phi_{x^{-1},y}\colon \quad A_{x,y}\to A_{x^{-1},y} \quad \text{with} \quad w_{x^{-1},y}(s,t)=w(1-s,t)^{-1}, \tag{2.5b}$$
$$\phi_{x^{-1},x^{-1}y}\colon \quad A_{x,y}\to A_{x^{-1},x^{-1}y} \quad \text{with} \quad w_{x^{-1},x^{-1}y}(s,t)=w(1-s-t,t)^{-1}, \tag{2.5c}$$
$$\phi_{x,y^{-1}}\colon \quad A_{x,y}\to A_{x,y^{-1}} \quad \text{with} \quad w_{x,y^{-1}}(s,t)=w(s,1-t)^{-1}, \tag{2.5d}$$
$$\phi_{x^{-1},y^{-1}}\colon \quad A_{x,y}\to A_{x^{-1},y^{-1}} \quad \text{with} \quad w_{x^{-1},y^{-1}}(s,t)=w(1-s,1-t), \tag{2.5e}$$
$$\phi_{y^{-1}x,y^{-1}}\colon \quad A_{x,y}\to A_{y^{-1}x,y^{-1}} \quad \text{with} \quad w_{y^{-1}x,y^{-1}}(s,t)=w(s,1-s-t)^{-1}. \tag{2.5f}$$

The involution (2.5c) appeared in the work of the second and third author in [9, Lemma 2].

It is straightforward to check that the simultaneous replacement of $w_{x,y}(s,t)$ $(s,t\in\mathbb{Z})$, x and y in (2.5a)–(2.5f) by $w_{x',y'}(s,t)$, x' and y', respectively, again satisfies the conditions in (2.1). As a consequence, given an identity in $w(s,t)$, x and y, a new valid identity can be obtained by applying the isomorphism ϕ to each of the occurring variables, where in both identities the variables satisfy the same commutation relations (2.1). We will apply such isomorphisms in the proofs of Lemma 2.3 and Theorem 2.10.

The following rule for interchanging powers of x and y is an extension to integer values of a corresponding lemma in [8, Lemma 1]:

Lemma 2.3. *For all $k,\ell\in\mathbb{Z}$ we have*

$$y^k x^\ell=\left(\prod_{i=1}^{\ell}\prod_{j=1}^{k}w(i,j)\right)x^\ell y^k=\left(\prod_{i=1}^{\ell}W(i,k)\right)x^\ell y^k.$$

Proof. The case $k,\ell\geq 0$ is already given in [8] and is easy to prove by induction; we therefore omit the proof. For $k\geq 0$ and $\ell<0$ we combine the involution (2.5b) with the $k,\ell\geq 0$ case and by the definition of products (2.2) we obtain:

$$y^k x^\ell=y^k(x^{-1})^{-\ell}=\left(\prod_{i=1}^{-\ell}\prod_{j=1}^{k}w(1-i,j)^{-1}\right)(x^{-1})^{-\ell}y^k=\left(\prod_{i=1}^{\ell}\prod_{j=1}^{k}w(i,j)\right)x^\ell y^k.$$

The remaining cases can be proved in the same manner by applying the involutions (2.5d) and (2.5e) to the $k,\ell\geq 0$ case. □

2.2 Weight-dependent binomial coefficients with integer values

Let the *weight-dependent binomial coefficients* or *w-binomial coefficients* be defined by

$$_w\begin{bmatrix}n\\0\end{bmatrix}={}_w\begin{bmatrix}n\\n\end{bmatrix}=1 \qquad \text{for } n\in\mathbb{Z}\,, \tag{2.6a}$$

and for $n, k \in \mathbb{Z}$, provided that $(n+1, k) \neq (0, 0)$,

$$ {}_w\begin{bmatrix} n+1 \\ k \end{bmatrix} = {}_w\begin{bmatrix} n \\ k \end{bmatrix} + {}_w\begin{bmatrix} n \\ k-1 \end{bmatrix} W(k, n+1-k). \tag{2.6b} $$

Example 2.4 ($n = -1$). Using induction for $k \geq 0$ and $k < 0$ we obtain that the weight-dependent binomial coefficient for $n = -1$ gives

$$ {}_w\begin{bmatrix} -1 \\ k \end{bmatrix} = (-1)^k \mathrm{sgn}(k) \prod_{j=1}^{k} W(j, -j), $$

where $\mathrm{sgn}(k)$ is defined in (2.3).

Before stating the general binomial theorem, let us extend the lattice path model of [7] to integer values $n, k \in \mathbb{Z}$. Let a (*weighted*) *hybrid lattice path* be a sequence of (weighted) steps in the xy-plane starting at the origin $(0, 0)$ and ending at (n, m) with $n, m \in \mathbb{Z}$ using the following steps:

1. If $n, m \geq 0$, we use north and east steps $(\uparrow, \rightarrow)$,
2. if $n \geq 0$ and $m < 0$, we use south steps and east-south step combinations ($\downarrow$, $\urcorner\!\!\downarrow$) and every path starts with a south step,
3. if $n < 0$ and $m \geq 0$, we use north-west step combinations and west steps ($\uparrow\!\!\urcorner$, $\leftarrow$) and every path starts with a west step,
4. if $n, m < 0$, there are no allowed steps.

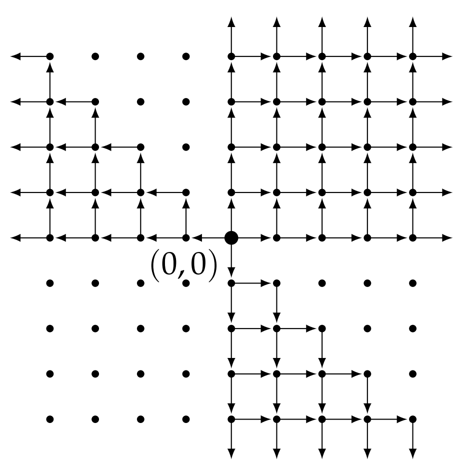
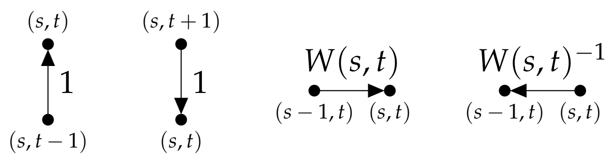
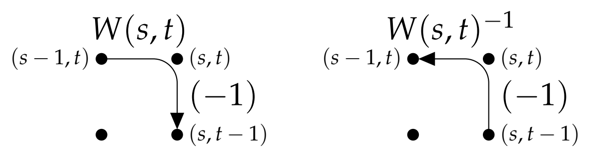


Figure 1: The possible steps of a hybrid lattice path.

Figure 1 shows the possible steps of a hybrid lattice path. The arrows indicate the direction of the steps. Note that in the region $m < 0 \leq n$ every east step has to be followed by a south step and in the region $n < 0 \leq m$ every north step has to be followed by a west step. We give weights to hybrid lattice paths by assigning the following weights to each of the respective steps



and, if we highlight the step combinations by rounded corners,



The weight of a path P, $w(P)$, is defined as the product of the weights of all its steps.

Example 2.5. Figure 2 shows a hybrid lattice path in the area $n, m \geq 0$ with weight $W(1,0) \cdot 1 \cdot W(2,1) \cdot 1 \cdot W(3,2) \cdot W(4,2) = w(2,1)w(3,1)w(3,2)w(4,1)w(4,2)$. Paths in this area correspond to (ordinary) weighted lattice paths. The left side of Figure 3 shows

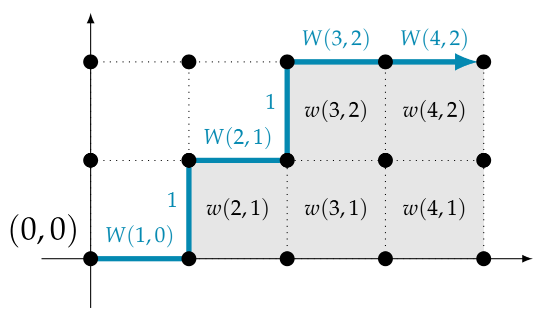


Figure 2: A hybrid lattice path in the area $n, m \geq 0$.

a hybrid lattice path in the area $m < 0 \leq n$ with weight $1 \cdot W(1,-1) \cdot (-1) \cdot 1 \cdot W(2,-3) \cdot (-1) = (-1)^2 w(1,0)^{-1} w(2,0)^{-1} w(2,-1)^{-1} w(2,-2)^{-1}$. The right side of Figure 3 shows a hybrid lattice path in the area $n < 0 \leq m$ with weight $W(0,0)^{-1} \cdot (-1) \cdot W(-1,1)^{-1} \cdot W(-2,1)^{-1} \cdot (-1) \cdot W(-3,2)^{-1} = (-1)^2 w(-1,1)^{-1} w(-2,1)^{-1} w(-3,1)^{-1} w(-3,2)^{-1}$.

Given two points $A, B \in \mathbb{Z}^2$, let $\mathcal{P}_{\mathbb{Z}}(A \to B)$ be the set of all hybrid lattice paths from A to B.

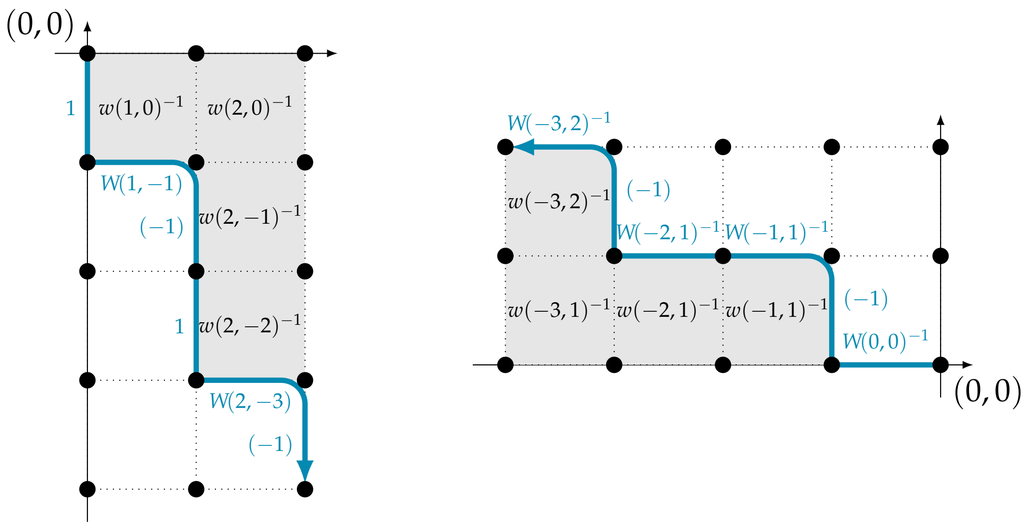


Figure 3: A hybrid lattice path in the area $m < 0 \leq n$ (left) and a path in the area $n < 0 \leq m$ (right). The (diagonal) step combinations are indicated by rounded corners.

Theorem 2.6. *Let $n, k \in \mathbb{Z}$. Then,*

$$\sum_{P \in \mathcal{P}_{\mathbb{Z}}((0,0)\to(k,n-k))} w(P) = {}_w\begin{bmatrix} n \\ k \end{bmatrix}. \tag{2.7}$$

This theorem can be proven by showing that the sum fulfills the relations (2.6). Figure 1 also shows that some points are not reachable by using the possible steps of a hybrid lattice path. In terms of the weight-dependent binomial coefficients, this corresponds to

$${}_w\begin{bmatrix} n \\ k \end{bmatrix} = 0 \qquad \text{for } 0 \leq n < k,\ k < 0 \leq n \text{ or } n < k < 0, \tag{2.8}$$

which can be proven by induction.

Example 2.7. By setting $w(s,t) = 1$ for all $s, t \in \mathbb{Z}$, the generating function of hybrid lattice paths is given by the binomial coefficients with integer values defined by (1.1) [5]. By setting $w(s,t) = q$ for all $s, t \in \mathbb{Z}$, we obtain a q-weighting of hybrid lattice paths with the generating function given by the q-binomial coefficients with integer values defined in [2]. Note that Loeb and also Formichella and Straub interpret ordinary and q-binomial coefficients with arbitrary integer values combinatorially in terms of subsets of hybrid sets [2, 5]. In the full article [4] we show that these are in one-to-one correspondence with hybrid lattice paths.

Remark 2.8. Note that we could also give weights to hybrid lattice paths by assigning weights to the boxes between the path and the x-axis (see Figures 2 and 3). In the

$w(s,t) = q$ case this corresponds to the q-counting of paths according to area. We refer the reader to [4] for more details. We further mention that the considered model makes it possible to establish bijective proofs of the following reflection and convolution formulae. We also omit these proofs here.

2.3 Reflection formulae

In order to generalize the reflection formulae in [2], we introduce the reflection weights

$$\widehat{w}(s,t) := w_{y,x}(s,t) = w(t,s)^{-1} \tag{2.9}$$

$$\widetilde{w}(s,t) := w_{x^{-1},x^{-1}y}(s,t) = w(1-s-t,t)^{-1}, \tag{2.10}$$

$$\breve{w}(s,t) := w_{y^{-1}x,y^{-1}}(s,t) = w(s,1-s-t)^{-1}. \tag{2.11}$$

for $s,t \in \mathbb{Z}$ from the involutions in Lemma 2.2.

Theorem 2.9. *Let $n,k \in \mathbb{Z}$ and $\widehat{w}(s,t)$, $\widetilde{w}(s,t)$ and $\breve{w}(s,t)$ as above with $s,t \in \mathbb{Z}$. Then,*

$$_w\begin{bmatrix} n \\ k \end{bmatrix} = {}_{\widehat{w}}\begin{bmatrix} n \\ n-k \end{bmatrix} \prod_{j=1}^{k} W(j,n-k) \tag{2.12}$$

$$= (-1)^{n-k}\mathrm{sgn}(n-k) \, {}_{\widetilde{w}}\begin{bmatrix} -k-1 \\ -n-1 \end{bmatrix} \prod_{j=1}^{n-k} W(n+1-j,j)^{-1} \tag{2.13}$$

$$= (-1)^{k}\mathrm{sgn}(k) \, {}_{\breve{w}}\begin{bmatrix} k-n-1 \\ k \end{bmatrix} \prod_{j=1}^{k} W(j,-j). \tag{2.14}$$

The theorem can be proven by showing that all expressions satisfy the same recurrence relations and initial conditions. One can also prove the theorem bijectively in terms of weighted hybrid lattice paths.

2.4 An extension of the noncommutative binomial theorem

We are now ready to state the weight-dependent noncommutative binomial theorem for integer values which generalizes the results of [2], [5] and [8]. The proof, which we omit, uses induction on n as well as the involution (2.5a) and the reflection formula (2.12).

Theorem 2.10. *Let $n,k \in \mathbb{Z}$ and x,y be invertible variables satisfying the commutation relations (2.1), then we have*

$$(x+y)^n = \sum_{k\geq 0} {}_w\begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k} \quad \text{in } \mathbb{C}_w[[x,y,y^{-1}]] \text{ or} \tag{2.15}$$

$$(x+y)^n = \sum_{k\leq n} {}_w\begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k} \quad \text{in } \mathbb{C}_w[[x,x^{-1},y]]. \tag{2.16}$$

2.5 Convolution formulae

In [8], the second author derived a weight-dependent generalization of the Chu–Vandermonde convolution formula

$$\binom{n+m}{k} = \sum_{j=0}^{k} \binom{n}{j} \binom{m}{k-j}$$

Given Theorem 2.10, we can expand $(x+y)^{n+m}$ and $(x+y)^n(x+y)^m$ using (2.15) for all $n, m \in \mathbb{Z}$ to obtain the following weight-dependent convolution formula by comparing the coefficients of $x^k y^{n+m-k}$ for $k \geq 0$. The proof is similar to the proof of [8, Corollary 1] and therefore we omit the details.

Corollary 2.11. *Let $n, m \in \mathbb{Z}$ and $k \geq 0$. For the binomial coefficients in (2.6), defined by the doubly-indexed sequence of indeterminate weights $(w(s,t))_{s,t\in\mathbb{Z}}$, we have the following formal identity in $\mathbb{C}[(w(s,t))_{s,t\in\mathbb{Z}}]$:*

$$_w\begin{bmatrix} n+m \\ k \end{bmatrix} = \sum_{j=0}^{k} {}_w\begin{bmatrix} n \\ j \end{bmatrix} \left(x^j y^{n-j} {}_w\begin{bmatrix} m \\ k-j \end{bmatrix} y^{j-n} x^{-j} \right) \prod_{i=1}^{k-j} W(i+j, n-j). \tag{2.17}$$

The above identity is formal because it contains noncommuting variables x and y defined by (2.1), which cancel after shifting all weights in ${}_w\left[\begin{smallmatrix} m \\ k-j \end{smallmatrix}\right]$.

By expanding $(x+y)^{n+m}$ and $(x+y)^n(x+y)^m$ using (2.16) and comparing coefficients as before, we obtain a second w-Chu–Vandermonde convolution formula.

Corollary 2.12. *Let $n, m \in \mathbb{Z}$ and $k \leq n+m$. For the binomial coefficients in (2.6), defined by the doubly-indexed sequence of indeterminate weights $(w(s,t))_{s,t\in\mathbb{Z}}$, we have the following formal identity in $\mathbb{C}[(w(s,t))_{s,t\in\mathbb{Z}}]$:*

$$_w\begin{bmatrix} n+m \\ k \end{bmatrix} = \sum_{j=k-m}^{n} {}_w\begin{bmatrix} n \\ j \end{bmatrix} \left(x^j y^{n-j} {}_w\begin{bmatrix} m \\ k-j \end{bmatrix} y^{j-n} x^{-j} \right) \prod_{i=1}^{k-j} W(i+j, n-j). \tag{2.18}$$

If $n+m < k < 0$, both sides of the equation vanish, therefore (2.18) is true for all $k < 0$. This equation generalizes an identity in [2, Lemma 4.9]. There, the corresponding identity is limited to the case $n, m, k < 0$ whereas Corollary 2.12 is even true if n, m are positive or have mixed signs.

In the case $\operatorname{sgn}(n) = \operatorname{sgn}(m)$, these identities translate into convolutions of hybrid lattice paths corresponding to a diagonal [4, 8]. In the case $\operatorname{sgn}(n) \neq \operatorname{sgn}(m)$, we were able to prove these identities combinatorially by applying a sign-reversing involution using hybrid lattice paths, see [4]. For brevity, the details are omitted here.

3 Specializations

In Example 2.7 we have already seen the specializations $w(s,t) = 1$ and $w(s,t) = q$. In these cases, the w-binomial coefficient corresponds to the ordinary and q-binomial coefficients, respectively. Corollaries include reflection formulae, a noncommutative binomial theorem, convolution formulae and a combinatorial model in terms of hybrid lattice paths for these specializations. We can also get corresponding corollaries for other specializations of the weights. In this section we list some interesting specializations; most of the details are omitted here but are provided in the full paper [4].

3.1 Elementary and complete symmetric functions

In [1], Damiani, D'Antona and Loeb define generalized complete homogeneous symmetric functions and elementary symmetric functions over hybrid sets. For our purposes it suffices to use the following simplified definitions. (See [1] for a more extensive discussion.) Let $\{a_1 \cdots a_n\}$ denote an n-element hybrid new set (defined in [5]) consisting of the elements $a_1, a_2, \ldots, a_n$ with multiplicity 1 if $n > 0$, consisting of the elements $a_0, a_{-1}, \ldots, a_{n+1}$ with multiplicity -1 if $n < 0$ or it is the empty set if $n = 0$. We define, for all $n, k \in \mathbb{Z}$, the *elementary symmetric function* $e_k(n) = e_k(\{a_1 \cdots a_n\})$ and the *complete homogeneous symmetric function* $h_k(n) = h_k(\{a_1 \cdots a_n\})$, provided $(n+1, k) \neq (0,0)$, by

$$e_k(n+1) = e_k(n) + a_{n+1} e_{k-1}(n) \quad \text{and} \quad h_k(n) = h_k(n-1) + a_n h_{k-1}(n)$$

with initial conditions, for $n \in \mathbb{Z}$,

$$e_0(n) = 1, \quad e_n(n) = \prod_{i=1}^{n} a_i \quad \text{and} \quad h_0(n) = 1, \quad h_n(1) = \operatorname{sgn}(n) a_1^n.$$

Note that in [1] these were only defined for $k \geq 0$. From these recurrence relations it is natural to choose as weights $w(s,t) = \frac{a_{s+t}}{a_{s+t-1}}$. In this case we have ${}_w\!\left[{n \atop k}\right] = e_k(n) \prod_{i=1}^{k} a_i^{-1}$. The second natural choice of the weights is $w(s,t) = \frac{a_{t+1}}{a_t}$. In this case we have ${}_w\!\left[{n \atop k}\right] = h_k(n-k+1) \operatorname{sgn}(k)\, a_1^{-k}$.

3.2 Generalized Stirling numbers

Here we define weighted integers ${}_w[n] := {}_w\!\left[{n \atop 1}\right]$, and consider α to be some parameter.

By generalizing the generating functions for the Stirling numbers, we define for $n \geq 0$

α-shifted weighted Stirling numbers of the first and the second kind, respectively, by

$$(t - {}_w[\alpha])(t - {}_w[\alpha+1]) \cdots (t - {}_w[\alpha+n-1]) = \sum_{k=0}^{n} s_w^\alpha(n,k)t^k \quad \text{and} \tag{3.1a}$$

$$t^n = \sum_{k=0}^{n} S_w^\alpha(n,k)(t - {}_w[\alpha])(t - {}_w[\alpha+1]) \cdots (t - {}_w[\alpha+k-1]), \tag{3.1b}$$

Note that we recover the original Stirling numbers when $\alpha = 0$ and $w(s,t) = 1$ for all $s,t \in \mathbb{Z}$ (so ${}_w[n] = n$). From (3.1), we can derive the recurrence relations

$$s_w^\alpha(n,k) = s_w^\alpha(n-1,k-1) - {}_w[\alpha+n-1]s_w^\alpha(n-1,k) \quad \text{and} \tag{3.2a}$$
$$S_w^\alpha(n,k) = {}_w[\alpha+k]S_w^\alpha(n-1,k) + S_w^\alpha(n-1,k-1). \tag{3.2b}$$

Given the initial conditions $s_w^\alpha(n,n) = S_w^\alpha(n,n) = 1$, $s_w^\alpha(n,0) = \prod_{i=1}^n(-{}_w[\alpha+i-1])$ and $S_w^\alpha(n,0) = \mathrm{sgn}(n)_w[\alpha]^n$ for $n \in \mathbb{Z}$ the recurrence relations (3.2), provided $(n,k) \neq (0,0)$, define $s_w^\alpha(n,k)$ and $S_w^\alpha(n,k)$ for all integer values of n and k. As a result, we obtain

$$\begin{aligned} s_w^\alpha(n,k) &= e_{n-k}(\{(-{}_w[\alpha]) \cdots (-{}_w[\alpha+n-1])\}), \\ S_w^\alpha(n,k) &= h_{n-k}(\{({}_w[\alpha]) \cdots ({}_w[\alpha+k])\}), \end{aligned}$$

for any $n,k \in \mathbb{Z}$.

3.3 Elliptic binomial coefficients

A very important specialization of the weights in [8], which also served as a major motivation for the present work, is the "elliptic" specialization. Define the *modified Jacobi theta function* by

$$\theta(x;p) := \prod_{j\geq 0}\left((1-p^jx)(1-p^{j+1}/x)\right), \quad \theta(x_1,\ldots,x_\ell;p) = \prod_{i=1}^{\ell}\theta(x_i;p),$$

where $x, x_1, \ldots, x_\ell \neq 0$ and $|p| < 1$. We define the theta shifted factorial as

$$(x;q,p)_k \;=\; \prod_{i=0}^{k-1} \theta(xq^i;p), \quad (x_1,x_2,\ldots,x_\ell;q,p)_k = \prod_{i=1}^{\ell}(x_i;q,p)_k,$$

where the first product is defined for all integers k by (2.2). An *elliptic function* is a function of a complex variable that is meromorphic and doubly periodic. It is well known that elliptic functions can be written as quotients of modified Jacobi theta functions [6].

We can now choose the weights

$$w_{a,b;q,p}(s,t) := \frac{\theta(aq^{s+2t}, bq^{2s+t-2}, aq^{t-s-1}/b;p)}{\theta(aq^{s+2t-2}, bq^{2s+t}, aq^{t-s+1}/b;p)}q \tag{3.3}$$

to obtain for all $n, k \in \mathbb{Z}$ the *elliptic binomial coefficients*, first defined for $n, k \geq 0$ in [7],

$$\begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} := \frac{(q^{1+k}, aq^{1+k}, bq^{1+k}, aq^{1-k}/b; q, p)_{n-k}}{(q, aq, bq^{1+2k}, aq/b; q, p)_{n-k}}. \tag{3.4}$$

Note that the elliptic versions of Corollaries 2.11 and 2.12 are equivalent to Frenkel and Turaev's ${}_{10}V_9$ summation [3] (see also [6, Theorem 2.3.1.]). This provides a combinatorial interpretation of this summation formula which is a fundamental identity in the theory of ellipic hypergeometric series.

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