Higher Lie Characters and Cyclic Descent Extension on Conjugacy Classes

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Abstract. A now-classical cyclic extension of the descent set of a permutation has been introduced by Klyachko and Cellini. Following a recent axiomatic approach to this notion, it is natural to ask which sets of permutations admit such a (not necessarily classical) extension. The main result of this paper is a complete answer in the case of conjugacy classes of permutations. It is shown that the conjugacy class of cycle type λ has such an extension if and only if λ is not of the form (r^s) for some square-free r. The proof involves a detailed study of hook constituents in higher Lie characters.

Keywords: cyclic descent, conjugacy class, symmetric group, higher Lie character

1 Introduction

Permutations, as well as standard Young tableaux, are equipped with a well-established notion of descent set. A cyclic extension of this concept was introduced in the study of Lie algebras [13] and descent algebras [5]. Surprising connections of the cyclic descent notion to a variety of mathematical areas were found later.

The *descent set* of a permutation $\pi = [\pi_1, \ldots, \pi_n]$ in the symmetric group S_n on n letters is

$$\mathrm{Des}(\pi) := \{1 \le i \le n-1 \,:\, \pi_i > \pi_{i+1}\} \quad \subseteq [n-1],$$

where $[m] := \{1, 2, \ldots, m\}$. Cellini [5] introduced a natural notion of *cyclic descent set*:

$$\mathrm{CDes}(\pi) := \{1 \le i \le n \,:\, \pi_i > \pi_{i+1}\} \quad \subseteq [n],$$

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with the convention $\pi_{n+1} := \pi_1$. The more restricted notion of cyclic descent number had been used previously by Klyachko [13]. This cyclic descent set was further studied by Dilks, Petersen and Stembridge [7] and others.

There exists an established notion of descent set for standard Young tableaux (SYT),

$$\text{Des}(T) := \{1 \le i \le n-1 : i+1 \text{ appears in a lower row of } T \text{ than } i\},$$

but it has no obvious cyclic analogue. In a breakthrough work, Rhoades [19] defined a notion of cyclic descent set for standard Young tableaux of rectangular shape. The properties common to Cellini's definition (for permutations) and Rhoades' construction (for SYT) appeared in other combinatorial settings as well [17, 16, 9]. This led to an abstract definition [3], as follows.

Definition 1.1. [3] Let $\mathcal{T}$ be a finite set, equipped with a map (called *descent map*) $\text{Des}: \mathcal{T} \longrightarrow 2^{[n-1]}$. Let $\text{sh}: 2^{[n]} \longrightarrow 2^{[n]}$ be the mapping on subsets of $[n]$ induced by the cyclic shift $i \mapsto i+1 \pmod{n}$ of elements $i \in [n]$. A *cyclic extension* of Des is a pair (cDes, p), where $\text{cDes}: \mathcal{T} \longrightarrow 2^{[n]}$ is a map and $p: \mathcal{T} \longrightarrow \mathcal{T}$ is a bijection, satisfying the following axioms: for all T in $\mathcal{T}$,

$$\begin{aligned} \text{(extension)} \quad & \text{cDes}(T) \cap [n-1] = \text{Des}(T), \\ \text{(equivariance)} \quad & \text{cDes}(p(T)) = \text{sh}(\text{cDes}(T)), \\ \text{(non-Escher)} \quad & \varnothing \subsetneq \text{cDes}(T) \subsetneq [n]. \end{aligned}$$

The term "non-Escher" refers to M. C. Escher's drawing "Ascending and Descending", which illustrates the impossibility of the cases $\text{CDes}(\pi) = \varnothing$ and $\text{CDes}(\pi) = [n]$ for permutations $\pi \in S_n$.

For connections of cyclic descents to Kazhdan–Lusztig theory see [19]; for topological aspects and connections to the Steinberg torus see [7]; for twisted Schützenberger promotion see [19, 12]; for cyclic quasi-symmetric functions and Schur-positivity see [1]; for Postnikov's toric Schur functions see [3]. The goal of this paper is to determine which conjugacy classes of the symmetric group carry a cyclic descent extension.

Cellini's cyclic descent set, denoted CDes, is a special case of a cyclic descent extension, denoted in general cDes, as attested by the following observation.

Observation 1.2. *Let* Des *and* CDes *denote the classical descent set and Cellini's cyclic descent set on permutations, respectively. Let* $p: S_n \to S_n$ *be the rotation*

$$[\pi_1, \pi_2, \ldots, \pi_{n-1}, \pi_n] \overset{p}{\longmapsto} [\pi_n, \pi_1, \pi_2, \ldots, \pi_{n-1}].$$

Then the pair (CDes, p) *is a cyclic descent extension of* Des *on* S_n *in the sense of Definition 1.1.*

Unlike the full symmetric group, for many conjugacy classes, Cellini's definition does not provide a cyclic extension.

Example 1.3. Consider the conjugacy class of 4-cycles in S_4,

$$\mathcal{C}_{(4)} = \{2341, 2413, 3142, 3421, 4123, 4312\}.$$

Cellini's cyclic descent sets are

$$\{3\}, \{2,4\}, \{1,3\}, \{2,3\}, \{1\}, \{1,2\},$$

respectively; this family is not closed under cyclic rotation. On the other hand, redefining the cyclic descent sets to be

$$\text{cDes}(2341) = \{3,4\}, \ \text{cDes}(2413) = \{2,4\}, \ \text{cDes}(3142) = \{1,3\},$$

$$\text{cDes}(3421) = \{2,3\}, \ \text{cDes}(4123) = \{1,4\}, \ \text{cDes}(4312) = \{1,2\}$$

and defining the map p by

$$2341 \rightarrow 4123 \rightarrow 4312 \rightarrow 3421 \rightarrow 2341$$

and

$$3142 \rightarrow 2413 \rightarrow 3142,$$

the pair (cDes, p) does determine a cyclic extension of Des on this conjugacy class.

The goal of this paper is to show that most conjugacy classes in S_n carry a cyclic descent extension. In fact, we obtain a full characterization.

Recall that an integer is *square-free* if no prime square divides it; in particular, 1 is square-free. Our main result is

Theorem 1.4. *Let λ be a partition of n, and let $\mathcal{C}_\lambda \subseteq S_n$ be the corresponding conjugacy class. The descent map* Des *on $\mathcal{C}_\lambda$ has a cyclic extension* (cDes, p) *if and only if λ is not of the form (r^s) for some square-free r.*

The proof of Theorem 1.4 is non-constructive and involves a detailed study of the hook constituents in higher Lie characters. In the rest of this extended abstract we describe the proof method. For a detailed full version see [2].

Acknowledgements

2   Higher Lie characters

For a partition λ of n, let $\mathcal{C}_\lambda$ be the conjugacy class consisting of all the permutations in S_n of cycle type λ, and let χ^λ denote the irreducible S_n-character corresponding to λ. Let Z_λ be the centralizer of a permutation in $\mathcal{C}_\lambda$ (defined up to conjugacy). If k_i denotes the number of parts of λ equal to i, then Z_λ is isomorphic to the direct product $\times_{i=1}^n \mathbb{Z}_i \wr S_{k_i}$. Here and in the rest of the paper $\mathbb{Z}_i$ denotes the cyclic group of order i.

   For each i, let ω_i be the linear character on $\mathbb{Z}_i \wr S_{k_i}$ indexed by the i-tuple of partitions $(\varnothing, (k_i), \varnothing, \ldots, \varnothing)$. In other words, letting ζ be a primitive irreducible character on the cyclic group $\mathbb{Z}_i$, extend it to the wreath product $\mathbb{Z}_i \wr S_{k_i}$ so that it is homogeneous on the base subgroup $\mathbb{Z}_i^{k_i}$ and trivial on the wreathing subgroup S_{k_i}. Denote this extension by ω_i. Now let

$$\omega^\lambda := \bigotimes_{i=1}^n \omega_i,$$

a linear character on Z_λ. Define the corresponding *higher Lie character* to be the induced character

$$\psi^\lambda := \omega^\lambda \uparrow_{Z_\lambda}^{S_n}.$$

   The study of higher Lie characters can be traced back to Schur [22]. An old problem of Thrall [27] is to provide an explicit combinatorial interpretation of the multiplicities of the irreducible characters in the higher Lie character, see also [23, Exercise 7.89(i)]. Only partial results are known: the case $\lambda = (n)$ was solved by Kraśkiewicz and Weyman [14]; Désarménien and Wachs [6] resolved a coarser version of Thrall's problem for the sum of higher Lie characters over all derangements, see also [18]. The best result so far is Schocker's expansion [21, Theorem 3.1], which however involves signs and rational coefficients. For recent discussions see, *e.g.*, [4, 26].

   A remarkable theorem of Gessel and Reutenauer [11, Theorem 2.1] applies higher Lie characters to describe the fiber sizes of the descent set map on conjugacy classes. It follows that higher Lie characters can be used to prove the existence of cyclic descent extensions as explained below.

3   The hook-multiplicity generating function

Recall the standard notation s_λ for the Schur function indexed by a partition λ, and $\mathcal{F}_{n,D}$ for the fundamental quasi-symmetric function indexed by a subset $D \subseteq [n-1]$

$$\mathcal{F}_{n,D}(\mathbf{x}) := \sum_{\substack{i_1 \le i_2 \le \ldots \le i_n \\ i_j < i_{j+1} \text{ if } j \in D}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

A symmetric function is called *Schur-positive* if all the coefficients in its expansion in the basis of Schur functions are non-negative. A subset $\mathcal{A} \subseteq S_n$ is *Schur-positive* if the associated quasi-symmetric function

$$\mathcal{Q}(\mathcal{A}) := \sum_{a \in \mathcal{A}} \mathcal{F}_{n,\mathrm{Des}(a)},$$

is symmetric and Schur-positive.

For an integer $0 \leq k < n$ and a Schur-positive subset $\mathcal{A} \subseteq S_n$ denote

$$m_{k,\mathcal{A}} := \langle \mathcal{Q}(\mathcal{A}), s_{(n-k,1^k)} \rangle,$$

where $s_{(n-k,1^k)}$ is the Schur function indexed by the hook partition $(n-k,1^k)$.

First we prove the following key lemma, which provides an algebraic criterion for the existence of a cyclic descent extension.

Lemma 3.1. *A Schur-positive set $\mathcal{A} \subseteq S_n$ has a cyclic descent extension if and only if the following two conditions hold:*

(divisibility) *the polynomial $\sum_{k=0}^{n-1} m_{k,\mathcal{A}} x^k$ is divisible by $1+x$;*
(non-negativity) *the quotient has non-negative coefficients.*

By the Gessel-Reutenauer theorem, for every conjugacy class $\mathcal{C}_\lambda$ the quasi-symmetric function $\mathcal{Q}(\mathcal{C}_\lambda)$ is the Frobenius image of the higher Lie character ψ^λ, thus $\mathcal{C}_\lambda$ is Schur-positive.

For a partition $\lambda \vdash n$ denote

$$m_{k,\lambda} := m_{k,\mathcal{C}_\lambda} = \langle \mathcal{Q}(\mathcal{C}_\lambda), s_{(n-k,1^k)} \rangle = \langle \psi^\lambda, \chi^{(n-k,1^k)} \rangle \qquad (0 \leq k \leq n-1)$$

and

$$M_\lambda(x) := \sum_{k=0}^{n-1} m_{k,\lambda} x^k,$$

the hook-multiplicity generating function of the higher Lie character ψ^λ.

In order to prove Theorem 1.4, we will show first that for conjugacy classes of cycle type λ, the hook-multiplicity generating function $M_\lambda(x)$ is divisible by $1+x$ if and only if $\lambda \neq (r^s)$ for any square-free integer r. Then we will show that the coefficients of the quotient $M_\lambda(x)/(1+x)$ are non-negative, whenever $\lambda \neq (r^s)$ for any square-free r.

4 Divisibility of the hook generating function

Proposition 4.1. *The hook-multiplicity generating function of the higher Lie character ψ^λ*

$$M_\lambda(x) := \sum_{k=0}^{n-1} m_{k,\lambda} x^k$$

is divisible by $1+x$ if and only if $\lambda \neq (r^s)$ for any square-free integer r.

This divisibility condition is proved using an explicit evaluation of the higher Lie character on n-cycles. By the Murnaghan–Nakayama rule [20, Lemma 4.10.3],

Lemma 4.2. *For every S_n-character ϕ, the hook-multiplicity generating function*

$$M_\phi(x) := \sum_{k=0}^{n-1} \langle \phi, \chi^{(n-k,1^k)} \rangle x^k$$

is divisible by $1 + x$ if and only if the value of ϕ on an n-cycle is zero: $\phi_{(n)} = 0$.

Letting $\phi = \psi^\mu$, the higher Lie character indexed by the partition μ, reduces Proposition 4.1 to the following character evaluation.

Recall the Möbius function $\mu(d)$, which is equal to the sum of the primitive d-th roots of 1. If d has a prime square divisor then $\mu(d) = 0$; otherwise, d is a product of k distinct primes and $\mu(d) = (-1)^k$. The following lemma is equivalent to a combinatorial identity due to Garsia, see Proposition 8.1 below. A direct algebraic proof is given in [2].

Lemma 4.3. *For $\lambda \vdash n$*

$$\psi^\lambda_{(n)} = \begin{cases} \mu(r), & \text{if } \lambda = (r^s); \\ 0, & \text{otherwise} , \end{cases} \tag{4.1}$$

where $\mu(r)$ is the Möbius function.

Proof of Proposition 4.1. By Lemma 4.2, $1 + x$ divides the hook-multiplicity generating function of the higher Lie character ψ^λ if and only if $\psi^\lambda_{(n)} = 0$. Lemma 4.3 completes the proof. $\qquad\square$

In the following sections we will prove the non-negativity of the coefficients of the quotient $\frac{M_\lambda(x)}{1+x}$ for partitions λ which are not equal to (r^s) for any square-free r.

5   Non-negativity: the case of more than one cycle length

Consider, first, the case of conjugacy classes with more than one cycle length. This is the easiest case to handle. In that case, we apply a factorization of the associated higher Lie character ψ^λ to prove

Lemma 5.1. *Let $\lambda = (r^s) \sqcup \mu$ be a partition of n, where $n > rs$ and μ is a partition of $n - rs$ with no part equal to r. Then*

$$\frac{M_\lambda(x)}{1+x} = M_{(r^s)}(x)M_\mu(x). \tag{5.1}$$

and its coefficients are thus non-negative.

The core of the proof of Theorem 1.4 is the case $\lambda = (r^s)$.

6 Non-negativity: the n-cycle case

6.1 A variant of the Witt transform

The greatest common divisor of two integers i, j is denoted by (i, j). Recall the Möbius function $\mu(d)$. We shall use here r instead of n, with an eye to the sequel.

Proposition 6.1. *For every* $0 \leq j \leq r$

$$\langle \psi^{(r)}, \chi^{(1^j) \oplus (r-j)} \rangle = \sum_{d | (r,j)} \frac{\mu(d)(-1)^{j+j/d}}{r} \binom{r/d}{j/d} \qquad (0 \leq j \leq r). \tag{6.1}$$

Remark 6.2. Proposition 6.1 is not new, see Section 8 below. Also, it is a special case of Proposition 7.1 below at $s = 1$.

Denote

$$f_j := \langle \psi^{(r)}, \chi^{(1^j) \oplus (r-j)} \rangle \qquad (0 \leq j \leq r) \tag{6.2}$$

and

$$F(x) = f_0 + f_1 x + f_2 x^2 + \cdots f_{r-1} x^{r-1} + f_r x^r.$$

It is easy to see that $f_0 = m_{0,(r)}$, $f_r = m_{r-1,(r)}$ and

$$f_j = m_{j-1,(r)} + m_{j,(r)} \qquad (1 \leq j \leq r-1),$$

so that $F(x) = (1 + x) M_{(r)}(x)$. Also, by Proposition 6.1,

$$
\begin{aligned}
F(x) &= \sum_j x^j \sum_{d | (r,j)} \frac{\mu(d)(-1)^{\frac{j(d+1)}{d}}}{r} \binom{r/d}{j/d} = \sum_{d | r} \frac{\mu(d)}{r} \sum_{k=0}^{r/d} (-1)^{k(d+1)} \binom{r/d}{k} x^{kd} \\
&= \sum_{d | r} \frac{\mu(d)}{r} (1 - (-x)^d)^{r/d}.
\end{aligned}
\tag{6.3}
$$

Recall from [15] that the r-th Witt transform of a polynomial $p(x)$ is defined by

$$\mathcal{W}_p^{(r)}(x) = \frac{1}{r} \sum_{d | r} \mu(d) p(x^d)^{r/d}.$$

In our case put $p(x) = 1 - x$ to get $F(x) = \mathcal{W}_p^{(r)}(-x)$. The proofs of Theorem 4 and Lemma 1 of [15] could have been used to prove that the coefficients of $F(x)$ are non-negative integers. However, this property of f_i also follows from its interpretation as an inner product of two characters.

We want to prove that the polynomial $M_{(r)}(x)/(1 + x) = F(x)/(1 + x)^2$ has non-negative coefficients.

6.2   Unimodality

A sequence $a_0, \ldots, a_n$ of real numbers is called *unimodal* if there exists an index $0 \leq i_0 \leq n$ such that the sequence is weakly increasing ($a_i \leq a_{i+1}$) for $i < i_0$ and weakly decreasing ($a_i \geq a_{i+1}$) for $i \geq i_0$.

Observation 6.3. *Let $a(x) = a_0 + a_1 x + \ldots + a_n x^n$ be a polynomial with real, non-negative and unimodal coefficients. Assume that $1 + x$ divides $a(x)$, and let $b(x) := a(x)/(1 + x)$. Then the coefficients of $b(x)$ are non-negative.*

The explicit description of the coefficients of $(1 + x)M_{(n)}(x)$ given in Proposition 6.1, combined with Equation (6.3), is applied to prove the following.

Proposition 6.4. *For every positive integer n the sequence $m_{0,(n)}, m_{1,(n)}, \ldots, m_{n-1,(n)}$ is unimodal.*

For a detailed proof, see [2].

By Lemmas 4.1 and 4.2, if n is not square-free then $1 + x$ divides $M_{(n)}(x)$.

Proposition 6.5. *If n is not square-free then the coefficients of $M_{(n)}(x)/(1 + x)$ are non-negative.*

Proof. Combine Proposition 6.4 with Observation 6.3.   □

7   Non-negativity: the case of cycle type (r^s)

In this section we consider the case $\mu = (r^s)$. We fix r, while s and hence $n = rs$ vary.

As in the previous section, instead of the hook-multiplicities $m_{k,s} = \langle \psi^{(r^s)}, \chi^{(n-k,1^k)} \rangle$, it will be easier to work with their consecutive sums $e_k = e_{k,s} := m_{k,s} + m_{k-1,s}$, which also have an inner product interpretation. For given i, r and s, let

$$P_{r,s}(i) := \{\gamma = (\gamma_1, \ldots, \gamma_s) \mid \sum_{\ell=1}^{s} \gamma_\ell = i, \, r \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_s \geq 0\} \qquad (7.1)$$

denote the set of all partitions of i into at most s parts, each of size at most r. Denote the multiplicity of j in $\gamma \in P_{r,s}(i)$ by $k_j(\gamma) := |\{1 \leq \ell \leq s \mid \gamma_\ell = j\}|$.

The main result of this section is the following extension of Proposition 6.1 to $s \geq 1$.

Proposition 7.1. *For every $s \geq 1$ and $i \geq 0$ we have*

$$e_i = \langle \psi^{(r^s)}, \chi^{(1^i) \oplus (n-i)} \rangle = \sum_{\gamma \in P_{r,s}(i)} \prod_{j \geq 0} (-1)^{(j+1)k_j(\gamma)} \binom{(-1)^{j+1} f_j}{k_j(\gamma)}$$

$$= \sum_{\substack{k_0, \ldots, k_r \geq 0 \\ \sum_j k_j = s \\ \sum_j j k_j = i}} \prod_{j=0}^{r} (-1)^{(j+1)k_j} \binom{(-1)^{j+1} f_j}{k_j}.$$
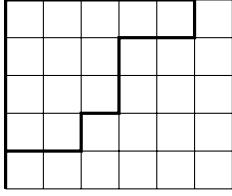
Figure 1: $\gamma = (5,3,3,2,0)$ in $P_{6,5}(13)$. The multiplicities of the parts are $k_1(\gamma) = 0$, $k_2(\gamma) = 1$, $k_3(\gamma) = 2$, $k_4(\gamma) = 0$, $k_5(\gamma) = 1$, and $k_6(\gamma) = 0$.

In particular, for $s = 1$ we have $e_i = f_i$.

We derive a formal power series product form of Proposition 7.1.

Corollary 7.2. *Let $E(x,y) = E_r(x,y)$ denote the formal power series in which the coefficient of $x^i y^s$ is $e_{i,s} = \langle \psi^{(r^s)}, \chi^{(1^i) \oplus (n-i)} \rangle$. Then*

$$E(x,y) = \prod_{j=0}^{r}(1 - (-x)^j y)^{(-1)^{j+1} f_j}. \tag{7.2}$$

For a fixed positive integer r, we consider the formal power series $M_r(x,y) := \sum_{i,s} m_{i,s}^{(r)} x^i y^s$, where $m_{i,s}^{(r)} := \langle \psi^{(r^s)}, \chi^{(rs-i,1^i)} \rangle$.

Theorem 7.3. *If r is not square-free then the polynomial*

$$\frac{M_r(x,y)}{1+x}$$

has non-negative integer coefficients.

Proof. Recall the notation $F(x) := \sum_j f_j x^j$. We may write $F(x) = (1+x)^2 \sum_i g_i x^i$, where, by Proposition 6.5, the coefficients $g_i = \sum_{i \leq j}(-1)^{i-j} m_j$ are non-negative.

Equation (7.2) can be rewritten in terms of the g_i's.

$$E(x,y) = \prod_{i \geq 0} \left(\frac{(1 - (-x)^i y)(1 - (-x)^{i+2} y)}{(1 - (-x)^{i+1} y)^2} \right)^{(-1)^{i+1} g_i}. \tag{7.3}$$

It suffices to show that all the factors in (7.3) are of the form $1 + (1+x)^2 p(x,y)$, with $p(x,y)$ a power series with non-negative coefficients. This can be achieved by separate arguments for even and odd indices i. □

Now we are ready to prove the main theorem.

Proof of Theorem 1.4. Combine Lemma 3.1 with Proposition 4.1, Lemma 5.1 and Theorem 7.3. □

8 Final remarks and open problems

Recall the major index of a permutation $\pi \in S_n$, $\mathrm{maj}(\pi) := \sum\limits_{i \in \mathrm{Des}(\pi)} i$. Let ζ be a primitive n-th root of unity. The following identity was proved by Garsia [10, Equation 5.8].

$$\sum_{\pi \in \mathcal{C}_\lambda} \zeta^{\mathrm{maj}(\pi)} = \begin{cases} \mu(r) & \text{if } \lambda = (r^s), \\ 0 & \text{otherwise,} \end{cases} \qquad \text{(for all } \lambda \vdash n\text{)}, \tag{8.1}$$

where $\mu(r)$ is the Möbius function. A purely combinatorial proof was given in [28] by Wachs. By [24, Lemma 3.4], we have the following result.

Proposition 8.1. *For every partition* $\lambda \vdash n$, $\psi^\lambda_{(n)} = \sum\limits_{\pi \in \mathcal{C}_\lambda} \zeta^{\mathrm{maj}(\pi)}$.

Corollary 8.2. *Equation* (4.1) *is equivalent to Equation* (8.1).

Proposition 6.1 is equivalent to the following equation

$$|\{\pi \in C_{(n)} : \mathrm{Des}(\pi) = [j]\}| = \frac{1}{n} \sum_{d \mid n} \mu(d)(-1)^{j - \lceil \frac{j}{d} \rceil} \binom{n-1}{j-1} \qquad (0 \le j < n),$$

which is an immediate consequence of [8, Theorem 3.1]. An older proof was presented to us by Sheila Sundaram, deducing Proposition 6.1 from [25, Lemma 2.7]. By the Kraśkiewicz–Weyman Theorem [10, Theorem 8.4],

Proposition 8.3. *For every* $0 \le k \le n$, *the multiplicity* $m_k = \langle \psi^{(n)}, \chi^{(n-k,1^k)} \rangle$ *is equal to the cardinality of the set*

$$\{1 \le a_1 < a_2 < \cdots < a_k \le n-1 \mid \sum_{i=1}^{k} a_i \equiv 1 \ (\mathrm{mod}\ n)\}.$$

Corollary 8.4. *Proposition* 6.1 *is equivalent to the following identity: For every* $0 \le k \le n$,

$$|\{1 \le a_1 < a_2 < \cdots < a_k \le n \mid \sum_{i=1}^{k} a_i \equiv 1 \ (\mathrm{mod}\ n)\}| = \sum_{d \mid (n,k)} \frac{\mu(d)(-1)^{k+k/d}}{n} \binom{n/d}{k/d}.$$

It remains a challenge to find such a direct link between Schocker's general description of the multiplicity and our version in Proposition 7.1.

Our proof of Theorem 1.4 is not constructive.

Problem 8.5. *Find an explicit combinatorial description of a cyclic descent extension on conjugacy classes of cycle type* λ, *not equal to* (r^s) *for any square-free* r.

By Proposition 6.4, the *hook-multiplicity sequence* $m_{0,(n)}, \ldots, m_{n-1,(n)}$ is unimodal.

Conjecture 8.6. *For every partition* $\lambda \vdash n$, *the hook-multiplicity sequence* $(m_{0,\lambda}, \ldots, m_{n-1,\lambda})$ *is unimodal.*

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