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Positive Tropical Flags and the Positive Tropical Dressian

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Abstract. We study the totally non-negative part of the complete flag variety and of its tropicalization. We start by showing that Lusztig's notion of non-negative complete flag variety coincides with the flags in the complete flag variety which have non-negative Plücker coordinates. This mirrors the characterization of the totally non-negative Grassmannian as those points in the Grassmannian with all non-negative Plücker coordinates. We then study the tropical complete flag variety and complete flag Dressian, which are two tropical versions of the complete flag variety, capturing realizable and abstract flags of tropical linear spaces, respectively. The complete flag Dressian properly contains the tropical complete flag variety. However, we show that the totally non-negative parts of these spaces coincide.

Keywords: flag varieties, tropical varieties, total positivity, Dressian

1 Introduction

The *Grassmannian* of k-planes in n-space describes k dimensional linear subspaces in n dimensional space. It is an algebraic variety cut out by the *Plücker relations*. We can *tropicalize* these relations to obtain the *tropical Plücker relations*. The set of points satisfying the tropical Plücker relations, called the *Dressian*, is the parameter space of abstract tropical linear spaces [16]. The set of points satisfying the tropicalizations of all polynomials in the ideal generated by the Plücker relations, called the *tropical Grassmannian*, is the parameter space of realizable tropical linear spaces [5]. In general, the Dressian properly contains the tropical Grassmannian (see, for instance, [6]). However, in [19], it is shown that if we restrict to positive solutions, for an appropriate notion of positivity, the situation is simpler: the *positive Dressian* equals the *positive tropical Grassmannian*. More explicitly, this means that a positive solution to the tropicalizations of the Plücker relations is also a positive solution to the tropicalization of any polynomial in the ideal

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generated by the Plücker relations. Our goal is to generalize this fact to the setting of the complete flag variety.

The *complete flag variety*, Fl_n, is the set of complete flags of linear subspaces $\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = \mathbb{R}^n$. Any point of this variety is determined by a set of coordinates called its *Plücker coordinates*. These are cut out by the *incidence-Plücker relations*, a set of polynomials which extends the Plücker relations, which generate an ideal called the *incidence-Plücker ideal*. We consider the set of points satisfying the tropicalizations of the incidence-Plücker relations, called the *complete flag Dressian*, $FlDr_n$, and the set of points satisfying the tropicalizations of all polynomials in the incidence-Plücker ideal, called the *tropical complete flag variety*, $TrFl_n$. These parameterize abstract flags of tropical linear spaces and realizable flags of tropical linear spaces, respectively [3].

The tropical spaces $FlDr_n$ and $TrFl_n$ are generally different [3]. Motivated by the example of the tropical Grassmannian, we will investigate the totally non-negative (TNN) parts of these spaces. We define the *totally non-negative complete flag Dressian* to be the set of simultaneous positive solutions to the tropicalizations of the incidence-Plücker relations and the *totally non-negative tropical complete flag variety* to be the set of simultaneous positive solutions to the tropicalizations of all the polynomials in the incidence-Plücker ideal. Our main result, Theorem 4.9, says the following:

Theorem. *The TNN tropical complete flag variety, $TrFl_n^{\geq 0}$, equals the TNN complete flag Dressian, $FlDr_n^{\geq 0}$.*

A number of authors, among them [20], [8] and [11], have proven that the TNN Grassmannian, in the sense of Lusztig [9], consists precisely of points in the Grassmannian where each Plücker coordinate is non-negative. We extend this result to the setting of the complete flag variety. Specifically, in proving theorem Theorem 4.9, we will need to carefully study the *totally non-negative complete flag variety*, denoted $Fl_n^{\geq 0}$. A construction based on the parameterization of $Fl_n^{\geq 0}$ by Marsh and Rietsch [12] will allow us to understand explicitly the Plücker coordinates $\{P_I(F)\}_{I\subset[n]}$ of an arbitrary flag F in $Fl_n^{\geq 0}$. In Theorem 3.15, we show:

Theorem. *The TNN complete flag variety $Fl_n^{\geq 0}$ equals the set $\{F \in Fl_n | \ P_I(F) \geq 0$ for all $I \subset [n]\}$.*

We have learned recently that this result has been independently proven in [1], where they show moreover that the only partial flag variety for which this theorem holds are those where the dimensions of the constituent subspaces are consecutive integers. This includes $FL_n^{\geq 0}$, with constituent dimensions $\{1, 2, \ldots, n\}$, and the TNN Grassmannian of k planes in n space, with constituent dimension $\{k\}$.

The structure of this extended abstract is as follows: In section 2, we introduce the TNN complete flag variety. In section 3, we give a parameterization of this space and study its Plücker coordinates. In section 4, we introduce two tropicalizations of the complete flag variety and demonstrate that the TNN parts of these spaces are equal.

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2 The Totally Non-Negative Complete Flag Variety

Definition 2.1. The **complete flag variety** Fl_n is the collection of all **complete flags** in $\mathbb{R}^n$, which are collections $(V_i)_{i=0}^n$ of linear subspaces satisfying $\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = \mathbb{R}^n$.

We first observe that Fl_n is a multi-projective variety. We can represent a flag $(V_i)_{i=1}^n$ by a full rank n by n matrix M such that V_i equals the span of the topmost i rows of M. Let GL_n be the group of invertible n by n matrices and $SL(n, \mathbb{R})$ be the *special linear group* of real matrices with determinant 1. Let B_- be the *Borel subgroup* of GL_n consisting of lower triangular matrices. One can check that two matrices M and M' represent the same flag if and only if they are related by left multiplication by some $B \in B_-$. Thus, we can think of the complete flag variety as $Fl_n = \{B_- u | u \in SL(n, \mathbb{R})\}$, where a flag in Fl_n represented by a matrix u is identified with the set $B_- u$.

For $I \subset [n] = \{1, \ldots, n\}$ and M an n by n matrix, the *Plücker coordinate* (or *flag minor*) $P_I(M)$ is the determinant of the submatrix of M in rows $\{1, 2, \ldots, |I|\}$ and columns I. To any flag F, associate the Plücker coordinates $(P_I(F))_{I \subset [n]}$, defined to be the Plücker coordinates of any matrix representative of that flag. By [13, Proposition 14.2], this is an embedding of Fl_n in $\mathbb{RP}^{\binom{n}{1}-1} \times \cdots \times \mathbb{RP}^{\binom{n}{n-1}-1}$. The Plücker coordinates of flags in Fl_n are cut out by multi-homogeneous polynomials, as shown in the following definition and theorem. Note that we will use shorthand such as $(S \setminus ab) \cup cd$ in place of $(S \setminus \{a, b\}) \cup \{c, d\}$.

Definition 2.2 ([4]). Consider $\mathbb{RP}^{\binom{n}{1}-1} \times \cdots \times \mathbb{RP}^{\binom{n}{n-1}-1}$, with coordinates indexed by proper subsets of $[n]$. For $1 \leq r \leq s \leq n$, the **incidence-Plücker relations** for indices of size r and s are

$$\mathfrak{P}_{r,s;n} = \left\{ \sum_{j \in J \setminus I} sign(j, I, J) P_{I \cup j} P_{J \setminus j} \,\middle|\, I \in \binom{n}{r-1}, J \in \binom{n}{s+1} \right\}, \tag{2.1}$$

where $sign(j, I, J) = (-1)^{|\{k \in J | k < j\}| + |\{i \in I | j < i\}|}$.

The full set of incidence-Plücker relations is given by $\mathfrak{P}_{IP;n} = \bigcup_{1 \leq r \leq s \leq n} \mathfrak{P}_{r,s;n}$. The ideal generated by $\mathfrak{P}_{IP;n}$, denoted $I_{IP;n}$, is called the **incidence-Plücker ideal**.

Note that the above definition allows for the option of $r = s$. The incidence-Plücker relations for which $r = s$ are called the (Grassmann) **Plücker relations**.

Theorem 2.3 ([4, Section 9, Proposition 1])**.** *Let* $P \in \mathbb{RP}^{\binom{n}{1}-1} \times \cdots \times \mathbb{RP}^{\binom{n}{n-1}-1}$. *Then* $P = P(F)$ *for some* $F \in Fl_n$ *if and only if* P *satisfies the incidence-Plücker relations* $\mathfrak{P}_{IP;n}$.

In particular, this means the incidence-Plücker relations are precisely the relations between the topmost minors of a generic full rank matrix.

Lusztig introduced the notion of non-negativity for flag varieties. We outline here the definition of the *totally non-negative complete flag variety*, following [10]. We work in type A and so the appropriate simplifications will be made in presenting the definition. Let s_i be the transposition $(i, i+1)$ in the symmetric group S_n and let w_0 be the longest permutation in S_n. For $1 \leq k < n$, let $x_k(a)$ be the n by n matrix which is the identity matrix with an a added in row k of column $k+1$. Explicitly,

$$x_k(a) = \begin{array}{c|cccccc|} & & & k & k+1 & & \\ \hline & 1 & & & & & \\ & & \ddots & & & & \\ k & & & 1 & a & & \\ k+1 & & & 0 & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{array},$$

where unmarked off-diagonal matrix entries are 0.

Definition 2.4 ([9])**.** Let $N = \binom{N}{2}$. Pick $(i_1, i_2, \ldots, i_N)$ such that $s_{i_1} \cdots s_{i_N} = w_0$. Then let

$$U^+_{>0} = \left\{ x_{i_1}(a_1) \cdots x_{i_N}(a_N) \,\middle|\, a_i \in \mathbb{R}_{>0} \text{ for all } i \right\}$$

This definition is independent of the choice of sequence $(i_1, \ldots, i_N)$.

Definition 2.5 ([9])**.** Let $\mathcal{B}_{>0} = \{B_- u | u \in U^+_{>0}\} \subset Fl_n$. The **totally non-negative complete flag variety** (of type A), $Fl_n^{\geq 0}$, is the closure of $\mathcal{B}_{>0}$.

3 Parameterization of the TNN Complete Flag Variety

3.1 The Marsh–Rietsch Parameterization

As shown by Rietsch [15], $Fl_n^{\geq 0}$ is a cell complex, whose cells $\mathcal{R}^{>0}_{v,w}$ are indexed by pairs of permutations $v \leq w$ in the Bruhat order on S_n. Each such $\mathcal{R}^{>0}_{v,w}$ is given an explicit parameterization in [12]. We will describe this parameterization here, making some choices that in principle are arbitrary but will be convenient for our purposes, and invite the reader to look at the above references for full generalities.

Any permutation w in S_n can be written as a product of simple transpositions s_i, called an *expression* for w. The *length* of w, $\ell(w)$, is the fewest number of transpositions in any expression for w. An expression for w consisting of $\ell(w)$ transpositions is called *reduced*. Let $\boldsymbol{w} = s_{i_1}s_{i_2}\cdots s_{i_k}$ be a reduced expression for w. If $v \leq w$ in the Bruhat order, then there is a reduced *subexpression* $\boldsymbol{v} = s_{i_{j_1}}s_{i_{j_2}}\cdots s_{i_{j_m}}$ for v in $\boldsymbol{w}$, where $1 \leq j_1 < j_2 < \cdots < j_m \leq k$. We will be interested in a special choice of subexpression which is called the *positive distinguished subexpression*. Intuitively, this can be thought of as the leftmost subexpression.

Definition 3.1. Let $v \leq w$. Choose a a reduced expression $\boldsymbol{w} = s_{i_1}s_{i_2}\cdots s_{i_k}$ for w and a reduced subexpression $\boldsymbol{v} = s_{i_{j_1}}\cdots s_{i_{j_m}}$ for v in $\boldsymbol{w}$. Then $\boldsymbol{v}$ is a **positive distinguished subexpression** if whenever $\ell(s_{i_p}s_{i_{j_r}}\cdots s_{i_{j_m}}) < \ell(s_{i_{j_r}}\cdots s_{i_{j_m}})$ for $j_{r-1} \leq p < j_r$, we have $p = j_{r-1}$.

Lemma 3.2 ([12, Lemma 3.5])**.** *For every $v \leq w$, and every reduced expression $\boldsymbol{w}$ of w, there is a unique positive distinguished subexpression for v in $\boldsymbol{w}$.*

Example 3.3. Let $n = 4$. Set $\boldsymbol{w} = s_1s_2s_3s_1s_2s_1$ and $v = s_1s_2s_1$. The leftmost subexpression for v in $\boldsymbol{w}$ is $j_1 = 1$, $j_2 = 2$ and $j_3 = 4$. Indeed, this choice satisfies the definition.

For $1 \leq k < n$, let $\dot{s}_k$ be the n by n identity matrix with the 2×2 submatrix in rows $\{k, k+1\}$ and columns $\{k, k+1\}$ replaced by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Explicitly,

$$\dot{s}_k = \begin{array}{c} \\ \\ \\ k \\ k+1 \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccc} & & k & k+1 & & \end{array} \\ \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & 1 & & \\ & & -1 & 0 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \end{array},$$

where unmarked off-diagonal matrix entries are 0.

We will describe each cell of $Fl_n^{\geq 0}$ as a product of matrices of the form x_k and $\dot{s}_{k'}$.

Definition 3.4. Fix $v \leq w$ in the Bruhat order. Fix a vector $\boldsymbol{a} \in \mathbb{R}^{\ell(w)-\ell(v)}$. Consider the reduced expression $\boldsymbol{w_0} = (s_1s_2\cdots s_{n-1})(s_1s_2\cdots s_{n-2})\cdots(s_1s_2)(s_1)$ for w_0, the longest permutation in the Bruhat order in S_n.[1] Choose the positive distinguished subexpression

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[1] This choice of expression is arbitrary in the context of the Marsh–Rietsch parameterization, but plays an important role in the proofs underlying later results in this abstract.

$\boldsymbol{w}$ for w in $\boldsymbol{w_0}$, and the positive distinguished subexpression $\boldsymbol{v}$ for v in $\boldsymbol{w}$, and write them as $\boldsymbol{w} = s_{i_1} \cdots s_{i_k}$ and $\boldsymbol{v} = s_{i_{j_1}} \cdots s_{i_{j_m}}$, respectively. Let $J = \{j \mid j = j_t \text{ for some } t\}$. In other words, J are those indices which correspond to transpositions that are used in $\boldsymbol{v}$. Then set

$$M_{v,w}(\boldsymbol{a}) := M_1 \cdots M_k, \quad \text{where} \quad M_j = \begin{cases} \dot{s}_{i_j}, & j \in J \\ x_{i_j}(a_j), & j \notin J \end{cases}.$$

Theorem 3.5 (Marsh–Rietsch Parametrization [12])**.** *Each cell $\mathcal{R}^{>0}_{v,w}$ of $Fl^{\geq 0}_n$ can be parameterized as*

$$\mathcal{R}^{>0}_{v,w} = \left\{ M_{v,w}(\boldsymbol{a}) \,\middle|\, \boldsymbol{a} \in \mathbb{R}^{\ell(w)-\ell(v)}_{>0} \right\}$$

In particular, each flag $F \in Fl^{\geq 0}_n$ is uniquely represented in some unique $\mathcal{R}^{>0}_{v,w}$. Moreover, each $\mathcal{R}^{>0}_{v,w}$ is a cell, meaning it is homeomorphic to an open ball.

Example 3.6. Let $n = 4$, $\boldsymbol{w} = s_1 s_3 s_2 s_1$ and $v = s_2$. The positive distinguished subexpression for v in w is the subexpression where $j_1 = 3$, so $J = \{3\}$. Thus, $M_1 = x_1(a_1)$, $M_2 = x_3(a_2)$, $M_3 = \dot{s}_2$ and $M_4 = x_1(a_3)$. The cell of the non-negative flag variety corresponding to $v \leq w$ is represented by matrices of the form

$$M = M_1 M_2 M_3 M_4 = \begin{pmatrix} 1 & a_3 & a_1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & a_2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where the a_i range over all positive real numbers.

We now give a useful property of the cells $\mathcal{R}^{>0}_{v,w}$.

Lemma 3.7. *Each cell $\mathcal{R}^{>0}_{v,w}$ of $Fl^{\geq 0}_n$ consists entirely of flags for which some fixed collection of Plücker coordinates is strictly positive and the rest are* 0*.*

3.2 Extremal Non-Zero Plücker Coordinates

We define a special subset of the Plücker coordinates of a flag which we call *extremal non-zero Plücker coordinates*. The set of indices of the extremal non-zero Plücker coordinates of a flag in $Fl^{\geq 0}_n$ will depend only on which cell $\mathcal{R}^{>0}_{v,w}$ that flag lies in. Further, in any given cell of $Fl^{\geq 0}_n$, the extremal non-zero Plücker coordinates will be chosen such that they determine all of the other Plücker coordinates.

For any $1 \leq k < n$ and any $P \in \mathbb{RP}^{\binom{n}{1}-1} \times \cdots \times \mathbb{RP}^{\binom{n}{n-1}-1}$, we define a map $\Xi_P \colon \binom{[n]}{k} \to \binom{[n]}{k}$. Intuitively, when applied to the index of a non-zero Plücker coordinate I, this map finds the largest member of I that can be increased without making the corresponding Plücker coordinate 0 and increases it maximally. Explicitly, given I, define $b = \max_{i \in I} \left\{ i \mid \text{there exists } j,\ i < j \notin I,\ P_{(I \setminus i) \cup j} \neq 0 \right\}$, if that set is non-empty. Otherwise, say b does not exist. If b exists, define $a = \max_{j \notin I} \left\{ j \mid P_{(I \setminus b) \cup j} \neq 0 \right\}$. Then,

$$\Xi_P(I) = \begin{cases} (I \setminus b) \cup a & \text{if } I \text{ is the index of a non-zero Plücker coordinate and } b \text{ exists,} \\ I & \text{otherwise.} \end{cases}$$

The indices of non-zero Plücker coordinates with index of some fixed size can be seen as the bases of a matroid. In this light, Ξ_P acts by basis exchange. Also note that for a TNN flag F, the map $\Xi_{P(F)}$ depends only on the cell $\mathcal{R}_{v,w}$ in which F lies by Lemma 3.7.

The extremal non-zero Plücker coordinates will be indexed by certain Ξ orbits. To properly define them, we first need a preliminary result on matroids:

Definition 3.8. The **Gale order** on subsets of $[n]$ of size k is a partial order such that, if $I = \{i_1 < \cdots < i_k\}$ and $J = \{j_1 < \cdots < j_k\}$, then we say $I \leq J$ if $i_r \leq j_r$ for every $r \in [k]$.

Lemma 3.9 ([2, Theorem 1.3.1])**.** *Any matroid has a unique Gale minimal basis.*

Note that the Gale minimal basis referenced in the previous lemma must simply be the lexicographically minimal and maximal bases, respectively.

Definition 3.10. Given a set of Plücker coordinates $\{P_I\}$ of a flag, let I_k be the Gale minimal index of size k such that $P_{I_k} \neq 0$. The set of indices of the **extremal non-zero Plücker coordinates** (referred to as **extremal indices**) of a point P in $\mathbb{RP}^{\binom{n}{1}-1} \times \cdots \times \mathbb{RP}^{\binom{n}{n-1}-1}$ is the set consisting of those indices which are in the Ξ_P orbit of I_k for some $k \in [n-1]$.

If F is a TNN flag, the extremal indices of the Plücker coordinates $P(F)$ depend only on the cell $\mathcal{R}^{>0}_{v,w}$ in which F lies, since $\Xi_{P(F)}$ depends only on the cell in which F lies.

Example 3.11. Let $a, b, c, d, e \in \mathbb{R}_{>0}$ and consider

$$M = \begin{pmatrix} 1 & a+e & ab+ad & abc \\ 0 & 1 & b+d & bc \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The minors of this matrix with indices of size 2 are all positive except for $P_{34} = 0$. Thus, the non-zero Plücker coordinate with Gale minimal index of size 2 is P_{12}. Then, $\Xi_{P(M)}(12) = 14$, replacing the 2 with a 4. Next, $\Xi_{P(M)}(14) = 24$, replacing the 1 with a 2. Thus, $P_{12} = 1$, $P_{14} = bc$ and $P_{24} = bce$ are the extremal non-zero Plücker coordinates of size 2 of this flag.

The next theorem highlights the importance of the extremal Plücker coordinates.

Theorem 3.12. *For any flag F with non-negative Plücker coordinates, the extremal non-zero Plücker coordinates of F uniquely determine the other non-zero Plücker coordinates of F by three-term incidence-Plücker relations.*

3.3 Plücker Coordinates of the TNN Flag Variety

Now, given a set of extremal non-zero Plücker coordinates for a flag lying in $\mathcal{R}_{v,w}^{>0}$, we want to understand how to construct a set of parameters a_i for which Theorem 3.5 yields a matrix agreeing with those coordinates.

Theorem 3.13. *For any $v \leq w$ with $r = \ell(w) - \ell(v)$, let $\Psi_{v,w}\colon \mathcal{R}_{v,w}^{>0} \to \mathbb{R}^r$ be the map $M_{v,w}(\boldsymbol{a}) \mapsto \boldsymbol{a}$, in the notation of Theorem 3.5. The map $\Psi_{v,w}$ consists of Laurent monomials in the extremal Plücker coordinates.*

In fact, by studying the relations between extremal Plücker coordinates, we can say something stronger.

Theorem 3.14. *Let S be any maximal algebraically independent subset of the extremal Plücker coordinates of $\mathcal{R}_{v,w}^{>0}$. The map $\Psi_{v,w}$, defined as above, can be expressed as Laurent monomials in the coordinates contained in S.*

We can use this theorem to prove the following, which is one of our main results:

Theorem 3.15. *The TNN flag variety defined in Definition 2.5 is precisely the set of flags with non-negative Plücker coordinates. In other words, $Fl_n^{\geq 0} = \{F \in Fl_n | \ P_I(F) \geq 0 \text{ for all } I \subset [n]\}$.*

It is shown in [7] that any flag in $Fl_n^{\geq 0}$ has non-negative Plücker coordinates. We now outline the strategy used to obtain the converse.

Definition 3.16. A (complete) **flag matroid** on a ground set E of size n is a sequence of matroids $\boldsymbol{\mathcal{M}} = (\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_{n-1})$ on the ground set E with the rank of $\mathcal{M}_i$ equal to i, called **constituent matroids**, such that for any $j < k$,

- each basis of $\mathcal{M}_j$ is contained in some basis of $\mathcal{M}_k$.
- each basis of $\mathcal{M}_k$ contains some basis of $\mathcal{M}_j$.

We identify a flag matroid with the collection of bases of its constituent matroids, collectively referred to as the *bases of the flag matroid*. Note that the indices of non-zero Plücker coordinates of an invertible square matrix are easily seen to form a flag matroid.

Definition 3.17. A flag matroid on $[n]$ is **realizable** if its bases are the non-zero Plücker coordinates of some $F \in Fl_n$.

We now define two types of flag positroid, mirroring the apparent difference between a flag in $Fl_n^{\geq 0}$ by Definition 2.5 and a flag with non-negative Plücker coordinates.

Definition 3.18. A **realizable flag positroid** on $[n]$ is the set of indices of non-zero Plücker coordinates of a flag $F \in Fl_n^{\geq 0}$ (as per Definition 2.5). A **synthetic flag positroid** on $[n]$ is the set of indices of non-zero Plücker coordinates of a flag F satisfying $P_I(F) \geq 0$ for all $I \subset [n]$.

A priori, one may expect that there could be more synthetic flag positroids than realizable flag positroids, but this is not the case.

Theorem 3.19. *The set of synthetic flag positroids on $[n]$ equals the set of realizable flag positroids on $[n]$.*

Proof of Theorem 3.15. Note that by Lemma 3.7, the realizable flag positroid arising from the non-zero Plücker coordinates of a TNN flag only depends on which cell $\mathcal{R}_{v,w}^{>0}$ that flag lies in. Thus, we can associate a cell $\mathcal{R}_{v,w}^{>0}$ to any realizable flag positroid. Let F be a flag whose Plücker coordinates P are all non-negative. Let $\boldsymbol{\mathcal{M}}$ be the synthetic (equivalently, realizable) flag positroid which has $I \subset [n]$ as a basis if and only if $P_I > 0$. As above, let $\mathcal{R}_{v,w}^{>0}$ be the cell associated to $\boldsymbol{\mathcal{M}}$. To prove Theorem 3.15, we are left to show that $F \in \mathcal{R}_{v,w}^{>0}$. By Theorem 3.14, $\Psi_{v,w}$ can be defined purely in terms of an algebraically independent subset of the extremal Plücker coordinates. Thus, one can apply $\Psi_{v,w}$ to the extremal coordinates of F and $M_{v,w}(\Psi_{v,w}(F))$ is a flag in $\mathcal{R}_{v,w}^{>0}$ which has the same extremal Plücker coordinates as F. Then, using Theorem 3.12, one may conclude that F itself lies in $\mathcal{R}_{v,w}^{>0}$, completing the proof of Theorem 3.15. □

4 Tropicalizing the Complete Flag Variety

We now discuss *tropical varieties* and introduce the precise definitions of the *TNN tropical complete flag variety* and the *TNN complete flag Dressian*.

Definition 4.1. Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\boldsymbol{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n$. We will use the notation $\boldsymbol{x}^{\boldsymbol{b}} = x_1^{b_1} \cdots x_n^{b_n}$. Let $p = \sum_i \pm a_i \boldsymbol{x}^{\boldsymbol{b}_i}$ be a polynomial, where each $a_i > 0$ and each $\boldsymbol{b}_i \in \mathbb{N}^n$. We define the **tropicalization of** p by $trop(p) = \min_i \{a_i + \boldsymbol{x} \cdot \boldsymbol{b}_i\}$. We say that a point $\boldsymbol{y} \in \mathbb{T}^n := (\mathbb{R} \cup \infty)^n$ is a **solution of the tropicalization of** p if

$$\min_i \{a_i + \boldsymbol{y} \cdot \boldsymbol{b}_i\} = \min_i \{a_i + y_1(b_i)_1 + \cdots + y_n(b_i)_n\}$$

is achieved at least twice. We further say that a point in $\mathbb{T}^n$ is a **positive solution of the tropicalization of** p if additionally, at least one of the minima comes from a term of p with a $+$ sign, and at least one of the minima comes from a term with a $-$ sign.

The tropical objects we are interested in will live in *projective tropical spaces*, which are spaces that interact nicely with homogeneous polynomials.

Definition 4.2. Projective tropical space $\mathbb{TP}^n$ is given by $\left(\mathbb{T}^{n+1} \setminus (\infty, \ldots \infty)\right) / \sim$ where the equivalence relation is $\boldsymbol{x} \sim \boldsymbol{y}$ if there exists $c \in \mathbb{R}$ such that $x_i = y_i + c$ for all $i \in [n]$.

The following is immediate from the definition:

Proposition 4.3. *If p is a homogeneous polynomial, then $\boldsymbol{x}$ is a (positive) solution of $trop(p)$ if and only if $\boldsymbol{y}$ is a (positive) solution of $trop(p)$ for all $\boldsymbol{y} \sim \boldsymbol{x}$.*

Definition 4.4. Given a set of multi-homogeneous polynomials $\mathcal{P}$, each of which is homogeneous with respect to sets of variables of sizes $\{n_i\}_{i=1}^{t}$, and the ideal I which they generate, we define the following sets in $\mathbb{TP}^{n_1-1} \times \cdots \times \mathbb{TP}^{n_t-1}$:

- The **tropical prevariety** $\overline{\text{trop}}(\mathcal{P})$ or $\overline{\text{trop}}(I)$ is the set of simultaneous solutions to the tropicalizations of all the polynomials in $\mathcal{P}$ or in I, respectively.
- The **non-negative tropical prevariety**, $\overline{\text{trop}}^{\geq 0}(\mathcal{P})$ or $\overline{\text{trop}}^{\geq 0}(I)$, is the set of simultaneous positive solutions of the tropicalizations of all the polynomials in $\mathcal{P}$ or in I, respectively.

Solutions of tropicalizations of polynomials can alternatively be described in a way that more clearly explains the term “positive solution”. Let $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$ be the field of *Puisseux series* over $\mathbb{C}$. A Puisseux series $p(t) \in \mathcal{C}$ has a term with a lowest exponent, say at^u with $a \in \mathbb{C}^*$ and $u \in \mathbb{Q}$. In this case, we define $\text{val}(p(t)) = u$. Also, we will define the semifield $\mathcal{R}^+$ to be the set of $p(t)$ in $\mathcal{C}$ where the coefficient of $t^{\text{val}(p(t))}$ is in $\mathbb{R}^+$. In fact, $\mathcal{R}^+$ and $\mathcal{C}$ can be thought of as analogous to $\mathbb{R}^+$ and $\mathbb{C}$, respectively. Given an ideal $I \trianglelefteq \mathbb{C}[x_1, \ldots x_n]$, let $V(I) \subseteq \mathcal{C}^n$ be the variety where all polynomials in I vanish. We define the *positive part* of this variety to be $V^+(I) = V(I) \cap (\mathcal{R}^+)^n$.

Proposition 4.5 ([17, Theorem 2.1], [18, Proposition 2.2])**.** *Let I be an ideal of* $\mathbb{C}[x_1, \ldots, x_n]$. *Then* $\overline{\text{trop}}(I) = \overline{\text{val}(V(I))}$ *and* $\overline{\text{trop}}^{\geq 0}(I) = \overline{\text{val}(V^+(I))}$*, where* $\overline{\text{val}(V(I))}$ *and* $\overline{\text{val}(V^+(I))}$ *are the closures of* $\text{val}(V(I))$ *and* $\text{val}(V^+(I))$*, respectively.*

Having introduced Fl_n, we now define two tropical analogues of this space along with their totally non-negative parts. Recall that $\mathfrak{P}_{IP;n}$ is the set of incidence-Plücker relations and $I_{IP;n}$ is the ideal generated by those relations.

Definition 4.6. We define the **tropical complete flag variety** to be $trFl_n = \overline{\text{trop}}(I_{IP;n})$ and the **totally non-negative tropical complete flag variety** to be $trFl_n^{\geq 0} = \overline{\text{trop}}^{\geq 0}(I_{IP;n})$. We define the **complete flag Dressian** to be $FlDr_n = \overline{\text{trop}}(\mathfrak{P}_{IP;n})$ and the **totally non-negative complete flag Dressian** to be $FlDr_n^{\geq 0} = \overline{\text{trop}}^{\geq 0}(\mathfrak{P}_{IP;n})$.

Theorem 4.9 will show that $trFl_n^{\geq 0}$ and $FlDr_n^{\geq 0}$ coincide. Note that this is not obvious, since a point in $trFl_n^{\geq 0}$ a priori satisfies more relations than a point in $FlDr_n^{\geq 0}$. In fact, in general, the tropical prevariety of a collection of polynomials will properly contain the tropical prevariety of the ideal those polynomials generate. In the specific case of the complete flag variety, it is shown in [3] that for $n \geq 6$, $FlDr_n$ properly contains $trFl_n$.

We now shift our attention to the non-negative parts of the tropical varieties we have introduced. For $v \leq w$ in the Bruhat order with $r = \ell(w) - \ell(v)$, let $\Phi_{v,w}\colon \mathbb{R}^r_{>0} \to \mathbb{RP}^{\binom{n}{1}-1} \times \cdots \times \mathbb{RP}^{\binom{n}{n-1}-1}$ be the map which takes a collection of $\boldsymbol{a} \in \mathbb{R}^r_{>0}$ to the Plücker coordinates of the matrix $M_{v,w}(\boldsymbol{a})$, in the notation of Theorem 3.5. Note that by construction, this map consists of a collection of polynomials in the a_i, and so we can tropicalize this map, obtaining a map Trop $\Phi_{v,w}\colon \mathbb{R}^r \to \mathbb{TP}^{\binom{n}{1}-1} \times \cdots \times \mathbb{TP}^{\binom{n}{n-1}-1}$. We now state the key connection between this map and $TrFl_n^{\geq 0}$.

Lemma 4.7 ([18, 14])**.** *The image of* Trop $\Phi_{v,w}$ *lies in* $trFl_n^{\geq 0}$.

We next make an observation relating the TNN complete flag variety and of the TNN complete flag Dressian. For $S \subset [n]$, and $a < b < c$ satisfying $a, c \notin S$ and $b \in S$, we have a three-term incidence-Plücker relation $P_S P_{(S\setminus b)\cup ac} = P_{(S\setminus b)\cup a} P_{S\cup c} + P_{(S\setminus b)\cup c} P_{S\cup a}$. Observe that if all the coordinates other than P_S are known and positive, then P_S is uniquely determined and is itself positive. Similarly, we can tropicalize this relation to get the three-term positive tropical incidence-Plücker relation $P_S + P_{(S\setminus b)\cup ac} = \min\{P_{(S\setminus b)\cup a} + P_{S\cup c}, P_{(S\setminus b)\cup c} + P_{S\cup a}\}$. Again, if all the coordinates other than P_S are known, then P_S is uniquely determined. One way to rephrase Theorem 3.15 is as follows: Every point in Fl_n with all non-negative Plücker coordinates lies in the image of $\Phi_{v,w}$ for some $v \leq w \in S_n$. From this, we will deduce a helpful corollary. In particular, the general idea is that whenever we determine certain values of P_S in the proof of Theorem 3.15, we are careful to do so using a three-term incidence-Plücker relation, as described earlier in this paragraph. This then translates nicely to the tropical context.

Corollary 4.8. *Every point in* ${FlDr_n}^{\geq 0}$ *lies in the image of* Trop $\Phi_{v,w}$ *for some* $v \leq w \in S_n$.

Using this corollary and Lemma 4.7, we come to our main result:

Theorem 4.9. *The TNN topical flag variety* $trFl_n^{\geq 0}$ *equals the TNN complete flag Dressian* $FlDr_n^{\geq 0}$.

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