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Log-Concave Poset Inequalities: Extended Abstract

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Abstract. We study combinatorial inequalities for various classes of set systems: matroids, morphisms of matroids, polymatroids, and poset antimatroids. We prove log-concave inequalities for counting certain weighted feasible words, which generalize and extend several previous results establishing *Mason conjectures* for the numbers of independent sets of matroids. Additionally, we rederive *Stanley's inequality* on the number of certain linear extensions, which we then also extend to the weighted case. Notably, we also prove matching equality conditions for all these inequalities.

Foreword: This extended abstract is concerned with log-concavity results for counting problems in the general context of posets, and is motivated by a large body of amazing recent work in area, see a survey by Huh [14] and also earlier surveys [3, 5, 6, 23]. We select a subset of these results for various combinatorial structures to be presented here. Their proofs and historical remarks can be found in the full version of the paper [8], and a simpler version of the method can be found in the expository version of the paper [9].

1 Matroids

A (finite) *matroid* $\mathcal{M}$ is a pair $(X, \mathcal{I})$ of a *ground set* X, $|X| = n$, and a nonempty collection of *independent sets* $\mathcal{I} \subseteq 2^X$ that satisfies the following:

- (*hereditary property*) $S \subset T$, $T \in \mathcal{I}$ implies $S \in \mathcal{I}$, and
- (*exchange property*) $S, T \in \mathcal{I}$, $|S| < |T|$ implies there exists $x \in T \setminus S$ such that $S + x \in \mathcal{I}$.

Rank of a matroid is the maximal size of the independent set: $\mathrm{rk}(\mathcal{M}) := \max_{S \in \mathcal{I}} |S|$. A *basis* of a matroid is an independent set of size $\mathrm{rk}(\mathcal{M})$. Finally, let $\mathcal{I}_k := \{S \in \mathcal{I}, |S| = k\}$, and let $\mathrm{I}(k) = |\mathcal{I}_k|$ be the *number of independent sets* in $\mathcal{M}$ of size k, $0 \leq k \leq \mathrm{rk}(\mathcal{M})$.

Mason's matroid log-concavity conjectures were stated in [17], motivated by the earlier work and conjectures in graph theory and combinatorial geometry. The strongest

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form of these conjectures were recently proved independently by Brändén and Huh [4] through *Lorentzian polynomials* approach, and by Anari *et al.* [2] through *completely log-concave polynomials* approach.

Theorem 1 (*Ultra-log-concavity for matroids* [2, Theorem 1.2] and [4, Theorem 4.14], formerly *strong Mason conjecture*)**.** *For a matroid* $\mathcal{M} = (X, \mathcal{I})$, $|X| = n$, *and integer* $1 \leq k < \mathrm{rk}(\mathcal{M})$,

$$\mathrm{I}(k)^2 \geq \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right) \mathrm{I}(k-1)\, \mathrm{I}(k+1). \tag{1.1}$$

Mason's conjectures were presented as a series of three log-concave inequalities, for which (1.1) is the strongest version of the inequalities. The other two inequalities were previously obtained in [1] and [15] by using the hard Lefschetz theorem and the Hodge–Riemann relations in a number of algebraic settings. The methods in [2, 4] use interrelated ideas, and avoid much of the algebraic technology in [1].

Equality conditions for (1.1) were recently established by Murai, Nagaoka and Yazawa in [18] using an algebraic argument built on [4], and can be described as follows. For a matroid $\mathcal{M} = (X, \mathcal{I})$ on $|X| = n$ elements, define $\mathrm{girth}(\mathcal{M}) := \min\{k : \mathrm{I}(k) < \binom{n}{k}\}$. The *girth of a matroid* is the size of the smallest non-independent sets in $\mathcal{M}$.

Theorem 2 (*Equality for matroids* [18, Corollary 1.2])**.** *Equality occurs in* (1.1) *if and only if we have* $\mathrm{girth}(\mathcal{M}) > (k+1)$.

We now present a refinement of log-concavity for matroids that is proved through our method. For an independent set $S \in \mathcal{I}$ of a matroid $\mathcal{M} = (X, \mathcal{I})$, denote by

$$\mathrm{Cont}(S) := \left\{x \in X \setminus S \ : \ S + x \in \mathcal{I}\right\} \tag{1.2}$$

the set of *continuations* of S. For all $x, y \in \mathrm{Cont}(S)$, we write $x \sim_S y$ when $S + x + y \notin \mathcal{I}$. Note that "$\sim_S$" is an equivalence relation. We call an equivalence class of the relation $\sim_S$ a *parallel class* of S, and we denote by $\mathrm{Par}(S)$ the set of parallel classes of S.

For every $0 \leq k < \mathrm{rk}(\mathcal{M})$, define the *k-continuation number* of a matroid $\mathcal{M}$ as the maximal number of parallel classes of independent sets of size k:

$$\mathrm{p}(k) := \max\left\{ \left|\mathrm{Par}(S)\right| \ : \ S \in \mathcal{I}_k \right\}. \tag{1.3}$$

Clearly, $\mathrm{p}(k) \leq n - k$.

Theorem 3 (*Refined log-concavity for matroids*)**.** *For a matroid* $\mathcal{M} = (X, \mathcal{I})$ *and integer* $1 \leq k < \mathrm{rk}(\mathcal{M})$, *we have:*

$$\mathrm{I}(k)^2 \geq \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{\mathrm{p}(k-1) - 1}\right) \mathrm{I}(k-1)\, \mathrm{I}(k+1). \tag{1.4}$$

Clearly, Theorem 3 implies Theorem 1. In particular, let us illustrate the power of this refinement in a special case.

Let $G = (V, E)$ be a connected graph with $|V| = \textsc{n}$ edges. The corresponding *graphical matroid* $\mathcal{M}_G = (E, \mathcal{I})$ is defined to have independent sets to be all *forests* in G. Then $\mathrm{I}(k)$ is the number of forests with k edges.

Proposition 4 (*Refinement for graphical matroids*)**.** *Let $G = (V, E)$ be a simple connected graph on $|V| = \textsc{n}$ vertices, and let $\mathrm{I}(k)$ be the number of forests with k edges. Then*

$$\frac{\mathrm{I}(\textsc{n}-2)^2}{\mathrm{I}(\textsc{n}-3)\cdot\mathrm{I}(\textsc{n}-1)} \;\geq\; \frac{3}{2}\left(1+\frac{1}{\textsc{n}-2}\right). \tag{1.5}$$

This is both numerically and asymptotically better than (1.1). For example, when $|E| - \textsc{n} \to \infty$, we have:

$$\frac{\mathrm{I}(\textsc{n}-2)^2}{\mathrm{I}(\textsc{n}-3)\cdot\mathrm{I}(n-1)} \;\geq_{(1.1)}\; \left(1+\frac{1}{|E|-\textsc{n}+2}\right)\left(1+\frac{1}{\textsc{n}-2}\right) \;\to\; 1 \quad \text{as} \quad \textsc{n}\to\infty.$$

We now present equality conditions for the refined log-concave inequality (1.4).

Theorem 5 (*Refined equality for matroids*)**.** *Equality occurs in* (3) *<u>if and only if</u> there exists* $\mathrm{s}(k-1) > 0$ *such that for every $S \in \mathcal{I}_{k-1}$ we have:*

$$\big|\operatorname{Par}(S)\big| \;=\; \mathrm{p}(k-1), \quad \textit{and} \tag{ME1}$$

$$|\,\mathcal{C}\,| \;=\; \mathrm{s}(k-1) \quad \textit{for every} \;\; \mathcal{C}\in\operatorname{Par}(S). \tag{ME2}$$

Condition (ME1) says that the $(k-1)$-continuation number is achieved on all independent sets $S \in \mathcal{I}_{k-1}$, while condition (ME2) is saying that all parallel classes $\mathcal{C} \in \operatorname{Par}(S)$ have the same size. In contrast to Theorem 2, there is a a rich family of examples satisfying equality conditions in Theorem 5, such as *paving matroids* and *Steiner system matroids*. In particular, we obtain the equality conditions for the special case of graphical matroids.

Proposition 6 (*Equality for graphical matroids*)**.** *Equality occurs in* (1.5) *<u>if and only if</u> G is an $\textsc{n}$-cycle.*

2 Log-concavity for morphisms

For a matroid $\mathcal{M} = (X, \mathcal{I})$, the *rank function* $f\colon 2^X \to \mathbb{R}_{>0}$ is defined by

$$f(S) \;:=\; \max\big\{|A| \,:\, A\subseteq S,\, A\in\mathcal{I}\big\}.$$

Note that $\operatorname{rk}(\mathcal{M}) = f(X)$. Let $\mathcal{M} = (X, \mathcal{I})$ and $\mathcal{N} = (Y, \mathcal{J})$ be two matroids with rank functions f and g, respectively. Let $\Phi\colon X \to Y$ be a function that satisfies

$$g\big(\Phi(T)\big) - g\big(\Phi(S)\big) \;\leq\; f(T) - f(S) \quad \text{for every} \;\; S\subseteq T\subseteq X. \tag{2.1}$$

In this case we say that $\Phi\colon \mathcal{M} \to \mathcal{N}$ is a *morphism* of matroids. A subset $S \in \mathcal{I}$ is said to be a *basis* of Φ if $g(\Phi(S)) = \mathrm{rk}(\mathcal{N})$. In other words, S is contained in a basis of $\mathcal{M}$, and $\Phi(S)$ contains a basis of $\mathcal{N}$. The notion of morphism of matroids generalizes many classical notions in combinatorics such as *graph coloring*, *graph embeddings*, *graph homomorphism*, *matroid quotients*, and are a special case of the *induced matroids*. We refer to [12] for a detailed overview and further references.

Denote by $\mathcal{B}$ the set of bases of $\Phi\colon \mathcal{M} \to \mathcal{N}$, let $\mathcal{B}_k := \mathcal{B} \cap \mathcal{I}_k$, and let $\mathrm{B}(k) := |\mathcal{B}(k)|$. The following log-concave inequality is due to Eur and Huh [12].

Theorem 7 (*Log-concavity for morphisms* [12, Theorem 1.3]). *Let $\mathcal{M} = (X, \mathcal{I})$ and $\mathcal{N} = (Y, \mathcal{J})$ be matroids, let $n := |X|$, and let $\Phi\colon \mathcal{M} \to \mathcal{N}$ be a morphism of matroids, and let $1 \le k < \mathrm{rk}(\mathcal{M})$. Then:*

$$\mathrm{B}(k)^2 \ \ge \ \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) \mathrm{B}(k-1)\, \mathrm{B}(k+1). \tag{2.2}$$

Note that when $Y = \{y\}$ and $\mathcal{N} = (Y, \varnothing)$ is defined by $g(y) = 0$, we have condition (2.1) holds trivially and $\mathcal{B} = \mathcal{I}$. Thus, the theorem generalizes Theorem 1 to the morphism of matroids setting.

Equality conditions for Theorem 7 was posed as an open problem in [18, Question 5.7], which we resolved through the following theorem.

Theorem 8 (*Equality for morphisms*). *Equality occurs in (2.2) <u>if and only if</u> $\mathrm{girth}(\mathcal{M}) > k+1$ and $g\big(\Phi(S)\big) = \mathrm{rk}(\mathcal{N})$ for all $S \in \mathcal{I}_{k-1}$.*

We now present a refinement of log-concavity for morphisms. Recall the equivalence relation "$\sim_S$" on the set $\mathrm{Cont}(S) \subseteq X \setminus S$ of continuations of $S \in \mathcal{I}$, see (1.2). Similarly, recall the set $\mathrm{Par}(S)$ of parallel classes of S, see (1.3). For every $1 \le k \le \mathrm{rk}(\mathcal{M})$, let

$$\mathrm{p}(k) \ := \ \max\big\{ \big|\mathrm{Par}(S)\big| \ : \ S \in \mathcal{B}_k \big\},$$

the maximum of the number of parallel classes of bases of morphism Φ of size k.

Theorem 9 (*Refined log-concavity for morphisms*). *Let $\mathcal{M} = (X, \mathcal{I})$ and $\mathcal{N} = (Y, \mathcal{J})$ be matroids, let $\Phi\colon \mathcal{M} \to \mathcal{N}$ be a morphism of matroids, and let $1 \le k < \mathrm{rk}(\mathcal{M})$. Then:*

$$\mathrm{B}(k)^2 \ \ge \ \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{\mathrm{p}(k-1) - 1}\right) \mathrm{B}(k-1)\, \mathrm{B}(k+1). \tag{2.3}$$

As before, since $\mathrm{p}(k-1) \le n - k + 1$, the theorem is an extension of Theorem 7. We also obtain equality conditions for (2.3), which we present below.

Theorem 10 (*Refined equality for morphisms*). *Equality occurs in (2.3) <u>if and only if</u> there exists $\mathrm{s}(k-1) > 0$, such that for every $S \in \mathcal{I}_{k-1}$ we have:*

$$\big|\mathrm{Par}_S\big| \ = \ \mathrm{p}(k-1), \tag{MME1}$$

$$|\,\mathcal{C}\,| \ = \ \mathrm{s}(k-1) \qquad \textit{for every } \ \mathcal{C} \in \mathrm{Par}(S), \ \textit{and} \tag{MME2}$$

$$g\big(\Phi(S)\big) = \mathrm{rk}(\mathcal{N}). \tag{MME3}$$

3 Discrete polymatroids

A *discrete polymatroid*,[1] also called *integral polymatroid*, $\mathcal{D}$ is a pair $([n], \mathcal{J})$ of a ground set $[n] := \{1, \ldots, n\}$ and a nonempty finite collection $\mathcal{J}$ of integer points $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ that satisfy the following:

- (*hereditary property*) $\boldsymbol{a} \in \mathcal{I}$, $\boldsymbol{b} \in \mathbb{N}^n$ such that $\boldsymbol{b} \leqslant \boldsymbol{a} \Rightarrow \boldsymbol{b} \in \mathcal{I}$, and
- (*exchange property*) $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{I}$, $|\boldsymbol{a}| < |\boldsymbol{b}|$ implies there exists $i \in [n]$ such that $a_i < b_i$ and $\boldsymbol{a} + \boldsymbol{e}_i \in \mathcal{J}$.

Here $\boldsymbol{b} \leqslant \boldsymbol{a}$ is a componentwise inequality, $|\boldsymbol{a}| := a_1 + \ldots + a_n$, and $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ is a standard linear basis in $\mathbb{R}^n$. When $\mathcal{J} \subseteq \{0,1\}^n$, discrete polymatroid $\mathcal{D}$ is a matroid. One can think of a discrete polymatroid as a set system where multisets are allowed, so we refer to $\mathcal{J}$ as *independent multisets* and to $|\boldsymbol{a}|$ as *size* of the multiset $\boldsymbol{a}$. We refer to [13] for the history and algebraic motivation.

Define $\mathrm{rk}(\mathcal{D}) := \max\{|\boldsymbol{a}| \,:\, \boldsymbol{a} \in \mathcal{J}\}$. For $0 \leq k \leq \mathrm{rk}(\mathcal{D})$, denote by $\mathcal{J}_k := \{\boldsymbol{a} \in \mathcal{J} : |\boldsymbol{a}| = k\}$ the subcollection of independent multisets of size k. Let $\omega \colon [n] \to \mathbb{R}_{>0}$ be a positive *weight function* on $[n]$. We extend weight function ω to all $\boldsymbol{a} \in \mathcal{J}$ as follows:

$$\omega(\boldsymbol{a}) \;:=\; \omega(1)^{a_1} \cdots \omega(n)^{a_n}.$$

For every $0 \leq k \leq \mathrm{rk}(\mathcal{D})$, define

$$\mathrm{J}_\omega(k) \;:=\; \sum_{\boldsymbol{a} \in \mathcal{J}_k} \frac{\omega(\boldsymbol{a})}{\boldsymbol{a}!}\,, \quad \text{where} \quad \boldsymbol{a}! \;:=\; a_1! \cdots a_n!.$$

The following log-concavity follows easily from the results in [4], which was stated in a somewhat different form.

Theorem 11 (*Log-concavity for polymatroids* [4, Theorem 3.10 (4) $\Leftrightarrow$ (7)])**.** *Let $\mathcal{D} = ([n], \mathcal{J})$ be a discrete polymatroid, and let $\omega \colon [n] \to \mathbb{R}_{>0}$ be a positive weight function. For every $1 \leq k < \mathrm{rk}(\mathcal{M})$, we have:*

$$\mathrm{J}_\omega(k)^2 \;\geq\; \left(1 + \frac{1}{k}\right)\, \mathrm{J}_\omega(k-1)\, \mathrm{J}_\omega(k+1). \tag{3.1}$$

Equality conditions for (3.1) was previously unknown, and we established them through the following theorem. A discrete polymatroid $\mathcal{D} = ([n], \mathcal{J})$ is called *non-degenerate* if $\mathbf{e}_i \in \mathcal{J}$ for every $i \in [n]$. Note that every discrete polymatroid can be made to be nondegenerate by restricting $[n]$ to integers i satisfying $\mathbf{e}_i \in \mathcal{J}$. Define

[1] *Discrete polymatroids* are related but should not to be confused with *polymatroids*, which is a family of convex polytopes, see, *e.g.*, [19, §44].

$\operatorname{polygirth}(\mathcal{D}) := \min\{k : |\mathcal{J}_k| < \binom{n+k-1}{k-1}\}$. Observe that $\boldsymbol{a} \in \mathcal{J}$ for all $\boldsymbol{a} \in \mathbb{N}^k$, $|\boldsymbol{a}| < \operatorname{polygirth}(\mathcal{D})$. Note that the *polygirth* of a discrete polymatroid does not coincide with the girth of a matroid. In fact, $\operatorname{polygirth}(\mathcal{D}) = 2$ when $\mathcal{D}$ is a matroid with more than one element.

Theorem 12 (*Equality for polymatroids*)**.** *Let $\mathcal{D}$ be a nondegenerate discrete polymatroid. Then equality occurs in* (3.1) *if and only if* $\operatorname{polygirth}(\mathcal{D}) > (k+1)$.

We now present a refinement of log-concavity for discrete polymatroids. Fix $t \in [0,1]$. For every $\boldsymbol{a} \in \mathcal{J}$ let

$$\pi(\boldsymbol{a}) := \sum_{i=1}^{n} \binom{a_i}{2}.$$

For every $0 \le k \le \operatorname{rk}(\mathcal{D})$, define

$$\mathrm{J}_{\omega,t}(k) := \sum_{\boldsymbol{a} \in \mathcal{J}_k} t^{\pi(\boldsymbol{a})} \, \frac{\omega(\boldsymbol{a})}{\boldsymbol{a}!}\,.$$

Note that $\binom{a}{2} = 0$ for $a \in \{0,1\}$, so $\pi(\boldsymbol{a}) = 0$ for all independent sets $\boldsymbol{a} \in \mathcal{I}$ in a matroid.

For an independent multiset $\boldsymbol{a} \in \mathcal{J}$ of a discrete polymatroid $\mathcal{D} = ([n], \mathcal{J})$, denote by

$$\operatorname{Cont}(\boldsymbol{a}) := \big\{ i \in [n] \ : \ \boldsymbol{a} + \boldsymbol{e}_i \in \mathcal{J} \big\}. \tag{3.2}$$

the set of *continuations* of $\boldsymbol{a}$. For all $i, j \in \operatorname{Cont}(\boldsymbol{a})$, we write $i \sim_{\boldsymbol{a}} j$ when $\boldsymbol{a} + \boldsymbol{e}_i + \boldsymbol{e}_j \notin \mathcal{J}$ or $i = j$. This is an equivalence relation again. We call an equivalence class of the relation $\sim_{\boldsymbol{a}}$ a *parallel class* of $\boldsymbol{a}$, and we denote by $\operatorname{Par}(\boldsymbol{a})$ the set of parallel classes of $\boldsymbol{a}$.

For every $0 \le k < \operatorname{rk}(\mathcal{D})$, define the *k-continuation number* of a discrete polymatroid $\mathcal{D}$ as the maximal number of parallel classes of independent multisets of size k:

$$\mathrm{p}(k) := \max\big\{ \, |\operatorname{Par}(\boldsymbol{a})| \ : \ \boldsymbol{a} \in \mathcal{J}_k \, \big\}. \tag{3.3}$$

For matroids, this is the same notion as defined above.

Theorem 13 (*Refined log-concavity for polymatroids*)**.** *Let $\mathcal{D} = ([n], \mathcal{J})$ be a discrete polymatroid, and let $\omega \colon [n] \to \mathbb{R}_{>0}$ be a positive weight function. For every $t \in [0,1]$ and $1 \le k < \operatorname{rk}(\mathcal{M})$, we have:*

$$\mathrm{J}_{\omega,t}(k)^2 \ \ge \left(1 + \frac{1}{k}\right) \left(1 + \frac{1-t}{\mathrm{p}(k-1) - 1 + t}\right) \mathrm{J}_{\omega,t}(k-1)\, \mathrm{J}_{\omega,t}(k+1). \tag{3.4}$$

When $t = 1$, this gives Theorem 11. When $\mathcal{D}$ is a matroid, $t = 0$, and ω is an uniform weight, this gives Theorem 3. For general discrete polymatroids $\mathcal{D}$ and $0 < t < 1$, this is a stronger result.

To get the equality conditions for (3.4), we separate the cases $t = 0$, $0 < t < 1$, and $t = 1$. The case $t = 0$ coincides with equality conditions for matroids given in Theorem 5; the case $t = 1$ coincides with equality conditions for polymatroids given in Theorem 12; and the case $0 < t < 1$ is contained in the following lemma.

Theorem 14 (*Refined equality for polymatroids* $0 < t < 1$ case)**.** *Let* $\mathcal{D}$ *be a nondegenerate discrete polymatroid, and let* $0 < t < 1$*. Then equality occurs in* (3.4) *<u>if and only if</u>* $k = 1$*,* $\operatorname{polygirth}(\mathcal{D}) > 2$*, and* ω *is uniform.*

One can view the dearth of nontrivial examples in Theorem 14 as suggesting that the bound in Theorem 13 can be further improved for $t > 0$. This is based on the reasoning that Theorem 3 sharply improves over Theorem 1 because there are only trivial equality conditions for the latter (see Theorem 2), when compared with rich equality conditions for the former (see Theorem 5).

4 Poset antimatroids

Let X be finite set we call *letters*, and let X^* be a set of *finite words* in the alphabet X. A *language over X* is a nonempty finite subset $\mathcal{L} \subset X^*$. A word is called *simple* if it contains each letter at most once; we consider only simple words from this point on. We write $x \in \alpha$ if word $\alpha \in \mathcal{L}$ contains letter x. Finally, let $|\alpha|$ be the length of the word, and denote $\mathcal{L}_k := \{\alpha \in \mathcal{L} : |\alpha| = k\}$.

A pair $\mathcal{A} = (X, \mathcal{L})$ is an *antimatroid*, if the language $\mathcal{L} \subset X^*$ satisfies:

- (*nondegenerate property*) every $x \in X$ is contained in at least one $\alpha \in \mathcal{L}$,
- (*normal property*) every $\alpha \in \mathcal{L}$ is simple,
- (*hereditary property*) $\alpha\beta \in \mathcal{L} \;\Rightarrow\; \alpha \in \mathcal{L}$, and
- (*exchange property*) $x \in \alpha$, $x \notin \beta$, and $\alpha, \beta \in \mathcal{L}$ implies there exists $y \in \alpha$ such that $\beta y \in \mathcal{L}$.

Note that for every antimatroid $\mathcal{A} = (X, \mathcal{L})$, it follows from the exchange property that

$$\operatorname{rk}(\mathcal{A}) := \max\{|\alpha| : \alpha \in \mathcal{L}\} = |X|.$$

Antimatroids is a subclass of greedoids named after the *anti-exchange property*, which is a key axiom in their definition via set systems. We refer to [16] for the history and geometric motivation.

In this extended abstract we use only one class of antimatroids which we now define. Let $\mathcal{P} = (X, \prec)$ be a poset on $|X| = n$ elements. A simple word $\alpha \in X^*$ is called *feasible* if α satisfies:

- (*poset property*) if α contains $x \in X$ and $y \prec x$, then letter y occurs before letter x in α.

A *poset antimatroid* $\mathcal{A}_{\mathcal{P}} = (X, \mathcal{L})$ is defined by the language $\mathcal{L}$ of all feasible words in X.

Let $\omega \colon X \to \mathbb{R}_{>0}$ be a positive weight function on X. Denote by $\mathrm{Cov}(x) := \{y \in X : x \lessdot y\}$ the set of elements which cover x. We assume the weight function ω satisfies the following (*cover monotonicity property*):

$$\omega(x) \,\geq\, \sum_{y \in \mathrm{Cov}(x)} \omega(y)\,, \quad \text{for all} \;\; x \in X. \tag{CM}$$

Note that when (CM) is equality for all $x \in X$, we have:

$$\omega(x) \;=\; \text{number of maximal chains in } \mathcal{P} \text{ starting at } x. \tag{4.1}$$

For all $\alpha \in \mathcal{L}$ and $0 \leq k \leq n$, let

$$\mathrm{L}_\omega(k) \;:=\; \sum_{\alpha \in \mathcal{L}_k} \omega(\alpha)\,, \quad \text{where} \quad \omega(\alpha) \;:=\; \prod_{x \in \alpha} \omega(x).$$

Theorem 15 (*Log-concavity for poset antimatroids*)**.** *Let $\mathcal{P} = (X, \prec)$ be a poset on $|X| = n$ elements, and let $\mathcal{A}_{\mathcal{P}} = (X, \mathcal{L})$ be the corresponding poset antimatroid. Let $\omega \colon X \to \mathbb{R}_{>0}$ be a positive weight function which satisfies* (CM)*. Then, for every integer $1 \leq k < n$, we have:*

$$\mathrm{L}_\omega(k)^2 \,\geq\, \mathrm{L}_\omega(k-1) \cdot \mathrm{L}_\omega(k+1). \tag{4.2}$$

Note that the weight function condition (CM) is necessary, as there are examples for which log-concavity fails to hold for uniform weight, see [8, Example 1.27] in the full version of this extended abstract.

5 Linear extensions

Let $\mathcal{P} := (X, \prec)$ be a poset on $n := |X|$ elements. A *linear extension* of $\mathcal{P}$ is a bijection $L \colon X \to \{1, \ldots, n\}$, such that $L(x) < L(y)$ for all $x \prec y$. We refer to [7, 24] for definitions and standard results on posets and linear extensions.

Fix an element $z \in X$. Denote by $\mathcal{E} := \mathcal{E}(P)$ the set of linear extensions of $\mathcal{P}$, let $\mathcal{E}_k := \{L \in \mathcal{E} \, : \, L(z) = k\}$. The following inequality was originally conjectured by Chung, Fishburn and Graham in [11], extending an earlier unimodality conjecture by R. Rivest (unpublished).

Theorem 16 (*Stanley inequality* [22, Theorem 3.1])**.** *Let $\mathcal{P} = (X, \prec)$ be a poset with $|X| = n$ elements, and let $z \in X$. Denote by $\mathrm{N}(k) := |\,\mathcal{E}_k\,|$ the number of linear extensions $L \in \mathcal{E}(P)$, such that $L(z) = k$. Then, for every $1 < k < n$, we have:*

$$\mathrm{N}(k)^2 \,\geq\, \mathrm{N}(k-1) \cdot \mathrm{N}(k+1). \tag{5.1}$$

The proof in [22] is a simple application of the *Alexandrov–Fenchel inequality*. In the same paper, Stanley asked for equality conditions for (5.1), which were recently obtained by Shenfeld and van Handel. Denote by $f(x) := \bigl|\{y \in X : y \prec x\}\bigr|$ and $g(x) := \bigl|\{y \in X : y \succ x\}\bigr|$ the sizes of lower and upper ideals of $x \in X$, respectively, excluding the element x.

Theorem 17 (*Equality condition for Stanley inequality* [20, Theorem 15.3])**.** *Suppose that* $\mathrm{N}(k) > 0$. *Then the following are equivalent:*

(a) $\mathrm{N}(k)^2 = \mathrm{N}(k-1) \cdot \mathrm{N}(k+1)$,

(b) $\mathrm{N}(k+1) = \mathrm{N}(k) = \mathrm{N}(k-1)$,

(c) *we have* $f(x) > k$ *for all* $x \succ z$, *and* $g(x) > n-k+1$ *for all* $x \prec z$.

The proof in [20] used a sophisticated geometric analysis to prove equality conditions of the Alexandrov–Fenchel inequality for convex polytopes.

We now give a weighted generalization of these results, which we proved combinatorially. Let $\omega\colon X \to \mathbb{R}_{>0}$ be a positive weight function on X. We say that ω is *order-reversing* if it satisfies

$$x \preccurlyeq y \quad \Rightarrow \quad \omega(x) \geq \omega(y). \tag{Rev}$$

Fix $z \in X$, as above. Define $\omega\colon \mathcal{E} \to \mathbb{R}_{>0}$ by

$$\omega(L) := \prod_{x \,:\, L(x)<L(z)} \omega(x), \tag{5.2}$$

and let

$$\mathrm{N}_\omega(k) := \sum_{L \in \mathcal{E}_k} \omega(L), \quad \text{for all } 1 \leq k \leq n. \tag{5.3}$$

Theorem 18 (*Weighted Stanley inequality*)**.** *Let $\mathcal{P} = (X, \prec)$ be a poset with $|X| = n$ elements, and let $\omega\colon X \to \mathbb{R}_{>0}$ be a positive order-reversing weight function. Fix an element $z \in X$. Then, for every $1 < k < n$, we have:*

$$\mathrm{N}_\omega(k)^2 \geq \mathrm{N}_\omega(k-1) \cdot \mathrm{N}_\omega(k+1), \tag{5.4}$$

where $\mathrm{N}_\omega(k)$ is defined by (5.3).

Prior to our work, no direct combinatorial proof of Stanley's inequality was known in full generality, although [11] gives a simple proof for posets of width two (see also [10]). Most recently, the authors and Panova obtained a q- and multivariate analogues of Stanley's inequality for posets of width two [10]. These notions are specific to the width two case and are incompatible with the weighted analogue above.

The equality conditions for (5.2) is a little more subtle and needs the following *(s,k)-cohesiveness property*:

$$\omega\bigl(L^{-1}(k-1)\bigr) = \omega\bigl(L^{-1}(k+1)\bigr) = \mathrm{s}, \quad \text{for all } L \in \mathcal{E}_k\,. \tag{Coh}$$

Theorem 19 (*Equality condition for weighted Stanley inequality*)**.** *Suppose that* $\mathrm{N}_\omega(k) > 0$. *Then the following are equivalent:*

(a) $\mathrm{N}_\omega(k)^2 \, = \, \mathrm{N}_\omega(k-1) \cdot \mathrm{N}_\omega(k+1)$,

(b) *there exists* $\mathrm{s} = \mathrm{s}(k,z) > 0$ *such that*

$$\mathrm{N}_\omega(k+1) \, = \, \mathrm{s}\,\mathrm{N}_\omega(k) \, = \, \mathrm{s}^2\,\mathrm{N}_\omega(k-1),$$

(c) *there exists* $\mathrm{s} = \mathrm{s}(k,z) > 0$ *such that* $f(x) > k$ *for all* $x \succ z$, $g(x) > n-k+1$ *for all* $x \prec z$, *and* (Coh).

6 Proof ideas

Although we prove multiple results, the proof of each log-concavity inequality uses the same approach and technology, so we refer to it as "the proof".

At the first level, the proof is an inductive argument proving a stronger claim about eigenvalues of certain matrices associated with the posets. The induction is not over posets of smaller size, but over other matrices which can in fact be larger, but correspond to certain parameters decreasing as we go along. The claim then reduces to the base of induction, which is the only part of the proof requiring a computation. The latter involves checking eigenvalues of explicitly written small matrices, making the proof fully elementary.

Delving a little deeper, we set up a new type of structure which we call a *combinatorial atlas*. In the special case of matroids, a combinatorial atlas $\mathbb{A}$ associated with a matroid $\mathcal{M} = (X, \mathcal{I})$, $|X| = n$, is comprised of:

- Acyclic digraph $\Gamma_{\mathcal{M}} = (X^*, \Theta)$, with the unique source at the empty word $\varnothing \in X^*$, and edges corresponding to multiplications by a letter: $\Theta = \{(\alpha, \alpha x) \, : \, \alpha, \alpha x \in X^*, \, x \in X\}$,
- Each vertex $\alpha \in X^*$ is associated with a pair $(\mathbf{M}_\alpha, \mathbf{h}_\alpha)$, where $\mathbf{M}_\alpha = (\mathrm{M}_{ij})$ is a nonnegative symmetric $d \times d$ matrix, $\mathbf{h}_\alpha = (\mathrm{h}_1, \ldots, \mathrm{h}_d)$ is a nonnegative vector, and $d = n+1$,
- Each edge $(\alpha, \alpha x) \in \Theta$ is associated with a linear transformation $\mathbf{T}_\alpha^{\langle x \rangle} : \mathbb{R}^d \to \mathbb{R}^d$.

The key technical observation is that under certain conditions on the atlas, we have every matrix $\mathbf{M} := \mathbf{M}_\alpha$, is *hyperbolic*:

$$\langle \mathbf{v}, \mathbf{M}\mathbf{w} \rangle^2 \, \geq \, \langle \mathbf{v}, \mathbf{M}\mathbf{v} \rangle \langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle \quad \text{for all } \mathbf{v}, \mathbf{w} \in \mathbb{R}^d \text{ such that } \langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle > 0. \qquad \text{(Hyp)}$$

Log-concavity inequalities now follow from (Hyp) for the matrix $\mathbf{M}_\varnothing$, by interpreting the inner products as numbers $\mathrm{L_q}(k)$, $\mathrm{L_q}(k-1)$ and $\mathrm{L_q}(k+1)$, respectively.

We prove (Hyp) by induction, reducing the claim for $\mathbf{M}_\alpha$ to that of $\mathbf{M}_{\alpha x}$, for all $x \in \mathrm{Cont}(\alpha)$. Proving (Hyp) for the base of induction required the eigenvalue interlacing argument. This is where our conditions for the weight function ω appear in the calculation. We also need a few other properties of the atlas. Notably, we require every matrix $\mathbf{M}_\alpha$ to be *irreducible* with respect to its support, but that is proved by a direct combinatorial argument.

For other log-concavity inequalities in the paper, we consider similar atlas constructions and similar claims. For the equalities, we works backwards and observe that we need equations (Hyp) to be equalities. These imply the local properties which must hold for certain edges $(\alpha, \alpha x) \in \Theta$. Analyzing these properties gives the equality conditions we present.

Although this can only be understood from the proofs, much of this work has been influenced by [21] which gives a new proof of the Alexandrov–Fenchel inequalities. The closest we come to [21] is in the most technical part of the paper on Stanley's inequality, but again the tools we employ are highly technical and go much beyond what can be described in the extended abstract. We refer to the full version [8] for all the details.

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