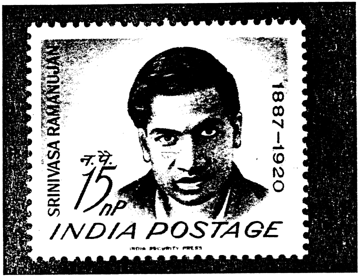
# Ramanujan's Lost Notebook in Five Volumes Thoughts and Comments

SLC (KrattenthalerFest) 2018
September 2018

This talk is dedicated to my good friend Christian Krattenthaler

# I. BACKGROUND


SRINIVASA RAMANUJAN
1887–1920
न.पै.
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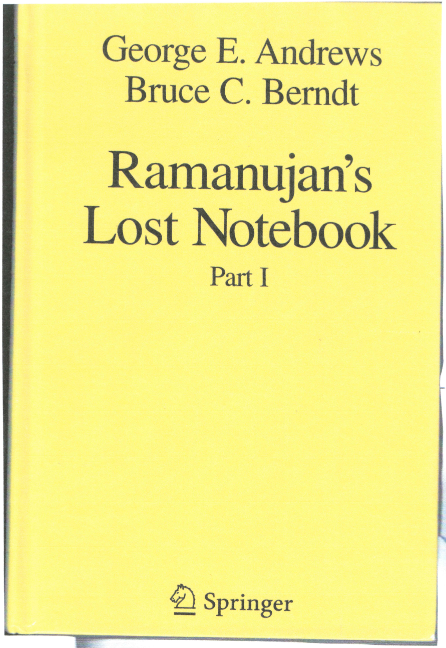
INDIA POSTAGE

“His memory, and his powers of calculation, were very unusual, but they could not be reasonably be called “abnormal”. If he had to multiply two very large numbers, he multiplied them in the ordinary way; he would do it with unusual rapidity and accuracy, but not more rapidly than any mathematician who is naturally quick and has the habit of computation. There is a table of partitions at the end of our paper. . . . This was, for the most part, calculated independently by Ramanujan and Major MacMahon; and Major MacMahon was, in general the slightly quicker and more accurate of the two.”


George E. Andrews
Bruce C. Berndt
Ramanujan's
Lost Notebook
Part I
Springer

THE PATH FROM THE LAST LETTER TO THE LOST NOTEBOOK

"He returned from England only to die, as the saying goes. He lived for less than a year. Throughout this period, I lived with him without break. He was only skin and bones. He often complained of severe pain. In spite of it he was always busy doing his Mathematics. That, evidently helped him to forget the pain. I used to gather the sheets of papers which he filled up. I would also give the slate whenever he asked for it. He was uniformly kind to me. In his conversation he was full of wit and humour. Even while mortally ill, he used to crack jokes. One day he confided in me that he might not live beyond thirty-five and asked me to meet the event with courage and fortitude. He was well looked after by his friends. He often used to repeat his gratitude to all those who had helped him in his life."

UNIVERSITY OF MADRAS,
12th January 1920.

I am extremely sorry for not writing you a single letter up to now .... I discovered very interesting functions recently which I call "Mock" $\theta$-functions. Unlike the "False" $\theta$-functions (studied partially by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as the ordinary $\theta$-functions. I am sending you with this letter some examples ....
Mock $\theta$-functions

$$\phi(q) = 1 + \frac{q}{1+q^2} + \frac{q^4}{(1+q^2)(1+q^4)} + \cdots,$$

$$\psi(q) = \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^3)} + \frac{q^9}{(1-q)(1-q^3)(1-q^5)} + \cdots.$$

......

Mock ϑ-functions

$$\phi(q) = 1 + \frac{q}{1+q^2} + \frac{q^4}{(1+q^2)(1+q^4)} + \cdots$$

$$\psi(q) = \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^3)} + \frac{q^9}{(1-q)(1-q^3)(1-q^5)} + \cdots$$

~~There are~~

$$\chi(q) = 1 + \frac{q}{1-q+q^2} + \frac{q^4}{(1-q+q^2)(1-q^2+q^4)} + \cdots$$

These are ~~substantially~~ related to $f(q)$ as shown below.

$$2\phi(-q) - f(q) = f(q) + 4\psi(-q) = \frac{1 - 2q + 2q^4 - 2q^9 + \cdots}{(1+q)(1+q^2)(1+q^3)\cdots}$$

$$4\chi(q) - f(q) =$$

These are of the 3rd order

Mock ϑ-functions (of 5th order)

$$f(q) = 1 + \frac{q}{1+q} + \frac{q^4}{(1+q)(1+q^2)} + \frac{q^9}{(1+q)(1+q^2)(1+q^3)} + \cdots$$

$$\phi(q) = 1 + q(1+q) + q^4(1+q)(1+q^3) + q^9(1+q)(1+q^3)(1+q^5) + \cdots$$

$$\psi(q) = q + q^3(1+q) + q^6(1+q)(1+q^2) + q^{10}(1+q)(1+q^2)(1+q^3) + \cdots$$

$$\chi(q) = 1 + \frac{q}{1-q^2} + \frac{q^2}{(1-q^3)(1-q^4)} + \frac{q^3}{(1-q^4)(1-q^5)(1-q^6)} + \cdots$$

$$= 1 + \left\{ \frac{q}{1-q} + \frac{q^3}{(1-q^2)(1-q^3)} + \frac{q^5}{(1-q^3)(1-q^4)(1-q^5)} + \cdots \right\}$$

$$F(q) = 1 + \frac{q^2}{1-q} + \frac{q^8}{(1-q)(1-q^3)} + \cdots$$

$$\phi(-q) + \chi(q) = 2F(q).$$

$$f(q) + 2F(q^2) - 2 = \phi(-q^2) + \psi(-q)$$

$$= 2\phi(-q^2) - f(q) = \frac{1 - 2q + 2q^4 - 2q^9 + \cdots}{(1-q)(1-q^4)(1-q^6)(1-q^9)\cdots}$$

$$\psi(q) - F(q^4) + 1 = q \cdot \frac{1 + q^2 + q^6 + q^{12} + \cdots}{(1-q^8)(1-q^{12})(1-q^{28})\cdots}$$

Mock $\vartheta$-functions (of 5th order)

$$f(q) = 1 + \frac{q^2}{1+q} + \frac{q^6}{(1+q)(1+q^2)} + \frac{q^{12}}{(1+q)(1+q^2)(1+q^3)} + \cdots$$

$$\phi(q) = q + q^4(1+q) + q^9(1+q)(1+q^2) + \cdots$$

$$\psi(q) = 1 + q(1+q) + q^3(1+q)(1+q^2) + q^6(1+q)(1+q^2)(1+q^3) \cdots$$

$$\chi(q) = \frac{1}{1-q} + \frac{q}{(1-q^2)(1-q^3)} + \frac{q^2}{(1-q^3)(1-q^4)(1-q^5)} + \frac{q^3}{(1-q^4)(1-q^5)(1-q^6)(1-q^7)}$$

$$F(q) = \frac{1}{1-q} + \frac{q^4}{(1-q)(1-q^3)} + \frac{q^{12}}{(1-q)(1-q^3)(1-q^5)}$$

have got similar relations as above.

Mock $\vartheta$-functions (of 7th order)

(i) $1 + \frac{q}{1-q^2} + \frac{q^4}{(1-q^3)(1-q^4)} + \frac{q^9}{(1-q^4)(1-q^5)(1-q^6)}$

(ii) $\frac{q}{1-q} + \frac{q^4}{(1-q^2)(1-q^3)} + \frac{q^9}{(1-q^3)(1-q^4)(1-q^5)}$

(iii) $\frac{1}{1-q} + \frac{q^2}{(1-q^2)(1-q^3)} + \frac{q^6}{(1-q^3)(1-q^4)(1-q^5)}$

These are not related to each other.

Ever yours sincerely
S. Ramanujan

When the Royal Society asked me to write G. N. Watson's obituary memoir I wrote to his widow to ask if I could examine his papers. She kindly invited me to lunch and afterwards his son took me upstairs to see them. They covered the floor of a fair sized room to a depth of about a foot, all jumbled together, and were to be incinerated in a few days. One could make lucky dips and, as Watson never threw anything away, the result might be a sheet of mathematics but more probably a receipted bill or a draft of his income tax for 1923. By extraordinary stroke of luck one of my dips brought up the Ramanujan material which Hardy must have passed on to him when he proposed to edit the earlier notebooks.

$$F(q) = \frac{(1-q)(1-q^2)(1-q^3)\ldots}{\left(1-2q\cos\frac{2n\pi}{5}+q^2\right)\left(1-2q^2\cos\frac{2n\pi}{5}+q^4\right)\ldots}$$

$$f(q) = 1 + \frac{q}{1-2q\cos\frac{2n\pi}{5}+q^2} + \frac{q^4}{\left(1-2q\cos\frac{2n\pi}{5}+q^2\right)\left(1-2q^2\cos\frac{2n\pi}{5}+q^4\right)} + \ldots\ldots, \quad n = 1 \text{ or } 2.$$

$$F(q^{\frac{1}{5}}) = A(q) - 4q^{\frac{1}{5}}\cos^2\frac{2n\pi}{5}\cdot B(q) + 2q^{\frac{2}{5}}\cos\frac{4n\pi}{5}\cdot C(q) - 2q^{\frac{3}{5}}\cos\frac{2n\pi}{5}\,D(q);$$

$$f(q^{\frac{1}{5}}) = \left\{A(q) - 4\sin^2\frac{n\pi}{5}\,\phi(q)\right\} + q^{\frac{1}{5}}B(q) + 2q^{\frac{2}{5}}\cos\frac{2n\pi}{5}\,C(q) - 2q^{\frac{3}{5}}\cos\frac{2n\pi}{5}\left\{D(q) + 4\sin^2\frac{2n\pi}{5}\,\frac{\psi(q)}{q}\right\}.$$

$$A(q) = \frac{1-q^2-q^3+q^9+\ldots}{(1-q)^2(1-q^4)^2(1-q^6)^2\ldots}$$

$$B(q) = \frac{(1-q^5)(1-q^{10})(1-q^{15})\ldots}{(1-q)(1-q^4)(1-q^6)\ldots}$$

$$C(q) = \frac{(1-q^5)(1-q^{10})(1-q^{15})\ldots}{(1-q^2)(1-q^3)(1-q^7)\ldots}$$

$$D(q) = \frac{1-q-q^4+q^7+\ldots}{(1-q^2)^2(1-q^3)^2(1-q^7)^2\ldots}$$

$$\phi(q) = -1 + \left\{\frac{1}{1-q} + \frac{q^5}{(1-q)(1-q^4)(1-q^6)} + \frac{q^{20}}{(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})} + \ldots\right\}.$$

$$\psi(q) = -1 + \left\{\frac{1}{1-q^2} + \frac{q^5}{(1-q^2)(1-q^3)(1-q^7)} + \frac{q^{20}}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)(1-q^{12})} + \ldots\right\}$$

$$\frac{q}{1-q} + \frac{q^3}{(1-q^2)(1-q^3)} + \frac{q^5}{(1-q^3)(1-q^4)(1-q^5)} + \ldots = 3\phi(q) + 1 - A(q).$$

$$\frac{q}{1-q} + \frac{q^2}{(1-q^2)(1-q^3)} + \frac{q^3}{(1-q^3)(1-q^4)(1-q^5)} + \ldots = 3\psi(q) + qD(q).$$

$$\frac{q^2}{1-q} + \frac{q^8}{(1-q)(1-q^3)} + \frac{q^{18}}{(1-q)(1-q^3)(1-q^5)} + \ldots = \phi(q) - q\cdot\frac{1+q^5+q^{15}+\ldots}{(1-q^4)(1-q^6)(1-q^{14})\ldots}$$

$$\frac{q}{1-q} + \frac{q^6}{(1-q)(1-q^3)} + \frac{q^{13}}{(1-q)(1-q^3)(1-q^5)} + \ldots = \psi(q) + q\cdot\frac{1+q^5+q^{15}+\ldots}{(1-q^2)(1-q^8)(1-q^{12})\ldots}$$

$$\frac{1}{1+q} - \frac{q^2(1-q)}{(1+q)(1+q^2)(1+q^3)} + \frac{q^6(1-q)(1-q^3)}{(1+q)(1+q^2)(1+q^3)(1+q^4)(1+q^5)} - \cdots$$

$$= 1 - q + q^3 - q^6 + \cdots$$

$$\frac{1}{1+q} + \frac{q(1-q)^2}{(1+q)(1+q^2)\cdots} + \frac{q^2(1-q)^2(1-q^3)^2}{(1+q)(1+q^2)\cdots(1+q^5)}$$

$$= 1 - q^2 + q^6 - q^{12} + \cdots$$

$$\frac{1}{1+q} + \frac{q^3(1-q)}{(1+q)(1+q^2)(1+q^3)} + \frac{q^6(1-q)(1-q^3)}{(1+q)(1+q^2)\cdots(1+q^5)}$$

$$= 1 - q^3 + q^9 - q^{18} + \cdots$$

$$\frac{1}{1+q} + \frac{q(1-q)}{(1+q)(1+q^3)} + \frac{q^2(1-q)(1-q^3)}{(1+q)(1+q^3)(1+q^5)} +$$

$$= 1 - q^4 + q^{12} - q^{24} + \cdots$$

$$\frac{1}{1+q} + \frac{q(1-q)}{(1+q)(1+q^3)} + \frac{q^2(1-q)(1+q^2)}{(1+q)(1+q^3)(1+q^5)} + \cdots$$

$$= 1 - q^6 + q^{18} - q^{36} + \cdots$$

$$\frac{q}{(1-q)^2} + \frac{q^3(1+q)(1+q^2)}{(1-q)^2(1-q^3)^2} + \frac{q^5(1+q)(1+q^2)(1+q^3)(1+q^4)}{(1-q)^2(1-q^3)^2(1-q^5)^2}$$

$$= \frac{1}{2}\cdot\frac{1-q+q^3-q^6+\cdots}{\{(1-q)(1-q^2)(1-q^3)\cdots\}^3} - \frac{1}{2}(1-q^2+q^6-q^{12}+\cdots)$$

$$\phi(q) = 1 - \frac{q(1-q)}{(1+q)(1+q^2)} + \frac{q^4(1-q)(1-q^3)}{(1+q)(1+q^2)(1+q^3)(1+q^4)} - \cdots$$

$$\psi(q) = \frac{q}{1+q} - \frac{q^4(1-q)}{(1+q)(1+q^2)(1+q^3)} + \frac{q^9(1-q)(1-q^3)}{(1+q)(1+q^2)(1+q^3)(1+q^4)(1+q^5)} - \cdots$$

$$\phi(q^3) - \psi(q^{\frac{1}{3}}) - q^{-1}\psi(q^3) = \frac{(1+q+q^3+\cdots)(1-q^2)^2(1-q^4)^2\cdots}{(1+q^{\frac{1}{3}}+q+\cdots)(1+q^3+q^9+\cdots)}$$

$$\frac{\psi(\omega q^{\frac{1}{3}}) - \psi(\omega^2 q^{\frac{1}{3}})}{(\omega-\omega^2)q^{\frac{1}{3}}} = \frac{(1+q^{\frac{1}{3}}+q+\cdots)(1+q^3+q^9+\cdots)(1-q)^2(1-q^2)^2\cdots}{1+q+q^3+q^6+\cdots}$$

$$q^{-1}\psi(q^2) + \frac{1}{1-q} + \frac{q(1+q)}{(1-q)(1-q^3)} + \frac{q^3(1+q)(1+q^2)}{(1-q)(1-q^3)(1-q^5)} + \cdots$$

$$= (1+q)^2(1+q^3)^2(1+q^5)^2\cdots(1+q+q^3+\cdots)$$

$$\frac{1}{2}\phi(q^2) + \frac{q}{1-q} + \frac{q^3(1+q)}{(1-q)(1-q^3)} + \frac{q^6(1+q)(1+q^2)}{(1-q)(1-q^3)(1-q^5)} + \cdots$$

$$= \frac{1}{2}(1+q)^2(1+q^3)^2(1+q^5)^2\cdots(1+2q^3+2q^{12}+\cdots)$$

$$= \phi(q^2) - \left\{1 - \frac{1+q}{1-q} + \frac{(1+q)(1+q^3)}{(1-q)(1+q^2)} - \cdots\right\}$$

$$= 2q^{-1}\psi(q^2) + \left\{1 + q\cdot\frac{1+q}{1-q} + q^2\cdot\frac{(1+q)(1+q^3)}{(1-q)(1+q^2)} + \cdots\right\}$$

SA18

-11. 25 $a_2(a_1-1)(a_5+a_4+a_3+a_2+a_1+1)$

-4. 26 $a_2(a_3+1)(a_3+a_2+a_1+1)$.

$$\frac{1}{1+a} - \left(\frac{q}{1+aq}+\frac{q}{a+q}\right) + \frac{q^3}{1+aq^2}+\frac{q^3}{a+q^2}$$

$$1-\{q(a_1+1)-q^2(a_2+a_1)+q^3(a_3+a_2)-$$
$$+q^3(a_1+1)-q^5(a_2+a_1)$$

$$1+q(a_1-1)+q^2(a_2-a_1)+q^3(a_3-a_2)+q^4(a_4-a_3)+$$
$$-\{q^3(a_1-1)+q^5(a_2-a_1)+q^7(a_3-a_2)+q^9(a_4-a_3)-\}$$
$$+\{q^6(a_1-1)+q^9(a_2-a_1)+q^{12}(a_3-a_2)+q^{15}(a_4-a_3)$$
$$-\{q^{10}(a_1-1)+q^{14}(a_2-a_1)+q^{18}(a_3-a_2)+q^{22}(a_4-a_3)$$
$$+\{q^{15}(a_1-1)+q^{20}(a_2-a_1)+q^{25}(a_3-a_2)$$
$$-\{q^{21}(a_1-1)+\}$$

$$1-q-q^2+q^5+q^7-q^{12}-q^{15}+q^{22}$$

$$1+q(a_1-1)+a_2q^2+(a_3+1)q^3+q^4(a_4+\dots)+q^5(a_3+a_2+a_1+1)$$

$a_3+1+a_2+a_4$

$a_3+a_1+1+a_2+x$

$$
\begin{aligned}
1 &+ q(a_1-1) + q^2 a_2 + q^3(a_3+1) + q^4(a_4+a_2+1) \\
&+ q^5(a_5+a_3+a_1+1) + q^6(a_6+a_4+a_3+a_2+a_1+1) \\
&+ q^7(a_3+1)(a_4+a_2+1) + q^8 a_2(a_6+a_4+a_3+a_2+a_1+1) \\
&+ q^9 a_2(a_3+1)(a_4+a_2+1) + q^{10} a_2(a_3+1)(a_5+a_3+a_1+1) \\
&+ q^{11} a_1 a_2(a_8+a_5+a_4+a_3+a_2+a_1+2) \\
&+ q^{12}(a_3+a_2+a_1+1)(a_5+a_4+a_3+a_2+a_1+1) \\
&\qquad \times (a_4-2a_3+2a_2-a_1+1). \\
&+ q^{13}(a_1-1)(a_2-a_1+1)(a_{10}+2a_9+2a_8+2a_7+3a_6 \\
&\qquad +4a_5+6a_4+8a_3+9a_2 \\
&\qquad +9a_1+9) \\
&+ q^{14}(a_2+1)(a_3+1)(a_4+a_2+1) \times (a_5-a_3+a_1+1) \\
&+ q^{15} a_1 a_2(a_5+a_4+a_3+a_2+a_1+1)(a_7-a_6+a_4+a_1) \\
&+ q^{16}(a_3+1)(a_3+a_2+a_1+1)(a_5+a_4+a_3+a_2+a_1+1) \\
&\qquad \times (a_5-2a_4+2a_3-2a_2+3a_1-3) \\
&+ q^{17}(a_2+1)(a_3+1)(a_5+a_4+a_3+a_2+a_1+1) \\
&\qquad \times (a_7-a_6+a_3+a_1-1) \\
&+ q^{18}(a_4+a_2+1)(a_3+a_2+a_1+1)(a_5+a_4+a_3+a_2+a_1+1) \\
&\qquad \times (a_6-2a_5+a_4+a_3-a_2+1) \\
&+ q^{19} a_2(a_1-1)(a_4+a_2+1)(a_3+a_2+a_1+1) \\
&\qquad \times (a_9-a_7+a_4+2a_3+a_2-1) \\
&+ q^{20}(a_2-a_1+1)(a_3+1)(a_5+a_4+a_3+a_2+a_1+1) \\
&\qquad \times (a_{10}+a_6+a \ldots
\end{aligned}
$$

THE MOCK THETA CONJECTURES

$$f_0(q) = 1 + \frac{q}{1+q} + \frac{q^4}{(1+q)(1+q^2)} + \cdots,$$

$$\phi_0(q) = 1 + q(1+q) + q^4(1+q)(1+q^3) + q^9(1+q)(1+q^3)(1+q^5) + \cdots,$$

$$\psi_0(q) = q + q^3(1+q) + q^6(1+q)(1+q^2) + q^{10}(1+q)(1+q^2)(1+q^3) + \cdots,$$

$$F_0(q) = 1 + \frac{q^2}{1-q} + \frac{q^8}{(1-q)(1-q^3)} + \cdots,$$

$$\chi_0(q) = 1 + \frac{q}{1-q^2} + \frac{q^2}{(1-q^3)(1-q^4)} + \frac{q^3}{(1-q^4)(1-q^5)(1-q^6)} + \cdots$$

$$= 1 + \frac{q}{1-q} + \frac{q^3}{(1-q^2)(1-q^3)} + \frac{q^5}{(1-q^3)(1-q^4)(1-q^5)} + \cdots$$

$$\phi(q) = -1 + \left\{ \frac{1}{1-q} + \frac{q^5}{(1-q)(1-q^4)(1-q^6)} + \frac{q^{20}}{(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})} + \cdots \right\},$$

$$\psi(q) = -1 + \left\{ \frac{1}{1-q^2} + \frac{q^5}{(1-q^2)(1-q^3)(1-q^7)} \right.$$

$$\phi_0(-q) = \prod_{n=0}^{\infty} \frac{(1-q^{5n+5})(1+q^{5n+2})(1+q^{5n+3})}{(1-q^{10n+2})(1-q^{10n+8})}$$

$$+1 - \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(1-q)(1-q^6)\cdots(1-q^{5n+1})(1-q^4)(1-q^9)\cdots(1-q^{5n-1})}.$$

$$\phi_0(-q) = \prod_{n=0}^{\infty} \frac{(1-q^{5n+5})(1+q^{5n+2})(1+q^{5n+3})}{(1-q^{10n+2})(1-q^{10n+8})}$$

$$+1 - \prod_{n=0}^{\infty}(1-q^{5n+5})^{-1} \left\{ \frac{1}{1-q} + (1-q^{-1}) \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(15n+5)/2}(1+q^{5n})}{(1-q^{5n+1})(1-q^{5n-1})} \right\}.$$

EQUIVALENTLY

$$\prod_{n=1}^{\infty}(1-q^{10n})\sum_{n=0}^{\infty}\frac{q^{n^2}}{(1+q)\cdots(1+q^n)}$$

$$=\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n}}{1-q^{5n+1}}$$

$$-2\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{15n^2+15n+2}}{1-q^{10n+2}}$$

MOCK THETA CONJECTURES

$r_a(n) = \#$ OF PTNS OF $n$ WITH RANK $\equiv a \pmod 5$

rank = (largest part) − (# of parts)

The rank of $5 + 4 + 2 + 1 + 1 + 1$ is $5 - 6 = -1$.

1ST MOCK CONJECTURE
(Proved by Hickerson)

$$r_1(5n) - r_0(5n)$$

EQUALS THE # OF PTNS OF $n$ IN WHICH THE LARGEST PART IS ODD AND EACH PART IS $\geqq \frac{1}{2}$(LARGEST PART)

SUMS-OF-TAILS

If $s = (1+q)(1+q^2)(1+q^3)\ldots$ then

$$1 + \frac{q}{1+q} + \frac{q^3}{(1+q)(1+q^2)} + \frac{q^6}{(1+q)(1+q^2)(1+q^3)} + \cdots$$

$$= 1 + q\{1 - q(1-q) + q^2(1-q)(1-q^2) - q^3(1-q)(1-q^2)(1-q^3) + \cdots\}$$

$$= 2\left\{\tfrac{1}{2}s + (s-1) + (s-(1+q)) + (s-(1+q)(1+q^2)) + \cdots\right\}$$

$$- 2s\left(\frac{q}{1-q} + \frac{q^2}{1-q^2} + \frac{q^3}{1-q^3} + \cdots\right)$$

$$= 2\left\{\tfrac{1}{2}s + \left(s - \frac{1}{1-q}\right) + \left(s - \frac{1}{(1-q)(1-q^3)}\right) + \cdots\right\}$$

$$- 2s\left(\frac{q^2}{1-q^2} + \frac{q^4}{1-q^4} + \frac{q^6}{1-q^6} + \cdots\right)$$

OR IN STANDARD NOTATION

IF $S = \prod_{n=1}^{\infty}(1+q^n) = \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n-1})}$

THEN

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(1+q)(1+q^2)\cdots(1+q^n)}$$

$$=1 = q\sum_{n=0}^{\infty}(-q)^n(1-q)(1-q^2)\cdots(1-q^n)$$

$$=2\{\frac{S}{2} + \sum_{n=1}^{\infty}(S-(1+q)(1+q^2)\cdots(1+q^n)) - 2S\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}$$

$$=2\{\frac{S}{2} + \sum_{n=0}^{\infty}(S - \frac{1}{(1-q)(1-q^3)\cdots(1-q^{2n+1})}) - 2S\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}$$

LET

$$
\begin{aligned}
&R(q) := \\
&\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(1+q)(1+q^2)\cdots(1+q^n)} \\
=&1 + q\sum_{n=0}^{\infty}(-q)^n(1-q)(1-q^2)\cdots(1-q^n)
\end{aligned}
$$

Dyson, Hickerson & I proved that if

$$R(q) = \sum_{n=0}^{\infty} c_n q^n,$$

then asymptotically 100% of the $c_n$ are ZERO, and for any integer $M$ between $-\infty$ and $\infty$, $c_n = M$ infinitely many times.

ALSO IT IS INFORMATIVE TO CONTRAST

$$R(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(1+q)(1+q^2)\cdots(1+q^n)}$$

with

$$1 + \sum_{n=1}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(1+q)(1+q^2)\cdots(1+q^n)}$$

ALSO IT IS INFORMATIVE TO CONTRAST

$$R(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(1+q)(1+q^2)\cdots(1+q^n)}$$

with

$$1 + \sum_{n=1}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(1+q)(1+q^2)\cdots(1+q^n)}$$

$$= 2$$

NATHAN FINE, IN HIS WONDERFUL, RAMANUJANESQUE BOOK, *BASIC HYPERGEOMETRIC SERIES AND APPLICATIONS*, INDEPENDENTLY FOUND ONE RESULT OF THIS NATURE:

LET

$$\Phi(q) = \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\frac{n(n+1)}{2}}}{(1-q)\cdots(1-q^{n-1})(1-q^n)}$$

THEN

$$\Phi(q) = \sum_{n=0}^{\infty} \left(1 - \prod_{j>n} (1-q^j)\right)$$

OR

$$\frac{\Phi(q)}{\prod_{n=1}^{\infty}(1-q^n)} = \sum_{j=0}^{\infty} \left( \frac{1}{\prod_{k=1}^{\infty}(1-q^k)} - \prod_{k=1}^{j} \frac{1}{(1-q^k)} \right)$$

SIMILARLY, D. ZAGIER HAS PROVED

$$\sum_{n \geqq 0} \left( \prod_{j=1}^{\infty} (1 - q^j) - \prod_{j=1}^{n} (1 - q^j) \right)$$
$$= \frac{1}{2} H(q) + D(q) \prod_{j=1}^{\infty} (1 - q^j),$$

where

$$H(q) = \sum_{n=1}^{\infty} n \chi(n) q^{(n^2 - 1)/24}$$

$\chi$ the unique primitive character of conductor 12.

ZAGIER STARTS FROM

$$\sum_{n=0}^{\infty}\left(\prod_{j=1}^{\infty}(1-q^j)-\prod_{j=1}^{n}(1-q^j)\right)$$
$$=-\sum_{n=1}^{\infty}(1-q)(1-q^2)\cdots(1-q^{n-1})nq^n$$

WHICH MAY BE PROVED DIRECTLY BY EXAMINING THE PARTIAL SUMS ON EACH SIDE.

THIS IS EXACTLY ANALOGOUS TO THE TREATMENT OF RAMANUJAN'S FORMULA

$$\sum_{n=0}^{\infty}\left(S-\prod_{j=1}^{n}(1+q^j)\right)=\sum_{n=1}^{\infty}(1+q)(1+q^2)\cdots(1+q^{n-1})nq^n$$

(Adv. in Math., 61(1986), p. 159).
ALSO EASILY PROVED

KEN ONO, JORGE JIMENEZ URROZ and I EXAMINED THE FOLLOWING QUESTION: WHAT IS GOING ON HERE?

IN EACH CASE, THERE IS A MISSING VARIABLE WHICH HAS BEEN DIFFERENTIATED AND THEN SET $= 1$.
IS THERE A GENERAL PRINCIPLE?

THEOREM. SUBJECT TO SOME TEDIOUS BUT NECESSARY CONVERGENCE CONDITIONS,
IF

$$\lim_{n\to\infty} \alpha_n = \alpha,$$

AND

$$f(z) = \sum_{n=0}^{\infty} z^n \alpha_n,$$

THEN

$$\lim_{z\to 1^-} \frac{d}{dz}(1-z)f(z) = \sum_{n=0}^{\infty}(\alpha - \alpha_n).$$

LET

$$(a, n) = a(a+1)(a+2)\cdots(a+n-1)$$

P. FREITAS AND I EXTENDED THIS RESULT TO

$$\frac{1}{p}\lim_{z\to 1^-}\left\{\frac{d^p}{dz^p}(1-z)f(z)\right\}$$
$$=\sum_{n=0}^{\infty}(n+1, p-1)(\alpha-\alpha_{n+p-1})$$

DELIGHTFULLY, THIS GENERAL THEOREM (INSPIRED BY RAMANUJAN) BECAME ESSENTIAL IN JOINT WORK WITH GARVAN AND LIANG CONCERNING spt($n$), THE NUMBER OF SMALLEST PARTS IN THE PARTITIONS ON $n$.

FOR EXAMPLE: $\mathrm{spt}(4) = 10$

$\underline{4}$, $3+\underline{1}$, $\underline{2}+\underline{2}$, $2+\underline{1}+\underline{1}$, $\underline{1}+\underline{1}+\underline{1}+\underline{1}$

spt($n$) CAME INTO PROMINENCE WHEN, IN THE SPIRIT OF RAMANUJAN'S CONGRUENCES FOR $p(n)$, IT WAS FOUND THAT

$$\begin{aligned} \mathrm{spt}(5n+4) &\equiv 0 \pmod 5 \\ \mathrm{spt}(7n+5) &\equiv 0 \pmod 7 \\ \mathrm{spt}(13n+6) &\equiv 0 \pmod{13} \end{aligned}$$

THE LAST BEING A GREAT SURPRISE.

GARVAN, LIANG & I PROVIDE COMBINATORIAL EXPLANATIONS OF THE CONGRUENCES $\mod 5$ AND 7 AS WELL AS A FULL ACCOUNT OF THE PARITY OF $\mathrm{spt}(n)$ (FIRST DONE BY FOLSOM & ONO).

CENTRAL TO OUR DEVELOPMENT ARE RESULTS LIKE:

$$\sum_{n=0}^{\infty}\frac{((q)_{2n}-(q)_{\infty})}{(q^2;q^2)_n} = \sum_{n=1}^{\infty}\frac{(-1)^n q^{n^2}}{(q;q^2)_n}$$

$$\sum_{n=0}^{\infty}\frac{((q)_{n}-(q)_{\infty})}{(q)_n^2} = \sum_{n=1}^{\infty}\frac{nq^{n^2}}{(q)_n^2},$$

where

$$(A)_n = (A;q)_n = \prod_{j=0}^{n-1}(1-Aq^j).$$

THE 40 IDENTITIES FOR THE ROGERS-RAMANUJAN FUNCTIONS AND GENERALIZATIONS

LET

$$G(q) := \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-4})(1-q^{5n-1})},$$

AND

$$H(q) := \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-3})(1-q^{5n-2})}.$$

In 1975, Brian Birch uncovered Ramanujan's manuscript on 40 identities relating $G(q)$ and $H(q)$. A typical example is

$$G(q)G(q^4) + qH(q)H(q^4) = \frac{\phi(q)}{(q^2;q^2)_\infty}$$

where $\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$.

PRIOR TO THE DISCOVERY OF THE LOST NOTEBOOK, EVERY KNOWN PROOF OF THESE FORMULAE RELIED HEAVILY ON THE MODULAR ASPECTS OF $G(q)$ AND $H(q)$. IN THE LOST NOTEBOOK, WE FIND A NON-MODULAR GENERALIZATION OF THE PREVIOUSLY STATED IDENTITY.

NAMELY,

$$\left(\sum_{n=0}^{\infty}\frac{a^n q^{n^2}}{(q^4;q^4)_n}\right)\left(\sum_{n=0}^{\infty}\frac{a^{-2n} q^{4n^2}}{(q^4;q^4)_n}\right)$$
$$+\left(\sum_{n=0}^{\infty}\frac{a^n q^{(n+1)^2}}{(q^4;q^4)_n}\right)\left(\sum_{n=0}^{\infty}\frac{a^{-2n-1} q^{4n^2+4n}}{(q^4;q^4)_n}\right)$$
$$=\frac{1}{(q^4;q^4)_\infty}\sum_{n=-\infty}^{\infty}a^n q^{n^2}$$

$a = 1$ YIELDS THE ORIGINAL FORMULA ONCE WE RECALL SOME RESULTS OF L. J. ROGERS

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n}$$
$$= (-q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n},$$

AND

$$H(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n}$$
$$= (-q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^4;q^4)_n}$$

SIMILAR GENERALIZATIONS HOLD FOR RELATED RESULTS OF G. N. WATSON

$$G(-q)\phi(q) - G(q)\phi(-q) = 2qH(q^4)\psi(q^2)$$

AND

$$H(-q)\phi(q) + H(q)\phi(-q) = 2G(q^4)\psi(q^2),$$

WHERE

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$$

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}$$

HOW RAMANUJAN PROVED THE RICHMOND-SZEKERES THEOREM

ROGERS-RAMANUJAN CONTINUED FRACTION

$$1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cfrac{q^4}{1+\ddots}}}}$$

$$= C(q)$$

$$= \frac{(1-q^2)(1-q^3)(1-q^7)(1-q^8)(1-q^{12})\cdots}{(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})\cdots}$$

G. Szekeres & B. Richmond examined the power series of $C(q)$

$$
\begin{aligned}
&C(q)\\
&= 1 + q - q^3\\
&+ q^5 + q^6 - q^7 - 2q^8\\
&+ 2q^{10} + 2q^{11} - q^{12} - 3q^{13} - q^1 4\\
&+ 3q^{15} + 3q^{16} - 2q^{17} - 5q^{18} - q^{19}\\
&+ 6q^{20} + 5q^{21} - 3q^{22} - 8q^{23} - 2q^{24}\\
&+ \cdots
\end{aligned}
$$

LET $C(q) = \sum_{n=0}^{\infty} c_n q^n$, R. and S. showed that *eventually* $c_n$ and $c_{n+5}$ have the same sign by showing

$$c_n = \frac{\sqrt{2}}{(5n)^{3/4}} e^{\frac{4\pi}{25}\sqrt{5n}} \times \left\{ \cos\left( \frac{2\pi}{5}(n - \frac{2}{5}) \right) + O(\frac{1}{\sqrt{n}}) \right\}.$$

IN THE LOST NOTEBOOK WE FIND

$$C(q) = \frac{\sum_{n=-\infty}^{\infty} (-1)^n q^{(15n^2+n)/2}(1+q^{5n+1})}{\prod_{n=1}^{\infty}(1-q^{5n})}$$

FROM THIS RAMANUJAN FINDS FORMULAE FOR

$$\sum_{m=0}^{\infty} c_{5m+j} q^m$$

$j = 0, 1, 2, 3, 4.$

FOR EXAMPLE

$$\sum_{m=0}^{\infty} c_{5m} q^m$$

$$= \prod_{\substack{n=1 \\ n \not\equiv 0, \pm 37 \pmod{75}}}^{\infty} \frac{1}{1-q^n}$$

$$+ q^4 \prod_{\substack{n=1 \\ n \not\equiv 0, \pm 13 \pmod{75}}}^{\infty} \frac{1}{1-q^n}$$

HOWEVER

$$\sum_{m=0}^{\infty} c_{5m+2} q^m =$$

$$-q \prod_{\substack{n=1 \\ n\not\equiv 0,\pm 23 \pmod{75}}}^{\infty} \frac{1}{1-q^n}$$

$$+q^8 \prod_{n\not\equiv 0,\pm 2 \pmod{75}}^{\infty} \frac{1}{1-q^n}$$

A PARTITION THEOREM OF B. GORDON ALLOWS US TO TREAT THE SIGN PROBLEM FULLY.

$$A_{k,a}(n) = B_{k,a}(n)$$

WHERE
$A_{k,a}(n) =$# OF PTNS OF $n$ INTO PARTS $\not\equiv 0, \pm a (\mod 2k+1)$.
$B_{k,a}(n) =$# OF PTNS OF $n$ INTO $b_1 + b_2 + \cdots + b_j$ WITH $b_i \geqq b_{i+1}$, $b_i - b_{i+k-1} \geqq 2$, & AT MOST $a-1$ OF THE $b_i$ ARE $= 1$.

FOR $1 < a \leqq k$

$$A_{k,a}(n)$$

is non-decreasing in $n$.

$$B_{k,a}(n)$$

is non-decreasing in $a$.

THUS

$$c_{5m+2} = -B_{37,23}(m-1) + B_{37,2}(m-8) < 0$$

for $m \geqq 1$.

INNOCENTS ABROAD

HERE ARE TWO SEEMINGLY BENIGN IDENTITIES

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{(-q;q)_n^2}=\sum_{n=0}^{\infty}\frac{(-q)^{n(n+1)/2}}{(-q^2;q^2)_n}-2\sum_{n=1}^{\infty}\frac{(-1)^n q^{2n^2}}{(-q;q^2)_{2n}},$$

$$\phi(-q)\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q^2;q^2)_n}=\sum_{n=0}^{\infty}\frac{(-q)^{n(n+1)/2}}{(-q^2;q^2)_n}+2\sum_{n=1}^{\infty}\frac{(-1)^n q^{2n^2}}{(-q;q^2)_{2n}}.$$

WHERE $\phi(q)=\sum_{n=-\infty}^{\infty}q^{n^2}$

THE PROOF REQUIRES TWO SEEMINGLY TRIVIAL FINITE IDENTITIES

$$(q;q^2)_m \sum_{n=0}^{2m} \frac{(-1)^n}{(\alpha;q)_n(q;q)_{2m-n}} = \sum_{n=0}^{m} \frac{(-1)^n q^{n^2}}{(q^2;q^2)_{m-n}(\alpha q;q^2)_n}.$$

$$(q;q^2)_m \sum_{n=0}^{2m-1} \frac{(-1)^n}{(q;q)_n(\alpha;q)_{2m-1-n}} = \left(1-\frac{\alpha}{q}\right) \sum_{n=1}^{m} \frac{(-1)q^{n^2}}{(q^2;q^2)_{m-n}(\alpha;q^2)_n}.$$

AND THREE $q$-HYPERGEOMETRIC SERIES IDENTITIES

$$ {}_{r+1}\phi_r\left(\begin{matrix} a_0, a_1, \ldots, a_2; q, t \\ b_1, \ldots, b_r \end{matrix}\right) = \sum_{n=0}^{\infty} \frac{(a_0, a_1, \ldots, a_r; q)_n t^n}{(q, b_1, \ldots, b_r; q)_n} $$

$$
\begin{aligned}
&\lim_{N\to\infty} {}_{10}\phi_9\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, r_1, -r_1, r_2, -r_2, q^{-N}, -q^{-N}; q, -\frac{a^3q^{2N+3}}{br_1^2r_2^2} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{r_1}, -\frac{aq}{r_1}, \frac{aq}{r_2}, -\frac{aq}{r_2}, aq^{N+1}, -aq^{N+1} \end{matrix}\right) \\
&\qquad = \frac{(a^2q^2;q^2)_\infty (a^2q^2/(r_1^2r_2^2);q^2)_\infty}{(a^2q^2/r_1^2;q^2)_\infty (a^2q^2/r_2^2;q^2)_\infty} \\
&\qquad \times \sum_{n=0}^{\infty} \frac{(r_1^2;q^2)_n (r_2^2;q^2)_n (-aq/b;q)_{2n}}{(q^2;q^2)_n (a^2q^2/b^2;q^2)_n (-aq;q)_{2n}} \left(\frac{a^2q^2}{r_1^2r_2^2}\right)^n
\end{aligned}
$$

$$\lim_{N\to\infty} {}_{10}\phi_9\left(\begin{matrix} a, q^2\sqrt{a}, -q^2\sqrt{a}, p_1, p_1q, p_2, p_2q, f, q^{-2N}, q^{-2N+1}; q^2, -\frac{a^3q^{4N+3}}{p_1^2p_2^2f} \\ \sqrt{a}, -\sqrt{a}, \frac{aq^2}{p_1}, \frac{aq}{p_1}, \frac{aq^2}{p_2}, \frac{aq}{p_2}, \frac{aq^2}{f}, aq^{2N+2}, aq^{2N+1} \end{matrix}\right)$$

$$\frac{(aq;q)_\infty(aq/(p_1p_2);q)_\infty}{(aq/p_1;q)_\infty(aq/p_2;q)_\infty}\sum_{n=0}^{\infty}\frac{(p_1;q)_n(p_2;q)_n(aq/f;q^2)_n}{(q;q)_n(aq;q^2)_n(aq/f;q)_n}\left(\frac{aq}{p_1p_2}\right)^n.$$

$$
{}_{10}\phi_9\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b_1, c_1, b_2, c_2, b_3, c_3, q^{-N}; q, -\frac{a^3 q^{N+3}}{b_1 b_2 b_3 c_1 c_2 c_3} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b_1}, \frac{aq}{c_1}, \frac{aq}{b_2}, \frac{aq}{c_2}, \frac{aq}{b_3}, \frac{aq}{c_3}, aq^{N+1} \end{matrix}\right)
$$

$$
= \frac{(aq;q)_N (aq/(b_3 c_3);q)_N}{(aq/b^3;q)_N (aq/c_3;q)_N}
$$

$$
\times \sum_{m_1, m_2 = 0}^{\infty} \frac{(aq/(b_1 c_1);q)_{m_1} (aq/(b_2 c_2);q)_{m_2} (b_2;q)_{m_1} (c_2;q)_{m_1}}{(q;q)_{m_1} (q;q)_{m_2} (aq/b_1;q)_{m_1} (aq/c_1;q)_{m_1}}
$$

$$
\times \frac{(b_3;q)_{m_1+m_2} (c_3;q)_{m_1+m_2} (q^{-N};q)_{m_1+m_2} (aq)^{m_1} q^{m_1+m_2}}{(aq/b_2;q)_{m_1+m_2} (aq/c_2;q)_{m_1+m_2} (b_3 c_3 q^{-N}/a;q)_{m_1+m_2} (b_3 c_3 q^{-N}/a;q)_{m_1+m_2} (b_2 c_2)^{m_1}}
$$

SURELY THERE MUST BE AN EASIER PROOF

TRINKETS

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{q^n}{(1+q^2)(1+q^4)\cdots(1+q^{2n})} \\
=&1+q+q^2+q^5-q^7-q^{12}-q^{15}-q^{22} \\
&+q^{26}+q^{35}+q^{40}+q^{51}-\cdots \\
=&\sum_{n=0}^{\infty}(-1)^n q^{n(6n+1)}(1+q^{4n+1}) \\
+&\sum_{n=0}^{\infty}(-1)^n q^{(2n+1)(3n+2)}(1+q^{4n+3})
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{q^n}{(1+q)(1+q^2)(1+q^3)\cdots(1+q^{2n})} \\
=&(1-q^{11}+q^{13}-\cdots)+q(1-q^5+q^{19}-\cdots) \\
=&\sum_{n=0}^{\infty} q^{12n^2+n}(1-q^{22n+11}) \\
&+q\sum_{n=0}^{\infty} q^{12n^2+7n}(1-q^{10n+5})
\end{aligned}
$$

$$\sum_{n=0}^{\infty} \frac{q^n}{(1+q)(1+q^3)\cdots(1+q^{2n+1})}$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{6n^2+4n}$$

$$\sum_{n=0}^{\infty} \frac{q^n}{(1+q)(1+q^2)(1+q^3)\cdots(1+q^{2n+1})}$$

$$= 1 - \sum_{n=1}^{\infty} q^{12n^2-n}(1-q^{2n})$$

$$+ q - \sum_{n=1}^{\infty} q^{12n^2-7n}(1-q^{14n})$$

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{q^n (q;q^2)_n}{(-q;q)_{2n}} \\
=&1 + q - q^2 - q^5 + q^7 + q^{12} - \cdots \\
=&\sum_{n=0}^{\infty} (-1) q^{n(3n+1)/2} (1 + q^{2n+1})
\end{aligned}
$$

Thank You!

Alles Gute zum Geburtstag!

Happy Birthday, Christian!