KrattenthalerFest

Strobl

12 September 2018

What this talk is about:

(1) Generalisations of the beta integral

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \qquad \mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0,$$

an integral first discovered by Euler in 1730.

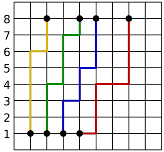
(2) The connections of such integrals with representation theory and conformal field theory.

(3) . . . but most importantly of all:

the celebration of 's birthday

and amazing career as

world-class pianist & combinatorialist.



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Asymptotic analysis of a Selberg-type integral via hypergeometrics

C. Krattenthaler[†]

Fakultät für Mathematik, Universität Wien,
Nordbergstraße 15, A-1090 Vienna, Austria.
WWW: http://www.mat.univie.ac.at/~kratt

Abstract. We show how to determine the asymptotics of a certain Selberg-type integral by means of tools available in the theory of (generalised) hypergeometric series. This provides an alternative derivation of a result of Carré, Deneufchâtel, Luque and Vivo [arχiv:1003.5996].

In [2], Carré, Deneufchâtel, Luque and Vivo consider the Selberg-type integral

$$S_k(a,b)\frac{1}{N!}\int_{[0,1]^N} x_1^k\left(\prod_{1\le i<j\le N}(x_i-x_j)^2\right)\left(\prod_{i=1}^N x_i^{a-1}(1-x_i)^{b-1}\,dx_i\right),$$

and they determine its asymptotic behaviour when N,a,b all tend to infinity so that $a\sim a_1N$ and $b\sim b_1N$, where a_1 and b_1 are given non-negative real numbers. The reader is referred to the introduction of [2] for information on motivation from random matrix theory connected to random scattering theory to investigate this question.

It should be noted that $S_0(a,b)$ *is* a Selberg integral, which can be evaluated in a product/quotient of gamma functions (cf. [3]). This being the case, the asymptotics of $S_0(a,b)$ is easily determined by means of known asymptotic formulae for the Barnes G-function (see [7]). Thus, it suffices to consider the quotient

$$J_k=\frac{S_k(a,b)}{S_0(a,b)}$$

(this quotient is denoted by I_k/N in [2]) and determine its asymptotic behaviour. By (now) classical identities in the theory of symmetric functions, it is shown in [2, Cor. II.3] that

$$J_k=\frac{1}{N\cdot k!}\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}\frac{(N-i)_k\,(a+N-i-1)_k}{(a+b+2N-i-2)_k}, \tag{1}$$

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Key words and phrases. Selberg-type integral, hypergeometric series, contiguous relation, balanced hypergeometric series, transformation formula.

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Discrete analogues of Macdonald–Mehta integrals

Richard P. Brent [a,1], Christian Krattenthaler [b,2], Ole Warnaar [c,3]

[a] *Australian National University, Canberra, ACT 2600, Australia*
[b] *Fakultät für Mathematik, Universität at Wien, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria*
[c] *School of Mathematics and Physics, The University of Queensland, Brisbane, QLD 4072, Australia*

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ABSTRACT

We consider discretisations of the Macdonald–Mehta integrals from the theory of finite reflection groups. For the classical groups, A_{r-1}, B_r and D_r, we provide closed-form evaluations in those cases for which the Weyl denominators featuring in the summands have exponents 1 and 2. Our proofs for the exponent-1 cases rely on identities for classical group characters, while most of the formulas for the exponent-2 cases are derived from a transformation formula for elliptic hypergeometric series for the root system BC_r. As a byproduct of our results, we obtain closed-form product formulas for the (ordinary and signed) enumeration of orthogonal and symplectic tableaux contained in a box.

E-mail addresses: Richard.Brent@anu.edu.au (R.P. Brent), Christian.Krattenthaler@univie.ac.at (C. Krattenthaler), O.Warnaar@maths.uq.edu.au (O. Warnaar).

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$\mathfrak{sl}_2$: the Selberg integral

Let

$$\Delta(t_1, \dots, t_k) := \prod_{1 \leqslant i < j \leqslant k} (t_i - t_j)$$

be the Vandermonde product.

In 1941/1944 Selberg extended the Euler beta integral to the k-dimensional hypergeometric integral

$$\int_{[0,1]^k} \prod_{i=1}^{k} t_i^{\alpha-1}(1 - t_i)^{\beta-1} \, |\Delta(t_1, \dots, t_k)|^{2\gamma} \, \mathrm{d}t_1 \cdots \mathrm{d}t_k$$

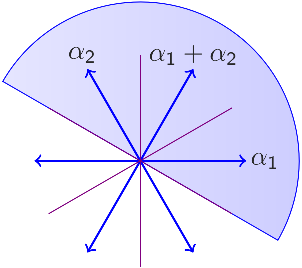
$$= \prod_{i=1}^{k} \frac{\Gamma(\alpha + (i-1)\gamma)\Gamma(\beta + (i-1)\gamma)\Gamma(1 + i\gamma)}{\Gamma(\alpha + \beta + (k+i-2)\gamma)\Gamma(1+\gamma)}$$

for $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$ and

$$\mathrm{Re}(\gamma) > -\min\{1/k, \mathrm{Re}(\alpha)/(k-1), \mathrm{Re}(\beta)/(k-1)\}$$

The Selberg integral plays an important role in the study of random matrices, Riemann zeros, hyperplane arrangements, orthogonal polynomials, integrable systems, conformal field theory, and more.

Because of the occurrence of the Vandermonde product, the Selberg integral is often associated with the A_{k-1} root system.



$$\prod_{\alpha>0} \alpha \cdot t = \prod_{1 \leqslant i<j \leqslant k} (\epsilon_i - \epsilon_j) \cdot t = \prod_{1 \leqslant i<j \leqslant k} (t_i - t_j) = \Delta(t_1, \ldots, t_k), \quad t \in \mathbb{R}^k$$

This viewpoint naturally leads to Selberg-type integrals for arbitrary finite reflection groups G, as first formulated as a conjecture by Macdonald:

$$\int_{\mathbb{R}^k} |\Delta_G(t_1, \ldots, t_k)|^{2\gamma} \, \mathrm{d}\varphi(t_1, \ldots, t_k) = \prod_{i=1}^{k} \frac{\Gamma(d_i\gamma + 1)}{\Gamma(\gamma + 1)}$$

Here the d_i are the degrees of the fundamental invariants of G, $\Delta_G(t_1, \ldots, t_k)$ is the type-G Vandermonde determinant

$$\Delta_G(t_1, \ldots, t_k) = \prod_{\alpha > 0} \alpha \cdot t$$

and φ is the k-dimensional Gaussian measure

$$\mathrm{d}\varphi(t_1, \ldots, t_k) = \prod_{i=1}^{k} \frac{\mathrm{e}^{-t_i^2/2}}{\sqrt{2\pi}} \, \mathrm{d}t_i$$

We will take a different algebraic point of view, connecting the Selberg integral to the representation theory of $\mathfrak{sl}_2$.

Recall that the beta integral arises as the solution of the hypergeometric differential equation

$$x(1-x)\frac{\mathrm{d}^2 F}{\mathrm{d}x^2} + \Big(c - \big(a+b+1\big)x\Big)\frac{\mathrm{d}F}{\mathrm{d}x} - abF = 0$$

at the regular singular point $x = 0$.

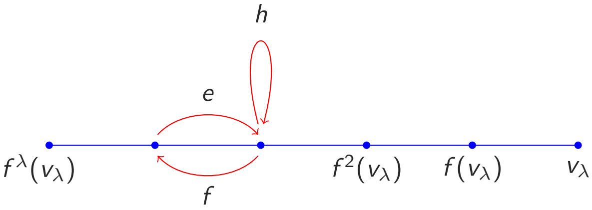
Indeed, more generally,

$$F(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,\mathrm{d}t$$

for $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ and $x \notin [1,\infty)$.

To also bring the full Selberg integral into the picture we reformulate the hypergeometric differential equation as a system of two first order partial differential equations.

Let λ be a nonnegative integer and V_λ an irreducible $\mathfrak{sl}_2$-module of highest weight λ.



$$h\, f^i(v_\lambda) = (\lambda - 2i)\, f^i(v_\lambda)$$

$$e\, f^i(v_\lambda) = i(\lambda - i + 1)\, f^{i-1}(v_\lambda)$$

$$f^{\lambda+1}(v_\lambda) = 0$$

Let V_λ and V_μ be two irreducible $\mathfrak{sl}_2$-modules, and denote by $\Omega \in \mathfrak{sl}_2 \otimes \mathfrak{sl}_2$ the Casimir element

$$\Omega = e \otimes f + f \otimes e + \frac{1}{2} h \otimes h$$

Then the Knizhnik–Zamolodchikov (KZ) equation for a function

$$u : \mathbb{C}^2 \to V_\lambda \otimes V_\mu$$

is the system of partial differential equations

$$\frac{\partial u}{\partial z} = \gamma \frac{\Omega}{z - w} u$$

$$\frac{\partial u}{\partial w} = \gamma \frac{\Omega}{w - z} u$$

where γ is a (complex) constant.

Fix an nonnegative integer k such that $0 \leqslant k \leqslant \lambda + \mu$.
Then Schechtman and Varchenko obtained the following solution of the KZ equation in the subspace of $V_\lambda \otimes V_\mu$ of weight $\lambda + \mu - 2k$ in terms of k-dimensional Selberg-type integrals

$$u(z,w) = \sum_{i=0}^{k} u_i(z,w) \left(f^i v_\lambda \otimes f^{k-i} v_\mu \right)$$

where

$$u_i(z,w) = (z-w)^{\lambda\mu\gamma} \int_C \prod_{i=1}^{k} (t_i - z)^{-\lambda\gamma} (t_i - w)^{-\mu\gamma} \, \Delta^{2\gamma}(t_1, \ldots, t_k) \\ \times A_i(z,w;t_1,\ldots,t_k) \, \mathrm{d}t_1 \cdots \mathrm{d}t_k$$

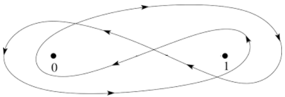
The functions A_i are explicitly known. For example,

$$A_0(z,w;t_1) = (t_1 - w)^{-1}, \quad A_1(z,w;t_1) = (z - t_1)^{-1}$$

The k-dimensional contour C in

$$u_i(z,w)=(z-w)^{\lambda\mu\gamma}\int_C \prod_{i=1}^{k}(t_i-z)^{-\lambda\gamma}(t_i-w)^{-\mu\gamma}\,\Delta^{2\gamma}(t_1,\dots,t_k)$$
$$\times A_i(z,w;t_1,\dots,t_k)\,\mathrm{d}t_1\cdots\mathrm{d}t_k$$

is a suitable deformation of $[0,1]^k$. For example, for $k=1$ it is the well-known Pochhammer double loop:



For $w=0$ and $z=1$ (and the real part of γ in the right range) one can deform C to $[0,1]^k$ to recover the Selberg integral.

We will return to the connection with KZ but first discuss two well-known and one not so well-known generalisations of the Selberg integral.

The Kadell integral

Macdonald conjectured and Kadell proved an extension of the Selberg integral obtained by adding a Jack polynomial

$$P_\eta^{(1/\gamma)}(t_1, \dots, t_k)$$

to the integrand:

$$\int\limits_{[0,1]^k} P_\eta^{(1/\gamma)}(t_1, \dots, t_k) \prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1} \left|\Delta(t_1, \dots, t_k)\right|^{2\gamma} \mathrm{d}t_1 \cdots \mathrm{d}t_k$$

$$= P_\eta^{(1/\gamma)}(1, 1, \dots, 1) \prod_{i=1}^{k} \frac{\Gamma(\alpha + (i-1)\gamma + \eta_i)\Gamma(\beta + (i-1)\gamma)\Gamma(1 + i\gamma)}{\Gamma(\alpha + \beta + (k+i-2)\gamma + \eta_i)\Gamma(1+\gamma)}$$

For $\eta = (1^r)$ this is also known as Aomoto's integral.

The Hua–Kadell integral

For $\beta = \gamma$ a second Jack polynomial may be added to the integrand

$$\int_{[0,1]^k} P_\eta^{(1/\gamma)}(t_1,\dots,t_k) P_\tau^{(1/\gamma)}(t_1,\dots,t_k) \times \prod_{i=1}^k t_i^{\alpha-1}(1-t_i)^{\gamma-1} \left|\Delta(t_1,\dots,t_k)\right|^{2\gamma} \mathrm{d}t_1\cdots\mathrm{d}t_k$$

$$= P_\eta^{(1/\gamma)}(1,1,\dots,1) P_\tau^{(1/\gamma)}(1,1,\dots,1) \times \prod_{i=1}^k \frac{\Gamma(\alpha+(i-1)\gamma+\eta_i)\Gamma(\gamma+(i-1)\gamma)\Gamma(1+i\gamma)}{\Gamma(\alpha+\gamma+(k+i-2)\gamma+\eta_i)\Gamma(1+\gamma)} \times \prod_{i,j=1}^k \frac{\Gamma(\alpha+\gamma+(2k-i-j-1)\gamma+\eta_i+\tau_j)}{\Gamma(\alpha+\gamma+(2k-i-j)\gamma+\eta_i+\tau_j)}$$

The AGT conjecture

In 2009 Alday, Gaiotto and Tachikawa conjectured a relation between conformal blocks in Liouville field theory and the Nekrasov partition function from $\mathcal{N} = 2$ supersymmetric gauge theory.

One does not have to understand any of the above jargon from string theory to appreciate that this is an important conjecture. It relates two seemingly unrelated notations and AGT paper has received well over 1000 citations to date.

One ingredient of the conjecture is an explicit combinatorial formula for the conformal blocks based on the closed form expression of the instanton part of the Nekrasov partition function.

Alba, Fateev, Litvinov, and Tarnopolskiy verified this combinatorial formula in the case of $\mathrm{SU}(2)$. To do so they had to compute a Selberg integral over two Jack symmetric functions, without the restriction $\beta = \gamma$ as given in the Hua–Kadell integral.

The Alba–Fateev–Litvinov–Tarnopolskiy integral

Let $X = \{x_1, x_2, \dots\}$ and $Y = \{y_1, y_2, \dots\}$ be arbitrary alphabets.

For f a symmetric function, we write $f[X+Y]$ and $f[X-Y]$ for f acting on the sum/difference of the alphabets X and Y.
This is most easily defined in terms of the Newton power sums p_r:

$$p_r[X] := x_1^r + x_2^r + \cdots$$

as follows

$$p_r[X+Y] := p_r[X] + p_r[Y] \quad \text{and} \quad p_r[X] - p_r[Y]$$

In particular

$$p_r[kX] := p_r[\underbrace{X + X + \cdots + X}_{k \text{ times}}] = k\, p_r[X]$$

which can be generalised further to $z \in \mathbb{C}$ by

$$p_r[zX] := z\, p_r[X]$$

Hence

$$p_r[X + zY] = p_r[X] + z\, p_r[Y]$$

When $Y = \{1\}$ we write this as

$$p_r[X + z] = p_r[X] + z \neq p_r[X + Z], \quad Z = \{z\}$$

With this notation, the proof of the $\mathrm{SU}(2)$ case of the AGT conjecture by Alba et al. uses the following integral over two Jack polynomials.

Let $t = \{t_1, \dots, t_k\}$ and μ a partition of length at most ℓ. Then

$$\int\limits_{[0,1]^k} P_\eta^{(1/\gamma)}[t] P_\tau^{(1/\gamma)}[t + \beta/\gamma - 1] \prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1} \, |\Delta(t)|^{2\gamma} \, \mathrm{d}t$$

$$= P_\eta^{(1/\gamma)}[k] P_\tau^{(1/\gamma)}[k + \beta/\gamma - 1]$$

$$\times \prod_{i=1}^{k} \frac{\Gamma(\alpha + (i-1)\gamma + \eta_i)\Gamma(\beta + (i-1)\gamma)\Gamma(1+i\gamma)}{\Gamma(\alpha+\beta+(k+i-2)\gamma+\eta_i)\Gamma(1+\gamma)}$$

$$\times \prod_{i=1}^{k} \prod_{j=1}^{\ell} \frac{\Gamma(\alpha+\beta+(2k-i-j-1)\gamma+\eta_i+\tau_j)}{\Gamma(\alpha+\beta+(2k-i-j)\gamma+\eta_i+\tau_j)}$$

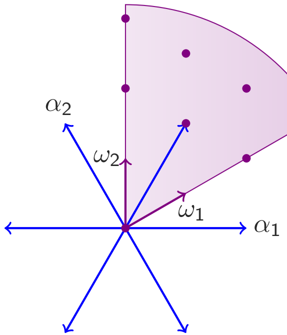
Note that for $\tau = 0$ this is the Kadell integral and for $\beta = \gamma$ the Hua–Kadell integral.

The $\mathfrak{sl}_3$ Selberg integral

To deal with the AGT conjecture for $\mathrm{SU(n)}$ an appropriate generalisation of the Selberg integral to $\mathfrak{sl}_n$ is required.

Such an integral arises by considering the KZ equation for $\mathfrak{sl}_n$ instead of $\mathfrak{sl}_2$.

Now V_λ and V_μ are two irreducible $\mathfrak{sl}_n$ highest-weight modules of highest weight $\lambda, \mu \in P_+$ respectively.



With Ω now in $\mathfrak{sl}_n \otimes \mathfrak{sl}_n$ we have exactly the same system of PDEs to solve:

$$\frac{\partial u}{\partial z} = \gamma \frac{\Omega}{z - w} u$$

$$\frac{\partial u}{\partial w} = \gamma \frac{\Omega}{w - z} u$$

for

$$u : \mathbb{C}^2 \to V_\lambda \otimes V_\mu$$

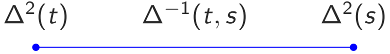
To keep some of the notation for this talk in check we will restrict ourselves to $\mathfrak{sl}_3$. Everything generalises in the obvious manner to higher rank.

The Schechtman and Varchenko solution of the KZ equation in the subspace of $V_\lambda \otimes V_\mu$ of weight $\lambda + \mu - k\alpha_1 - \ell\alpha_2$ at $w = 0$ and $z = 1$ is now a linear combination of $\mathfrak{sl}_3$ Selberg integrals of the form

$$\int \prod_{i=1}^{k} t_i^{-\gamma(\lambda,\alpha_1)}(1-t_i)^{-\gamma(\mu,\alpha_1)} \prod_{i=1}^{\ell} s_i^{-\gamma(\lambda,\alpha_2)}(1-s_i)^{-\gamma(\mu,\alpha_2)} \\ \times \Delta^{2\gamma}(t)\Delta^{2\gamma}(s)\Delta^{-\gamma}(t,s)\,\mathrm{d}t\,\mathrm{d}s$$

where $t = (t_1, \dots, t_k)$, $s = (s_1, \dots, s_\ell)$ and

$$\Delta(t,s) := \prod_{i=1}^{k}\prod_{j=1}^{\ell}(t_i - s_j)$$



$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

In general we do not know how to compute the above integral but Mukhin and Varchenko conjectured that if the subspace of $V_\lambda \otimes V_\mu$ of weight $\lambda + \mu - k\alpha_1 - \ell\alpha_2$ is one-dimensional then an evaluation as a product of gamma functions should exist.

But the dimension of the subspace is exactly given by the Littlewood–Richardson coefficient

$$c_{\lambda\mu}^{\lambda+\mu-k\alpha_1-\ell\alpha_2}$$

Stembridge classified for which weights λ and μ <u>all</u>

$$c_{\lambda\mu}^{\nu} \leqslant 1$$

Below we will focus on a well-known special case of this classification corresponding to what is known as the Pieri rule:

$$\lambda = \lambda_1\omega_1, \quad \mu = \mu_1\omega_1 + \mu_2\omega_2, \quad \nu = \lambda + \mu - k\alpha_1 - \ell\alpha_2$$

where $0 \leqslant k \leqslant \ell$ (plus some further conditions bounding k and ℓ).

In accordance with the above we will consider $\mathfrak{sl}_3$ Selberg integrals of the form

$$\int f(s,t) \prod_{i=1}^{k} t_i^{-\alpha_1-1} \prod_{i=1}^{\ell} s_i^{-\alpha_2-1}(1-s_i)^{\beta_2-1} \\ \times |\Delta(t)|^{2\gamma} |\Delta(s)|^{2\gamma} |\Delta(t,s)|^{-\gamma} \, \mathrm{d}t \, \mathrm{d}s$$

for $0 \leqslant k \leqslant \ell$ and $f(s,t)$ a bisymmetric function.

Warning: In the above $\alpha_1, \alpha_2, \beta_2, \gamma \in \mathbb{C}$.

Back to AGT

For the $\mathrm{SU}(3)$ case of AGT we need

$$f(t,s) = P_\nu^{(1/\gamma)}[t]\, P_\eta^{(1/\gamma)}[s-t]\, P_\tau^{(1/\gamma)}[s+\beta/\gamma-1]$$

for partitions ν, η, τ such that $l(\nu) \leqslant k$. In other words, the task is:

For $0 \leqslant k \leqslant \ell$, compute

$$\begin{aligned} I^{k,\ell}_{\nu,\eta,\tau}(\alpha_1,\alpha_2,\beta,\gamma) := \int P_\nu^{(1/\gamma)}[t]\, P_\eta^{(1/\gamma)}[s-t]\, P_\tau^{(1/\gamma)}[s+\beta/\gamma-1] \\ \times \prod_{i=1}^{k} t_i^{-\alpha_1-1} \prod_{i=1}^{\ell} s_i^{-\alpha_2-1}(1-s_i)^{\beta-1} \\ \times |\Delta(t)|^{2\gamma} |\Delta(s)|^{2\gamma} |\Delta(t,s)|^{-\gamma}\, \mathrm{d}t\, \mathrm{d}s \end{aligned}$$

Note that for $k=0$ this reduces to the $\mathfrak{sl}_2$ AFLT integral.

Theorem (Seamus Albion, SOW). For m any integer such that $m \geqslant l(\tau)$,

$$
\begin{aligned}
& I^{k,\ell}_{\nu,\color{red}{0}\color{black}{},\tau}(\alpha_1,\alpha_2,\beta,\gamma) \\
&\quad = P^{(1/\gamma)}_{\nu}[k] P^{(1/\gamma)}_{\tau}[\beta/\gamma+\ell-1] \\
&\qquad \times \prod_{i=1}^{\ell-k} \frac{\Gamma(\alpha_2+(\ell-k-i)\gamma)}{\Gamma(\alpha_2+\beta+(2\ell-k-m-i-1)\gamma)} \prod_{i=1}^{m} \frac{\Gamma(\alpha_2+\beta+\tau_i+(\ell-i-1)\gamma)}{\Gamma(\alpha_2+\beta+\tau_i+(2\ell-k-i-1)\gamma)} \\
&\qquad \times \prod_{i=1}^{k} \frac{\Gamma(\alpha_1+\nu_i+(k-i)\gamma)\Gamma(\alpha_1+\alpha_2+\nu_i+(k-i-1)\gamma)}{\Gamma(1+\alpha_1+\nu_i+(2k-\ell-i-1)\gamma)\Gamma(\alpha_1+\alpha_2+\beta+\nu_i+(k+\ell-m-i-2)\gamma)} \\
&\qquad \times \prod_{i=1}^{k} \frac{\Gamma(1+(i-\ell-1)\gamma)\Gamma(i\gamma)}{\Gamma(\gamma)} \prod_{i=1}^{\ell} \frac{\Gamma(\beta+(i-1)\gamma)\Gamma(i\gamma)}{\Gamma(\gamma)} \\
&\qquad \times \prod_{i=1}^{k} \prod_{j=1}^{m} \frac{\Gamma(\alpha_1+\alpha_2+\beta+\nu_i+\tau_j+(k+\ell-i-j-2)\gamma)}{\Gamma(\alpha_1+\alpha_2+\beta+\nu_i+\tau_j+(k+\ell-i-j-1)\gamma)}
\end{aligned}
$$

It is as yet an open problem to compute $I^{k,\ell}_{\nu,\eta,\tau}(\alpha_1,\alpha_2,\beta,\gamma)$ in full.

However, for $\gamma=1$ (the Schur function case) we have

Near-Theorem (SA, SOW). For m an arbitrary integer such that $m \geqslant l(\tau)$,

$$\frac{I^{k,\ell}_{\nu,\eta,\tau}(\alpha_1,\alpha_2,\beta,1)}{I^{k,\ell}_{\nu,0,\tau}(\alpha_1,\alpha_2,\beta,1)}$$

$$= s_\eta[\ell-k]\prod_{i\geqslant 1}\Bigg(\frac{(\alpha_2+\ell-k-i)_{\eta_i}}{(\alpha_2+\beta+2\ell-k-m-i-1)_{\eta_i}}$$

$$\times\prod_{j=1}^{k}\frac{(\alpha_1-\eta_i+\nu_j+2k-\ell+i-j-1)}{(\alpha_1+\nu_j+2k-\ell+i-j-1)}$$

$$\times\prod_{j=1}^{m}\frac{(\alpha_2+\beta+\tau_j+2\ell-k-i-j-1)}{(\alpha_2+\beta+\eta_i+\tau_j+2\ell-k-i-j-1)}\Bigg)$$

Proof.

- Step 0. In the integrand we have the symmetric function

$$s_\nu[t]\, s_\eta[s-t]\, s_\tau[s+\beta-1]$$

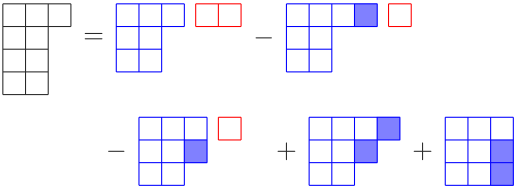
- Step 1. Use induction on the length of η using the inverse Pieri rule.

  Let $\eta = (\eta_1, \ldots, \eta_n)$ be a partition of length n such that $\eta_n \geqslant r$.

  Then

$$s_{(\eta_1,\ldots,\eta_n,r)} = \sum_{\lambda=(\lambda_1,\ldots,\lambda_n)} (-1)^{|\lambda|-|\eta|}\, s_\lambda\, h_{r+|\eta|-|\lambda|}$$

  where $\lambda - \eta$ is a vertical strip.



- Step 2. Absorb the unwanted $h_r[s-t]$ in

$$s_\nu[t]\, s_\lambda[s-t]\, h_r[s-t]\, s_\tau[s+\beta-1]$$

  into the $s_\nu[t]$ and $s_\tau[s+\beta-1]$ using the e- and h-Pieri rules plus some plethystic gymnastics.

- Step 3. Prove some (very) complicated rational function identities. Hopefully Seamus is very busy doing this as we speak.

Some concluding remarks

First the good news:

- More generally, we have proved the analogue of the AFLT integral for $\mathfrak{sl}_n$. This integral contains two Jack polynomials in the integrand.

Now the bad news:

- Our inductive methods fails for $\mathfrak{sl}_n$ when $n \geqslant 4$.

  For example, it is not clear how to obtain

  $$s_\omega[r]\, s_\nu[t-r]\, s_\eta[s-t]\, s_\tau[s+\beta-1]$$

  from

  $$s_\omega[r]\, s_\tau[s+\beta-1]$$

- We do not know how to lift

  $$s_\nu[t]\, s_\eta[s-t]\, s_\tau[s+\beta-1]$$

  to

  $$P_\nu^{(1/\gamma)}[t]\, P_\eta^{(1/\gamma)}[s-t]\, P_\tau^{(1/\gamma)}[s+\beta/\gamma-1]$$

  For example, there is no simple analogue of the inverse Pieri rule for Jack polynomials.

The End