SLC81 (KrattenthalerFest)
Strobl, September 9–12, 2018

Ramanujan's congruences modulo powers of 5, 7, and 11 revisited

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Prelude

1	1	2	3	5
7	11	15	22	30
42	56	77	101	135
176	231	297	385	490
627	792	1002	1255	1575
1958	2436	3010	3718	4565
5604	6842	8349	10143	12310
14883	17977	21637	26015	31185
37338	44583	53174	63261	75175
89134	105558	124754	147273	173525
204226	239943	281589	329931	386155
451276	526823	614154	715220	831820
966467	1121505	1300156	1505499	1741630
2012558	2323520	2679689	3087735	3554345
4087968	4697205	5392783	6185689	7089500
8118264	9289091	10619863	12132164	13848650

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Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} p(n) q^n &= \prod_{j=1}^{\infty} \frac{1}{1-q^j} = \frac{1}{(q;q)_\infty} \\
&= (1 + q^1 + q^{1+1} + q^{1+1+1} + \dots) \\
&\times (1 + q^2 + q^{2+2} + q^{2+2+2} + \dots) \\
&\times (1 + q^3 + q^{3+3} + q^{3+3+3} + \dots) \\
&\times \text{etc.} \\
&= \dots + q^{1+1+1+2+2+3+\cdots} + \dots
\end{aligned}$$

Holonomic functions have only finitely many singularities. □

Back to the $p(n)$ table: ANY PATTERN?

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Some properties of $p(n)$, the number of partitions of n*

Proceedings of the Cambridge Philosophical Society, XIX, 1919, 207 – 210

1. A recent paper by Mr Hardy and myself † contains a table, calculated by Major MacMahon, of the values of $p(n)$, the number of unrestricted partitions of n, for all values of n from 1 to 200. On studying the numbers in this table I observed a number of curious congruence properties, apparently satisfied by $p(n)$. Thus

(1)	$p(4)$,	$p(9)$,	$p(14)$,	$p(19)$,	$\cdots$	$\equiv 0$	(mod 5),
(2)	$p(5)$,	$p(12)$,	$p(19)$,	$p(26)$,	$\cdots$	$\equiv 0$	(mod 7),
(3)	$p(6)$,	$p(17)$,	$p(28)$,	$p(39)$,	$\cdots$	$\equiv 0$	(mod 11),
(4)	$p(24)$,	$p(49)$,	$p(74)$,	$p(99)$,	$\cdots$	$\equiv 0$	(mod 25),
(5)	$p(19)$,	$p(54)$,	$p(89)$,	$p(124)$,	$\cdots$	$\equiv 0$	(mod 35),
(6)	$p(47)$,	$p(96)$,	$p(145)$,	$p(194)$,	$\cdots$	$\equiv 0$	(mod 49),
(7)	$p(39)$,	$p(94)$,	$p(149)$,	$\cdots$		$\equiv 0$	(mod 55),
(8)	$p(61)$,	$p(138)$,	$\cdots$			$\equiv 0$	(mod 77),
(9)	$p(116)$,	$\cdots$				$\equiv 0$	(mod 121),
(10)	$p(99)$,	$\cdots$				$\equiv 0$	(mod 125).

From these data I conjectured the truth of the following theorem: if $\delta = 5^a 7^b 11^c$ and $24\lambda \equiv 1 \pmod{\delta}$ *then*

Ramanujan's Congruences

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$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11} \end{aligned}$$

Ramanujan's congruences modulo prime powers

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(5^2 n+24) &\equiv 0 \pmod{5^2}, \\ p(5^3 n+99) &\equiv 0 \pmod{5^3}, \end{aligned}$$

etc.

Ramanujan's congruences modulo prime powers

$$\begin{aligned} p(7n+5) &\equiv 0 \pmod{7}, \\ p(7^2 n+47) &\equiv 0 \pmod{7^2}, \\ p(7^3 n+243) &\equiv 0 \pmod{7^2}, \end{aligned}$$

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etc.

NOTE. Correction for $\ell = 7$: Chowla (1934) using a table of Gupta:

$p(243) = 2^3 \cdot 7^2 \cdot 97 \cdot 5783 \cdot 609289$:

$$
\begin{aligned}
p(11n+6) &\equiv 0 \pmod{11},\\
p(11^2 n+116) &\equiv 0 \pmod{11^2},\\
p(11^3 n+721) &\equiv 0 \pmod{11^3},
\end{aligned}
$$

etc.

Ramanujan's Congruences (modified conjecture 1919)

For $\ell \in \{5, 7, 11\}$ and $\alpha \in \{1, 2, 3, \dots\}$:

$$p(\ell^\alpha n + \mu_{\alpha,\ell}) \equiv \begin{cases} 0 \pmod{\ell^\alpha}, & \ell = 5, 11, \\ 0 \pmod{7^{\lceil \frac{\alpha+1}{2} \rceil}}, & \ell = 7, \end{cases}$$

with $\mu_{\alpha,\ell} \in \{0, \dots, \ell^\alpha - 1\}$ such that

$$24\,\mu_{\alpha,\ell} \equiv 1 \pmod{\ell^\alpha}.$$

A Bit of History

Watson's proof (1938) for $\ell = 5$ *and* $\ell = 7$ *(modified version)*

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- Regarding $\ell = 11$ Watson states: "Da die Untersuchung der Aussage über 11^α recht langweilig ist, verschiebe ich den Beweis dieses Falles auf eine spätere Abhandlung".

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Proof by P. and Radu for $\ell = 5, 7, 11$:

- One common framework – motivated by algorithmics.
- The case $\ell = 11$ is interesting!

P. and S. Radu: A unified algorithmic framework for Ramanujan's congruences modulo powers of 5, 7, and 11; preprint, 2018

Ramanujan's Most Beautiful Formula

PROOF (mod 5 congruence).

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{j=1}^{\infty} \frac{(1-q^{5j})^5}{(1-q^j)^6}$$

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"It would be difficult to find more beautiful formulae than the 'Rogers-Ramanujan' identities . . . ; but here Ramanujan must take second place to Prof. Rogers; and, if I had to select one formula from all Ramanujan's work, I would agree with Major MacMahon in selecting . . ." [G.H. Hardy]

http://www-history.mcs.st-andrews.ac.uk/BigPictures/Hardy_5.jpeg

PROOF (mod 7 congruence).

$$\sum_{n=0}^{\infty} p(7n+5)q^n$$

$$= 7\prod_{j=1}^{\infty}\frac{(1-q^{7j})^3}{(1-q^j)^4} + 49q\prod_{j=1}^{\infty}\frac{(1-q^{7j})^7}{(1-q^j)^8}.$$

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⇝ What about the 11 congruence?

Witness identities for $\mathbf{11 \mid p(11n+6)}$

PROOF (mod 11). Joseph Lehner [1943] found that

$$q \sum_{n=0}^{\infty} p(11n+6)q^n = 11 \, \frac{11AC^2 - 11^2C + 2AC - 32C - 2}{\prod_{k=1}^{\infty}(1-q^{11k})}$$

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where

$$A = \frac{\left(\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+3n^2}\right)^2}{q(q;q)_\infty^2(q^{11};q^{11})_\infty^2}, C = \frac{A^2 - 10A - B - 22}{2\cdot 11^2},$$

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and

$$B = \frac{11^2E_4(11\tau) - E_4(\tau),}{q^2(q^{11};q^{11})_\infty^4(q;q^{11})_\infty^4}$$

with

$$120E_4(\tau) = 1 + 240\sum_{n=1}^{\infty}\Big(\sum_{1\le d|n} d^3\Big)q^n.$$

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Lehner used modular function machinery.

PROOF (mod 11). Atkin [1967] found that

$$q \prod_{k=1}^{\infty}(1-q^{11k}) \sum_{n=0}^{\infty} p(11n+6)q^n$$
$$= 11^4 z_{11} + 11 g_2 + 2 \cdot 11^2 g_3 + 11^3 g_4,$$

where

$$z_{11} := q^5 \frac{(q^{11};q^{11})_\infty^{12}}{(q;q)_\infty^{12}} \in M_{\mathbb{Z}}^0(11), \quad g_4 := g_2^2 - g_3,$$

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with

$$\bar{u}_{11} := q^5 \frac{(q^{121};q^{121})_\infty}{(q;q)_\infty} \text{ and } (q;q)_\infty^r =: \sum_{n=0}^{\infty} p_r(n) q^n,$$

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$$14\ g_3 := -14 g_2 + 11^3 \bar{u}_{11}^7 - \frac{\sum_{n=0}^\infty \left(1 + \left(\frac{2-n}{11}\right)\right) p_7(n) q^n}{(q;q)_\infty^7}.$$

PROOF (mod 11). Radu's Ramanujan-Kolberg package computes

$$\sum_{n=0}^{\infty} p(11n+6)q^n = q^{14} \prod_{j=1}^{\infty} \frac{(1-q^{22j})^{22}}{(1-q^j)^{10}(1-q^{2j})^2(1-q^{11j})^{11}}$$

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$$t:=\frac{3}{88}w_1+\frac{1}{11}w_2-\frac{1}{8}w_3,\ s_1:=-\frac{5}{88}w_1+\frac{2}{11}w_2-\frac{1}{8}w_3-3,$$

$$s_2:=\frac{1}{44}w_1-\frac{3}{11}w_2+\frac{5}{4}w_3,$$ where

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$$[r_2, r_{11}, r_{22}] := \prod_{\delta|22} \left(\frac{\eta(\delta\tau)}{\eta(\tau)}\right)^{r_\delta} \in E^{\infty}(22).$$

Recall

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Use freshman's dream: $(1-x)^n \equiv 1 - x^n$. But one can do better!

PROOF (mod 11). Ralf Hemmecke, using his SAMBA algorithm, computed

$$\sum_{n=0}^{\infty} p(11n+6)q^n = q^{14} \prod_{j=1}^{\infty} \frac{(1-q^{22j})^{22}}{(1-q^j)^{10}(1-q^{2j})^2(1-q^{11j})^{11}}$$

$$\times \big(11^2 \cdot 3068 w_7 + 11^2(3w_1 + 4236)w_6 + 11(285w_1 + 11 \cdot 5972)w_5$$

$$+ \frac{11}{8}(w_1^2 + 11 \cdot 4497 w_1 + 11^2 \cdot 3156)w_3$$

$$+ 11(1867 w_1 + 11 \cdot 2476)w_4$$

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$$w_1 = [-3, 3, -7], w_2 = [8, 4, -8], w_3 = [1, 11, -11], w_4 = [1, 11, -11]$$
$$w_5 = [4, 8, -12], w_6 = [2, 10, -14], w_7 = [0, 12, -16] \text{ where}$$

$$[a_1, a_2, a_3] := q^{\square} \prod_{k=1}^{\infty} \frac{(1-q^{2k})^{a_1}(1-q^{11k})^{a_2}(1-q^{22k})^{a_3}}{(1-q^k)^{a_1+a_2+a_3}} \in M_{\mathbb{Z}}^{\infty}(22).$$

PROOF (mod 11). P. & Radu, using the "NUGGETS algorithm", found

$$
\begin{aligned}
f^5 &= 5 \cdot 11^4 f^4 + 11^4(-2 \cdot 5 \cdot 11^4 + 251\,\bar{z}_{11}) f^3 \\
&\quad + 11^3(2 \cdot 5 \cdot 11^9 + 2 \cdot 3 \cdot 5 \cdot 11^5 \cdot 31\,\bar{z}_{11} + 4093\,\bar{z}_{11}^2) f^2 \\
&\quad + 11^4(-5 \cdot 11^{12} + 2 \cdot 5 \cdot 11^8 \cdot 17\,\bar{z}_{11} - 2^2 \cdot 3 \cdot 11^3 \cdot 1289\,\bar{z}_{11}^2 \\
&\qquad\qquad + 3 \cdot 41\,\bar{z}_{11}^3) f \\
&\quad + 11^5(11^4 + \bar{z}_{11})(11^{11} - 3 \cdot 7 \cdot 11^7\,\bar{z}_{11} + 11^2 \cdot 1321\,\bar{z}_{11}^2 + \bar{z}_{11}^3)
\end{aligned}
$$

$$f := q\,\bar{z}_{11} \prod_{k=1}^{\infty}(1-q^{11k}) \sum_{n=0}^{\infty} p(11n+6) q^n,$$

where

$$\bar{z}_{11} := \frac{1}{z_{11}} = q^{-5} \prod_{k=1}^{\infty} \left(\frac{1-q^k}{1-q^{11k}} \right)^{12}.$$

"NUGGETS algorithm": Take

$$t := q^{-5} \prod_{k=1}^{\infty} \left(\frac{1-q^k}{1-q^{11k}} \right)^{12} \in M^{\infty}(11)$$

and

$$f := q\, t \prod_{k=1}^{\infty} (1-q^{11k}) \sum_{n=0}^{\infty} p(11n+6) q^n \in M^{\infty}(11).$$

Determine polynomials p_j such that

$$f^5 = p_0(t) + p_1(t) f + p_2(t) f^2 + p_3(t) f^3 + p_4(t) f^4.$$

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NOTE. For the pole orders of f, f^2, f^3, f^4 with respect to q:

$$\{4, 8, 12, 16\} \equiv \{4, 3, 2, 1\} \pmod 5.$$

The pole order of t is 5.

"NUGGETS algorithm" (cont'd): Recall

$$t := q^{-5} \prod_{k=1}^{\infty} \left(\frac{1-q^k}{1-q^{11k}} \right)^{12} \in M^{\infty}(11),$$

$$f := q\,t \prod_{k=1}^{\infty} (1-q^{11k}) \sum_{n=0}^{\infty} p(11n+6) q^n \in M^{\infty}(11).$$

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In[1]:= $\mathbf{Tquot[k\_] := \dfrac{1-q^k}{1-q^{11k}};}$

In[2]:= $\mathbf{t = \dfrac{1}{q^5} Product[Series[Tquot[k]^{12}, q, 0, 26], k, 1, 26]}$

Out[2]=

$$\frac{1}{q^5} - \frac{12}{q^4} + \frac{54}{q^3} - \frac{88}{q^2} - \frac{99}{q} + 540 - 418q - 648q^2 + \cdots - 22176q^{20} + 61656q^{21} + O[q]^{22}$$

"NUGGETS algorithm" (cont'd): Recall

$$t := q^{-5} \prod_{k=1}^{\infty} \left(\frac{1-q^k}{1-q^{11k}} \right)^{12} \in M^{\infty}(11),$$

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TASK: Determine polynomials p_j such that

$$f^5 = p_0(t) + p_1(t) f + p_2(t) f^2 + p_3(t) f^3 + p_4(t) f^4.$$

In[3]:= $\mathbf{f = qtSeries[(1-q^{11})(1-q^{22}), q, 0, 21] *}$
$\mathbf{Sum[PartitionsP[11n+6]q^n, n, 0, 21]}$

Out[3]= $\frac{11}{q^4} + \frac{165}{q^3} + \frac{748}{q^2} + \frac{1639}{q} + 3553 + 4136q + 6347q^2 + \cdots + 12738q^{16} - 51216q^{17} + O[q]^{18}$

"NUGGETS algorithm" (cont'd): Recall

In[4]:= $\mathbf{t}$

Out[4]=

$$\frac{1}{q^5}-\frac{12}{q^4}+\frac{54}{q^3}-\frac{88}{q^2}-\frac{99}{q}+540-418q-648q^2+\cdots-22176q^{20}+61656q^{21}+O[q]^{22}$$

In[5]:= $\mathbf{f}$

Out[5]= $\frac{11}{q^4}+\frac{165}{q^3}+\frac{748}{q^2}+\frac{1639}{q}+3553+4136q+6347q^2+\cdots+12738q^{16}-51216q^{17}+O[q]^{18}$

In[6]:= $\mathbf{F=\frac{f}{11};\quad F^5}$

Out[6]= $\frac{1}{q^{20}}+\frac{75}{q^{19}}+\frac{2590}{q^{18}}+\cdots+\frac{298958660282220}{q}+530018316923711+877706745683995q+O[q]^2$

In[7]:= $\mathbf{F^5-t^4}$

Out[7]= $\frac{123}{q^{19}}+\frac{1510}{q^{18}}+\frac{69935}{q^{17}}+\cdots+530018316923711+877706745683995q+O[q]^2$

"NUGGETS algorithm" (cont'd): Recall

In[8]:= $\mathbf{F^5 - t^4}$

Out[8]= $$\frac{123}{q^{19}} + \frac{1510}{q^{18}} + \frac{69935}{q^{17}} + \cdots + 530018316923711 + 877706745683995q + O[q]^2$$

In[9]:= $\mathbf{F^5 - t^4 - 3 * 41 * t^3 F}$

Out[9]= $$\frac{4093}{q^{18}} + \frac{54929}{q^{17}} + \frac{570947}{q^{16}} + \cdots + 530565611750339 + 877363195058527q + O[q]^2$$

"NUGGETS algorithm" (cont'd): Recall

In[11]:= $\mathbf{F^5 - t^4}$

Out[11]= $\frac{123}{q^{19}} + \frac{1510}{q^{18}} + \frac{69935}{q^{17}} + \cdots + 530018316923711 + 877706745683995q + O[q]^2$

In[12]:= $\mathbf{F^5 - t^4 - 3 * 41 * t^3 F}$

Out[12]=

$$\frac{4093}{q^{18}} + \frac{54929}{q^{17}} + \frac{570947}{q^{16}} + \cdots + 530565611750339 + 877363195058527q + O[q]^2$$

In[13]:= $\mathbf{F^5 - t^4 - 3 * 41 t^3 F - 4093 t^2 F^2}$

Out[13]= $\frac{30371}{q^{17}} + \frac{1008898}{q^{16}} + \frac{12509585}{q^{15}} + \cdots + 536556550241327 + 873666097417069q + O[q]^2$

"NUGGETS algorithm" (cont'd): Recall

In[14]:= $\mathbf{F^5 - t^4}$

Out[14]= $\frac{123}{q^{19}} + \frac{1510}{q^{18}} + \frac{69935}{q^{17}} + \cdots + 530018316923711 + 877706745683995q + O[q]^2$

In[15]:= $\mathbf{F^5 - t^4 - 3 * 41 * t^3 F}$

Out[15]=

$\frac{4093}{q^{18}} + \frac{54929}{q^{17}} + \frac{570947}{q^{16}} + \cdots + 530565611750339 + 877363195058527q + O[q]^2$

In[16]:= $\mathbf{F^5 - t^4 - 3 * 41 t^3 F - 4093 t^2 F^2}$

Out[16]= $\frac{30371}{q^{17}} + \frac{1008898}{q^{16}} + \frac{12509585}{q^{15}} + \cdots + 536556550241327 + 873666097417069q + O[q]^2$

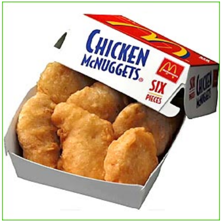
a.s.o. until $\mathbf{F^5}$ is reduced to $O[q]^2$.

What is the "nuggets algorithm"?

Nuggets Monoids

Nuggets Partitions

GIVEN



TASK: buy exactly 43 nuggets.

Nuggets Partitions

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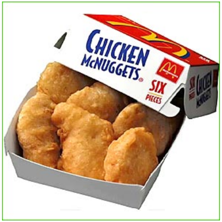

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IMPOSSIBLE!

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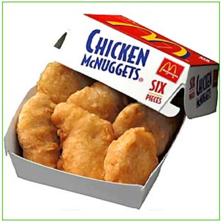

TASK: buy exactly 43 nuggets.

IMPOSSIBLE!

See: "How to order 43 Chicken McNuggets - Numberphile"
www.youtube.com/watch?v=vNTSugyS038

The (additive) nuggets monoid

$$43 \overset{?}{\in} M := \langle 6, 9, 20 \rangle := \{6a + 9b + 20c : a, b, c \in \mathbb{Z}_{\geq 0}\}$$

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Many solution strategies, for example,

$$\begin{aligned} 43 &= 6 \cdot 7 + 1 \\ &= 6 \cdot 6 + 7 \\ &= 6 \cdot 5 + 13 \\ &= 6 \cdot 4 + 19 \\ &= \ldots \end{aligned}$$

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E.g., $19 = 9 \cdot b + 20 \cdot c$: no solution

Decomposition into residue classes

$$\begin{aligned} M := \langle 6, 9, 20\rangle := \{6a + 9b + 20c : a, b, c \in \mathbb{Z}_{\geq 0}\} \\ = [M]_0 \cup [M]_1 \cup \cdots \cup [M]_5 \end{aligned}$$

where

$$[M]_i := \{x \in M : x \equiv i \pmod 6\}$$

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$$[M]_0 = \{0, 6, 12, \dots\},$$
$$[M]_1 = \{\cancel{1}, \cancel{7}, \cancel{13}, \cancel{19}, \cancel{25}, \cancel{31}, \cancel{37}, \cancel{43}, 49, 54, \dots\},$$
$$[M]_2 = \{\cancel{2}, \cancel{8}, \cancel{14}, 20, 26, \dots\},$$
$$[M]_3 = \{\cancel{3}, 9, 15, \dots\},$$
$$[M]_4 = \{\cancel{4}, \cancel{10}, \cancel{16}, \cancel{22}, \cancel{28}, \cancel{34}, 40, 46, \dots\},$$
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$$M = \langle 6, 9, 20 \rangle = \langle 6, 49, 20, 9, 40, 29 \rangle$$

QUESTION. What happens if we add more generators?

$M^+ := \langle 4, 6, 9, 20 \rangle$

$$[M^+]_i := \{x \in M : x \equiv i \pmod 6\}$$

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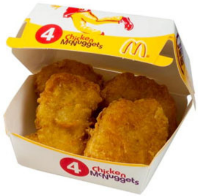

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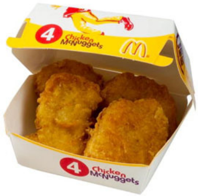

NOW $43 = 2\textcircled{4} + \textcircled{6} + \textcircled{9} + 1\textcircled{20}$

$= \textcircled{4} + 2\textcircled{6} + 3\textcircled{9} =$ [four more].

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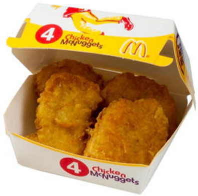

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Back to our original TASK:
Given a number x, present it in terms of the generators $\textcircled{6}$, $\textcircled{9}$, and $\textcircled{20}$.

Solution: use the residue class generators

Given

$$M = \langle 6, 9, 20\rangle = \langle 6, 49, 20, 9, 40, 29\rangle$$
$$= (0 + 6\mathbb{Z}_{\geq 0}) \cup (49 + 6\mathbb{Z}_{\geq 0}) \cup (20 + 6\mathbb{Z}_{\geq 0})$$
$$\cup (9 + 6\mathbb{Z}_{\geq 0}) \cup (40 + 6\mathbb{Z}_{\geq 0}) \cup (29 + 6\mathbb{Z}_{\geq 0}),$$

find a representation $x = u_i + \ell \, \textcircled{6}$.

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find a representation $x = u_i + \ell \textcircled{6}$.

EXAMPLE. $x = 62$:

$62 \equiv 2 \pmod 6$; hence by

$$[M]_2 = \{\not{2}, \not{8}, \not{14}, 20, 26, \dots\};$$

one has

$$62 = 20 + 7 \textcircled{6}.$$

NOTE. Given $M = \langle 6, 9, 20 \rangle$, for this STRATEGY we need the "residue class generators"

$$\langle 6, 49, 20, 9, 40, 29 \rangle = M.$$

QUESTION. How to compute the "residue class generators"

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$$\langle 6, 49, 20, 9, 40, 29 \rangle = M.$$

QUESTION. How to compute the "residue class generators"

ANSWER (e.g., for 49). Notice, $49 \in M$ is minimal such that

$$49 \equiv 1 \pmod{6};$$

i.e.; it is the smallest number of the form $1 + 6\,a = 49$ such that

$$9\,b + 20\,c = 1 + 6\,a, \quad a, b, c \in \mathbb{N}.$$

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This, e.g., can be computed with the Omega package:

In[3]:=OEqR[

OEqSum[$x^a y^b z^c, \{-6a + 9b + 20c = 1\}, \lambda$]]

Out[3]= $\dfrac{x^8 y z^2}{\left(1-x^3y^2\right)\left(1-x^{10}z^3\right)}$.

```
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```

$$\texttt{Out[3]=}\ \frac{x^8yz^2}{\left(1-x^3y^2\right)\left(1-x^{10}z^3\right)}.$$

Out[3] means that

$$\sum_{\substack{a,b,c\geq 0 \\ s.t.\ -6a+9b+20c=1}} x^a y^b z^c = \frac{x^8yz^2}{(1-x^3y^2)(1-x^{10}z^3)};$$

in other words,

$$\begin{pmatrix} 8 \\ 1 \\ 2 \end{pmatrix} + \mathbb{N}\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + \mathbb{N}\begin{pmatrix} 10 \\ 0 \\ 3 \end{pmatrix}$$

is the solution set of $-6a + 9b + 20c = 1$.

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In particular, $1\,\text{⑨} + 2\,\text{⑳} = 1 + 8\,\text{⑥} = 49.$

Application: Representations of Subalgebras

Representing $\mathbb{K}$-subalgebras as $\mathbb{K}[z]$-modules

GIVEN polynomials $f_0, f_1, f_2 \in \mathbb{K}[z]$, e.g. as in EXAMPLE II,

$$\begin{aligned} t := f_0 &= z^6 + \alpha\, z^5 + \ldots, \\ f_1 &= z^9 + \beta\, z^8 + \ldots, \\ f_2 &= z^{20} + \gamma\, z^{19} + \ldots; \end{aligned}$$

FIND $g_1, \ldots, g_5 \in \mathbb{K}[z]$ such that

$$\begin{aligned} \mathbb{K}[t, f_1, f_2] &= \left\{ \sum_{k_0,k_1,k_2} c(k_0,k_1,k_2) t^{k_0} f_1^{k_1} f_2^{k_2} : c(k_0,k_1,k_2) \in \mathbb{K} \right\} \\ &= \langle 1, g_1, \ldots, g_5 \rangle_{\mathbb{K}[t]} \end{aligned}$$

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$$\begin{aligned} \langle 1, g_1, \dots, g_5 \rangle_{\mathbb{K}[t]} &:= \mathbb{K}[t] + \mathbb{K}[t]\, g_1 + \dots + \mathbb{K}[t]\, g_5 \\ &:= \Big\{ p_0(t) + p_1(t) g_1 + \dots + p_5(t) g_5 : p_j(x) \in \mathbb{K}[x] \Big\}. \end{aligned}$$

Application: $\mathbb{K}$-subalgebra membership

GIVEN polynomials f and $f_0, f_1, \dots, f_n \in \mathbb{K}[z]$;

DECIDE whether $f \overset{?}{\in} \mathbb{K}[f_0, f_1, \dots, f_n]$.

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FIND polynomials $p_0, p_1, \ldots, p_k \in \mathbb{K}[z]$ such that

$$f \quad = \quad p_0(t) + p_1(t)\, g_1 + \cdots + p_k(t)\, g_k.$$

APPLICATION to modular functions in

$$M^\infty(N) = \{f \in M(N) : \tilde{f} \text{ has a pole only at } [\infty]\} :$$

Application: $\mathbb{K}$-subalgebra membership

GIVEN polynomials f and $f_0, f_1, \ldots, f_n \in \mathbb{K}[z]$;

DECIDE whether $f \overset{?}{\in} \mathbb{K}[f_0, f_1, \ldots, f_n]$.

Let $t := f_0$ and $\mathbb{K}[f_0, f_1, \ldots, f_n] = \langle 1, g_1, \ldots, g_k \rangle_{\mathbb{K}[t]}$,
FIND polynomials $p_0, p_1, \ldots, p_k \in \mathbb{K}[z]$ such that

$$f = p_0(t) + p_1(t)\, g_1 + \cdots + p_k(t)\, g_k.$$

APPLICATION to modular functions in

$$M^\infty(N) = \{f \in M(N) : \tilde{f} \text{ has a pole only at } [\infty]\} :$$

In this context, $z = \frac{1}{q}$. Therefore functions like

$$f_1 = \frac{a_6}{q^6} + \frac{a_5}{q^5} + \cdots, f_2 = \frac{b_9}{q^9} + \frac{b_8}{q^8} + \cdots,$$
$$f_3 = \frac{c_{20}}{q^{20}} + \frac{c_{19}}{q^{19}} + \cdots, \text{ etc.},$$

can be treated like polynomials.

Recall: APPLICATION to modular functions in

$$M^\infty(N) = \{f \in M(N) : \tilde{f} \text{ has a pole only at } [\infty]\} :$$

In this context, $z = \frac{1}{q}$. Therefore functions like

$$t = f_0 = \frac{a_6}{q^6} + \frac{a_5}{q^5} + \cdots, f_1 = \frac{b_9}{q^9} + \frac{b_8}{q^8} + \cdots,$$

$$f_2 = \frac{c_{20}}{q^{20}} + \frac{c_{19}}{q^{19}} + \cdots,$$ etc., can be treated like polynomials.

EXAMPLE I. $t := q^{-5} \prod_{k=1}^{\infty} \left(\frac{1-q^k}{1-q^{11k}} \right)^{12} \in M^\infty(11)$ and

$$f := q\,t \prod_{k=1}^{\infty} (1-q^{11k}) \sum_{n=0}^{\infty} p(11n+6) q^n \in M^\infty(11),$$

satisfy

$$\mathbb{C}[t,f] = \mathbb{C}[t] + \mathbb{C}[t]\,g_1 + \cdots + \mathbb{C}[t]\,g_4$$

with

$$(g_1, g_2, g_3, g_4) := (f, f^2, f^3, f^4).$$

EXAMPLE II.

In[109]:=

```
t = z^6 - 1; f1 = z^9 + 2; f2 = z^20 + 1;
```

WE FIND $g_1, \ldots, g_5 \in \mathbb{C}[z]$ such that

$$\begin{aligned} \mathbb{C}[t, f_1, f_2] &= \langle 1, g_1, \ldots, g_5 \rangle_{\mathbb{C}[t]} \\ &= \mathbb{C}[t] + \mathbb{C}[t]\, g_1 + \cdots + \mathbb{C}[t]\, g_5. \end{aligned}$$

NOTE.

$$t(z)^a f_1(z)^b f_2(z)^c = z^{6\,a+9\,b+20\,c} + \ldots$$

EXAMPLE II.

In[109]:=

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t = z^6 - 1;  f1 = z^9 + 2;  f2 = z^20 + 1;
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WE FIND $g_1, \ldots, g_5 \in \mathbb{C}[z]$ such that

$$\begin{aligned} \mathbb{C}[t, f_1, f_2] &= \langle 1, g_1, \ldots, g_5 \rangle_{\mathbb{C}[t]} \\ &= \mathbb{C}[t] + \mathbb{C}[t]\, g_1 + \cdots + \mathbb{C}[t]\, g_5. \end{aligned}$$

NOTE.

$$t(z)^a f_1(z)^b f_2(z)^c = z^{6\,a+9\,b+20\,c} + \ldots$$

In[110]:=

```
{g1 = f1 f2^2, g2 = f2, g3 = f1, g4 = f2^2, g5 = f1 f2}
```

Out[110]=

$$\left\{\left(2+z^9\right)\left(1+z^{20}\right)^2,\ 1+z^{20},\ 2+z^9,\ \left(1+z^{20}\right)^2,\ \left(2+z^9\right)\left(1+z^{20}\right)\right\}$$

EXAMPLE II contd. (1)

In[110]:=

```
{g1 = f1 f2^2, g2 = f2, g3 = f1, g4 = f2^2, g5 = f1 f2}
```

Out[110]=

$$\left\{\left(2+z^{9}\right)\left(1+z^{20}\right)^{2},\ 1+z^{20},\ 2+z^{9},\ \left(1+z^{20}\right)^{2},\ \left(2+z^{9}\right)\left(1+z^{20}\right)\right\}$$

Recall: $\{49, 20, 9, 40, 29\} \equiv \{1, 2, 3, 4, 5\} \pmod 6$.

EXAMPLE II contd. (1)

In[110]:=

```
{g1 = f1 f2^2, g2 = f2, g3 = f1, g4 = f2^2, g5 = f1 f2}
```

Out[110]=

$$\left\{\left(2+z^9\right)\left(1+z^{20}\right)^2,\ 1+z^{20},\ 2+z^9,\ \left(1+z^{20}\right)^2,\ \left(2+z^9\right)\left(1+z^{20}\right)\right\}$$

Recall: $\{49, 20, 9, 40, 29\} \equiv \{1, 2, 3, 4, 5\} \pmod 6.$

NOTE.

$$\mathbb{C}[t, f_1, f_2] = \mathbb{C}[t] + \mathbb{C}[t]\, g_1 + \cdots + \mathbb{C}[t]\, g_5$$

iff

$$f_i = p_0^i(t) + p_1^i(t) g_1 + \cdots + p_5^i(t) g_5$$

and

EXAMPLE II contd. (1)

In[110]:=

```
{g1 = f1 f2^2, g2 = f2, g3 = f1, g4 = f2^2, g5 = f1 f2}
```

Out[110]=

$$\left\{\left(2+z^{9}\right)\left(1+z^{20}\right)^{2},\ 1+z^{20},\ 2+z^{9},\ \left(1+z^{20}\right)^{2},\ \left(2+z^{9}\right)\left(1+z^{20}\right)\right\}$$

Recall: $\{49, 20, 9, 40, 29\} \equiv \{1, 2, 3, 4, 5\} \pmod 6$.

NOTE.

$$\mathbb{C}[t, f_1, f_2] = \mathbb{C}[t] + \mathbb{C}[t]\, g_1 + \cdots + \mathbb{C}[t]\, g_5$$

iff

$$f_i = p_0^i(t) + p_1^i(t) g_1 + \cdots + p_5^i(t) g_5$$

and

$$g_i\, g_j = p_0^{i,j}(t) + p_1^{i,j}(t) g_1 + \cdots + p_5^{i,j}(t) g_5,$$

EXAMPLE II contd. (1)

In[110]:=

```
{g1 = f1 f2^2, g2 = f2, g3 = f1, g4 = f2^2, g5 = f1 f2}
```

Out[110]=

$$\{(2+z^9)(1+z^{20})^2,\ 1+z^{20},\ 2+z^9,\ (1+z^{20})^2,\ (2+z^9)(1+z^{20})\}$$

Recall: $\{49, 20, 9, 40, 29\} \equiv \{1, 2, 3, 4, 5\} \pmod 6.$

NOTE.

$$\mathbb{C}[t, f_1, f_2] = \mathbb{C}[t] + \mathbb{C}[t]\, g_1 + \cdots + \mathbb{C}[t]\, g_5$$

iff

$$f_i = p_0^i(t) + p_1^i(t) g_1 + \cdots + p_5^i(t) g_5$$

and

$$g_i\, g_j = p_0^{i,j}(t) + p_1^{i,j}(t) g_1 + \cdots + p_5^{i,j}(t) g_5,$$

Here: $f_1 = g_3, f_2 = g_2$; what, e.g., is $g_3^2 = ?$

EXAMPLE II contd. (2)

In[110]:=

```
{g1 = f1 f2^2, g2 = f2, g3 = f1, g4 = f2^2, g5 = f1 f2}
```

Out[110]=

$$\left\{\left(2+z^{9}\right)\left(1+z^{20}\right)^{2},\ 1+z^{20},\ 2+z^{9},\ \left(1+z^{20}\right)^{2},\ \left(2+z^{9}\right)\left(1+z^{20}\right)\right\}$$

RECALL: $t = z^6 - 1.$

In[137]:=

```
{g3^2, g3^2 - t^3} // Expand
```

Out[137]=

$$\left\{4+4\,z^{9}+z^{18},\ 5-3\,z^{6}+4\,z^{9}+3\,z^{12}\right\}$$

EXAMPLE II contd. (2)

In[110]:=

```
{g1 = f1 f2^2, g2 = f2, g3 = f1, g4 = f2^2, g5 = f1 f2}
```

Out[110]=

$$\left\{\left(2+z^{9}\right)\left(1+z^{20}\right)^{2},\ 1+z^{20},\ 2+z^{9},\ \left(1+z^{20}\right)^{2},\ \left(2+z^{9}\right)\left(1+z^{20}\right)\right\}$$

RECALL: $t = z^6 - 1$.

In[138]:=

```
{g3^2, g3^2 - t^3, g3^2 - t^3 - 3 t^2} // Expand
```

Out[138]=

$$\left\{4+4z^{9}+z^{18},\ 5-3z^{6}+4z^{9}+3z^{12},\ 2+3z^{6}+4z^{9}\right\}$$

EXAMPLE II contd. (3)

In[110]:=

```
{g1 = f1 f2^2, g2 = f2, g3 = f1, g4 = f2^2, g5 = f1 f2}
```

Out[110]=

$$\left\{ \left(2 + z^9\right) \left(1 + z^{20}\right)^2, 1 + z^{20}, 2 + z^9, \left(1 + z^{20}\right)^2, \left(2 + z^9\right) \left(1 + z^{20}\right) \right\}$$

RECALL: $t = z^6 - 1.$

In[141]:=

```
{g3^2 - t^3 - 3 t^2, g3^2 - t^3 - 3 t^2 - 4 g3} // Expand
```

Out[141]=

$$\left\{2 + 3 z^6 + 4 z^9, -6 + 3 z^6\right\}$$

EXAMPLE II contd. (3)

In[110]:=

```
{g1 = f1 f2^2, g2 = f2, g3 = f1, g4 = f2^2, g5 = f1 f2}
```

Out[110]=

$$\left\{\left(2+z^{9}\right)\left(1+z^{20}\right)^{2},\ 1+z^{20},\ 2+z^{9},\ \left(1+z^{20}\right)^{2},\ \left(2+z^{9}\right)\left(1+z^{20}\right)\right\}$$

RECALL: $t = z^6 - 1.$

In[143]:=

```
{g3^2 - t^3 - 3 t^2, g3^2 - t^3 - 3 t^2 - 4 g3 - 3 t + 3} // Expand
```

Out[143]=

$$\left\{2+3\,z^{6}+4\,z^{9},\ 0\right\}$$

EXAMPLE II contd. (4)

In[110]:=

```
{g1 = f1 f2^2, g2 = f2, g3 = f1, g4 = f2^2, g5 = f1 f2}
```

Out[110]=

$$\left\{\left(2+z^{9}\right)\left(1+z^{20}\right)^{2},\ 1+z^{20},\ 2+z^{9},\ \left(1+z^{20}\right)^{2},\ \left(2+z^{9}\right)\left(1+z^{20}\right)\right\}$$

RECALL: $t = z^6 - 1$.

In EXAMPLE II this works for all $g_i g_j$. E.g., to reduce $g_2\, g_4$ we need

In[157]:=

```
p24 = t^10 + 10 t^9 + 45 t^8 + 120 t^7 + 210 t^6 +
   252 t^5 + 210 t^4 + 120 t^3 + 45 t^2 + 10 t + 2;
```

In[159]:=

```
g2 g4 - (3 g4 - 3 g2 + p24) // Expand
```

Out[159]=

0

MONOID PROPERTY PROBLEM. To construct the g_i from the given $f_0(=t), f_1, \ldots, f_n$ such that

$$\mathbb{K}[t, f_1, \ldots, f_n] = \mathbb{K}[t] + \mathbb{K}[t]\, g_1 + \cdots + \mathbb{K}[t]\, g_k \qquad (\star)$$

we require the MONOID PROPERTY:

$$\langle \deg(f) : f \in \mathbb{K}[t, f_1, \ldots, f_n] \rangle = \langle \deg(t), \deg(f_1), \ldots, \deg(f_n) \rangle.$$

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NOTE. $(\star)$ implies

$$\langle \deg(f) : f \in \mathbb{K}[t, f_1, \ldots, f_n] \rangle = \langle \deg(t), \deg(g_1), \ldots \deg(g_k) \rangle.$$

MONOID PROPERTY PROBLEM. To construct the g_i from the given $f_0(=t), f_1, \ldots, f_n$ such that

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NOTE. $(\star)$ implies

$$\langle \deg(f) : f \in \mathbb{K}[t, f_1, \ldots, f_n] \rangle = \langle \deg(t), \deg(g_1), \ldots \deg(g_k) \rangle.$$

Back to EXAMPLE II: Here

$$\mathbb{C}[t, f_1, f_2] = \mathbb{C}[t] + \mathbb{C}[t]\, g_1 + \cdots + \mathbb{C}[t]\, g_5$$

with $f_1 = g_3$ and $f_2 = g_2$. Hence the MONOID PROPERTY holds.

WARNING. The MONOID PROPERTY

$$\langle \deg(f) : f \in \mathbb{C}[t, f_1, \ldots, f_n] \rangle = \langle \deg(t), \deg(f_1), \ldots, \deg(f_n) \rangle$$

is not always satisfied!

EXAMPLE III.

In[161]:=

```
t = z^6 - 1;  f1 = z^9 + 2;  f2 = z^20 + 1;  f3 = z^18 + z^4;
```

Here we have

$$f_3 - (t^3 + 3t^2 + 3t) = z^4 + 1.$$

WARNING. The MONOID PROPERTY

$$\langle \deg(f) : f \in \mathbb{C}[t, f_1, \dots, f_n] \rangle = \langle \deg(t), \deg(f_1), \dots, \deg(f_n) \rangle$$

is not always satisfied!

EXAMPLE III.

In[161]:=

```
t = z^6 - 1;  f1 = z^9 + 2;  f2 = z^20 + 1;  f3 = z^18 + z^4;
```

Here we have

$$f_3 - (t^3 + 3t^2 + 3t) = z^4 + 1.$$

SOLUTION: add $f_4 := z^4 + 1$ (i.e., $\mathbb{K}[t, f_1, f_2] \rightsquigarrow \mathbb{K}[t, f_1, f_2, f_4]$) and update the module basis as in the nuggets example:

In[184]:=

```
f4 = z^4 + 1;
```

In[185]:=

```
{G1 = f1 f4, G2 = f4^2, G3 = f1, G4 = f4, G5 = f1 f4^2}
```

Out[185]=

$$\{(1 + z^4)(2 + z^9), (1 + z^4)^2, 2 + z^9, 1 + z^4, (1 + z^4)^2 (2 + z^9)\}$$

EXAMPLE III contd. (1)

SOLUTION: add $f_4 := z^4 + 1$ (i.e., $\mathbb{K}[t, f_1, f_2] \rightsquigarrow \mathbb{K}[t, f_1, f_2, f_4]$); i.e., update the generators as in the nuggets problem:

In[184]:=

```
f4 = z^4 + 1;
```

In[185]:=

```
{G1 = f1 f4, G2 = f4^2, G3 = f1, G4 = f4, G5 = f1 f4^2}
```

Out[185]=

$$\{(1+z^4)(2+z^9),\ (1+z^4)^2,\ 2+z^9,\ 1+z^4,\ (1+z^4)^2(2+z^9)\}$$

EXAMPLE III contd. (1)

SOLUTION: add $f_4 := z^4 + 1$ (i.e., $\mathbb{K}[t, f_1, f_2] \rightsquigarrow \mathbb{K}[t, f_1, f_2, f_4]$);
i.e., update the generators as in the nuggets problem:

In[184]:=

```
f4 = z^4 + 1;
```

In[185]:=

```
{G1 = f1 f4, G2 = f4^2, G3 = f1, G4 = f4, G5 = f1 f4^2}
```

Out[185]=

$$\{(1+z^4)(2+z^9),\ (1+z^4)^2,\ 2+z^9,\ 1+z^4,\ (1+z^4)^2(2+z^9)\}$$

NOTE. Now all the f_i and G_iG_j are in the $\mathbb{C}[t]$-module. —
The G_i-degrees are much smaller

$$\{13, 8, 9, 4, 17\} \equiv \{1, 2, 3, 4, 5\} \pmod 6;$$

in comparison to the g_i-degrees

$$\{49, 20, 9, 40, 29\} \equiv \{1, 2, 3, 4, 5\} \pmod 6.$$

The Modular Module Point of View

$$\Gamma_0(\ell) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \ell \mid c \right\}$$

$f \in M(\ell)$; i.e., f is a MODULAR FUNCTION for $\Gamma_0(\ell)$:$\Leftrightarrow$

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$f \in M(\ell)$; i.e., f is a MODULAR FUNCTION for $\Gamma_0(\ell)$:$\Leftrightarrow$

- f is analytic on the upper half complex plane.

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$f \in M(\ell)$; i.e., f is a MODULAR FUNCTION for $\Gamma_0(\ell)$:⇔

- f is analytic on the upper half complex plane.
- f satisfies the modular transformation property

$$(f \mid \gamma)(\tau) := f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau) \text{ for all } \gamma \in \begin{pmatrix} a\ b \\ c\ d \end{pmatrix} \in \Gamma_0(\ell).$$

$$\Gamma_0(\ell) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \ell \mid c \right\}$$

$f \in M(\ell)$; i.e., f is a MODULAR FUNCTION for $\Gamma_0(\ell)$:⇔

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$$(f \mid \gamma)(\tau) := f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau) \text{ for all } \gamma \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\ell).$$

- For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{Im}(\tau)$ big:

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \sum_{n=\mathrm{ord}_{a/c} f}^{\infty} a_n q_w^n \text{ where } q_w := q_w(\tau) = e^{2\pi i\tau/w}$$

and $w = w(a/c) := \ell / \gcd(c^2, \ell)$.

Example. $q := q_1(\tau) = e^{2\pi i\tau}$, $W := \left(\begin{smallmatrix} 0 & -1 \\ 11 & 0 \end{smallmatrix}\right)$,

$f(\tau) := z_{11}(\tau) = q^5 \dfrac{(q^{11};q^{11})_\infty^{12}}{(q;q)_\infty^{12}} \in M_{\mathbb{Z}}^0(11)$ and $f \mid W \in M^\infty(11)$:

Example. $q := q_1(\tau) = e^{2\pi i\tau}$, $W := \begin{pmatrix} 0 & -1 \\ 11 & 0 \end{pmatrix}$,

$$f(\tau) := z_{11}(\tau) = q^5 \frac{(q^{11};q^{11})_\infty^{12}}{(q;q)_\infty^{12}} \in M_{\mathbb{Z}}^0(11) \text{ and } f \mid W \in M^\infty(11):$$

$$f\left(\frac{1\tau+0}{0\tau+1}\right) = q^5 + 12q^6 + O(q^9), \qquad \operatorname{ord}_\infty f = 5;$$

$$f\left(\frac{0\tau-1}{1\tau+0}\right) = \frac{1}{11^6}\Big(\frac{1}{q^{5/11}} - \frac{12}{q^{4/11}} + O\big((q^{1/11})^{-2}\big)\Big), \quad \operatorname{ord}_0 f = -5.$$

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$$f\left(\frac{0\tau-1}{11\tau+0}\right) = \frac{1}{11^6}\left(\frac{1}{q^5} - \frac{12}{q^4} + O\big(q^{-2}\big)\right), \quad \operatorname{ord}_\infty f \mid W = -5$$

$$(z_{11} \mid W)(\tau) := z_{11} \mid \begin{pmatrix} 0 & -1 \\ 11 & 0 \end{pmatrix}(\tau) = z_{11}\left(-\frac{1}{11\tau}\right) = \frac{11^{-6}}{z_{11}(\tau)}.$$

Zero test in $M^\infty(\ell)$, ℓ prime

$$f = \sum_{n=-M}^{\infty} a(n)q^n \in M^\infty(\ell), \quad g = \sum_{n=-N}^{\infty} b(n)q^n \in M^\infty(\ell)$$

$f = g \Leftrightarrow$

Zero test in $M^\infty(\ell)$, ℓ prime

$$f = \sum_{n=-M}^{\infty} a(n) q^n \in M^\infty(\ell), \quad g = \sum_{n=-N}^{\infty} b(n) q^n \in M^\infty(\ell)$$

$f = g \Leftrightarrow M = N$ and

$$\frac{a_{-N}}{q^N} + \frac{a_{-N+1}}{q^{N-1}} + \cdots + a_0 = \frac{b_{-N}}{q^N} + \frac{b_{-N+1}}{q^{N-1}} + \cdots + b_0$$

Short notation for partition generating functions

$\ell = 5, 7, 11:$

$$L_\ell := q \prod_{k=1}^{\infty}(1 - q^{\ell k}) \sum_{n=0}^{\infty} p(\ell n + \mu_\ell) q^n \in M_{\mathbb{Z}}^0(\ell).$$

where

$$\mu_\ell := \begin{cases} 4, \text{if } \ell = 5 \\ 5, \text{if } \ell = 7 \\ 6, \text{if } \ell = 11 \end{cases}$$

Ramanujan's witness identities for $\ell = 5, 7$:

$$L_5 = 5z_5$$

where

$$z_5 := q\frac{(q^5;q^5)_\infty^6}{(q;q)_\infty^6} \in M_{\mathbb{Z}}^0(5)$$

Ramanujan's witness identities for $\ell = 5, 7$:

$$L_5 = 5z_5$$

where

$$z_5 := q\frac{(q^5;q^5)_\infty^6}{(q;q)_\infty^6} \in M_{\mathbb{Z}}^0(5)$$

$$L_7 = 7z_7 + 49z_7^2$$

where

$$z_7 := q\frac{(q^7;q^7)_\infty^4}{(q;q)_\infty^4} \in M_{\mathbb{Z}}^0(7)$$

Ramanujan's witness identities for $\ell = 5, 7$:

$$L_5 = 5z_5$$

where

$$z_5 := q\frac{(q^5;q^5)_\infty^6}{(q;q)_\infty^6} \in M_{\mathbb{Z}}^0(5)$$

$$L_7 = 7z_7 + 49z_7^2$$

where

$$z_7 := q\frac{(q^7;q^7)_\infty^4}{(q;q)_\infty^4} \in M_{\mathbb{Z}}^0(7)$$

Module point of view for $\ell = 5, 7$:

$$\begin{aligned} M_{\mathbb{Z}}^0(\ell) = \mathbb{Z}[z_\ell] &= \langle 1 \rangle_{\mathbb{Z}[\ell]} \\ &= \{p(z_\ell)\cdot 1 : p(z_\ell) \in \mathbb{Z}[\ell]\} \end{aligned}$$

Lehner's witness identity for $\ell = 11$:

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Module point of view:

$$\begin{aligned} M^0_{\mathbb{Z}}(11) &= \langle 1, AC \rangle_{\mathbb{Z}[C]} \\ &= \{p_0(C) \cdot 1 + p_1(C) \cdot AC : p_i(C) \in \mathbb{Z}[C]\} \end{aligned}$$

Atkin's witness identity for $\ell = 11$:

$$L_{11} = 11^4 z_{11} + 11 g_2 + 2 \cdot 11^2 g_3 + 11^3 g_4$$

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Module point of view:

$$\begin{aligned} M_{\mathbb{Z}}^0(11) &= \langle 1, g_2, g_3, g_4, g_6 \rangle_{\mathbb{Z}[z_{11}]} \\ &= \{p_0(z_{11}) \cdot 1 + p_2(z_{11}) \cdot g_2 + p_3(z_{11}) \cdot g_3 \\ &\qquad + p_4(z_{11}) \cdot g_4 + p_6(z_{11}) \cdot g_6 : \\ &\qquad p_i(z_{11}) \in \mathbb{Z}[z_{11}]\} \end{aligned}$$

NOTE. $g_6 := g_2 g_4$.

Radu's witness identity for $\ell = 11$:

$$\frac{w_6^2 w_7}{w_3^2} L_{11} = 1078t^4 + 13893t^3 + 31647t^2 + 11209t - 21967 + s_1(187t^3 + 5390t^2 + 594t - 9581) + s_2(11t^3 + 2761t^2 + 5368t - 6754);$$

t, s_1, s_2 are $\mathbb{Q}$-linear combinations of 3 eta quotients in $M_{\mathbb{Z}}^{\infty}(22)$:

$$\left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^a \left(\frac{\eta(11\tau)}{\eta(\tau)}\right)^b \left(\frac{\eta(22\tau)}{\eta(\tau)}\right)^c = q^{\square} \prod_{k=1}^{\infty} \frac{(1-q^{2k})^a (1-q^{11k})^b (1-q^{22k})^c}{(1-q^k)^{a+b+c}}$$

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t, s_1, s_2 are $\mathbb{Q}$-linear combinations of 3 eta quotients in $M_{\mathbb{Z}}^{\infty}(22)$:

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Module point of view:

$$M_{\mathbb{Z}}^{\infty}(22) = \langle 1, s_1, s_2 \rangle_{\mathbb{Z}[t]}$$
$$= \{p_0(t) \cdot 1 + p_1(t) \cdot s_1 + p_2(t) \cdot s_2 : p_i(t) \in \mathbb{Z}[t]\}$$

Hemmecke's witness identity for $\ell = 11$:

$$F := \frac{w_6^2 w_7}{w_3^2} L_{11} = 11^2 \cdot 3068 w_7 + 11^2 (3w_1 + 4236) w_6$$
$$+ 11(285 w_1 + 11 \cdot 5972) w_5 + \frac{11}{8}(w_1^2 + 11 \cdot 4497 w_1 + 11^2 \cdot 3156) w_3$$
$$+ 11(1867 w_1 + 11 \cdot 2476) w_4$$
$$- \frac{11}{8}(w_1^3 + 1011 w_1^2 + 11 \cdot 6588 w_1 + 11^2 \cdot 10880),$$

$$w_j := q^{\square} \prod_{k=1}^{\infty} \frac{(1-q^{2k})^{a_1(j)}(1-q^{11k})^{a_2(j)}(1-q^{22k})^{a_3(j)}}{(1-q^k)^{a_1(j)+a_2(j)+a_3(j)}} \in M_{\mathbb{Z}}^{\infty}(22).$$

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Module point of view: choosing $t := w_1$, h_1, h_2 [→ next page],

$$F \in C[w_1, \ldots, w_7] = \langle 1, h_1, h_2, w_3, w_4, w_6 \rangle_{C[t]},$$

$C :=$ rational numbers with denominators not divisible by 11; an Euclidean domain with "degree" function the 11-adic evaluation

$$v\left(\frac{a}{b}\right) = v(\pm 2^{\alpha} 3^{\beta} \ldots 11^{\gamma} \ldots) := \gamma.$$

$$
\begin{aligned}
h_1 &= \frac{1}{2^{10}}(8w_7 - 40w_6 + 168w_5 + (2343 - w_1)w_3 \\
&\qquad - 680w_4 + w_1^2 + 505w_1), \\
&= 11q^{-3} + 11q^{-1} - \frac{11^2}{2^5} + O(q), \\
h_2 &= \frac{1}{2^3}(w_3 - w_1) \\
&= q^{-4} - 2q^{-3} + 2q^{-2} + 3 + O(q)
\end{aligned}
$$

The witness identity by P. and Radu for $\ell = 11$:

$$f^5 = p_0(\bar{z}_{11}) + p_1(\bar{z}_{11})f + p_2(\bar{z}_{11})f^2 + p_3(\bar{z}_{11})f^3 + p_4(\bar{z}_{11})f^4$$

where

$$f := \bar{z}_{11} L_{11} = q\, \bar{z}_{11} \prod_{k=1}^{\infty}(1-q^{11k}) \sum_{n=0}^{\infty} p(11n+6) q^n$$

with

$$\bar{z}_{11} := \frac{1}{z_{11}} = q^{-5} \prod_{k=1}^{\infty} \left(\frac{1-q^k}{1-q^{11k}} \right)^{12} \in M_{\mathbb{Z}}^{\infty}(11).$$

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Module point of view:

$$\begin{aligned}\mathbb{C}[\bar{z}_{11}, f] &= \mathbb{C}[\bar{z}_{11}] + \mathbb{C}[\bar{z}_{11}]\, f + \mathbb{C}[\bar{z}_{11}]\, f^2 + \mathbb{C}[\bar{z}_{11}]\, f^3 + \mathbb{C}[\bar{z}_{11}]\, f^4 \\ &= \langle 1, f, f^2, f^3, f^4 \rangle_{\mathbb{C}[\bar{z}_{11}]}\end{aligned}$$

Atkin's Generators Revisited

$$M_{\mathbb{Z}}^{0}(11) = \langle 1, g_2, g_3, g_4, g_6 \rangle_{\mathbb{Z}[z_{11}]}.$$

NOTE. g_2, g_3, g_4 are as in Atkin's 11-witness identity;
in addition, $g_6 := g_2 g_4$.

Recall: $z_{11} := q^5 \frac{(q^{11};q^{11})_\infty^{12}}{(q;q)_\infty^{12}} \in M_{\mathbb{Z}}^{0}(11), \quad g_4 := g_2^2 - g_3,$

$$M^0_{\mathbb{Z}}(11) = \langle 1, g_2, g_3, g_4, g_6 \rangle_{\mathbb{Z}[z_{11}]}.$$

NOTE. g_2, g_3, g_4 are as in Atkin's 11-witness identity;
in addition, $g_6 := g_2 g_4$.

Recall: $$z_{11} := q^5 \frac{(q^{11}; q^{11})^{12}_\infty}{(q;q)^{12}_\infty} \in M^0_{\mathbb{Z}}(11), \quad g_4 := g_2^2 - g_3,$$

$$10\ g_2 := 11^2 \bar{u}^5_{11} - \frac{\sum_{n=0}^\infty \left(1 + \left(\frac{n-3}{11}\right)\right) p_5(n) q^n}{(q;q)^5_\infty}$$

with

$$\bar{u}_{11} := q^5 \frac{(q^{121}; q^{121})_\infty}{(q;q)_\infty} \quad \text{and} \quad (q;q)^r_\infty =: \sum_{n=0}^\infty p_r(n) q^n,$$

$$14\ g_3 := -14 g_2 + 11^3 \bar{u}^7_{11} - \frac{\sum_{n=0}^\infty \left(1 + \left(\frac{2-n}{11}\right)\right) p_7(n) q^n}{(q;q)^7_\infty}.$$

$\rightsquigarrow$ Are there alternative functions or better representations of the g_i?

$$(\operatorname{ord}_\infty g_2, \operatorname{ord}_\infty g_3, \operatorname{ord}_\infty g_4, \operatorname{ord}_\infty g_6) = (1, 2, 3, 4):$$

$$g_2 = q + 5q^2 + O(q^3), \qquad g_3 = q^2 + 9q^3 + O(q^4),$$

$$g_4 = q^3 + 14q^4 + O(q^5), \qquad g_6 = q^4 + 19q^5 + O(q^6)$$

$(\operatorname{ord}_\infty g_2, \operatorname{ord}_\infty g_3, \operatorname{ord}_\infty g_4, \operatorname{ord}_\infty g_6) = (1, 2, 3, 4):$

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$$g_4 = q^3 + 14q^4 + O(q^5), \qquad g_6 = q^4 + 19q^5 + O(q^6)$$

$(\operatorname{ord}_0 g_2, \operatorname{ord}_0 g_3, \operatorname{ord}_0 g_4, \operatorname{ord}_0 g_6) = (-2, -3, -4, -6):$

$$g_2 \mid W = \frac{1}{11^2}\Big(\frac{1}{q^2} + \frac{2}{q} - \dots\Big), \quad g_3 \mid W = \frac{1}{11^3}\Big(\frac{1}{q^3} - \frac{3}{q^2} - \dots\Big),$$
$$g_4 \mid W = \frac{1}{11^4}\Big(\frac{1}{q^4} - \frac{1}{q^3} + \dots\Big), \quad g_6 \mid W = \frac{1}{11^6}\Big(\frac{1}{q^6} - \frac{5}{q^3} - \dots\Big).$$

Recall,

$$W = \begin{pmatrix} 0 & -1 \\ 11 & 0 \end{pmatrix} \text{ and } \operatorname{ord}_0 f = \operatorname{ord}_\infty f \mid W.$$

Which $J_i = \eta(\tau)^{r_1(i)}\eta(11\tau)^{r_{11}(i)} \in M^0_{\mathbb{Z}}(11)$ satisfy

$$(\mathrm{ord}_0\, J_2, \mathrm{ord}_0\, J_3, \mathrm{ord}_0\, J_4, \mathrm{ord}_0\, J_6) = (-2, -3, -4, -6)?$$

Which $J_i = \eta(\tau)^{r_1(i)}\eta(11\tau)^{r_{11}(i)} \in M^0_{\mathbb{Z}}(11)$ satisfy

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By Ligozat's formula and Newman's Lemma this translates into

$11r_1(i) + r_{11}(i) = -120, r_1(i) + r_{11}(i) = 0, r_1(i) + 11r_{11}(i) = 24a(i),$
$11r_1(i) + r_{11}(i) = 24b(i),$ and $r_{11}(i) = 2c(i),\ i = 2, 3, 4, 6.$

Which $J_i = \eta(\tau)^{r_1(i)}\eta(11\tau)^{r_{11}(i)} \in M_{\mathbb{Z}}^0(11)$ satisfy

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- No integer solution for $i = 2, 3, 4, 6$.
- For $i = 5$: exactly one solution, z_{11} with $\mathrm{ord}_0\, z_{11} = -5$; $(r_1(5), r_{11}(5)) = (-12, 12)$ and $(a(5), b(5), c(5)) = (5, -5, 6)$.

Which $J_i = \eta(\tau)^{r_1(i)}\eta(11\tau)^{r_{11}(i)} \in M_{\mathbb{Z}}^0(11)$ satisfy

$$(\mathrm{ord}_0 J_2, \mathrm{ord}_0 J_3, \mathrm{ord}_0 J_4, \mathrm{ord}_0 J_6) = (-2, -3, -4, -6)?$$

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$\rightsquigarrow$ Increase the "search space" to find alternatives J_i to Atkin's g_i:

Which $J_i = \eta(\tau)^{r_1(i)}\eta(11\tau)^{r_{11}(i)} \in M^0_{\mathbb{Z}}(11)$ satisfy

$$(\mathrm{ord}_0\, J_2, \mathrm{ord}_0\, J_3, \mathrm{ord}_0\, J_4, \mathrm{ord}_0\, J_6) = (-2,-3,-4,-6)?$$

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$\rightsquigarrow$ Increase the "search space" to find alternatives J_i to Atkin's g_i:

- by allowing non-eta quotients;

Which $J_i = \eta(\tau)^{r_1(i)}\eta(11\tau)^{r_{11}(i)} \in M_{\mathbb{Z}}^0(11)$ satisfy

$$(\mathrm{ord}_0\, J_2, \mathrm{ord}_0\, J_3, \mathrm{ord}_0\, J_4, \mathrm{ord}_0\, J_6) = (-2, -3, -4, -6)?$$

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$\rightsquigarrow$ Increase the "search space" to find alternatives J_i to Atkin's g_i:

- by allowing non-eta quotients;
- by extending from $M(11)$ to $M(22)$; etc.

TASK 1. Construct $F_i \in M^{\infty}(11)$, $i = 2, 3, 4, 6$, such that

$$\operatorname{ord}_{\infty} F_i = -i \text{ and } \operatorname{ord}_0 F_i \geq 0.$$

Then $\operatorname{ord}_0 J_i = -i$ for $J_i := F_i \mid W \in M^0(11)$.

TASK 1. Construct $F_i \in M^{\infty}(11)$, $i = 2, 3, 4, 6$, such that

$$\operatorname{ord}_{\infty} F_i = -i \text{ and } \operatorname{ord}_0 F_i \geq 0.$$

Then $\operatorname{ord}_0 J_i = -i$ for $J_i := F_i \mid W \in M^0(11)$.

TASK 2. Turn the J_i into $g_i \in M^0(11)$ such that $\operatorname{ord}_0 g_i = -i$ remains valid, and

$$(\operatorname{ord}_{\infty} g_2, \operatorname{ord}_{\infty} g_3, \operatorname{ord}_{\infty} g_4, \operatorname{ord}_{\infty} g_6) = (1, 2, 3, 4).$$

Recall TASK 1. Construct $F_i \in M^\infty(11)$, $i = 2, 3, 4, 6$, such that

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$$\Gamma_0(11) = \Gamma_0(22) \cup \Gamma_0(22)V \cup \Gamma_0(22)V^2 \text{ for } V = \begin{pmatrix} 1 & 1 \\ 11 & 12 \end{pmatrix}.$$

Recall TASK 1. Construct $F_i \in M^\infty(11)$, $i = 2, 3, 4, 6$, such that

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$$\Gamma_0(11) = \Gamma_0(22) \cup \Gamma_0(22)V \cup \Gamma_0(22)V^2 \text{ for } V = \begin{pmatrix} 1 & 1 \\ 11 & 12 \end{pmatrix}.$$

F_2: Hence for any $f_2 \in M^\infty(22)$,

$$F_2 := f_2 + f_2 \mid V + f_2 \mid V^2 \in M^\infty(11).$$

To have $\operatorname{ord}_\infty F_2 = -2$, we solve the linear Ligozat-Newman system with ansatz $f_2 = \eta(\tau)^{r_1}\eta(2\tau)^{r_2}\eta(11\tau)^{r_{11}}\eta(22\tau)^{r_{22}}$ s.t.

$$f_2 \in M^\infty(22) \text{ and } \operatorname{ord}_\infty f_2 = -2 \text{ and } \operatorname{ord}_{1/11} f_2 \geq -4 :$$

Solution: $$f_2(\tau) := \frac{\eta(\tau)\eta(2\tau)^3}{\eta(11\tau)^3\eta(22\tau)} = q^{-2}\prod_{k=1}^{\infty}\frac{(1-q^k)(1-q^{2k})^3}{(1-q^{11k})^3(1-q^{22k})}.$$

F_3: Next for $f_3 \in M^\infty(22)$,

$$F_3 := f_3 + f_3 \mid V + f_3 \mid V^2 \in M^\infty(11).$$

To have $\operatorname{ord}_\infty F_3 = -3$, we solve the linear Ligozat-Newman system with ansatz $f_3 = \eta(\tau)^{r_1}\eta(2\tau)^{r_2}\eta(11\tau)^{r_{11}}\eta(21\tau)^{r_{22}}$ s.t.

$$f_3 \in M^\infty(22) \text{ and } \operatorname{ord}_\infty f_3 = -3 \text{ and } \operatorname{ord}_{1/11} f_3 \geq -6 :$$

Solution: $$f_3(\tau) := \frac{\eta(\tau)^3\eta(2\tau)}{\eta(11\tau)\eta(22\tau)^3} = q^{-3}\prod_{k=1}^{\infty}\frac{(1-q^k)^3(1-q^{2k})}{(1-q^{11k})(1-q^{22k})^3}.$$

F_3: Next for $f_3 \in M^\infty(22)$,

$$F_3 := f_3 + f_3 \mid V + f_3 \mid V^2 \in M^\infty(11).$$

To have $\operatorname{ord}_\infty F_3 = -3$, we solve the linear Ligozat-Newman system with ansatz $f_3 = \eta(\tau)^{r_1}\eta(2\tau)^{r_2}\eta(11\tau)^{r_{11}}\eta(21\tau)^{r_{22}}$ s.t.

$$f_3 \in M^\infty(22) \text{ and } \operatorname{ord}_\infty f_3 = -3 \text{ and } \operatorname{ord}_{1/11} f_3 \geq -6 :$$

Solution: $f_3(\tau) := \dfrac{\eta(\tau)^3\eta(2\tau)}{\eta(11\tau)\eta(22\tau)^3} = q^{-3}\prod_{k=1}^{\infty}\dfrac{(1-q^k)^3(1-q^{2k})}{(1-q^{11k})(1-q^{22k})^3}.$

F_4 and F_6: Finally, for $i = 4$ and $i = 6$,

$$F_i := f_i + f_i \mid V + f_i \mid V^2 \in M^\infty(11)$$

with

$$f_4 := f_2^2 = \frac{1}{q^4} + O(q^{-3}) \text{ and } f_6 := f_3^2 = \frac{1}{q^6} + O(q^{-5}) \text{ in } M^\infty(22).$$

$V = \begin{pmatrix} 1 & 1 \\ 11 & 12 \end{pmatrix}$, $W = \begin{pmatrix} 0 & -1 \\ 11 & 0 \end{pmatrix}$, $U_2 \sum_{n=N}^{\infty} a(n) q^n := \sum_{2n \geq N}^{\infty} a(2n) q^n$:

$$\begin{aligned} F_i(\tau) &:= f_i(\tau) + (f_i \mid V)(\tau) + (f_i \mid V^2)(\tau) \\ &= f_i(\tau) + 2\, U_2((f_i \mid V)(2\tau)) \in M^{\infty}(11) \end{aligned}$$

$V = \begin{pmatrix} 1 & 1 \\ 11 & 12 \end{pmatrix}$, $W = \begin{pmatrix} 0 & -1 \\ 11 & 0 \end{pmatrix}$, $U_2 \sum_{n=N}^{\infty} a(n)q^n := \sum_{2n \geq N}^{\infty} a(2n)q^n$:

$$\begin{aligned} F_i(\tau) &:= f_i(\tau) + (f_i \mid V)(\tau) + (f_i \mid V^2)(\tau) \\ &= f_i(\tau) + 2\, U_2((f_i \mid V)(2\tau)) \in M^{\infty}(11) \end{aligned}$$

$$(f_2 \mid V)(2\tau) = -\frac{1}{2} f_3(\tau) \text{ and } (f_3 \mid V)(2\tau) = -2 f_2(\tau), \text{ and}$$

$$(f_2 \mid W)(2\tau) = \frac{11^2}{2} \frac{1}{f_2(\tau)} \text{ and } (f_3 \mid W)(2\tau) = 2 \cdot 11^2 \frac{1}{f_3(\tau)}.$$

This gives compact representations for the F_i, resp. J_i:

Recall

$$f_2(\tau) := \frac{\eta(\tau)\eta(2\tau)^3}{\eta(11\tau)^3\eta(21\tau)} = q^{-2}\prod_{k=1}^{\infty}\frac{(1-q^k)(1-q^{2k})^3}{(1-q^{11k})^3(1-q^{22k})}.$$

$$f_3(\tau) := \frac{\eta(\tau)^3\eta(2\tau)}{\eta(11\tau)\eta(21\tau)^3} = q^{-3}\prod_{k=1}^{\infty}\frac{(1-q^k)^3(1-q^{2k})}{(1-q^{11k})(1-q^{22k})^3}.$$

$$U_2\sum_{n=N}^{\infty}a(n)q^n := \sum_{2n\geq N}^{\infty}a(2n)q^n.$$

$$F_2(\tau) = f_2(\tau) - (U_2f_3)(\tau) = q^{-2} + 2q^{-1} - 12 + 5q + 8q^2 + \dots;$$
$$F_3(\tau) = f_3(\tau) - 4(U_2f_2)(\tau) = q^{-3} - 3q^{-2} - 5q^{-1} + 24 - 13q - \dots;$$
$$F_4(\tau) = f_2(\tau)^2 + \frac{1}{2}(U_2f_3^2)(\tau) = q^{-4} - \frac{3}{2}q^{-3} - \frac{7}{2}q^{-2} - \frac{21}{2}q^{-1} + \dots;$$
$$F_6(\tau) = f_3(\tau)^2 + 8(U_2f_2^2)(\tau) = q^{-6} - 6q^{-5} + 7q^{-4} + 22q^{-3} + \dots.$$

Recall $J_i := F_i \mid W$:

$$J_2(\tau) = -11^2\Big(\frac{1}{f_3} - U_2\frac{1}{f_2}\Big)(\tau) = 11^2(q + 5q^2 + \dots);$$

$$J_3(\tau) = -11^2\Big(\frac{1}{f_2} - 4\,U_2\frac{1}{f_3}\Big)(\tau) = 11^2(11q^2 + 99q^3 + \dots);$$

$$J_4(\tau) = 11^4\Big(\frac{1}{f_3^2} + \frac{1}{2}\,U_2\frac{1}{f_2^2}\Big)(\tau) = 11^4\Big(\frac{1}{2}q^2 + \frac{11}{2}q^3 + \dots\Big);$$

$$J_6(\tau) = 11^4\Big(\frac{1}{f_2^2} + 8\,U_2\frac{1}{f_3^2}\Big)(\tau) = 11^4(8q^3 + 233q^4 + \dots).$$

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Recall TASK 2. Turn the J_i into $g_i \in M^0(11)$ such that $\mathrm{ord}_0\, g_i = -i$ remains valid, and

$$(\mathrm{ord}_\infty\, g_2, \mathrm{ord}_\infty\, g_3, \mathrm{ord}_\infty\, g_4, \mathrm{ord}_\infty\, g_6) = (1, 2, 3, 4).$$

Recall $J_i := F_i \mid W$:

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$$g_4 := \frac{J_4}{11^4} - \frac{1}{2}\frac{J_3}{11^3} = q^3 + \dots \in M^0_{\mathbb{Z}}(11).$$

Proof. $$\frac{J_4}{11^4} - \frac{1}{2}\frac{J_3}{11^3} = \frac{F_4 \mid W}{11^4} - \frac{1}{2}\frac{F_3 \mid W}{11^3} = z_{11}F_2 \in M^0_{\mathbb{Z}}(11).$$

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New representations of Atkin's g_i:

$$g_2(\tau) = \frac{J_2(\tau)}{11^2} = q + q^5 + \dots \in M^0_{\mathbb{Z}}(11);$$

$$g_3(\tau) = \frac{J_3(\tau)}{11^3} = q^2 + 9q^3 + \dots \in M^0_{\mathbb{Z}}(11);$$

$$g_4(\tau) = \frac{J_4(\tau)}{11^4} - \frac{1}{2}\,\frac{J_3(\tau)}{11^3} = q^3 + O(q^4) \in M^0_{\mathbb{Z}}(11).$$

$$g_6(\tau) = \frac{J_6(\tau)}{11^6} - 8\,\frac{J_4(\tau)}{11^6} + 4\,\frac{J_3(\tau)}{11^5} = q^4 + O(q^5) \in M^0_{\mathbb{Z}}(11).$$

Atkin's functions:

$$g_2(\tau) = \frac{J_2(\tau)}{11^2} = q + q^5 + \cdots \in M^0_{\mathbb{Z}}(11);$$

$$g_3(\tau) = \frac{J_3(\tau)}{11^3} = q^2 + 9q^3 + \cdots \in M^0_{\mathbb{Z}}(11);$$

$$g_4(\tau) = \frac{J_4(\tau)}{11^4} - \frac{1}{2}\ \frac{J_3(\tau)}{11^3} = q^3 + O(q^4) \in M^0_{\mathbb{Z}}(11).$$

$$g_6(\tau) = \frac{J_6(\tau)}{11^6} - 8\ \frac{J_4(\tau)}{11^6} + 4\ \frac{J_3(\tau)}{11^5} = q^4 + O(q^5) \in M^0_{\mathbb{Z}}(11).$$

NOTE.

$$\left(\frac{J_4}{11^4} - \frac{1}{2}\ \frac{J_3}{11^3}\right)\bar{z}_{11} = F_2;$$

$$\left(\frac{J_6}{11^6} - 8\ \frac{J_4}{11^6} + 4\ \frac{J_3}{11^5}\right)\bar{z}_{11}^2 = F_6 - 8F_4 + 44F_3.$$

This can be algorithmically derived and proved.

A New 11-Witness Identity

Recall: Atkin found

$$\begin{aligned} L_{11} &= q \prod_{k=1}^{\infty}(1-q^{11k})\sum_{n=0}^{\infty}p(11n+6)q^n \\ &= 11^4 z_{11} + 11 g_2 + 2\cdot 11^2 g_3 + 11^3 g_4 \\ &= \frac{1}{11}J_4 - \frac{7}{22}J_3 + \frac{1}{11}J_2 + 11^4 z_{11}. \end{aligned}$$

We express the J_i in terms of f_2 and f_3:

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$$L_{11} = q \prod_{k=1}^{\infty}(1-q^{11k}) \sum_{n=0}^{\infty} p(11n+6)q^n$$
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$$= \frac{1}{11} J_4 - \frac{7}{22} J_3 + \frac{1}{11} J_2 + 11^4 z_{11}.$$

We express the J_i in terms of f_2 and f_3:

$$L_{11} = 11^3\Big(\frac{1}{f_3^2} + \frac{1}{2} U_2 \frac{1}{f_2^2}\Big) + \frac{7}{2} 11 \Big(\frac{1}{f_2} - 4 U_2 \frac{1}{f_3}\Big) - 11\Big(\frac{1}{f_3} - U_2 \frac{1}{f_2}\Big) + 11^4 z_{11}$$

$$L_{11} = 11^3\Big(\frac{1}{f_3^2} + \frac{1}{2}U_2\frac{1}{f_2^2}\Big) + \frac{7}{2}11\Big(\frac{1}{f_2} - 4U_2\frac{1}{f_3}\Big) - 11\Big(\frac{1}{f_3} - U_2\frac{1}{f_2}\Big) + 11^4 z_{11}$$

Recall: $$z_{11} := q^5\frac{(q^{11};q^{11})_\infty^{12}}{(q;q)_\infty^{12}} \in M_{\mathbb{Z}}^0(11),$$

$$f_2(\tau) := \frac{\eta(\tau)\eta(2\tau)^3}{\eta(11\tau)^3\eta(21\tau)} = q^{-2}\prod_{k=1}^{\infty}\frac{(1-q^k)(1-q^{2k})^3}{(1-q^{11k})^3(1-q^{22k})}.$$

$$f_3(\tau) := \frac{\eta(\tau)^3\eta(2\tau)}{\eta(11\tau)\eta(21\tau)^3} = q^{-3}\prod_{k=1}^{\infty}\frac{(1-q^k)^3(1-q^{2k})}{(1-q^{11k})(1-q^{22k})^3}.$$

$$U_2\sum_{n=N}^{\infty}a(n)q^n := \sum_{2n\geq N}^{\infty}a(2n)q^n.$$

Conclusion

Recall:

- Regarding $\ell = 11$ Watson states: "Da die Untersuchung der Aussage über 11^α recht langweilig ist, verschiebe ich den Beweis dieses Falles auf eine spätere Abhandlung".
- Atkin ends with the comment: "We may observe finally that, in comparison with $\ell = 5$ and $\ell = 7$, this proof is indeed "langweilig" [i.e., "boring"] as Watson suggested."

We disagree!

What about $p(25) = 1958$?

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Kolberg [Mathematica Scandinavica, 1957]:

$$\Big(\sum_{n=0}^{\infty} p(5n) q^n\Big)\Big(\sum_{n=0}^{\infty} p(5n+3) q^n\Big) = 3\prod_{j=1}^{\infty} \frac{(1-q^{5j})^6}{(1-q^j)^4} + 25q \prod_{j=1}^{\infty} \frac{(1-q^{5j})^{10}}{(1-q^j)^{12}}.$$

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Given the left side, the right side is produced by Radu's Ramanujan-Kolberg algorithm.

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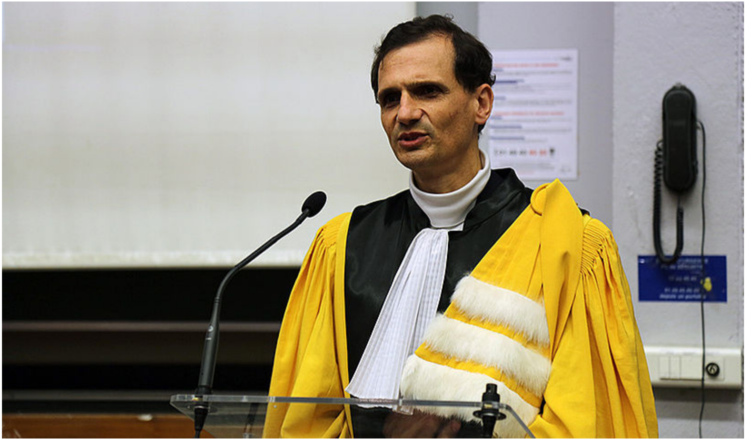
Mathematica implementation by Nicolas Smoot.

Happy Birthday, Christian!



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