The essence of bijections:
from growth diagrams to Laguerre heaps of segments for the PASEP

81th SlC, Strobl, Austria
KrattenthalerFest
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growth diagrams

Laguerre

essence

heaps of segments

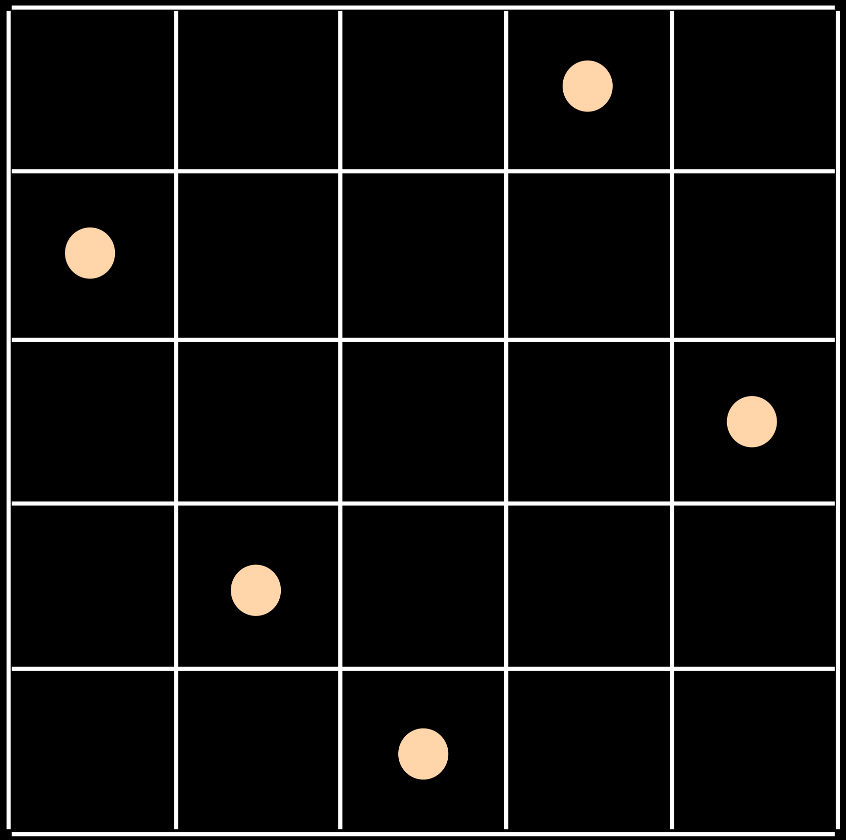
PASEP

growth diagrams

S. Fomin, 1986, 1994

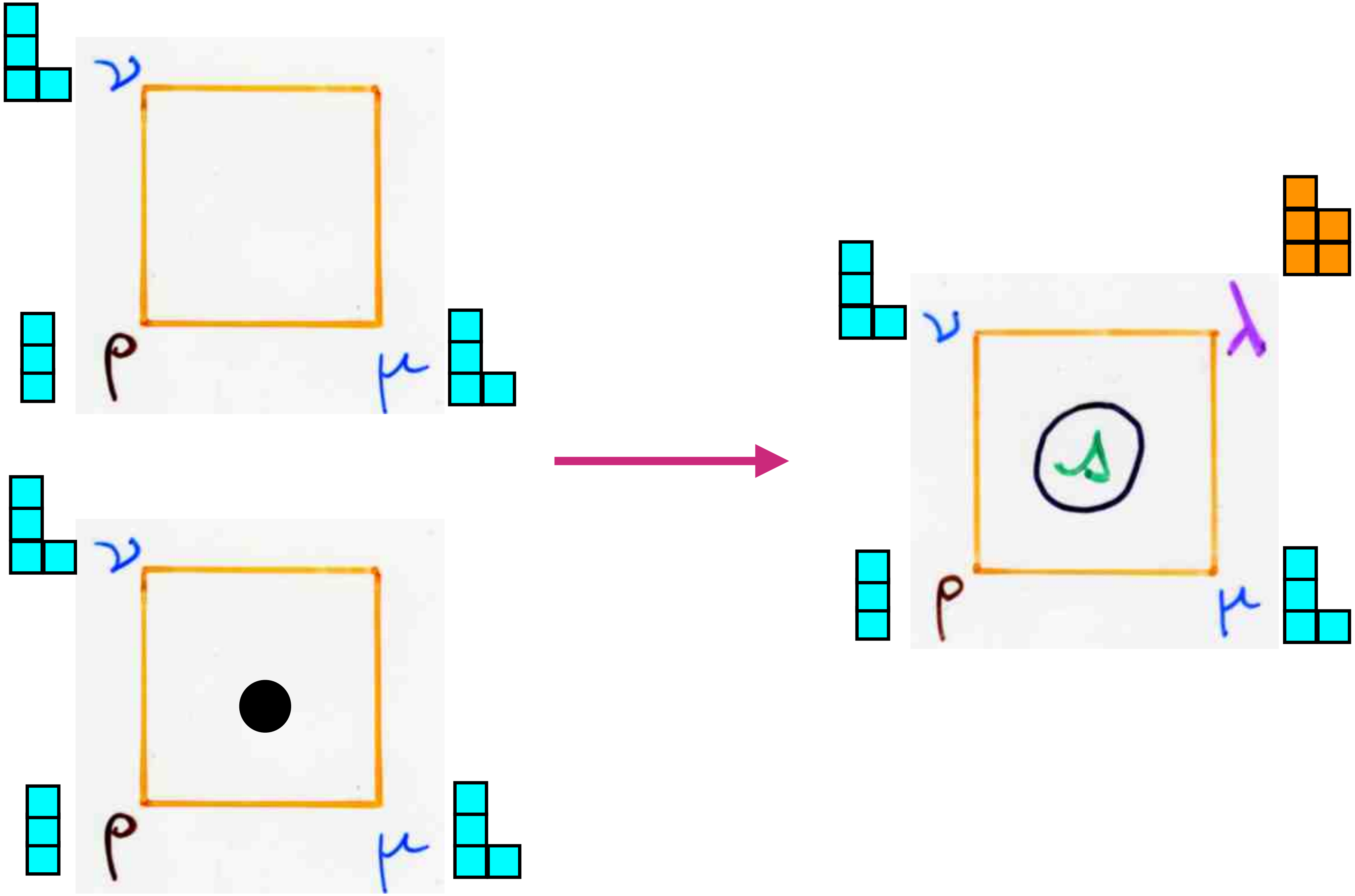
C.Krattenthaler

Banff, 2005

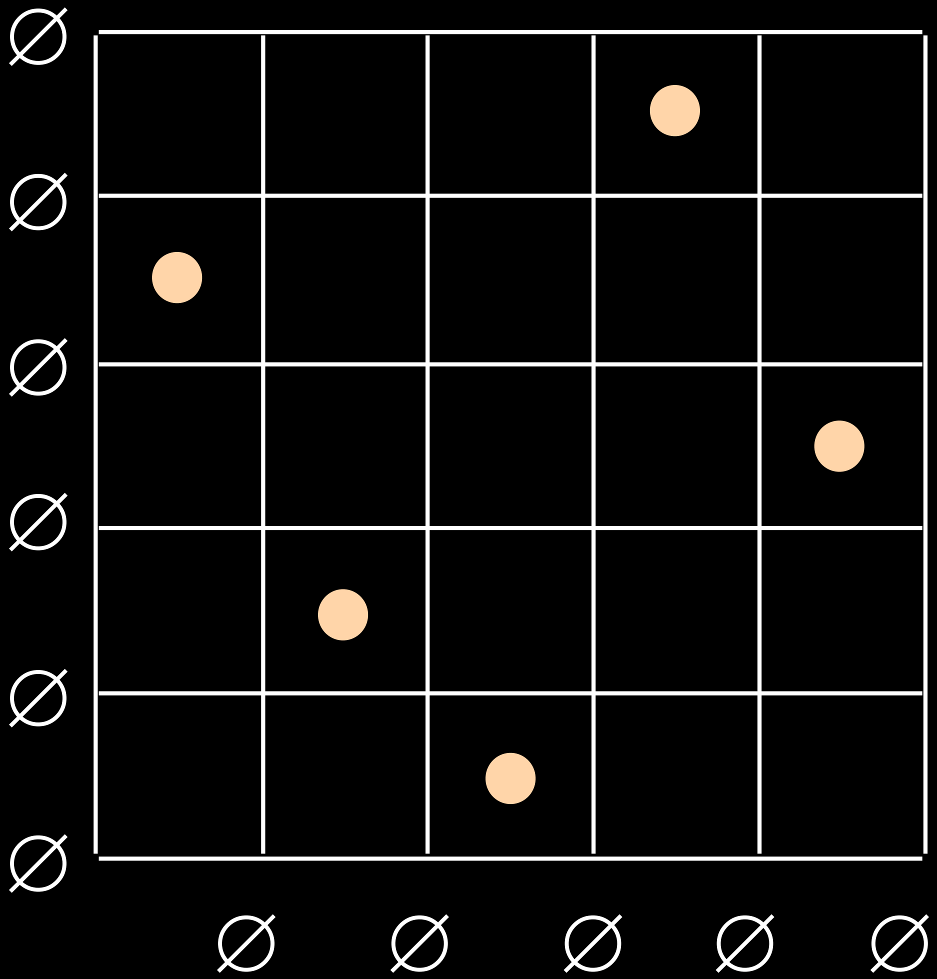


$\sigma = 4,2,1,5,3$

Fomin's "local rules"
"growth diagrams"

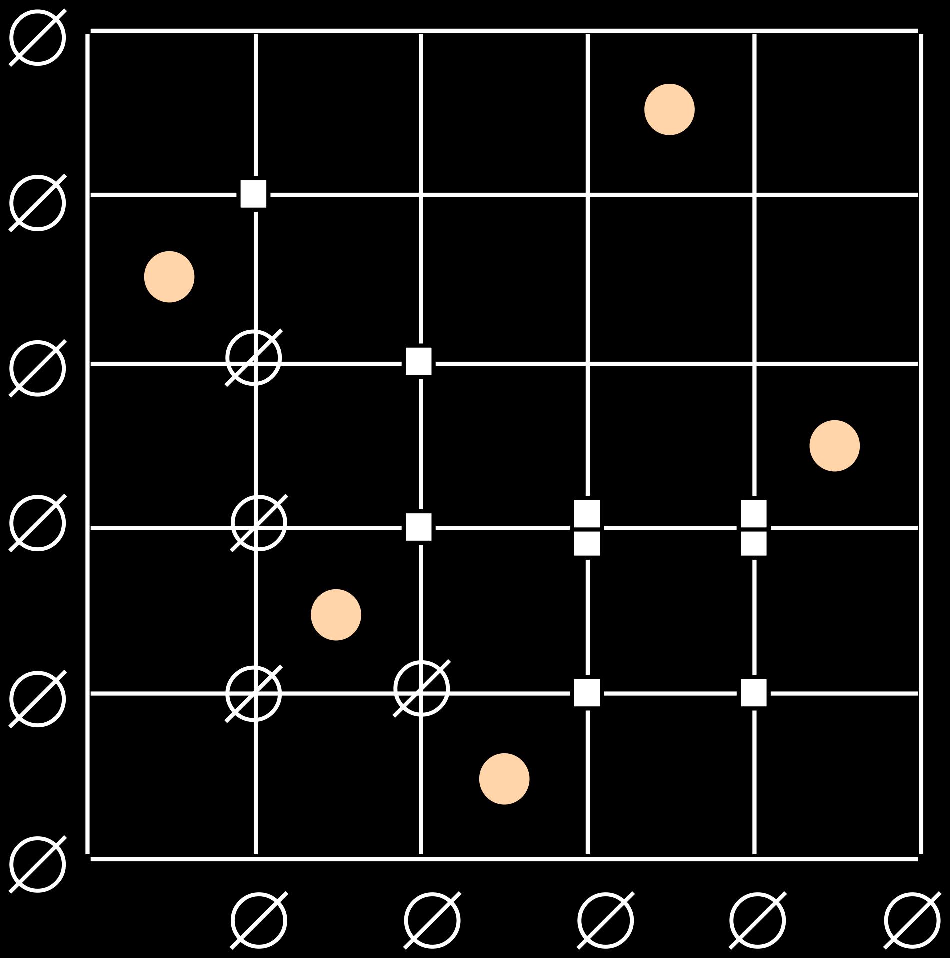


initial state

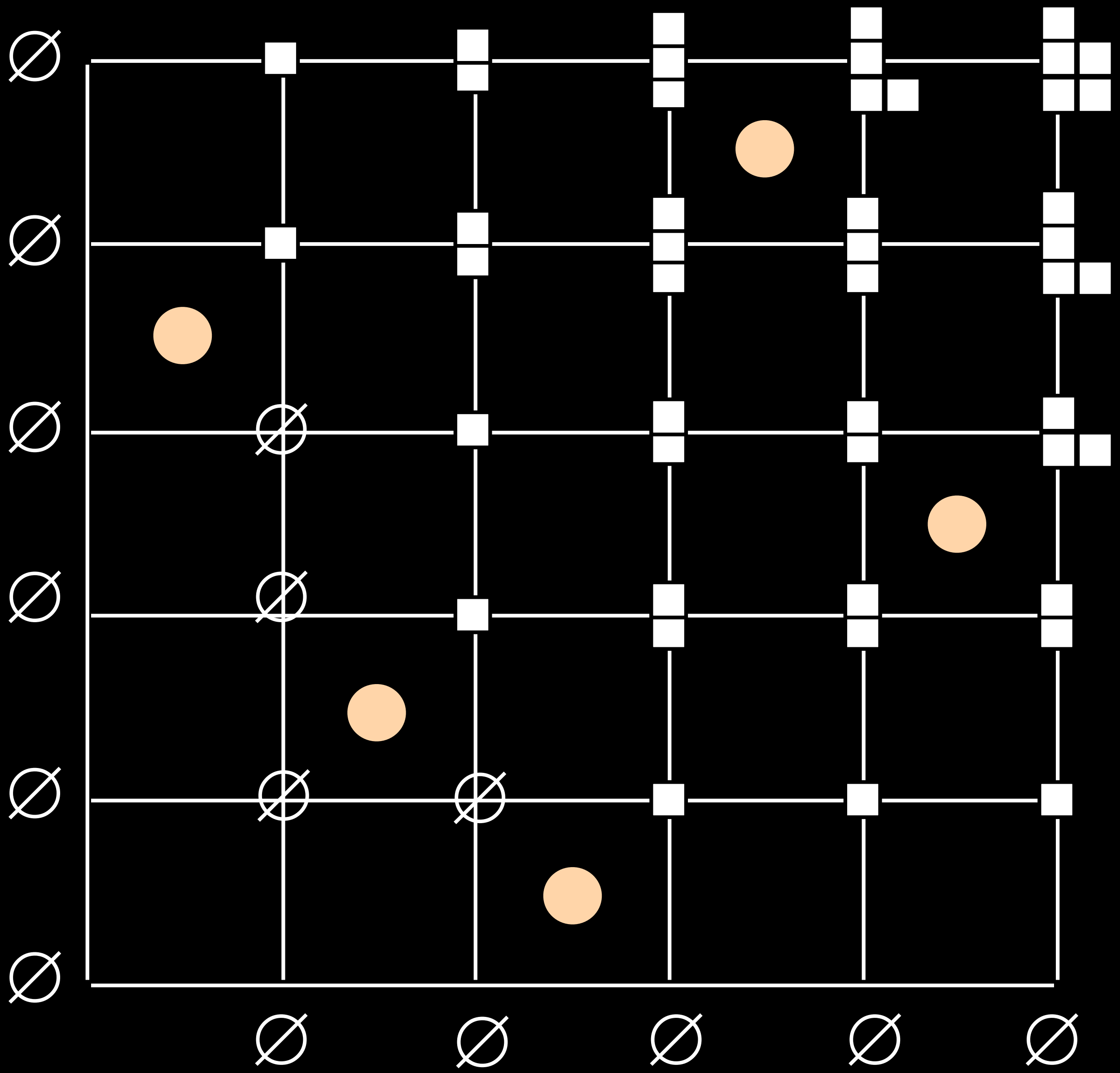


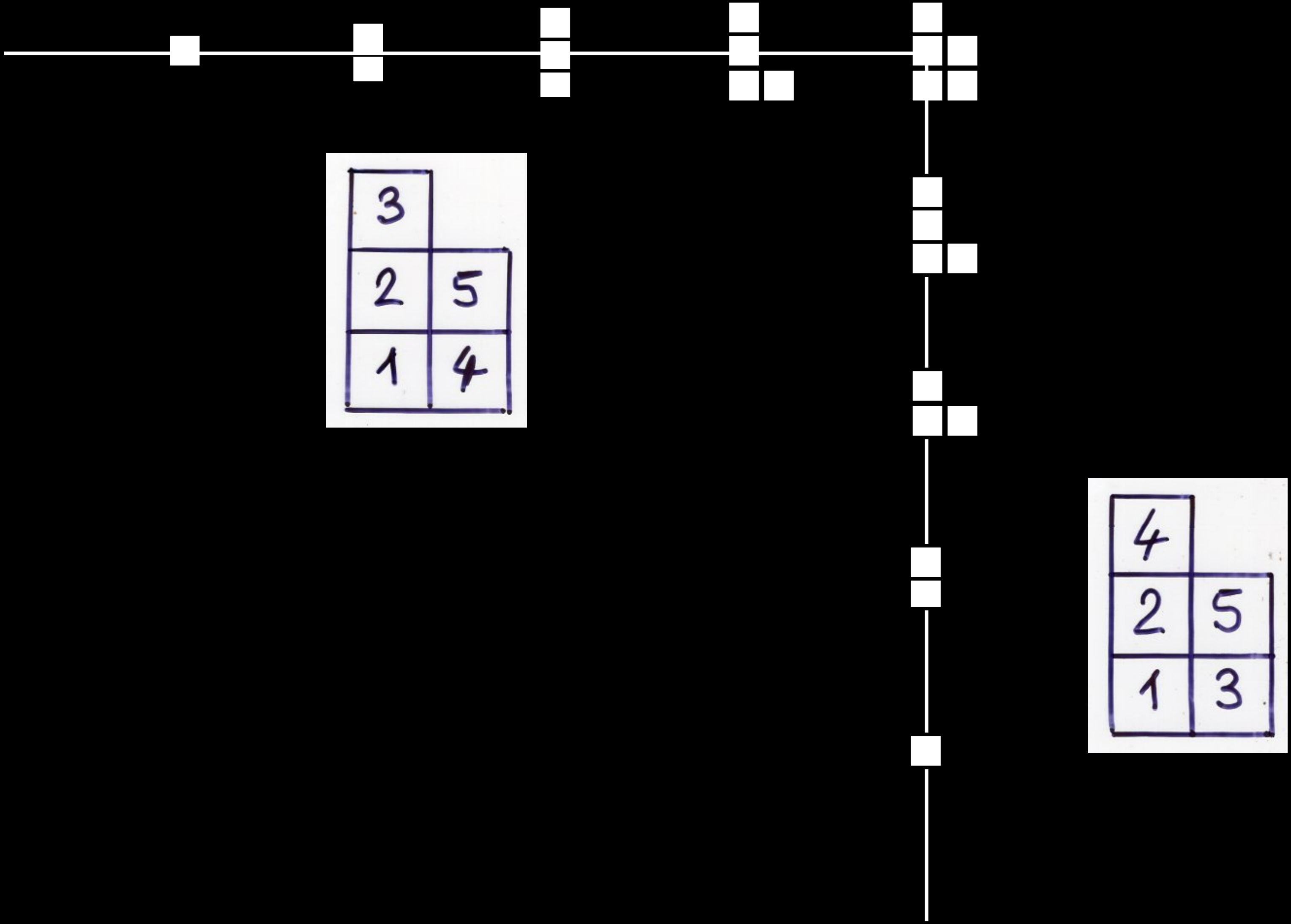
$\sigma = 4, 2, 1, 5, 3$

during the labeling process

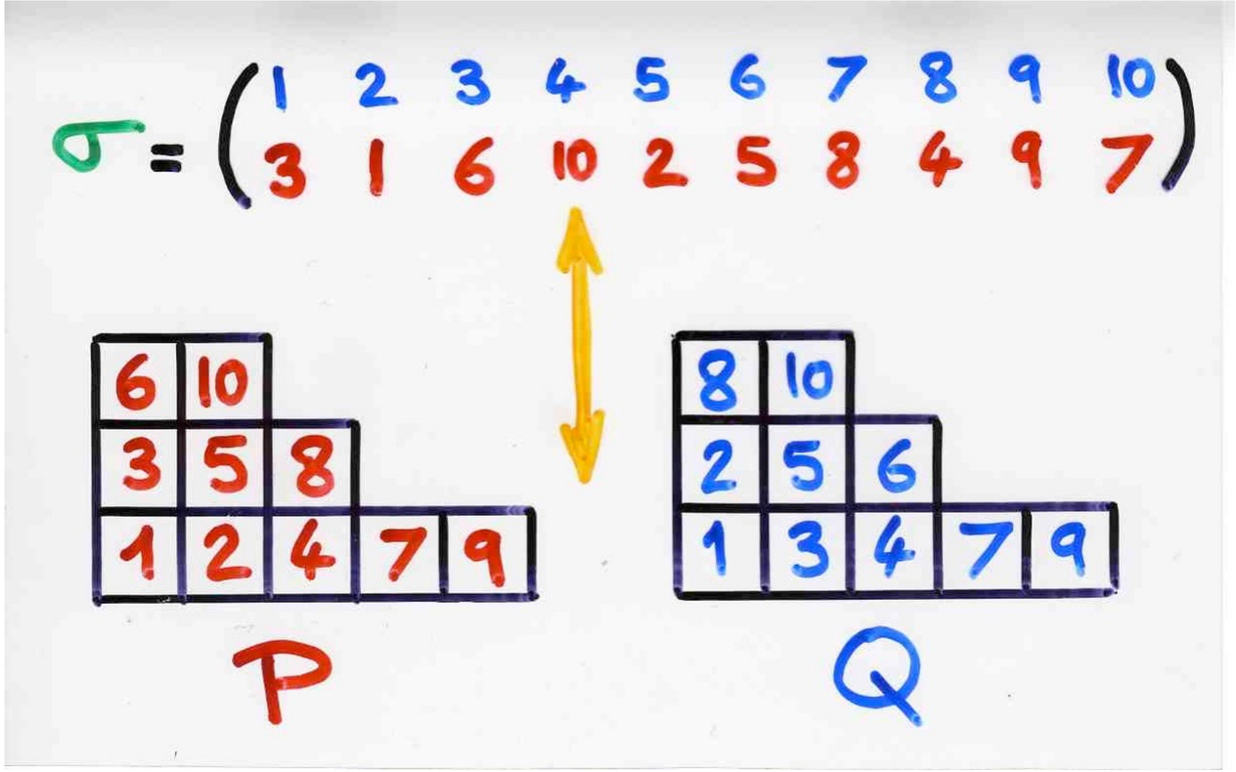


final
state





$\sigma = 4, 2, 1, 5, 3$



$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 1 & 6 & 10 & 2 & 5 & 8 & 4 & 9 & 7 \end{pmatrix}$$

$$\updownarrow$$

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3	5	8		
1	2	4	7	9

P

8	10			
2	5	6		
1	3	4	7	9

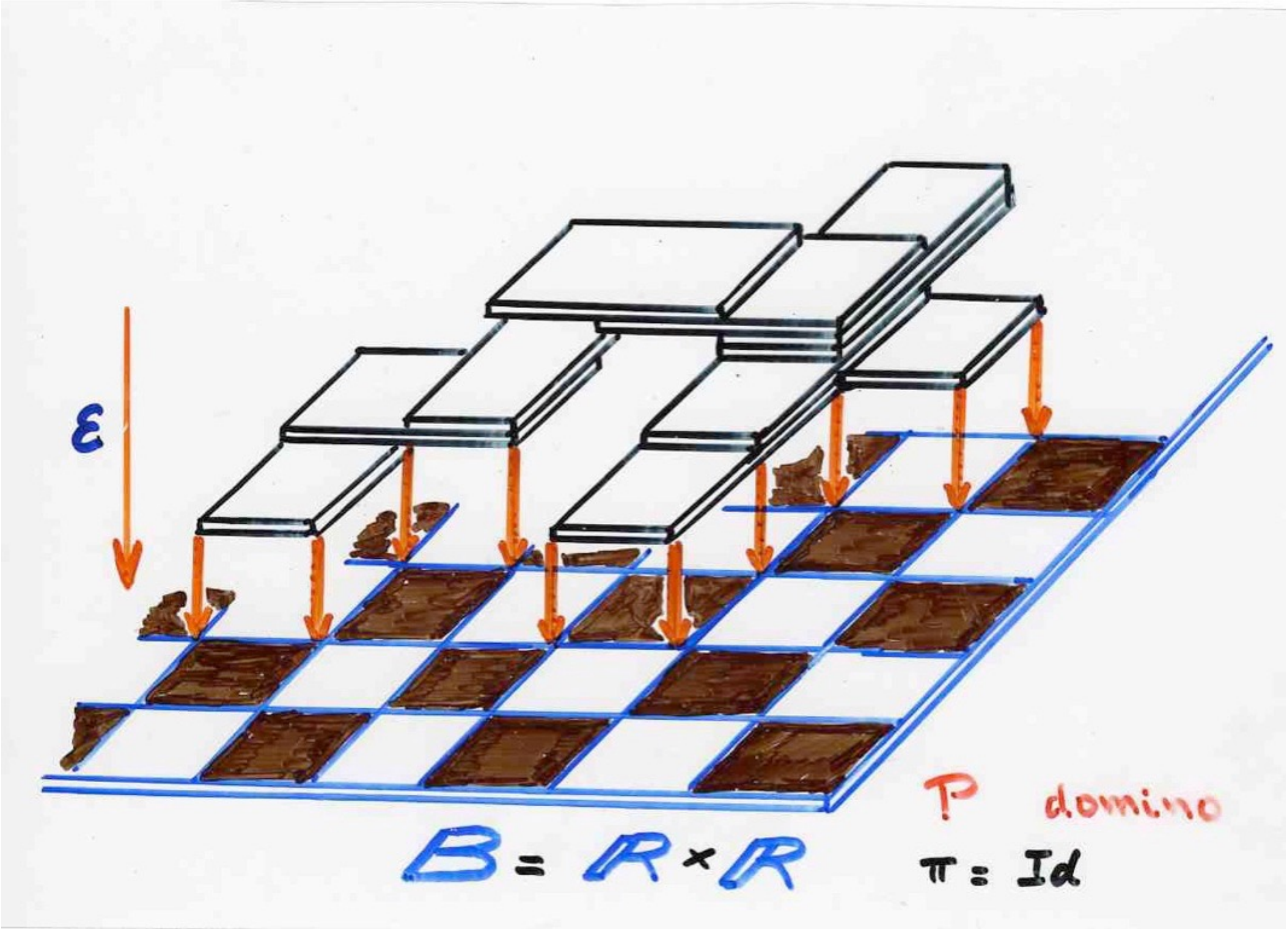
Q

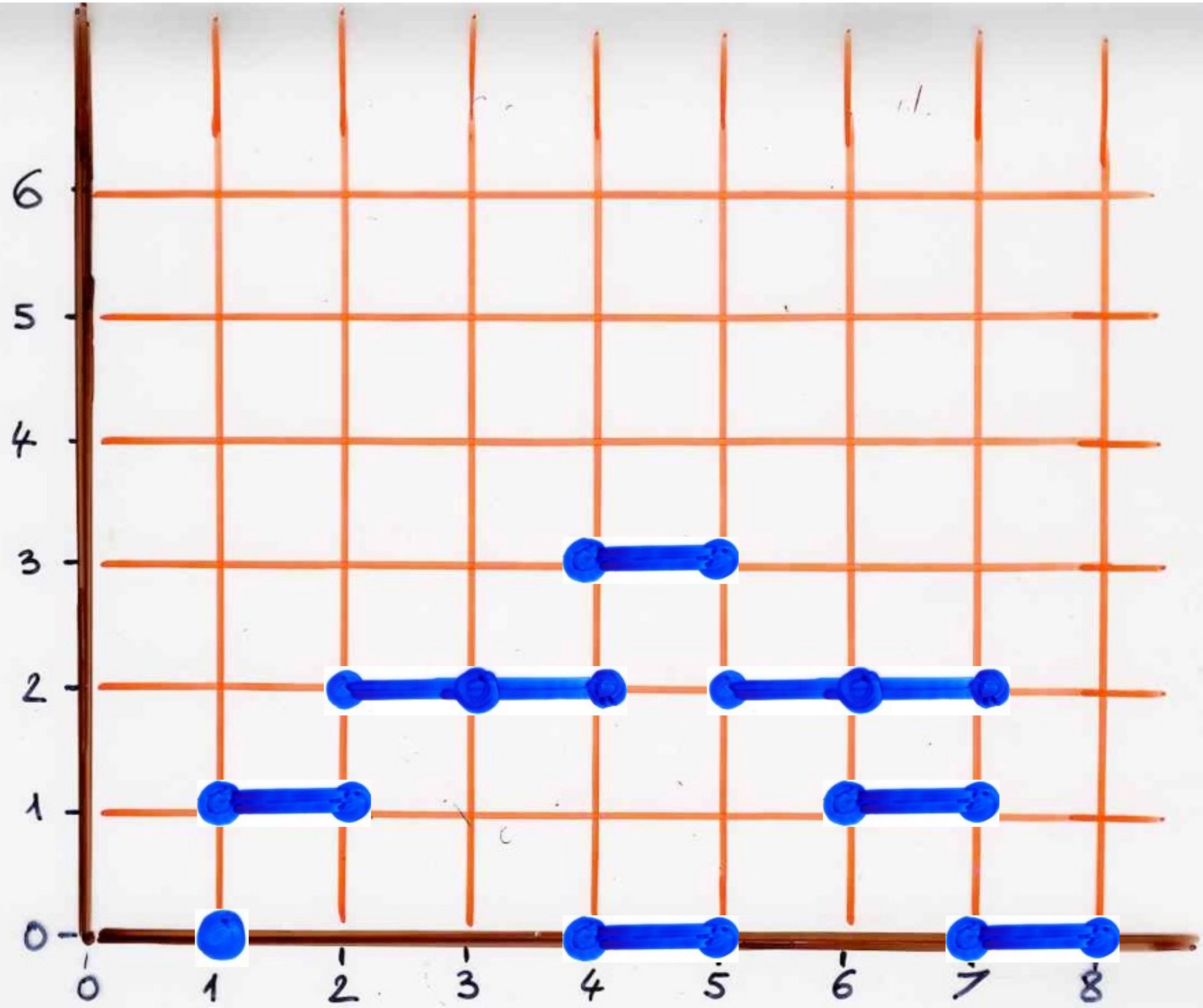
The Robinson-Schensted correspondence
betwen permutations and pairs of
(standard) Young tableaux with the same shape

heaps of segments

Introduction

Heaps





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Laguerre

Laguerre

polynomial

formal
orthogonality

$$\mathfrak{f}(x^n) = \mu_n$$

$$\mathfrak{f}(P_k P_l) = 0$$

moments

$k \neq l$

$$\mathfrak{f}(PQ) = \int_a^b P(x)\, Q(x)\, d\mu$$

classical
analysis

measure

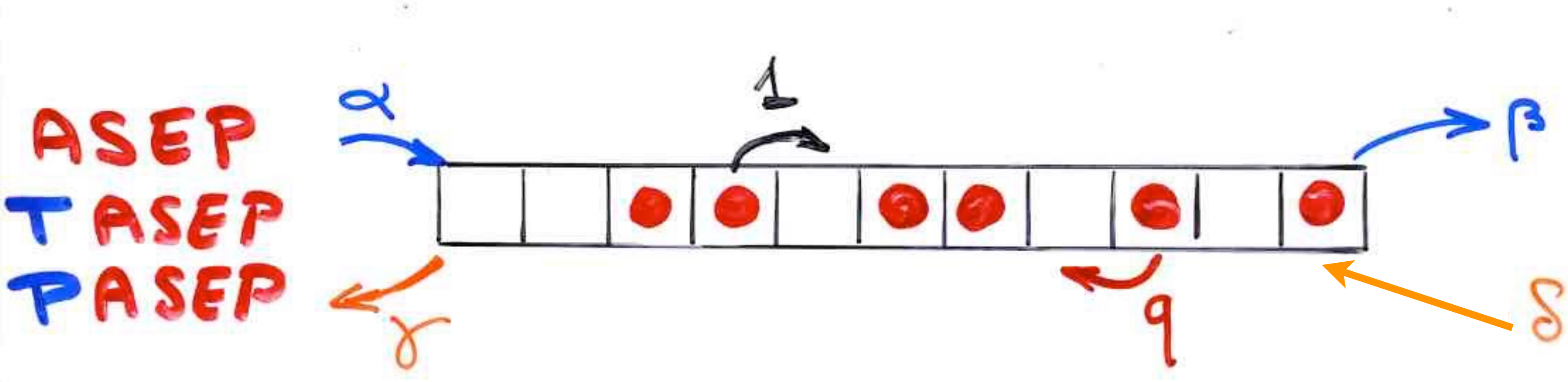
Laguerre
polynomial

$$\mu_n = (n+1)!$$

moments

PASEP

toy model in the physics of
dynamical systems far from equilibrium



computation of the
"stationary probabilities"

seminal paper "matrix ansatz"

Derrida, Evans, Hakim, Pasquier (1993)

D, E matrices (may be ∞)

$$\begin{cases} DE = qED + E + D \\ \langle W | (\alpha E - \gamma D) = \langle W | \\ (\beta D - \delta E) | V \rangle = | V \rangle \end{cases}$$

colum vector V

row vector W

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q- Hermite polynomial

α, β, q $\gamma = \delta = 1$

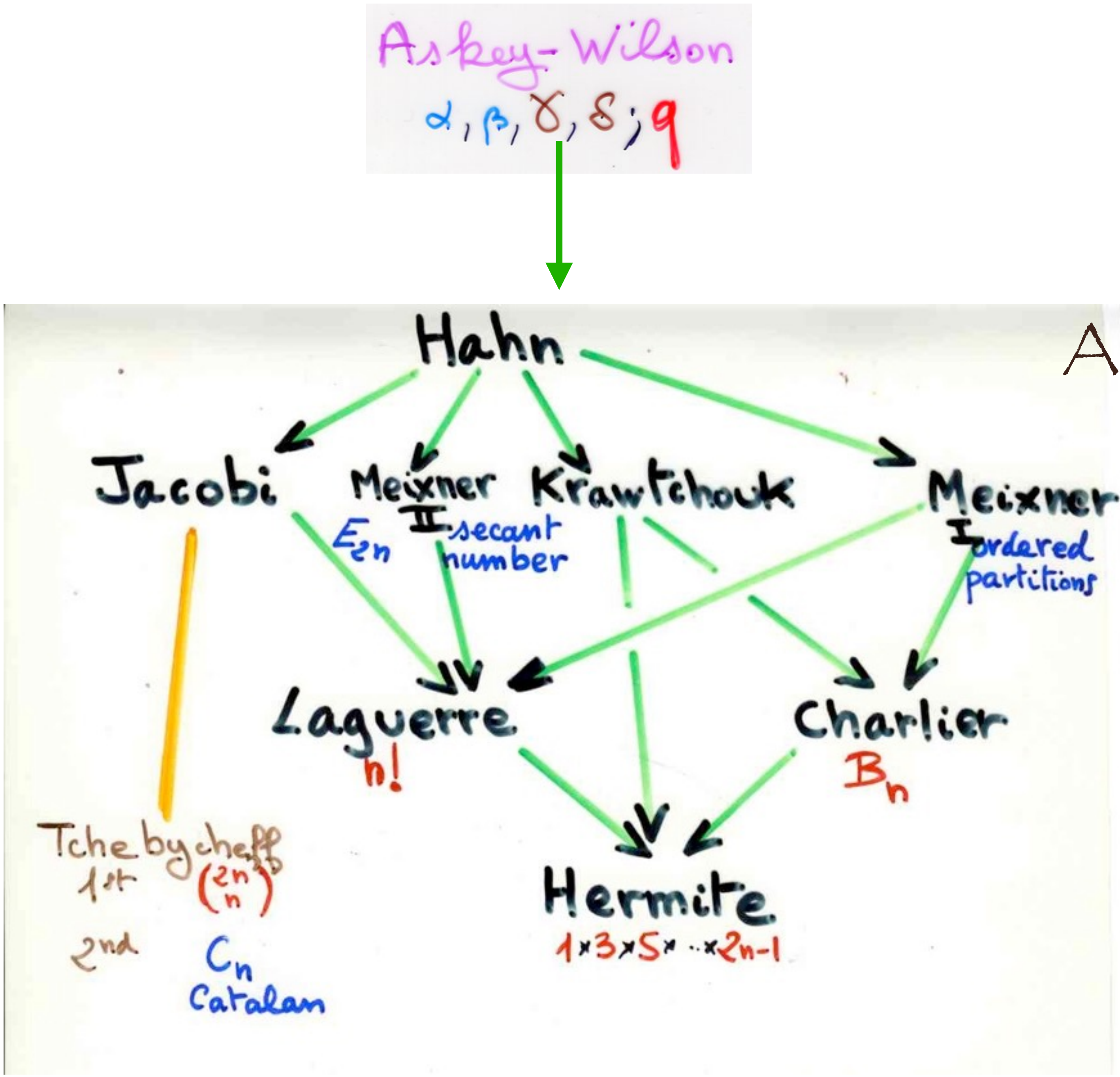
$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}} \hat{a}$$
$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}} \hat{a}^+$$
$$\hat{a} \hat{a}^+ - q \hat{a}^+ \hat{a} = 1$$

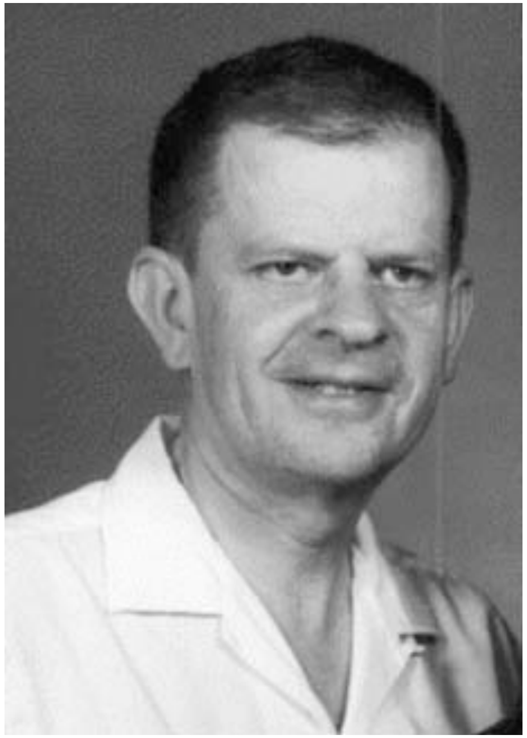
→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta,$ q

Askey-Wilson polynomials

Askey tableau





Combinatorial theory of orthogonal polynomials

combinatorial
theory of
orthogonal polynomials

moments X.V. (1983)

Françon, X.V. (1978)

and

continued fractions

Flajolet (1980)

Thm. (Favard)

- $\{P_n(x)\}_{n\geqslant 0}$ sequence of monic polynomials, $\deg(P_n)=n$
- $\{b_k\}_{k\geqslant 0}$, $\{\lambda_k\}_{k\geqslant 1}$ coeff. in $\mathbb{K}$

orthogonality $\Longleftrightarrow$

$$P_{k+1}(x) = (x-b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

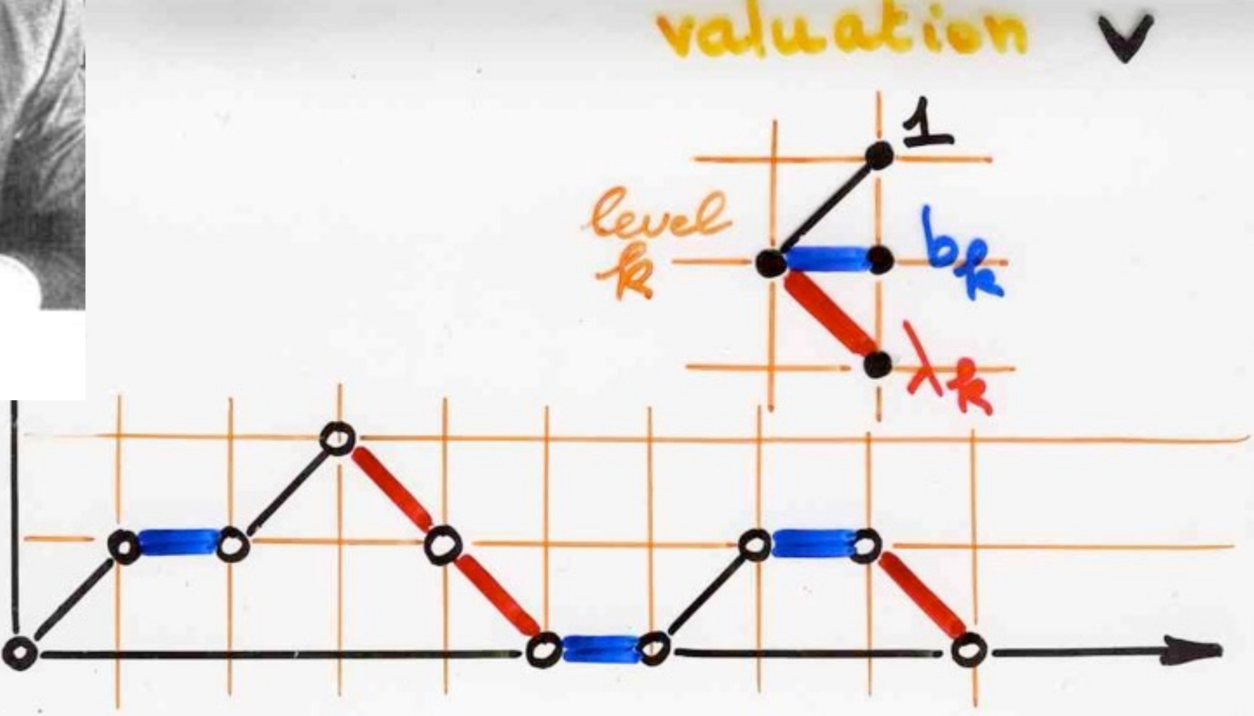
$(\forall k\geqslant 1)$

3 terms linear recurrence relation

$\{b_k\}_{k \geq 0}$ $\{\lambda_k\}_{k \geq 1}$

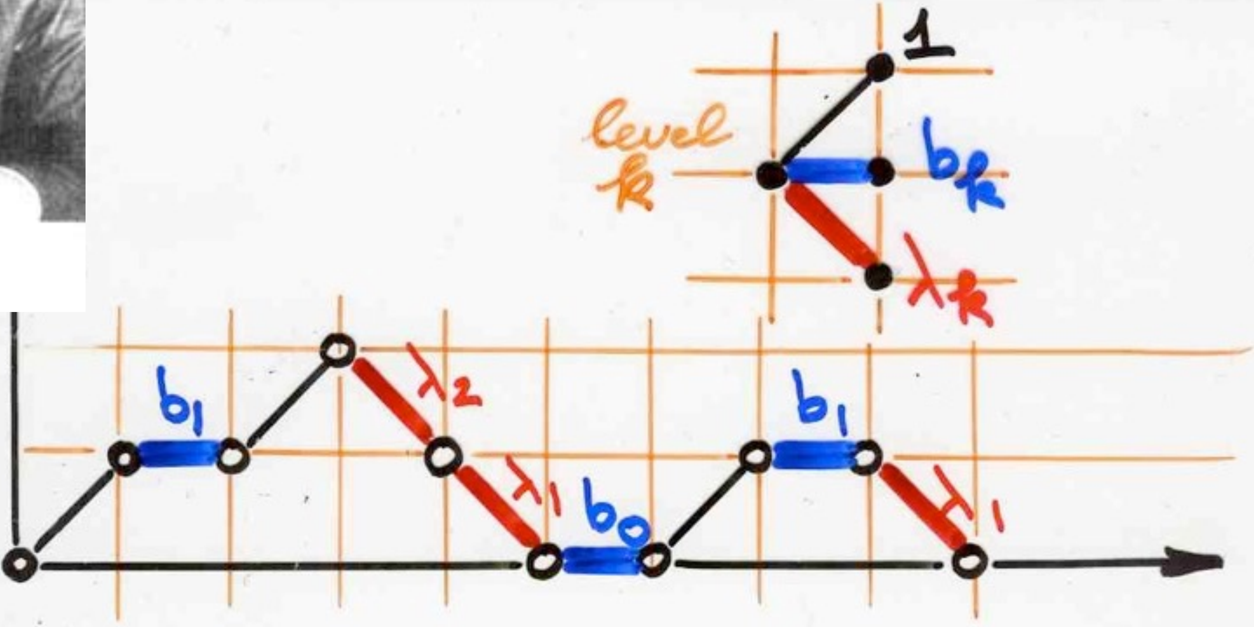
$b_k, \lambda_k \in \mathbb{K}$ ring

μ_n ?



ω Motzkin path

valuation v



ω Motzkin path

$$v(\omega) = b_0\, b_1^2\, \lambda_1^2\, \lambda_2$$

$$f(x^n) = \mu_n$$

moments

$$\mu_n = \sum_{\substack{\omega \\ \text{Motzkin path} \\ |\omega| = n}} v(\omega)$$

Jacobi continued fraction

$$\sum_{\substack{\omega \\ \text{Motzkin} \\ \text{path}}} v(\omega)\, t^{|\omega|} = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{\cfrac{\cdots\cdots}{1 - b_k t - \cfrac{\lambda_{k+1} t^2}{\cfrac{\cdots}{\cdots}}}}}}$$

Philippe Flajolet

fundamental

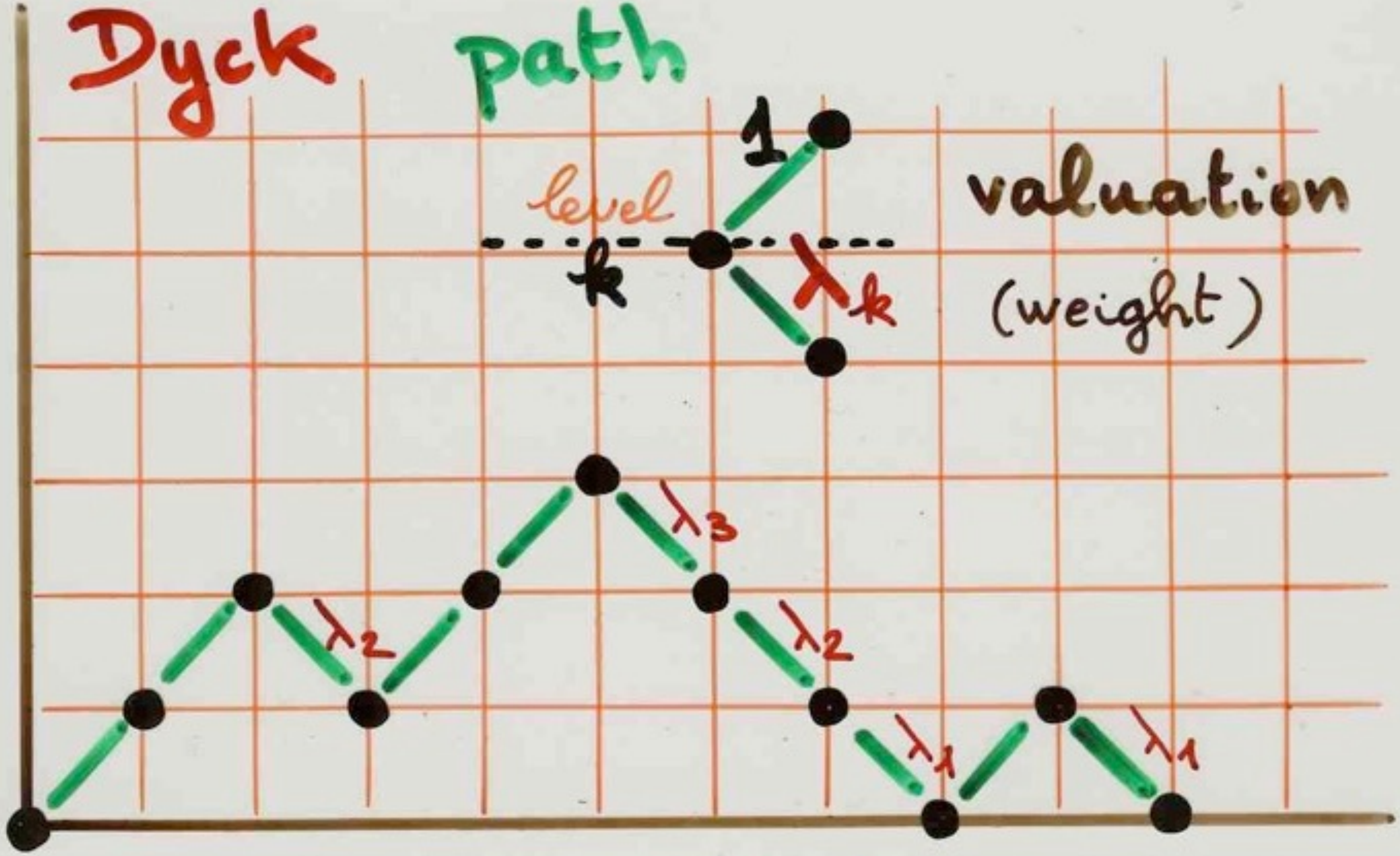
Lemma

continued fractions

$$\sum_{n\geqslant 0} \mu_n t^n = \underbrace{\cfrac{1}{1-\cfrac{\lambda_1 t}{1-\cfrac{\lambda_2 t}{\cdots\cdots\cdot\cfrac{}{1-\cfrac{\lambda_k t}{\cdots\cdots}}}}}}_{S(t;\lambda)}$$

$\mu_0 = 1$

Stieltjès continued fraction



weight $v(\omega) = \lambda_1^2 \lambda_2^2 \lambda_3$

The notion of histories

example: Hermite histories

Hermite $\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$

Hermite
polynomials

$$\cfrac{1}{1-1t\,\cfrac{}{1-2t\,\cfrac{}{1-3t\,\cfrac{}{\cdots\cdots}}}}$$

moments
Hermite
polynomials

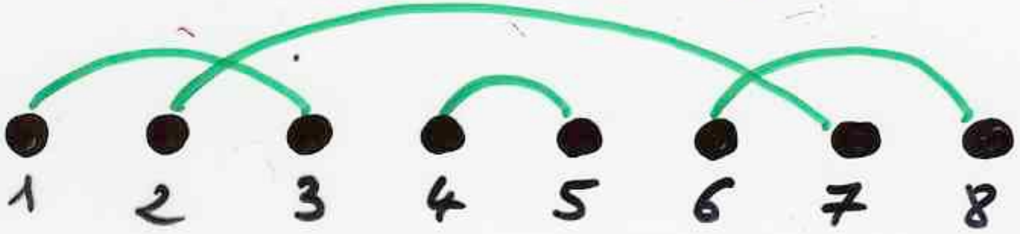
Hermite $\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$

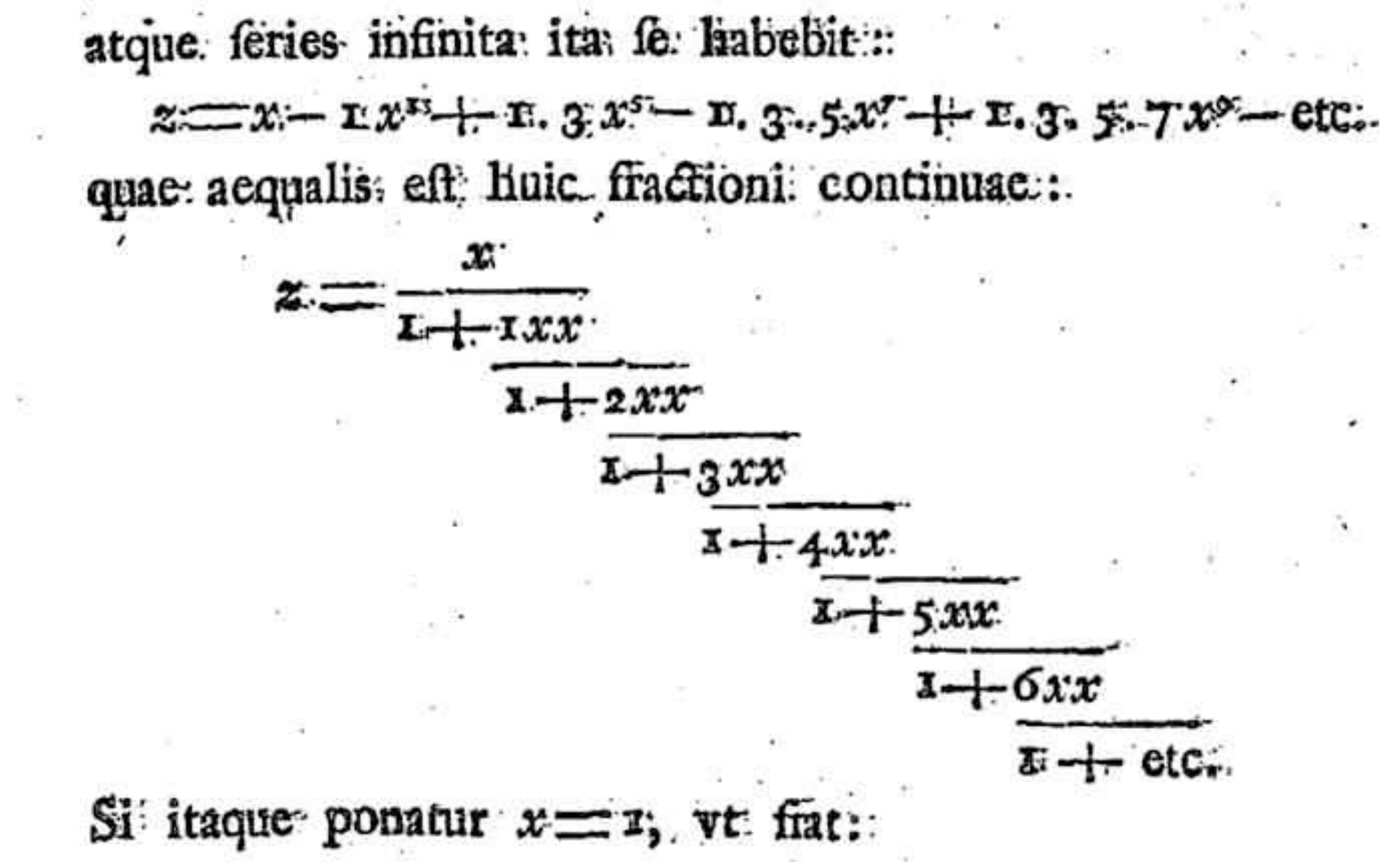
$\mu_{2n+1} = 0$

$\mu_{2n} = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$

number of
involutions
no fixed point
on $\{1, 2, \ldots, 2n\}$

chord diagrams
perfect matching





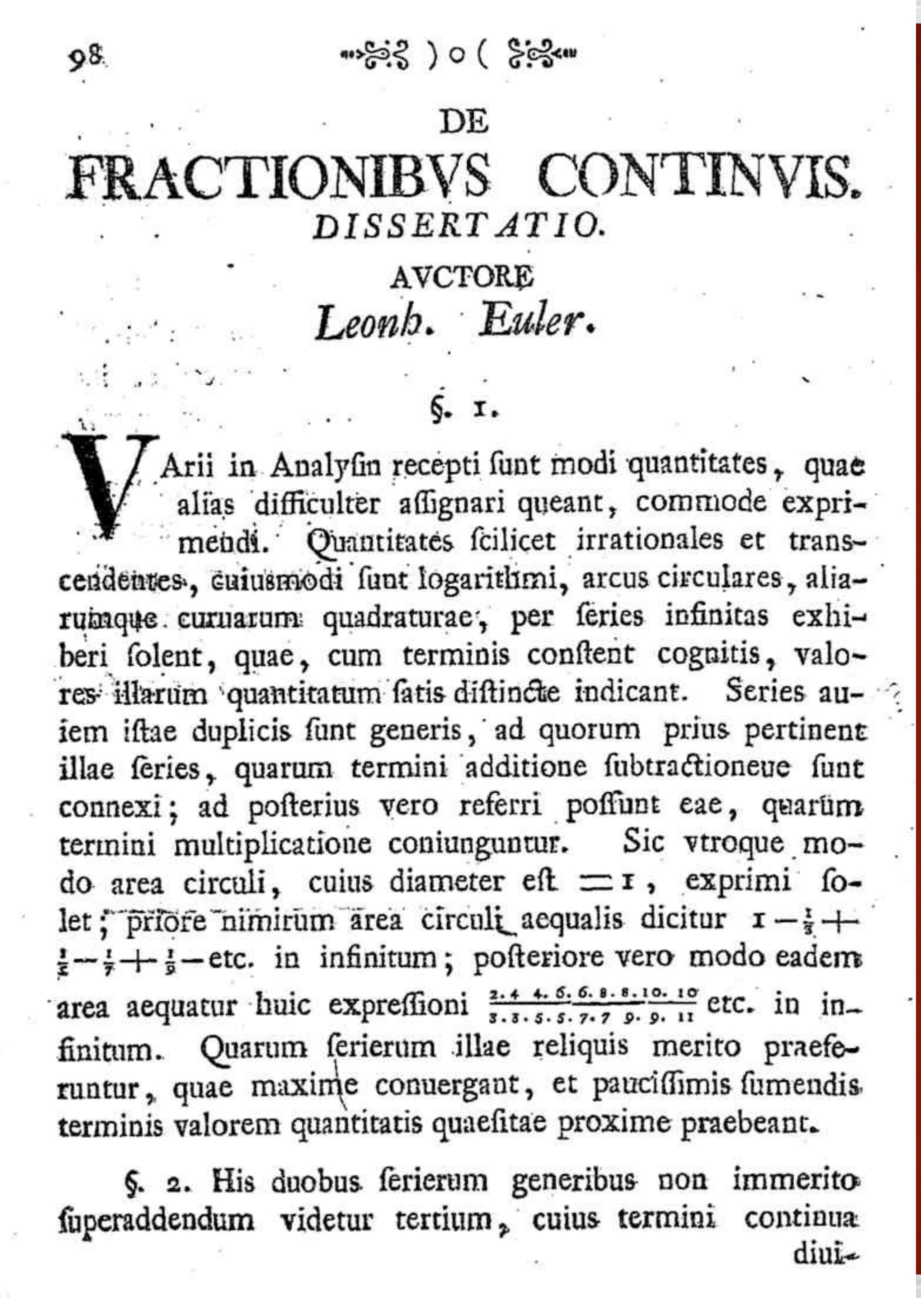
atque ſeries infinita ita ſe habebit:

$$z = x - 1x^3 + 1.3x^5 - 1.3.5x^7 + 1.3.5.7x^9 - \text{etc.}$$

quae aequalis eſt huic fractioni continuae:

$$z = \cfrac{x}{1 + \cfrac{1xx}{1 + \cfrac{2xx}{1 + \cfrac{3xx}{1 + \cfrac{4xx}{1 + \cfrac{5xx}{1 + \cfrac{6xx}{1 + \text{etc.}}}}}}}}$$

Si itaque ponatur $x = 1$, vt fiat:



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DE

FRACTIONIBVS CONTINVIS.

DISSERTATIO.

AVCTORE

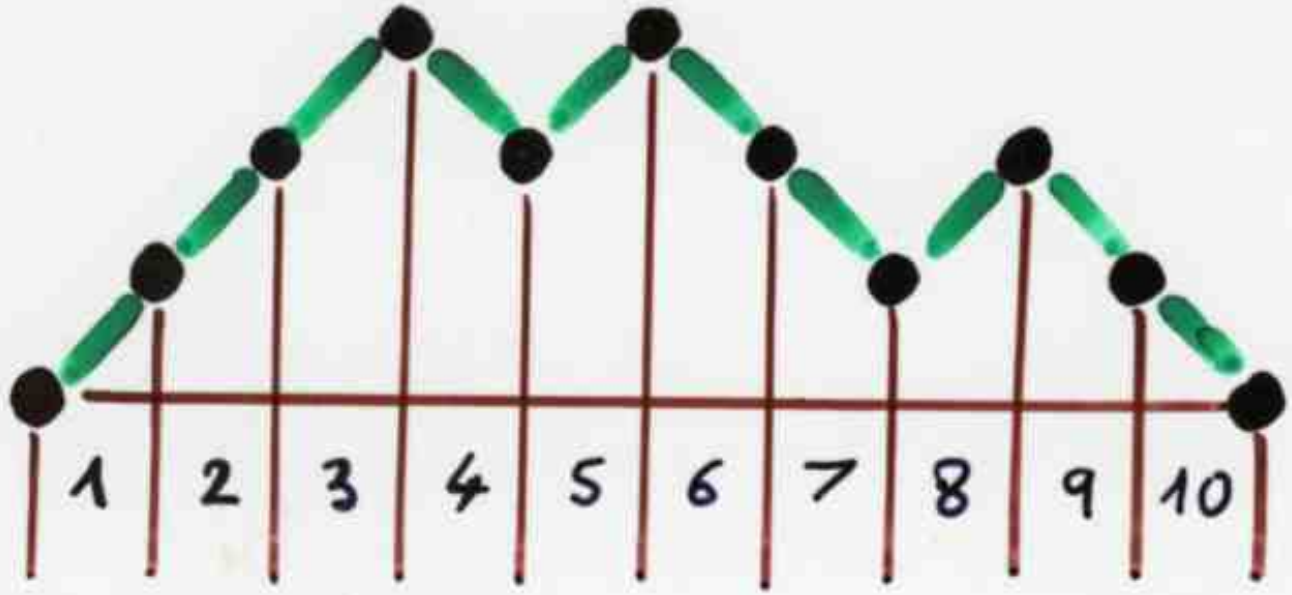
Leonh. Euler.

§. 1.

VArii in Analyſin recepti ſunt modi quantitates, quae alias difficulter aſſignari queant, commode exprimendi. Quantitates ſcilicet irrationales et transcendentes, cuiusmodi ſunt logarithmi, arcus circulares, aliarumque curuarum quadraturae, per ſeries infinitas exhiberi ſolent, quae, cum terminis conſtent cognitis, valores illarum quantitatum ſatis diſtincte indicant. Series autem iſtae duplicis ſunt generis, ad quorum prius pertinent illae ſeries, quarum termini additione ſubtractioneue ſunt connexi; ad poſterius vero referri poſſunt eae, quarum termini multiplicatione coniunguntur. Sic vtroque modo area circuli, cuius diameter eſt $= 1$, exprimi ſolet; priore nimirum area circuli aequalis dicitur $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} -$ etc. in infinitum; poſteriore vero modo eadem area aequatur huic expreſſioni $\frac{2.4.4.6.6.8.8.10.10}{3.3.5.5.7.7.9.9.11}$ etc. in infinitum. Quarum ſerierum illae reliquis merito praeferuntur, quae maxime conuergant, et pauciſſimis ſumendis terminis valorem quantitatis quaeſitae proxime praebeant.

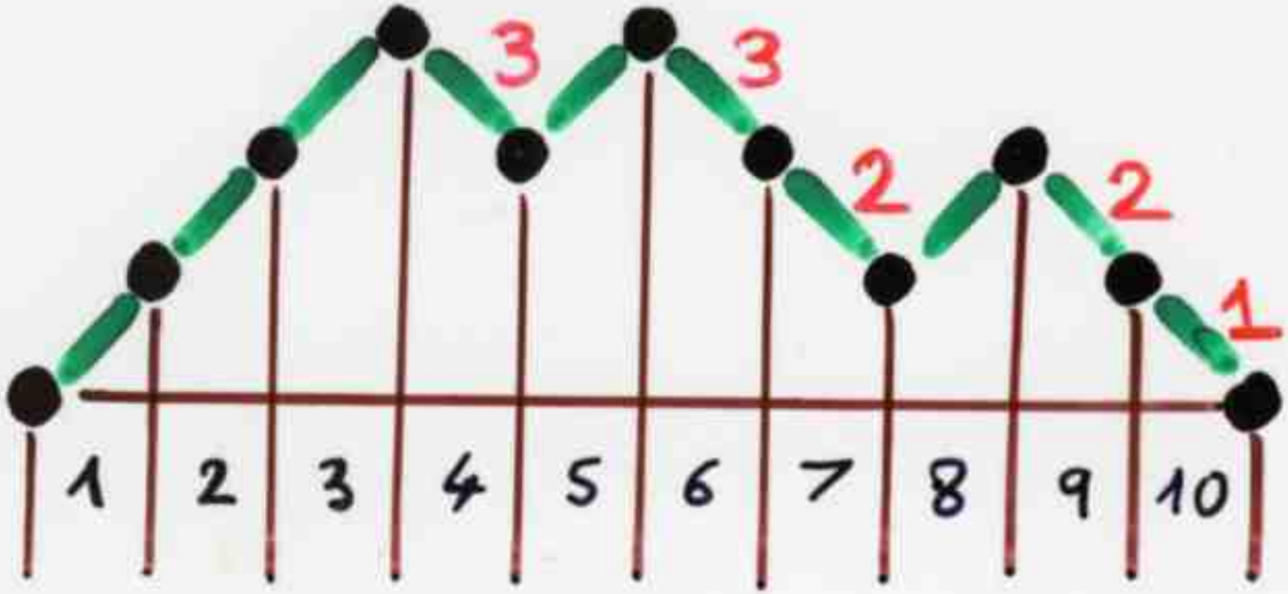
§. 2. His duobus ſerierum generibus non immerito ſuperaddendum videtur tertium, cuius termini continua diui-

Hermite history



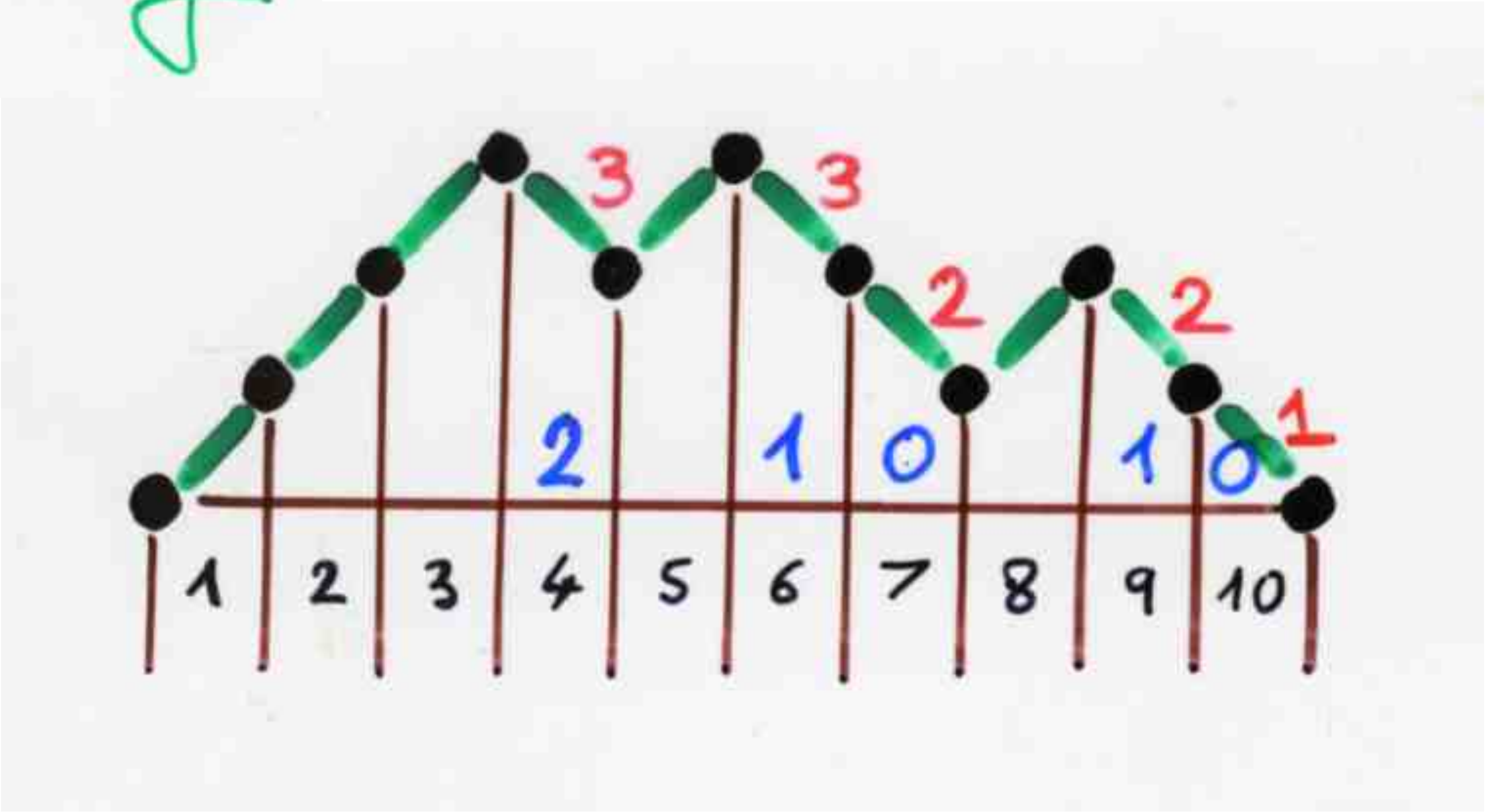
Hermite
history

Hermite $\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$

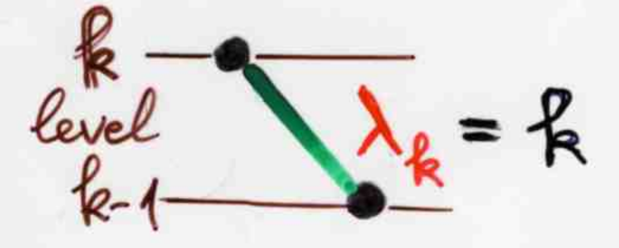


Hermite history

Hermite $\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$



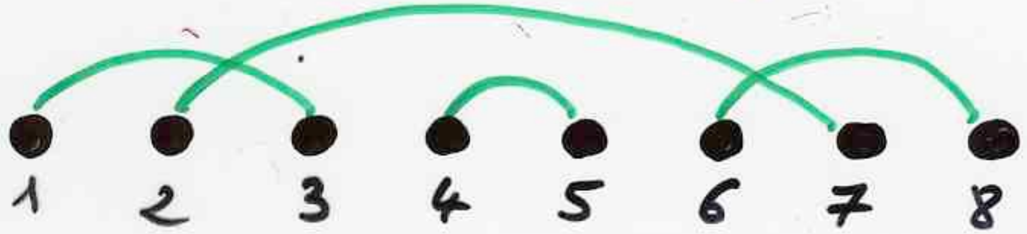
$0 \leq i < \lambda_k = k$

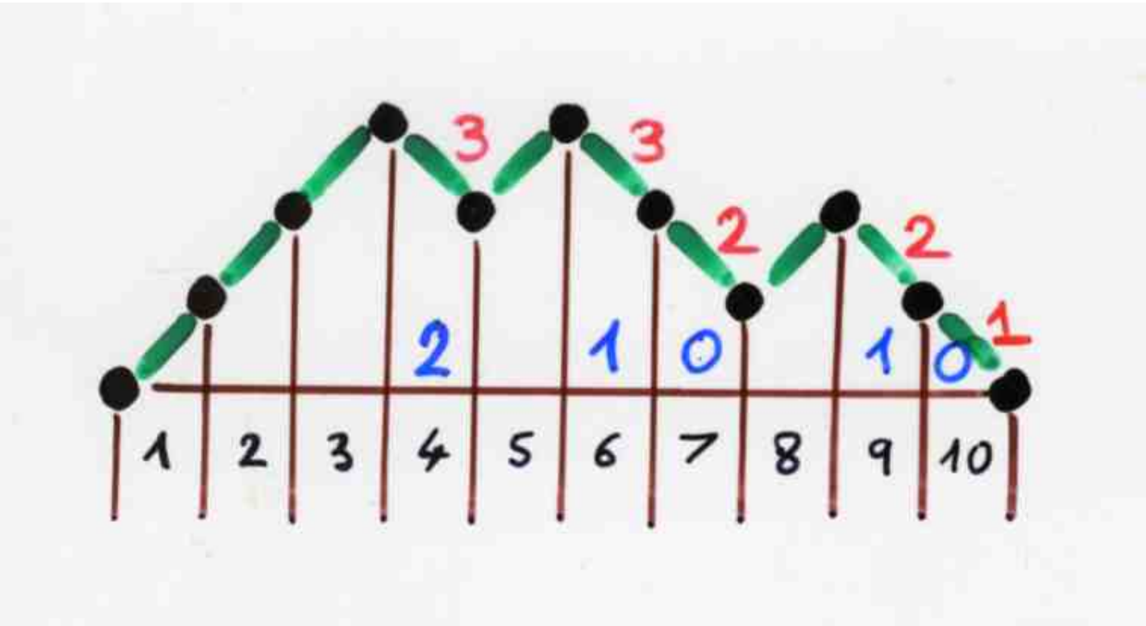


bijection

Hermite history ↔ chord diagrams

perfect matching

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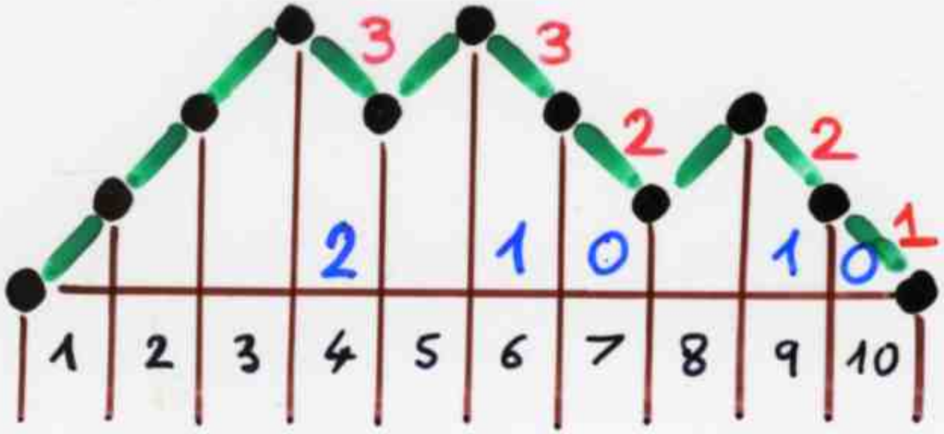
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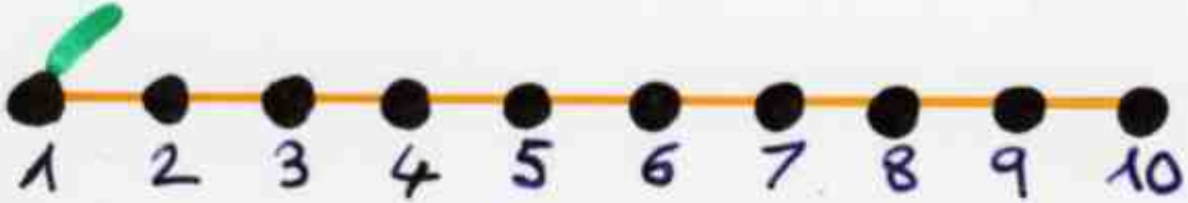
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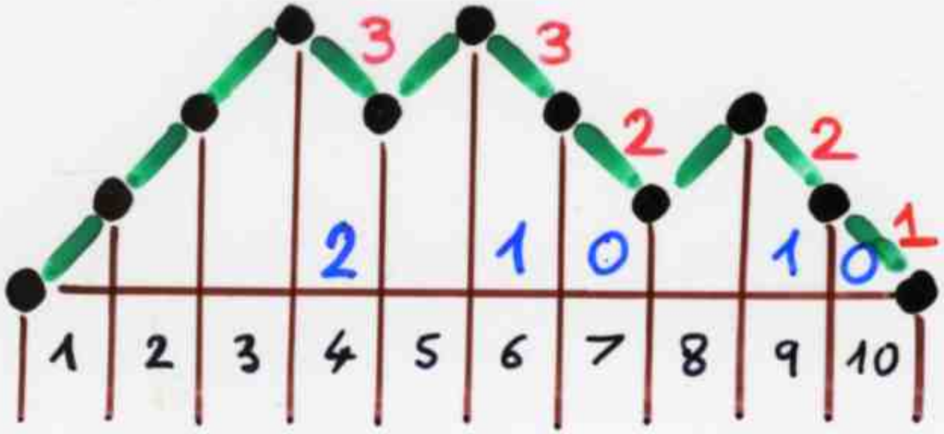
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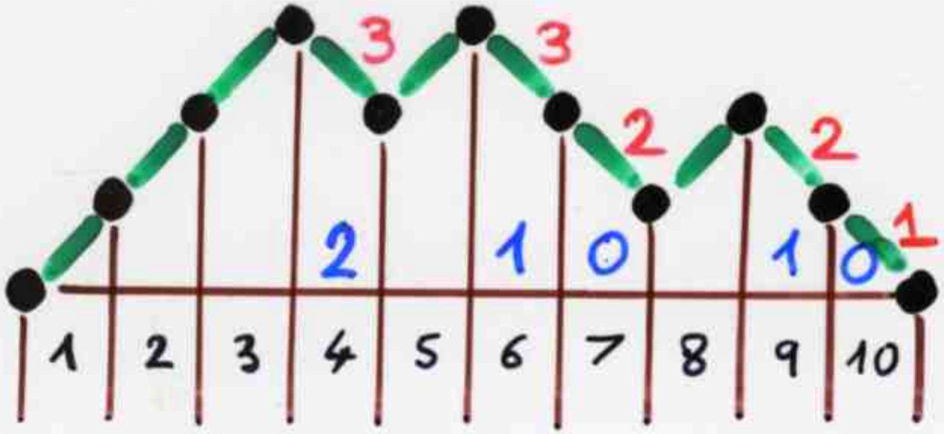
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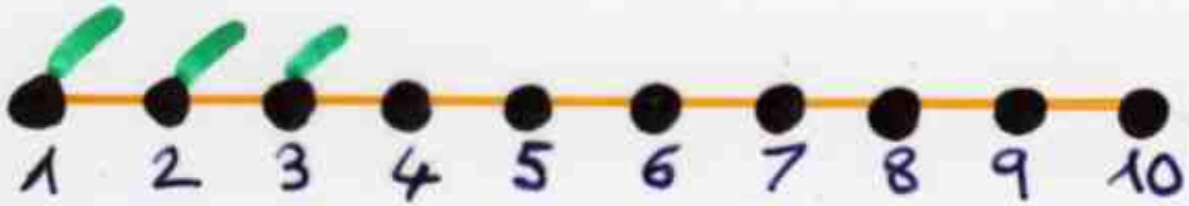
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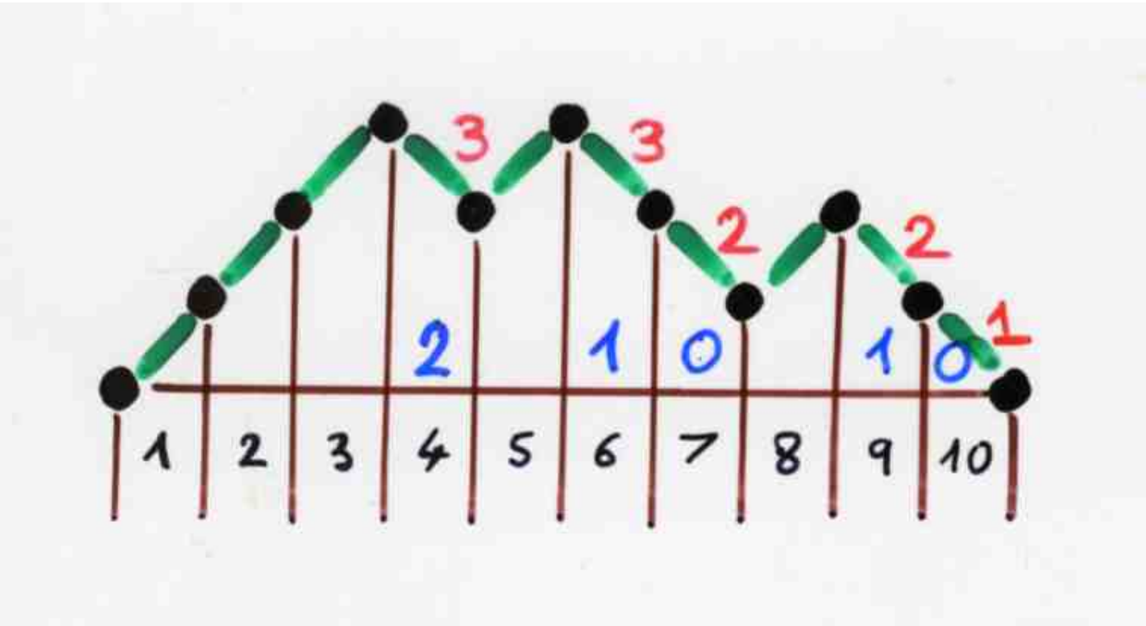
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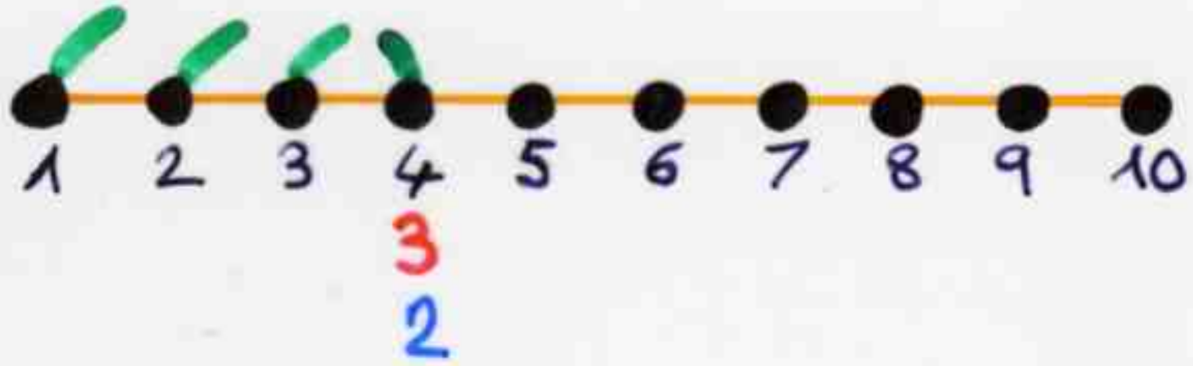
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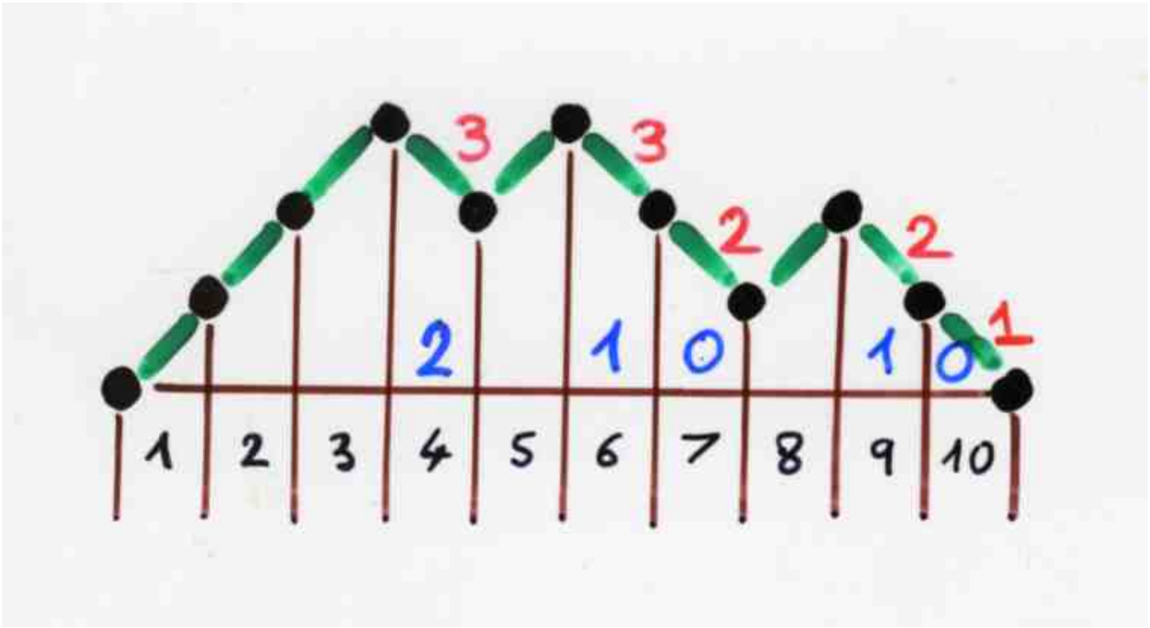
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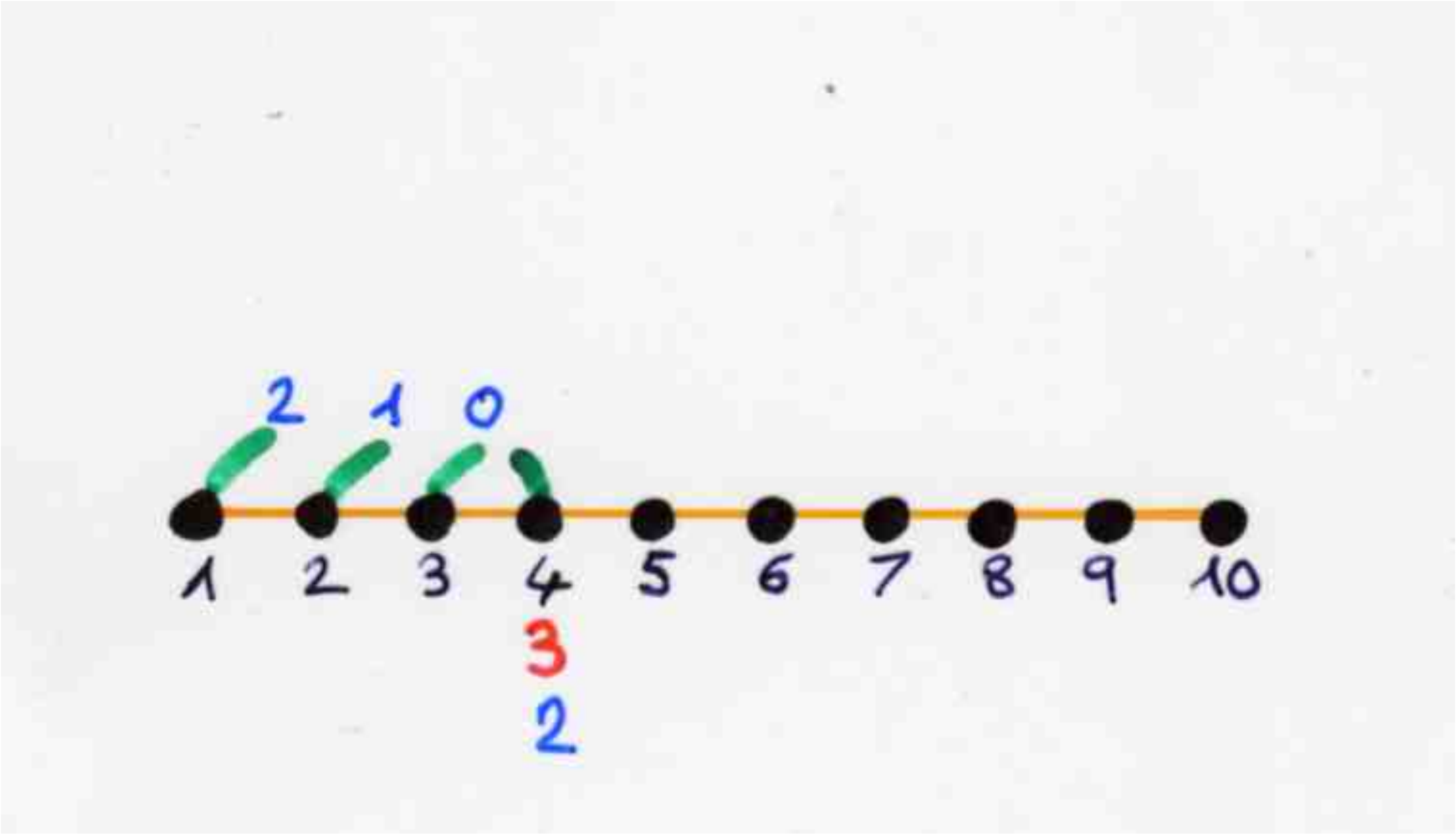
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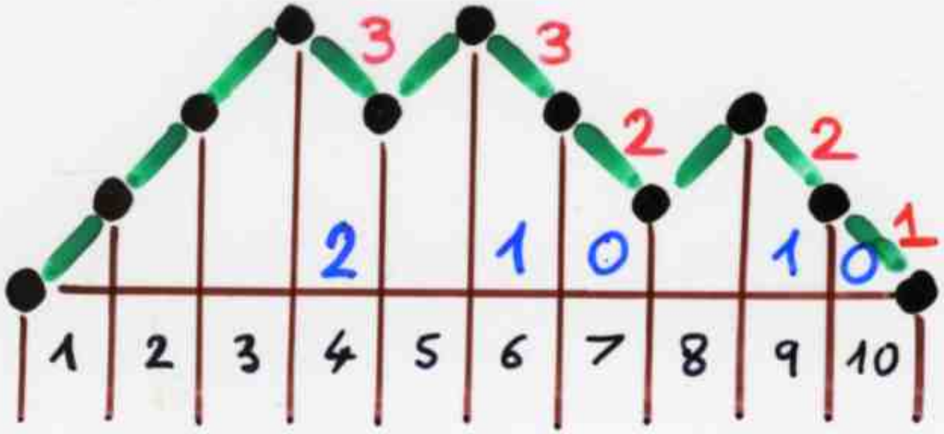
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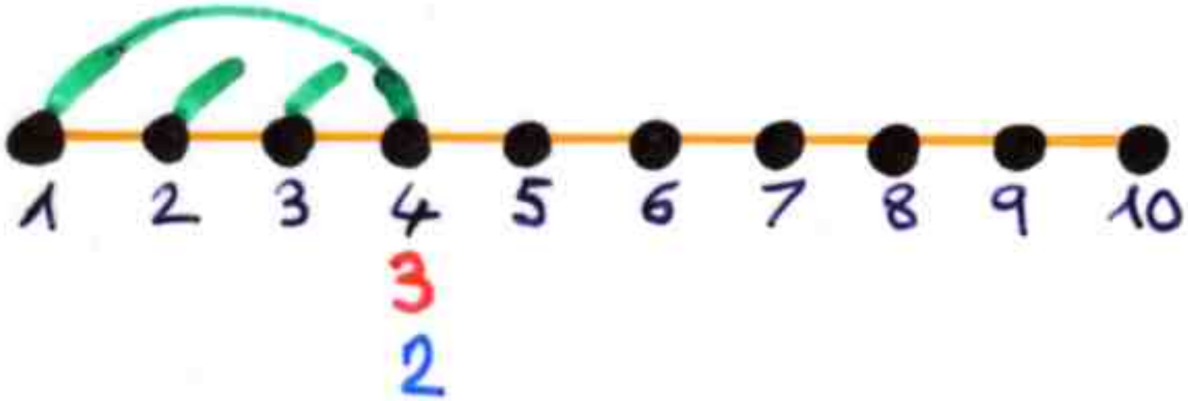
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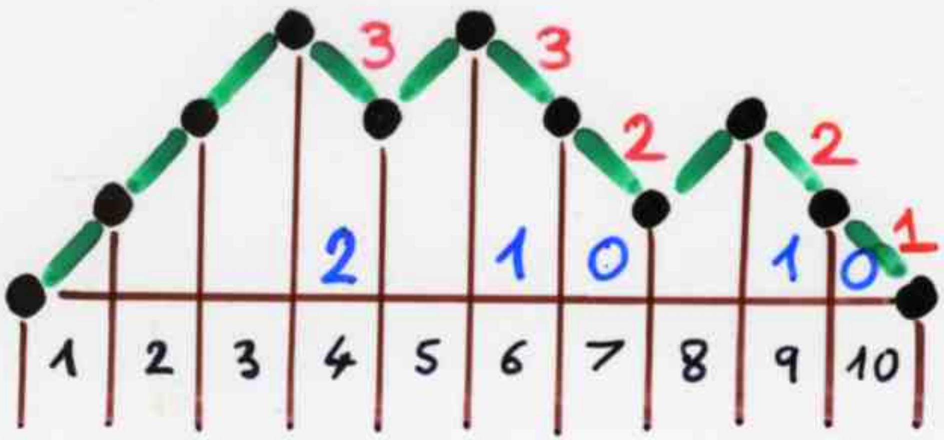
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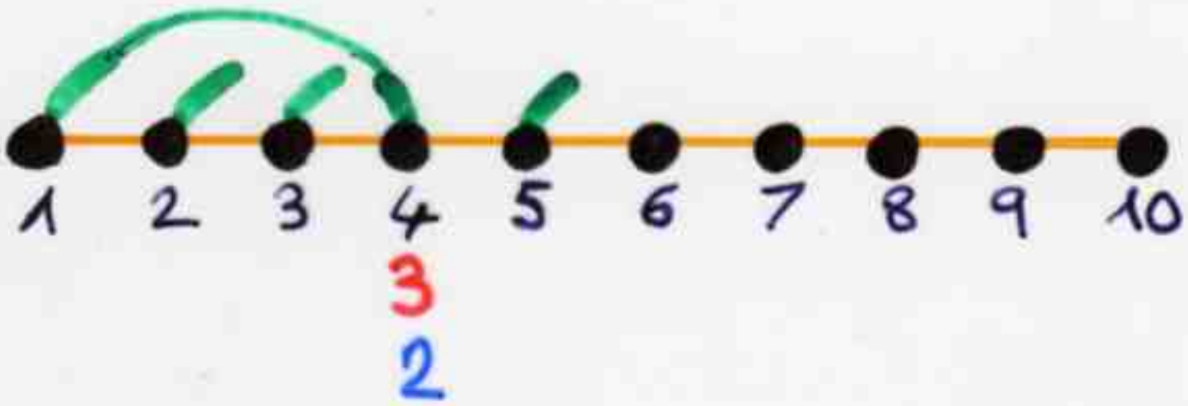
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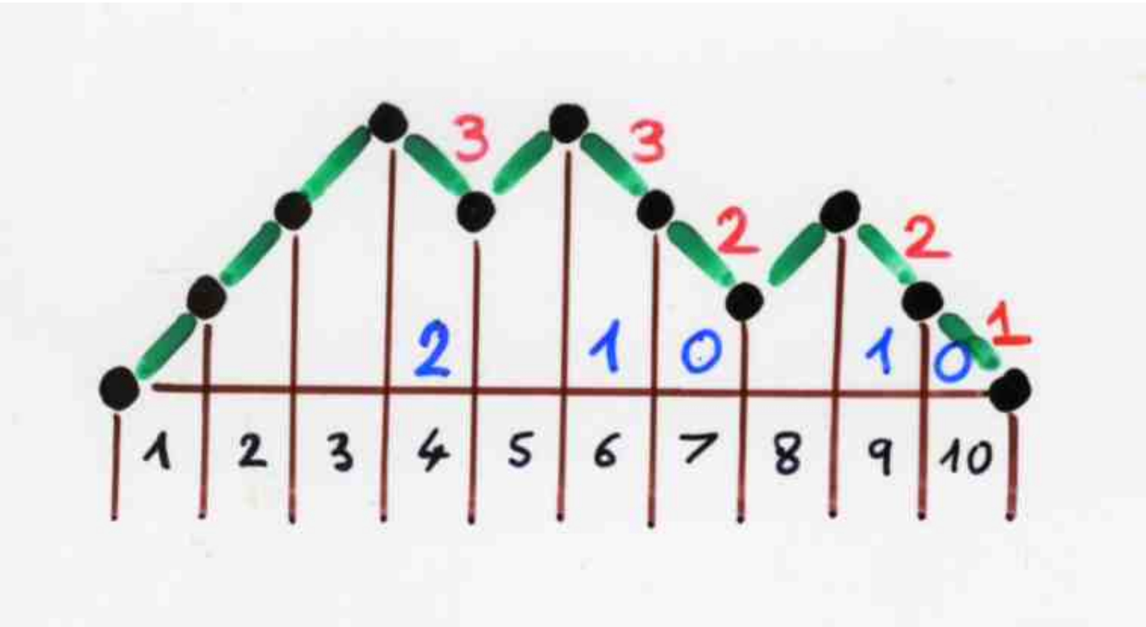
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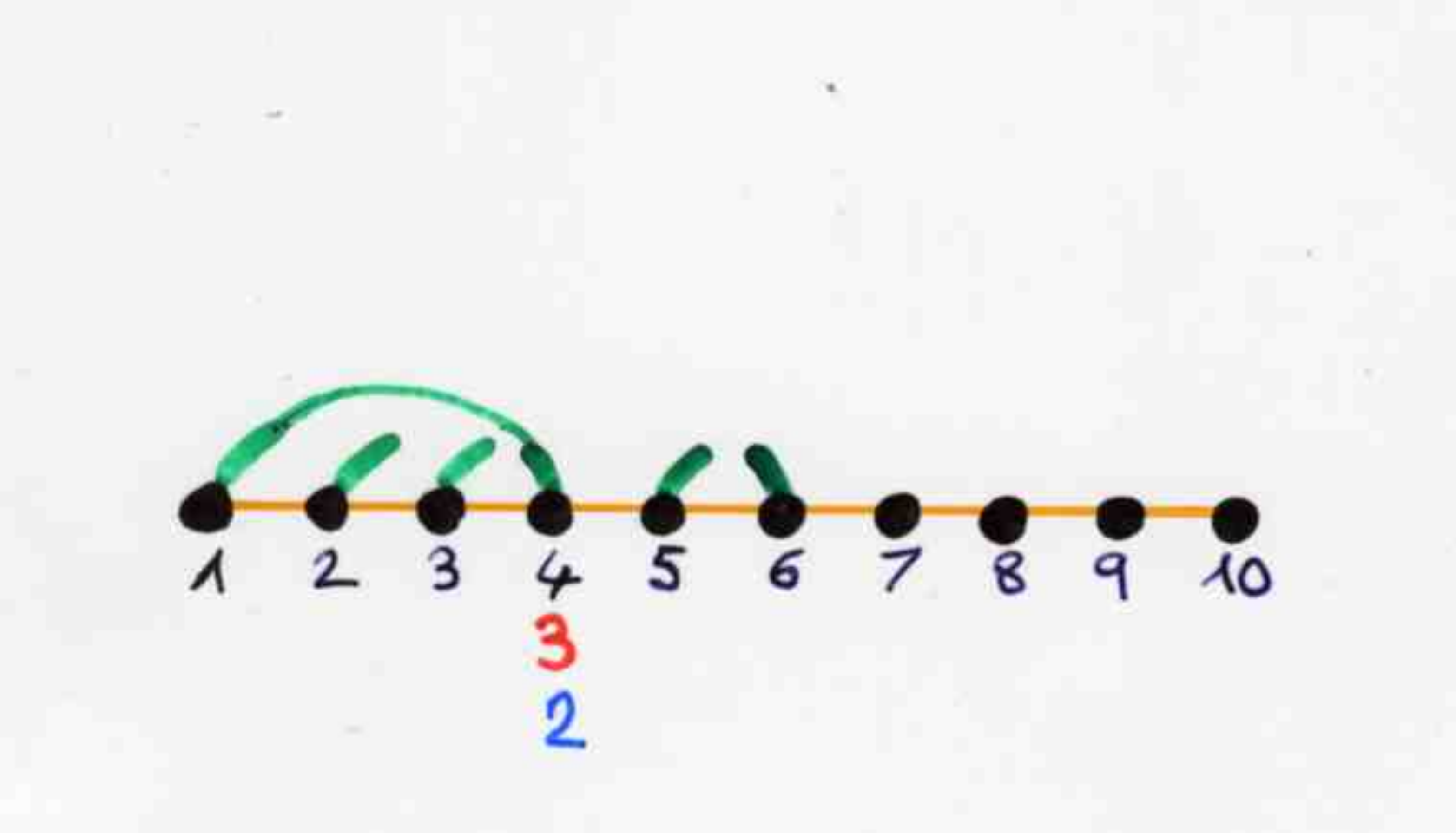
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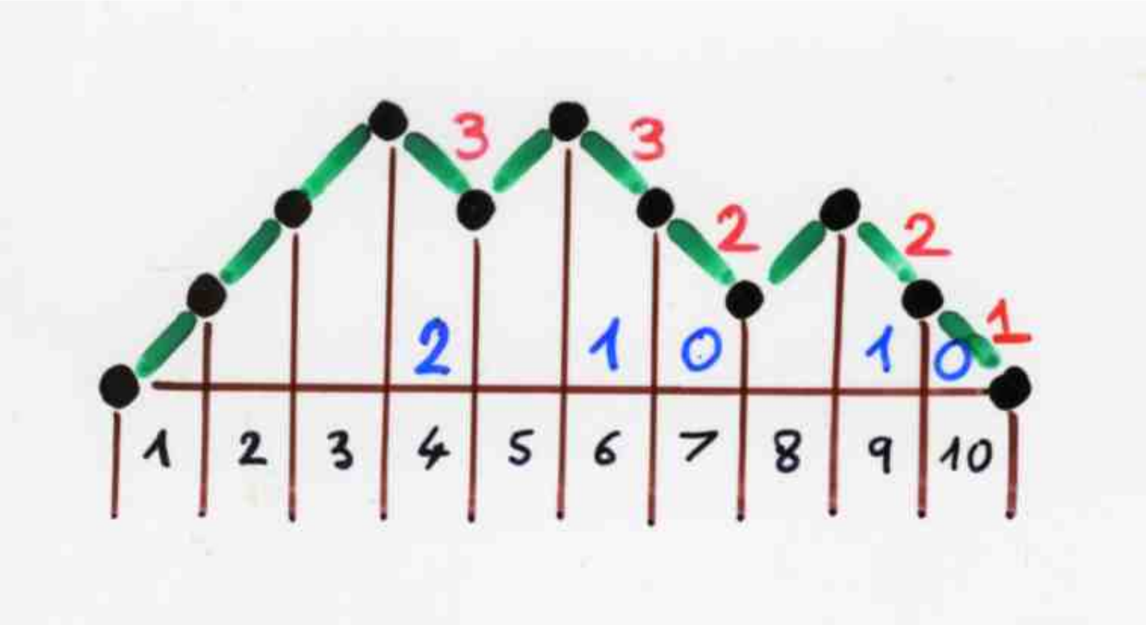
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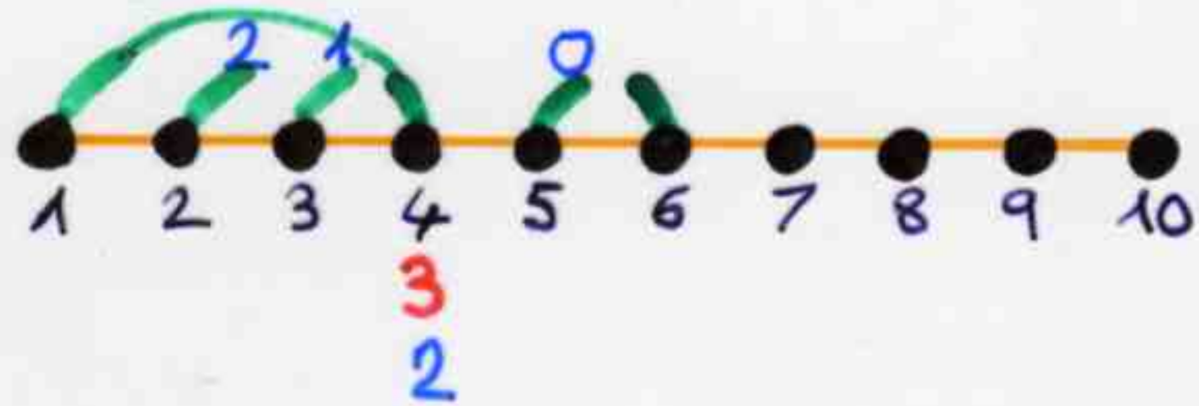
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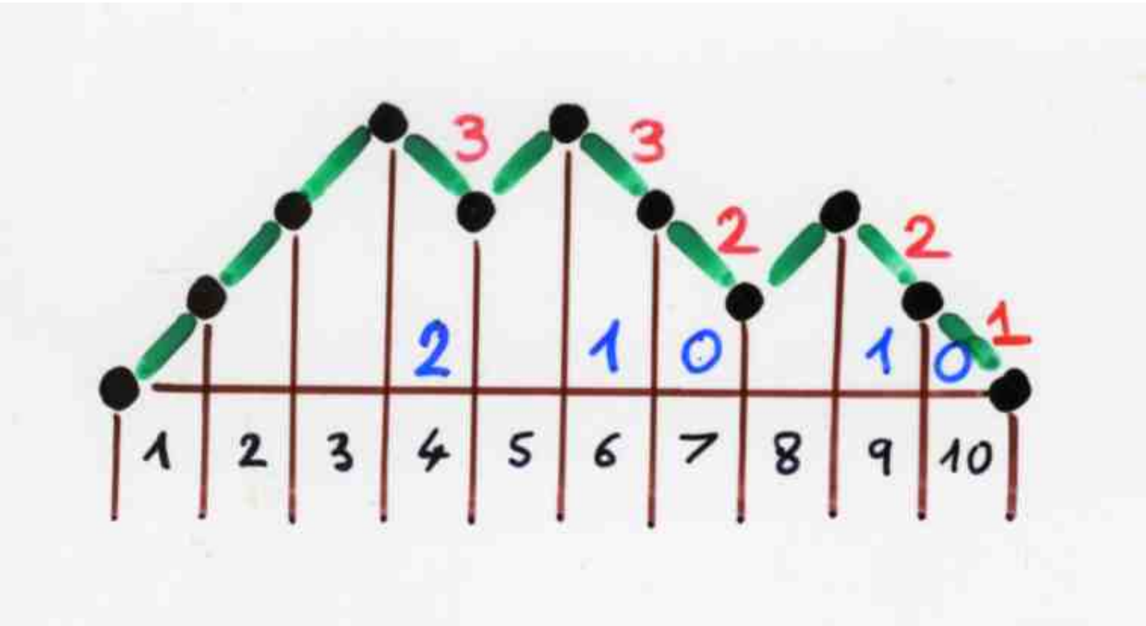
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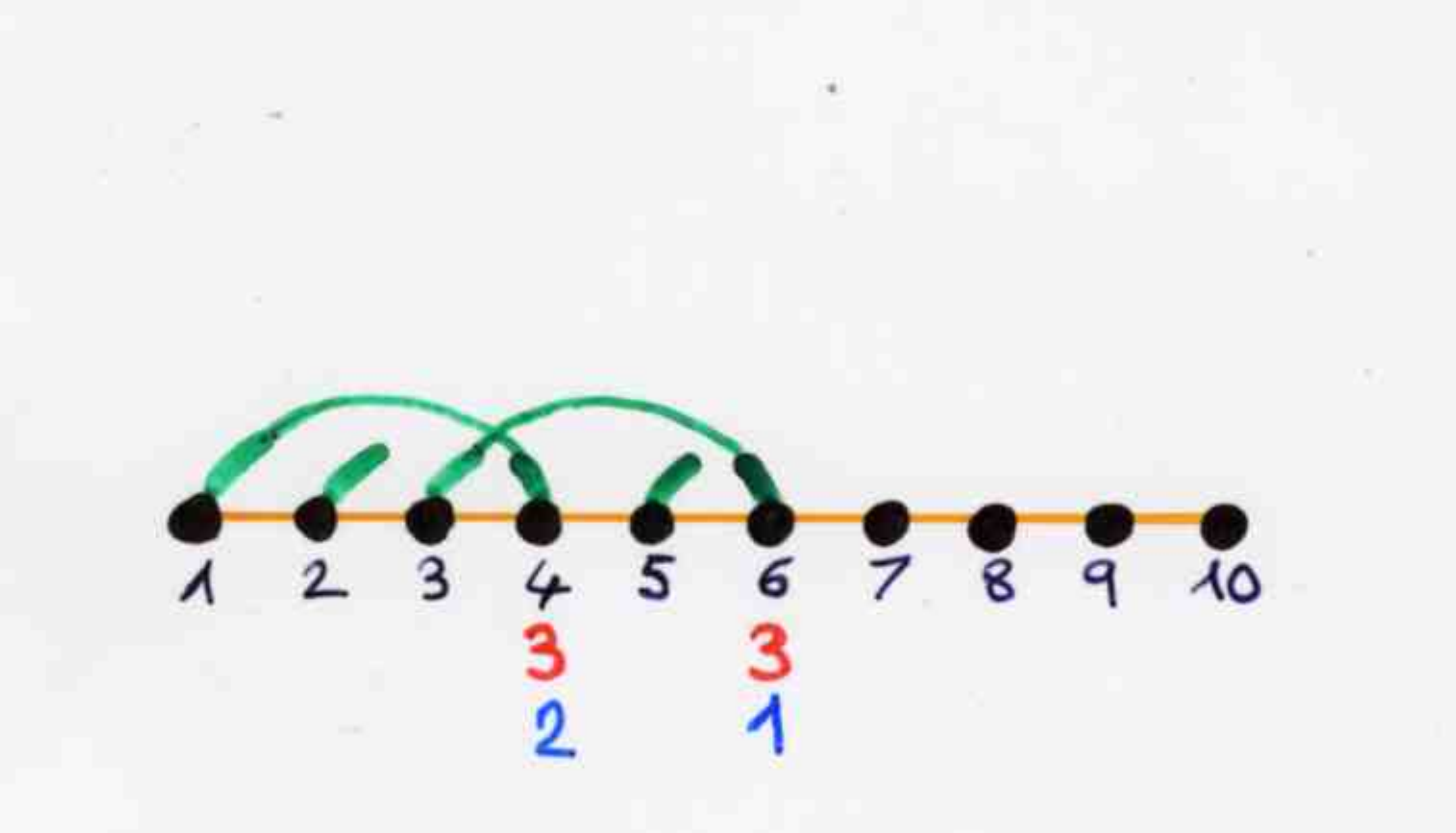
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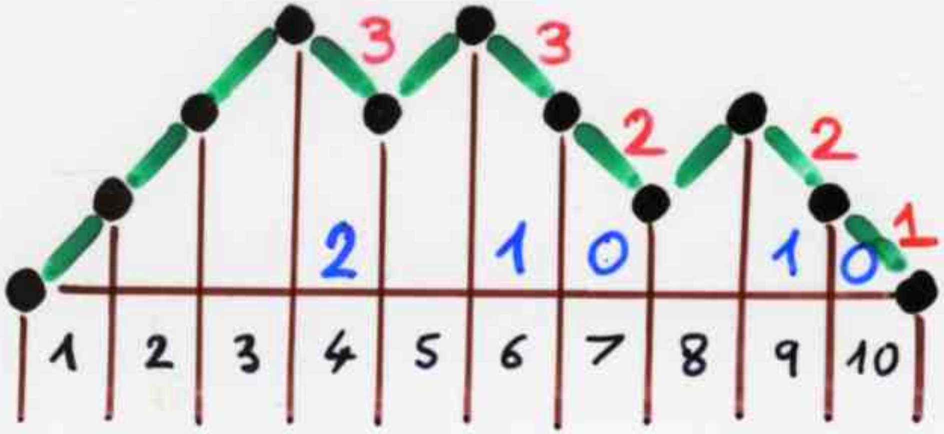
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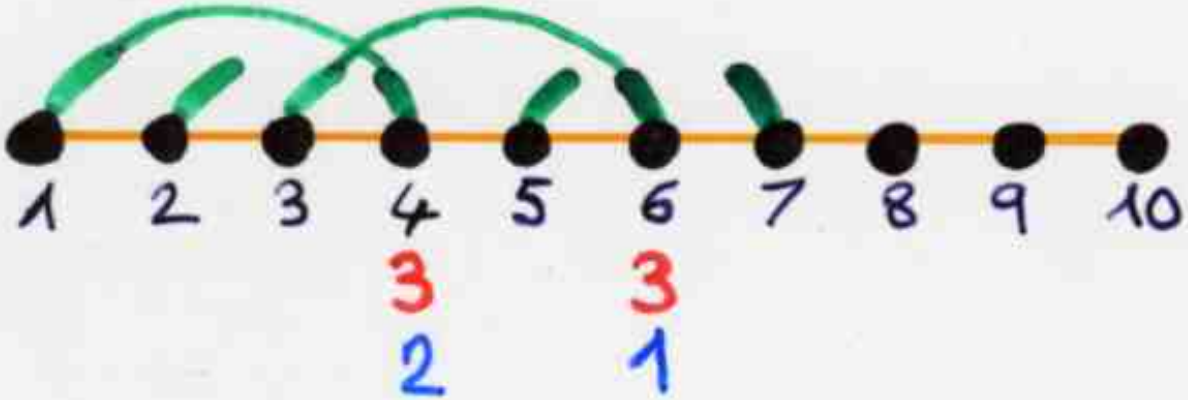
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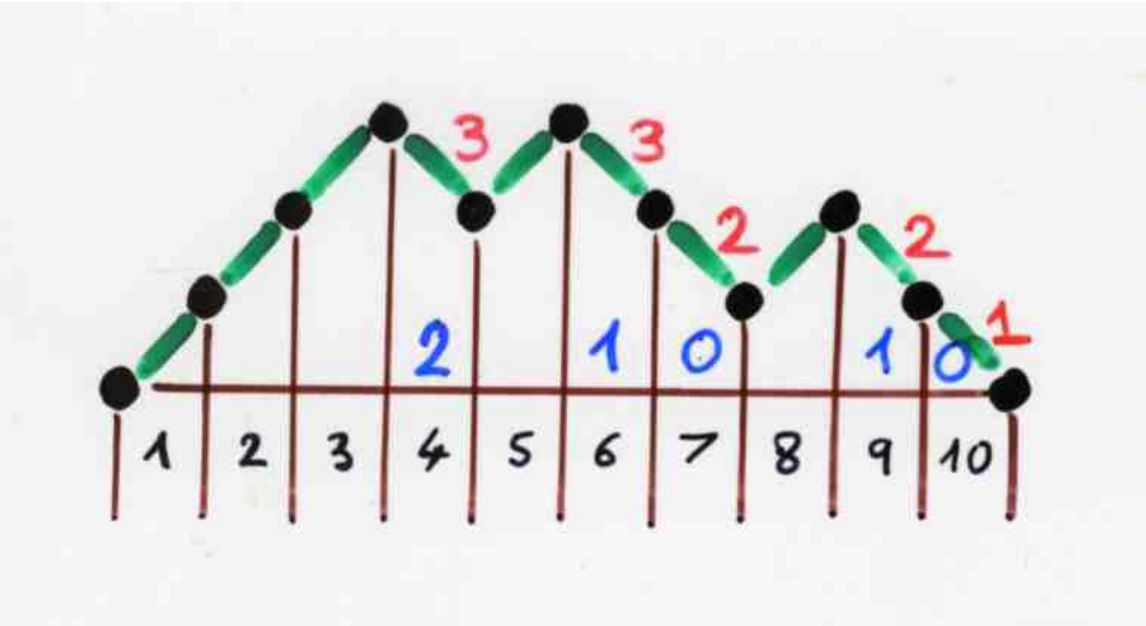
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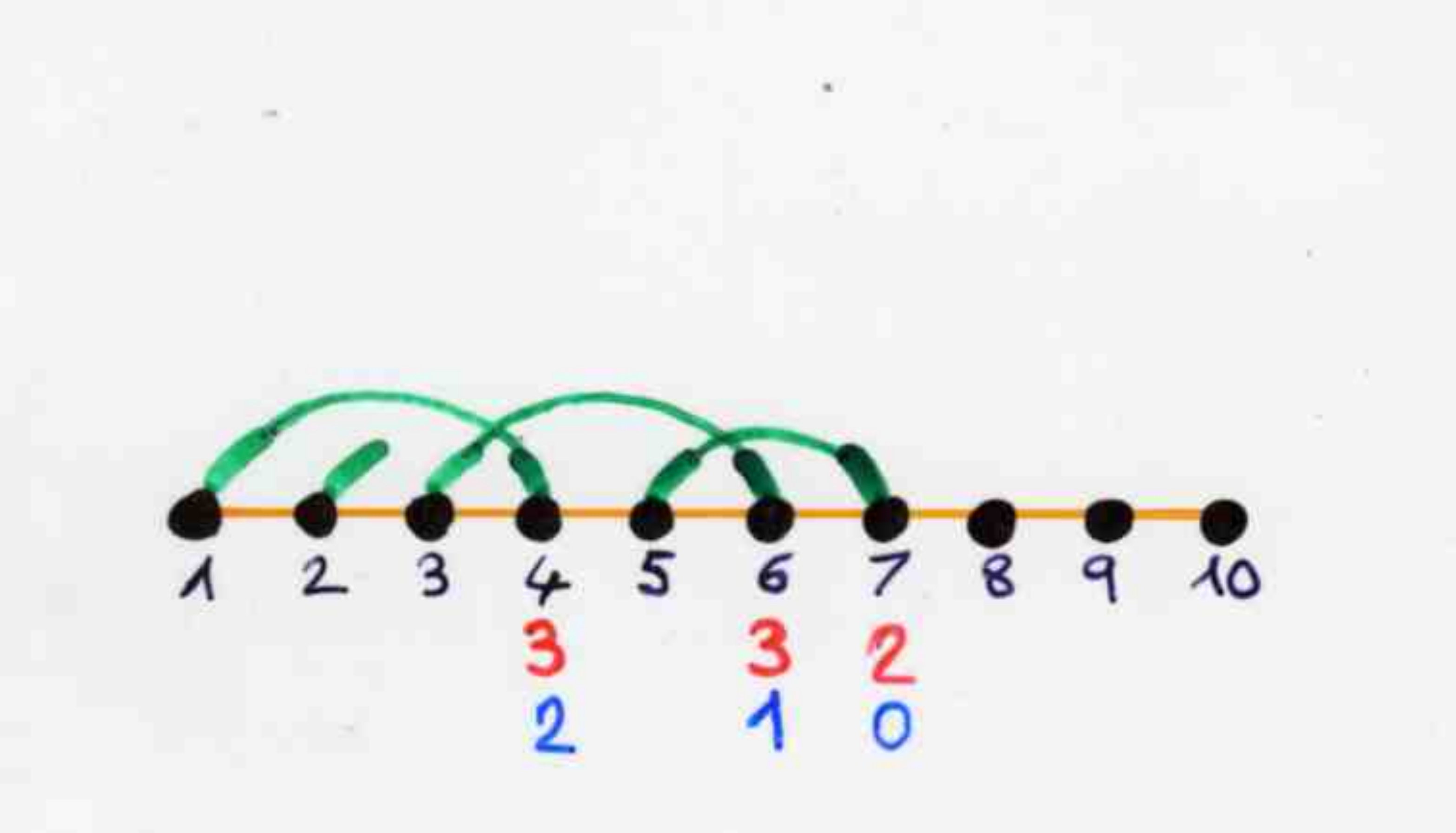
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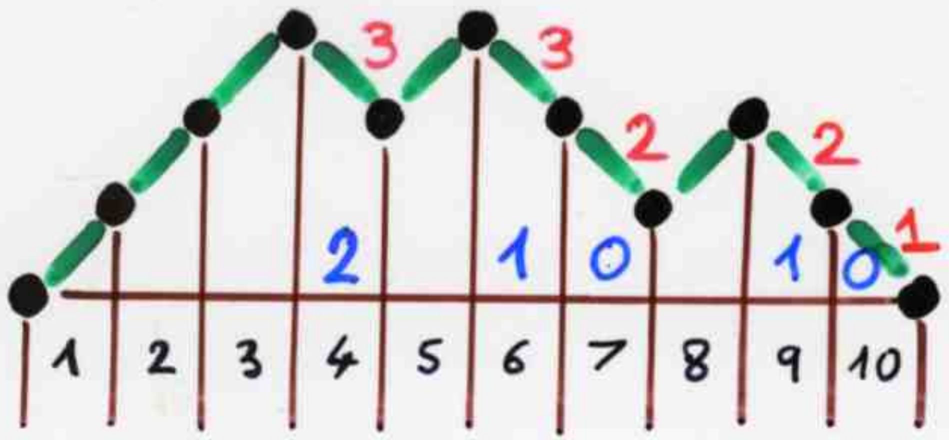
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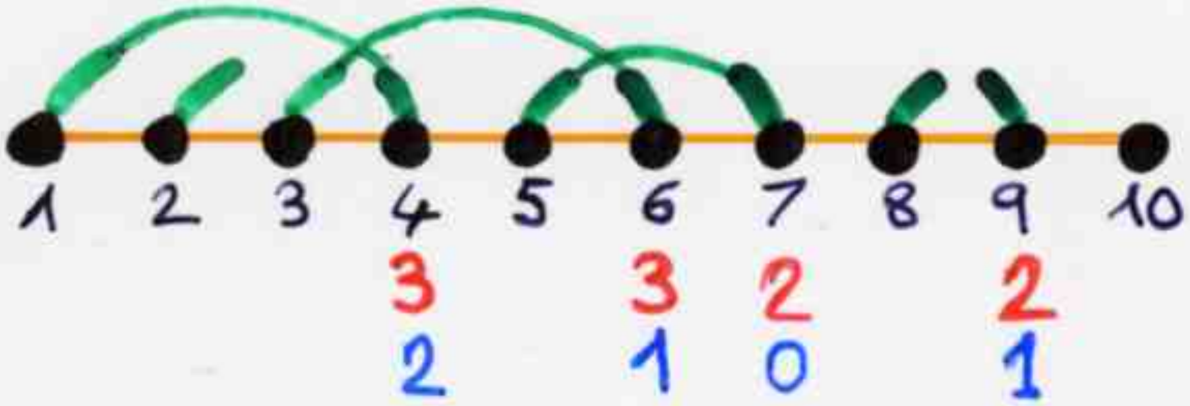
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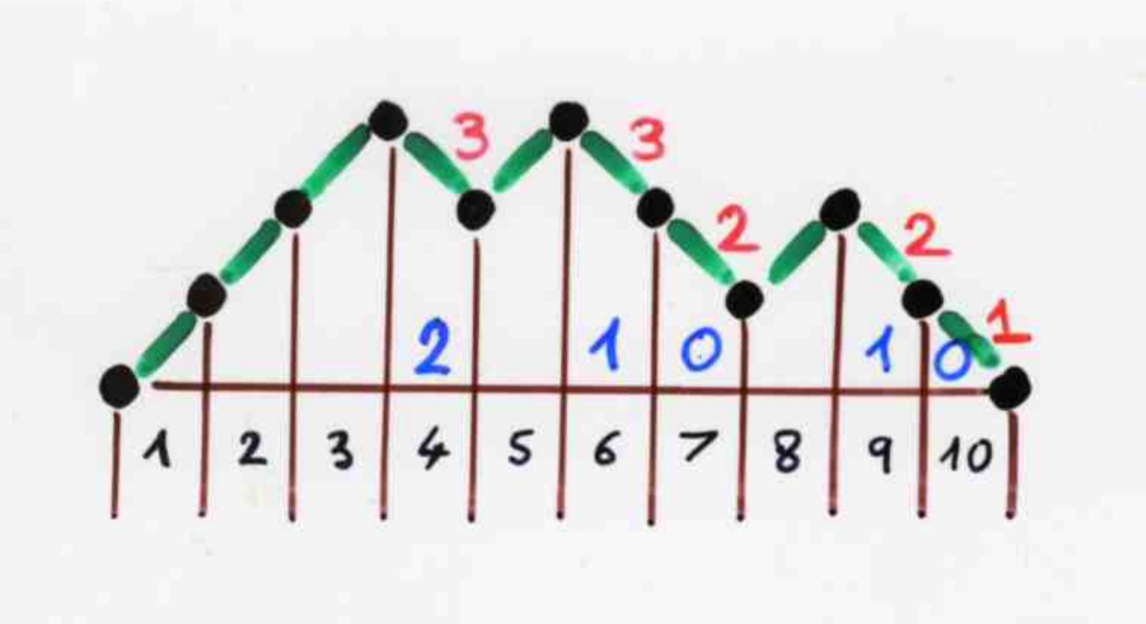
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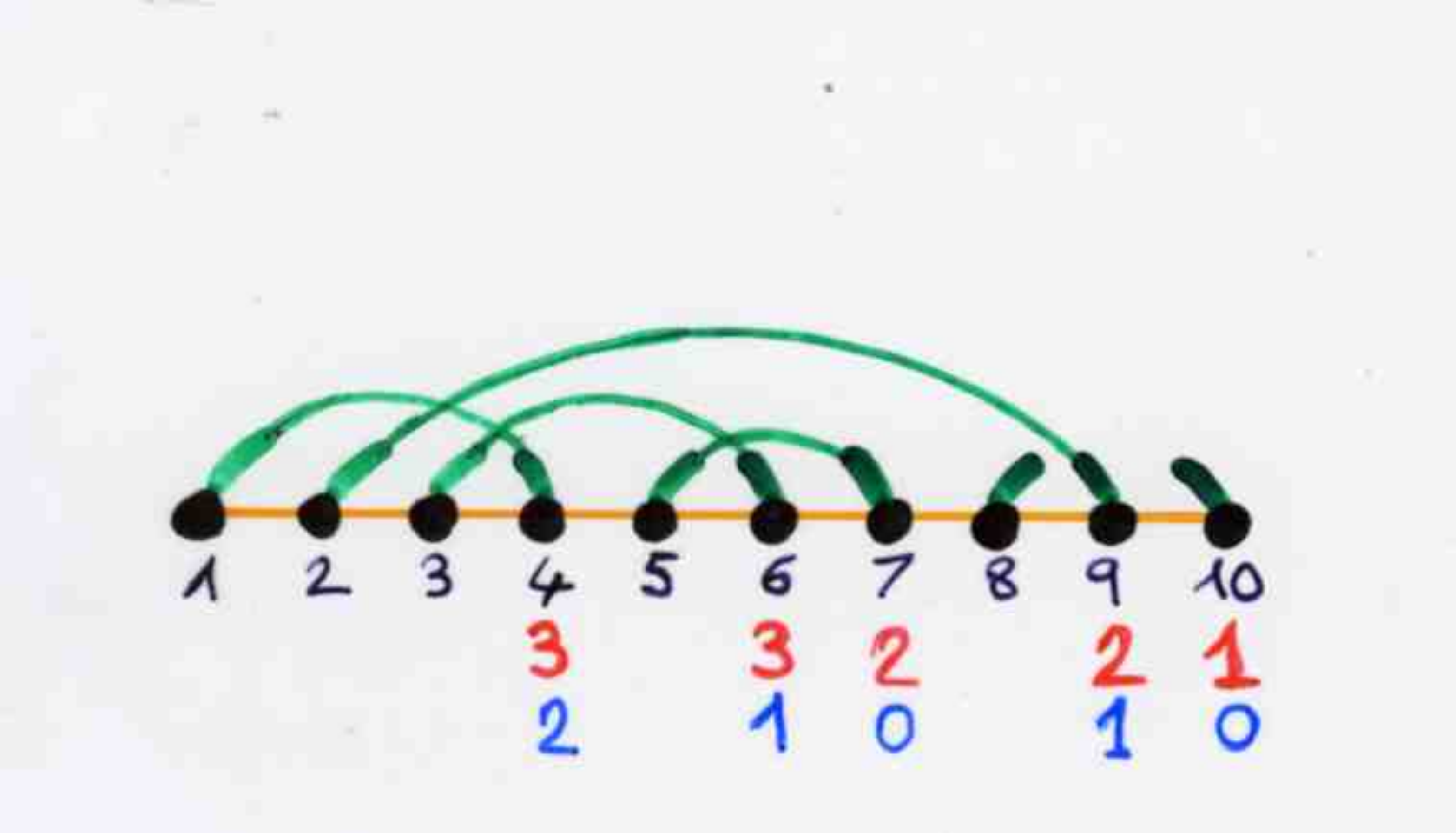
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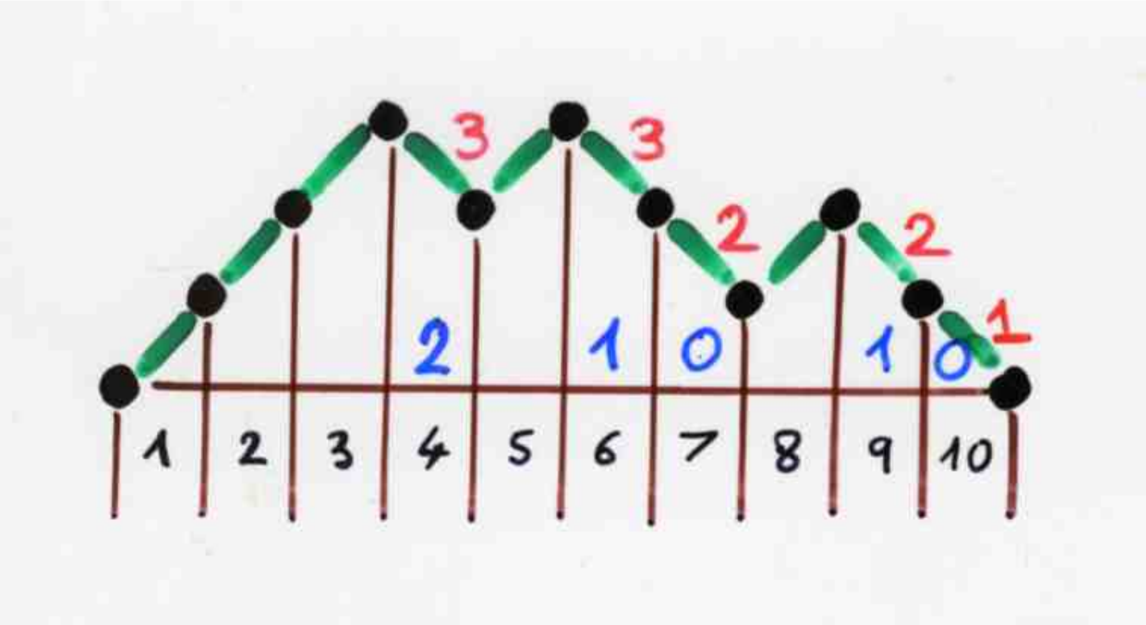
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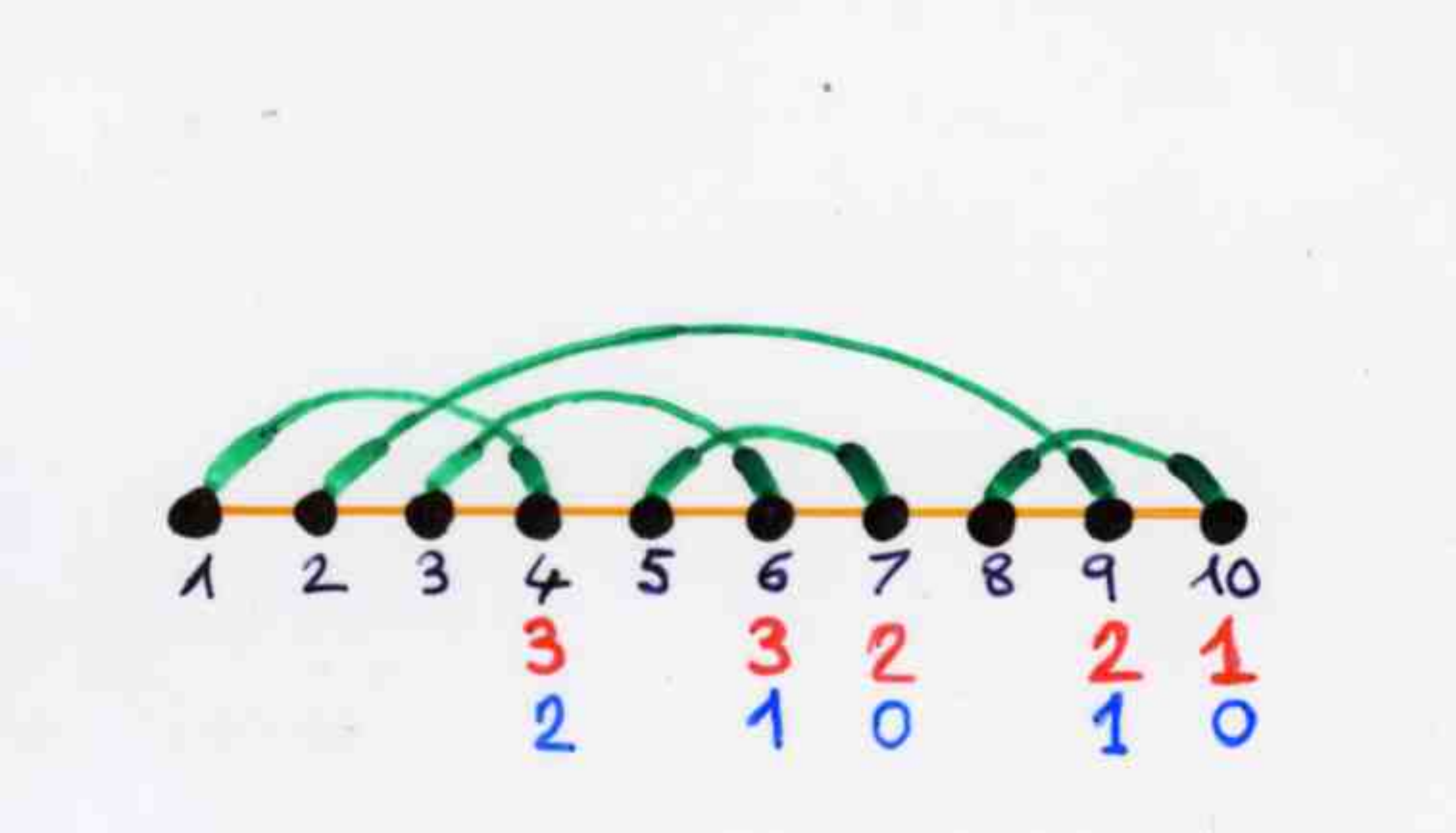
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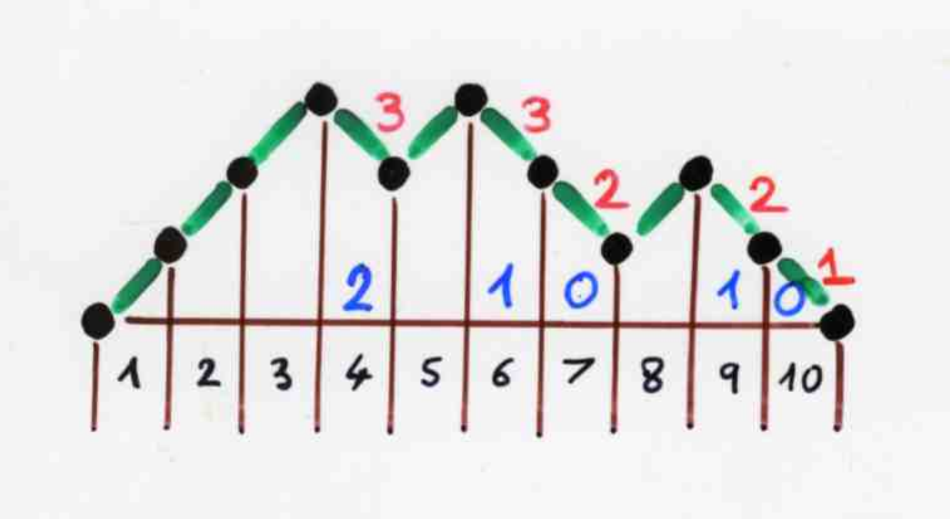
q-analog of Hermite histories

$\lambda_k = [k]_q$

$[k]_q = 1 + q + q^2 + \dots + q^{k-1}$

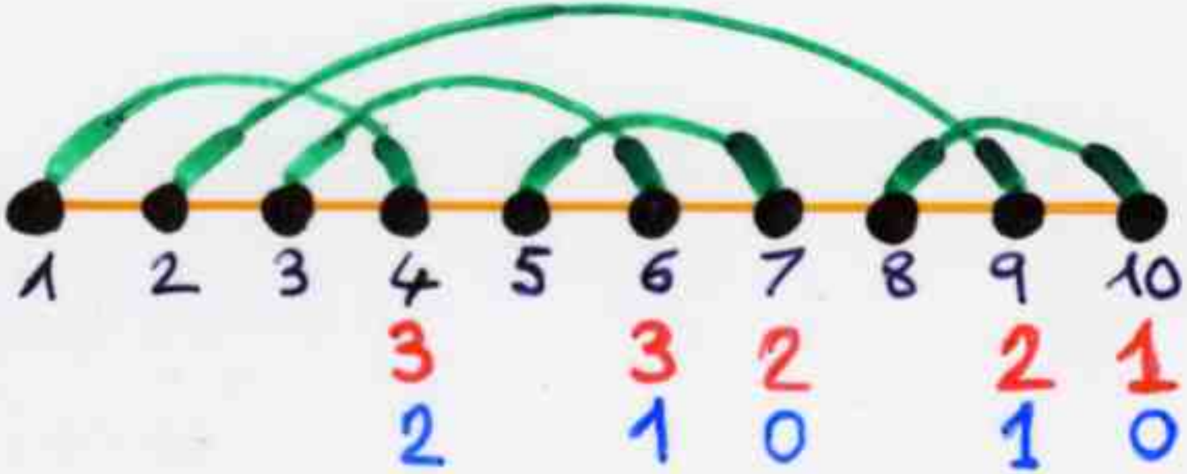


Hermite history related to ω

ω

Dyck path

$v_q(h)$ $q^{2+1+0+1+0}$ $= q^4$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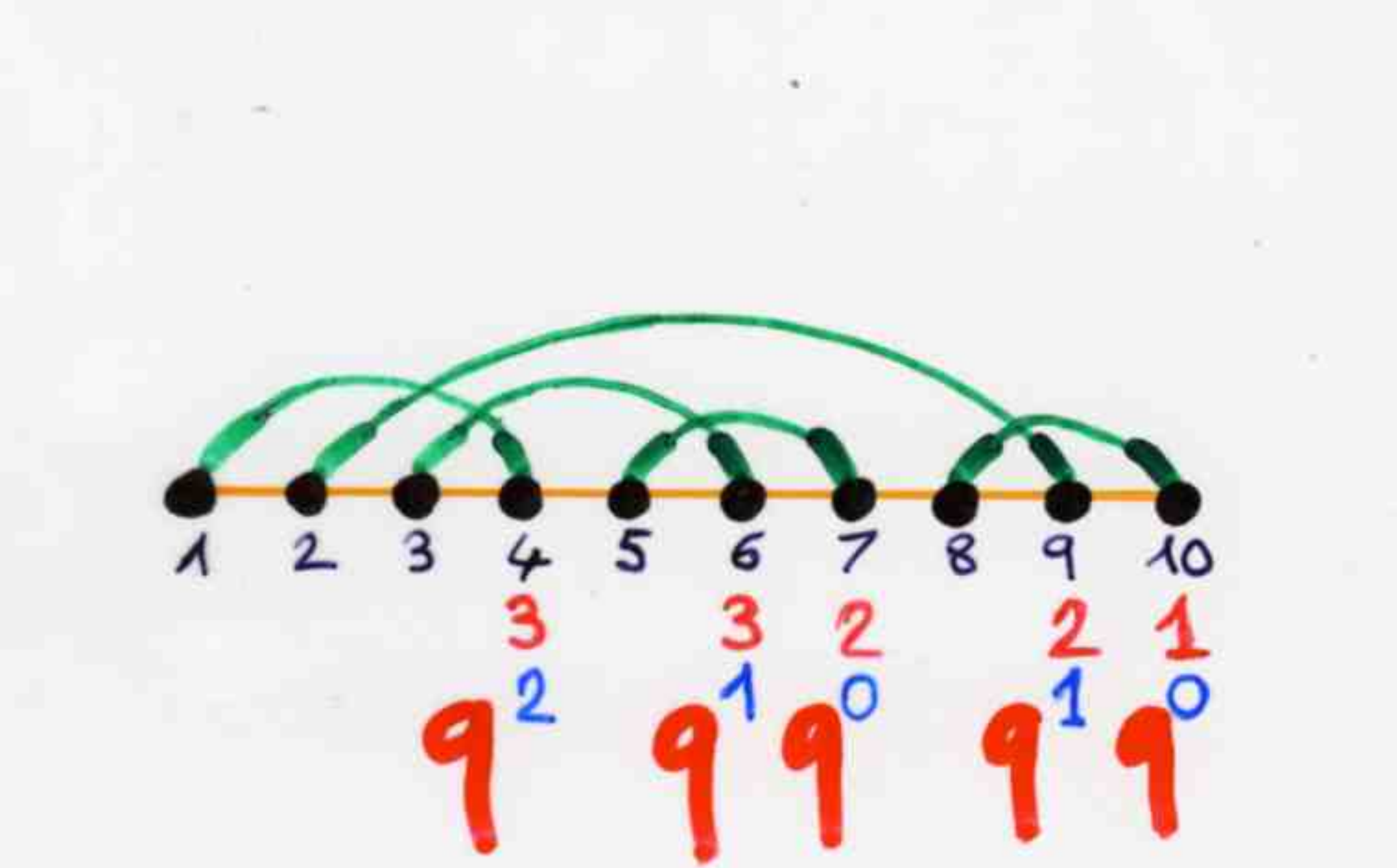
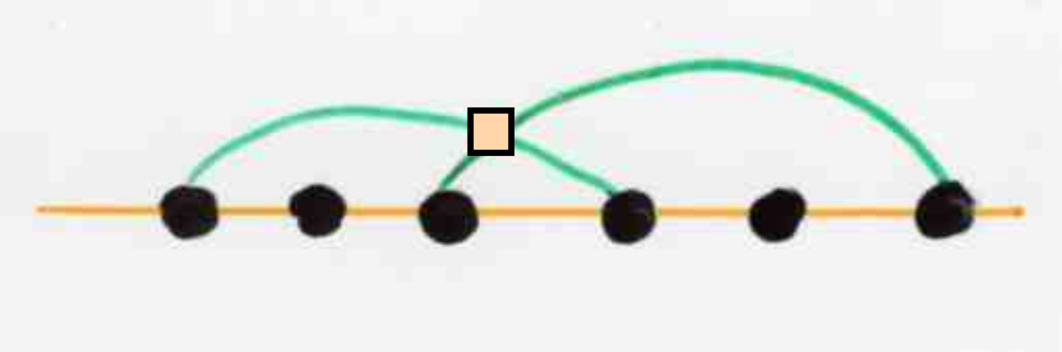
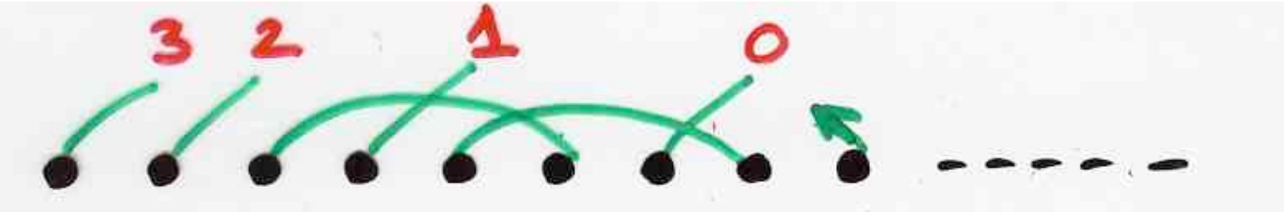


$$q^{2+1+0+1+0} = q^4$$

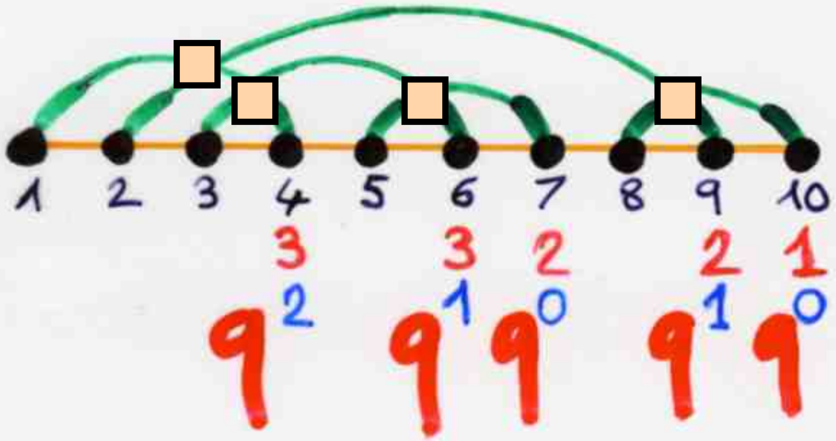
$$v_q(h)$$



crossing



crossing



$v_q(h)$

$q^{2+1+0+1+0} = q^4$

Combinatorics of the PASEP

(1982) Shapiro, Zeilberger

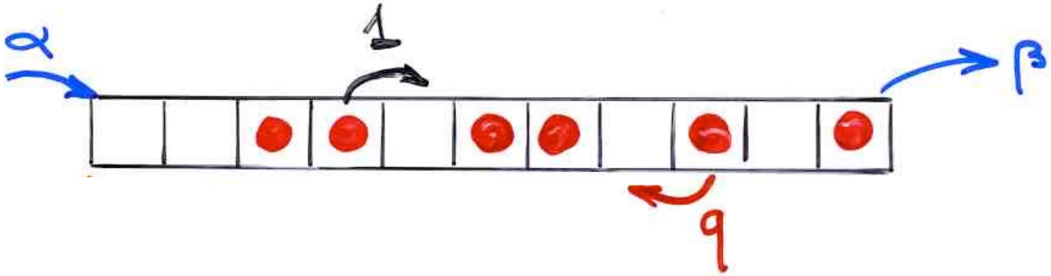
(2004) Brak, Essam
(2005) Duchi, Schaeffer
(2006) Corteel   Burnstein
Brak, Corteel, Essam, Paviainen, Rechnitzer
Corteel, Williams
(2007) Corteel, Nadeau   Corteel, Williams
Steingrimsson, Williams   X.V.
(2008) X.V.
(2009) Corteel, Josuat-Vergès, Prellberg, Rubey
Josuat-Vergès   Nadeau
(2010) Corteel, Williams

(2011) Josuat-Vergès   Corteel, Kim
Corteel, Dasse-Hartaut
Corteel, Josuat-Vergès, Williams
Aval, Boussicault, Nadeau
(2012) Corteel, Stanley, Stanton, Williams
(2013) Aval, Boussicault, Bouvel, Silimbani
Aval, Boussicault, Nadeau
Aval, Boussicault, Dasse-Hartaut
(2014) E. Jin
(2016) Aval, Boussicault, Delcroix-Oger, Hivert,
Laborde-Zubieta   Mandelshtam, X.V.
Corteel, Kim, Stanton
(2017) Corteel, Williams   Mandeshtam, X.V.
Corteel, Mandelshtam, Williams
Corteel, Nunge
Laborde-Zubieta

alternative tableaux

PASEP with 3 parameters

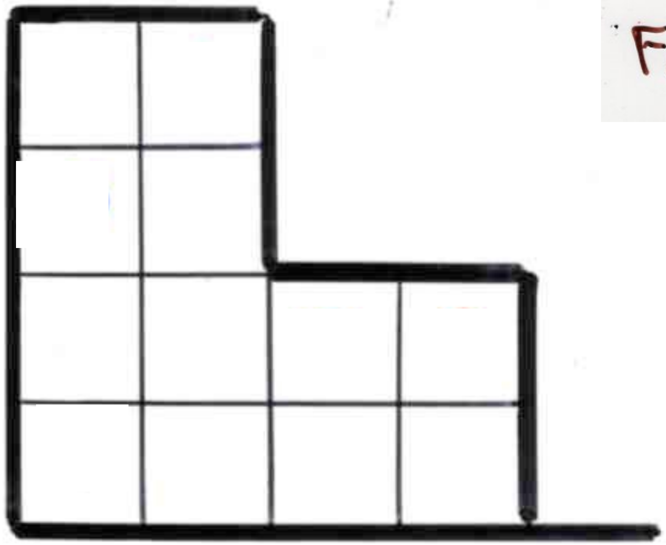
$\gamma = \delta = 0$ $\quad q, \alpha, \beta$



PASEP

alternative tableau

Definition



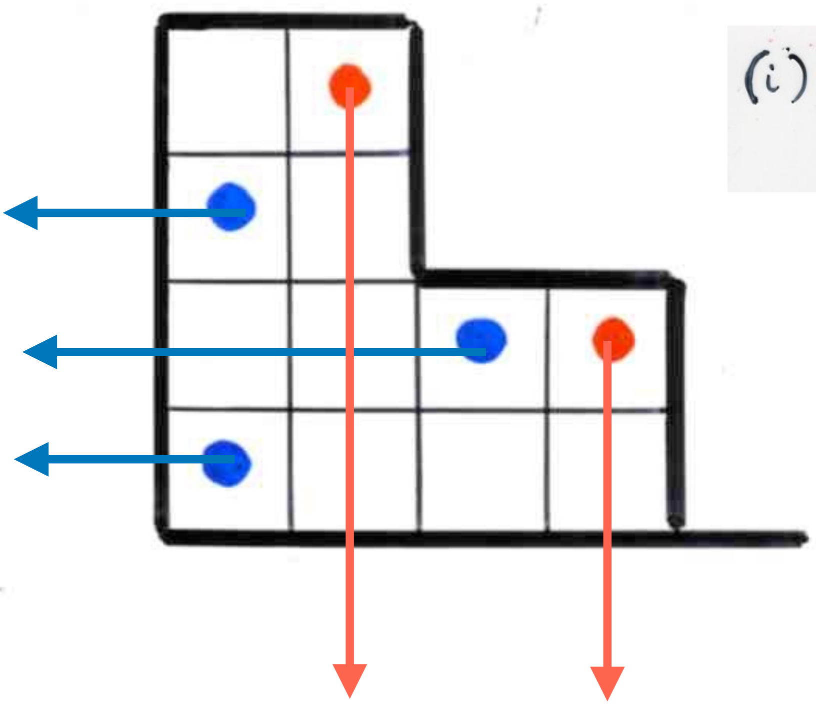
Ferrers diagram F

with possibly empty rows or columns

size of F

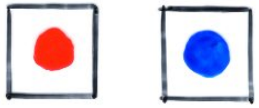
n = (number of rows) + (number of columns)

alternative tableau

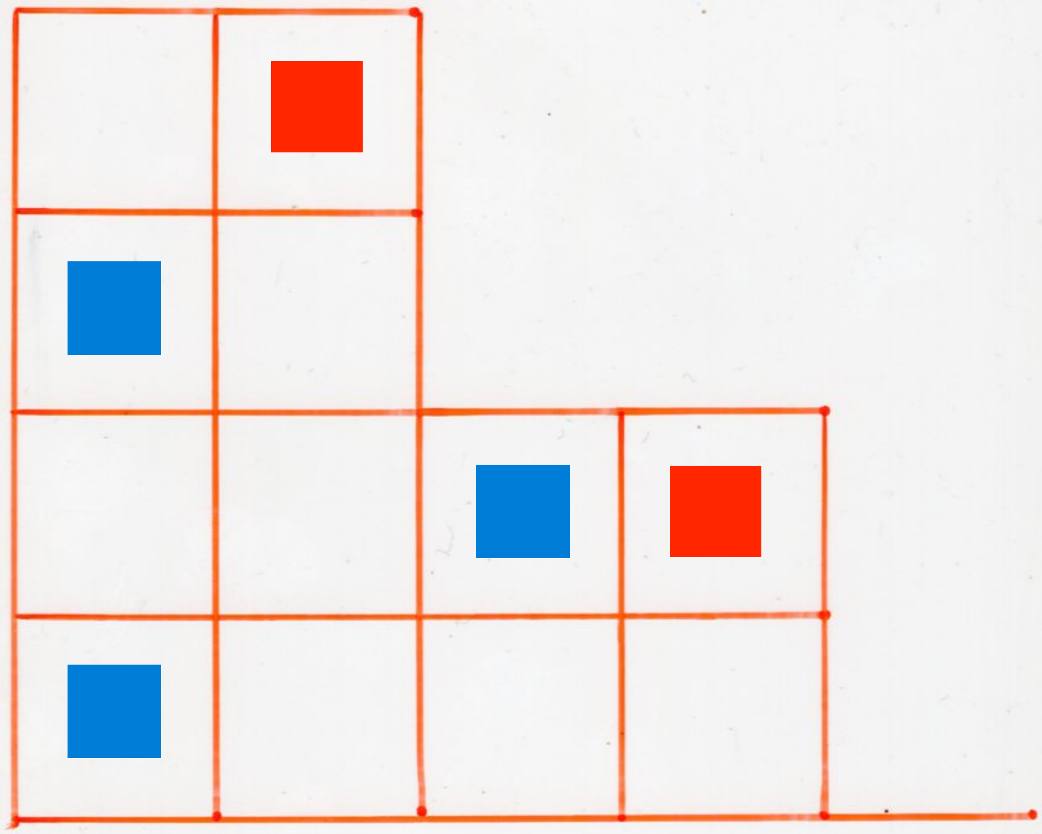


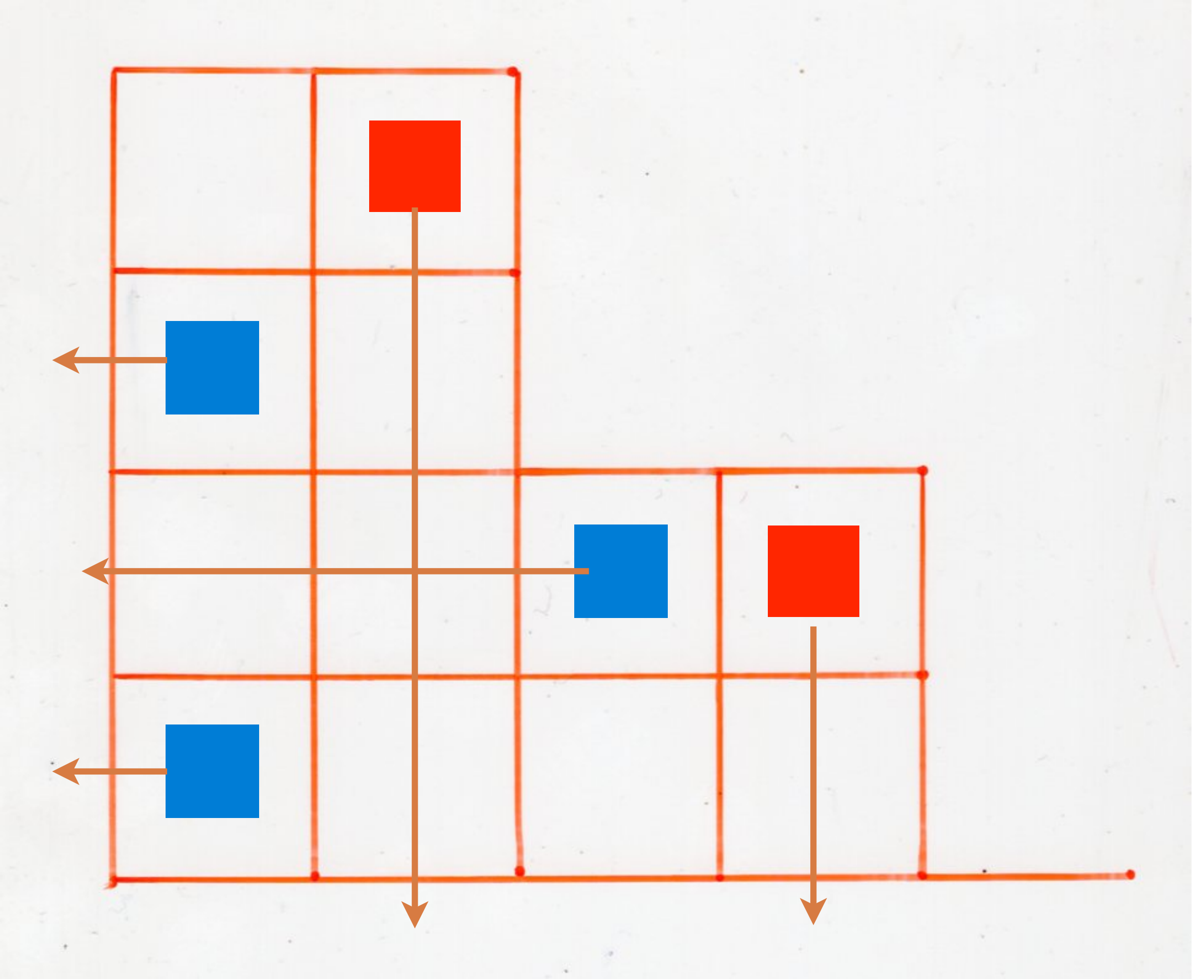
Definition

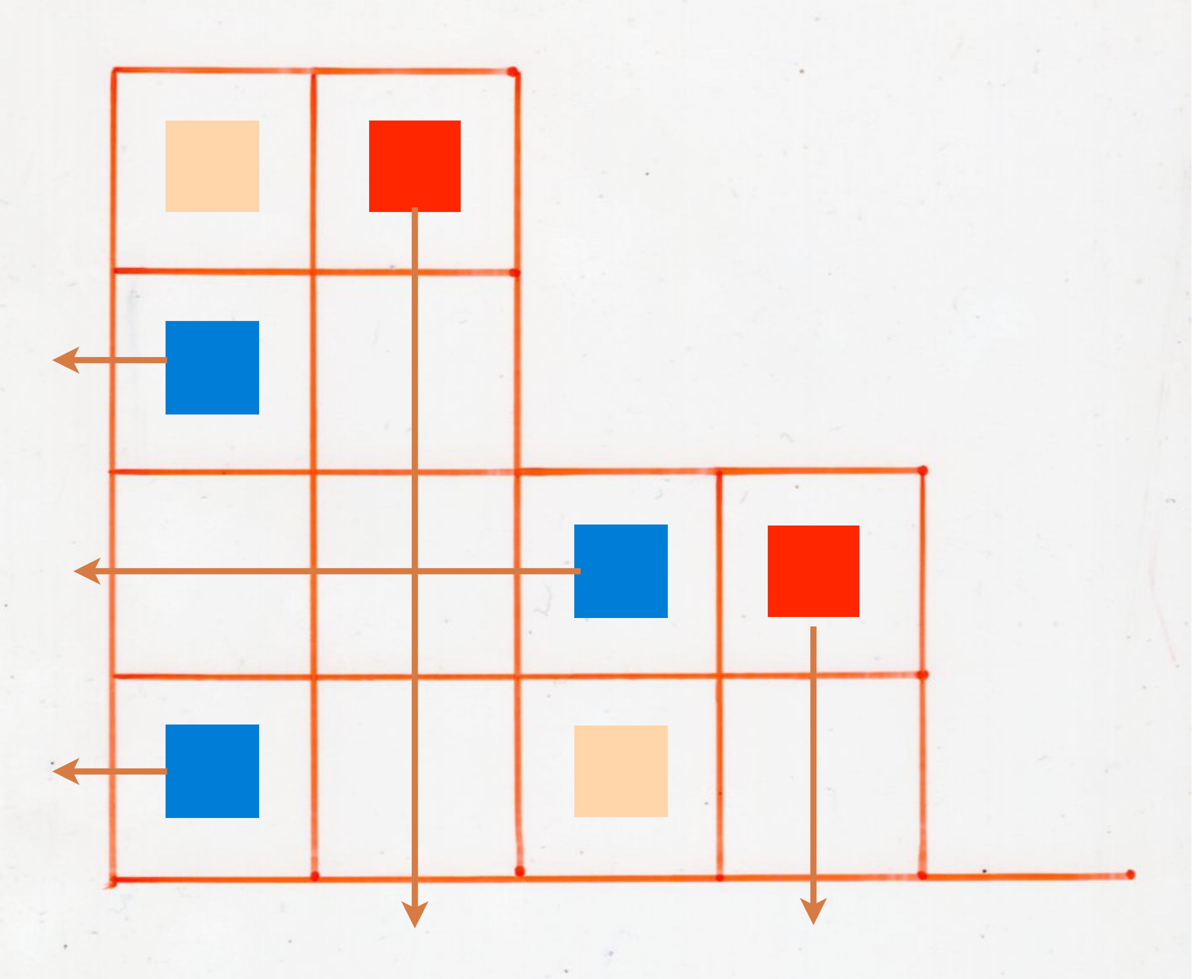
(i) some cells are coloured **red** or **blue**



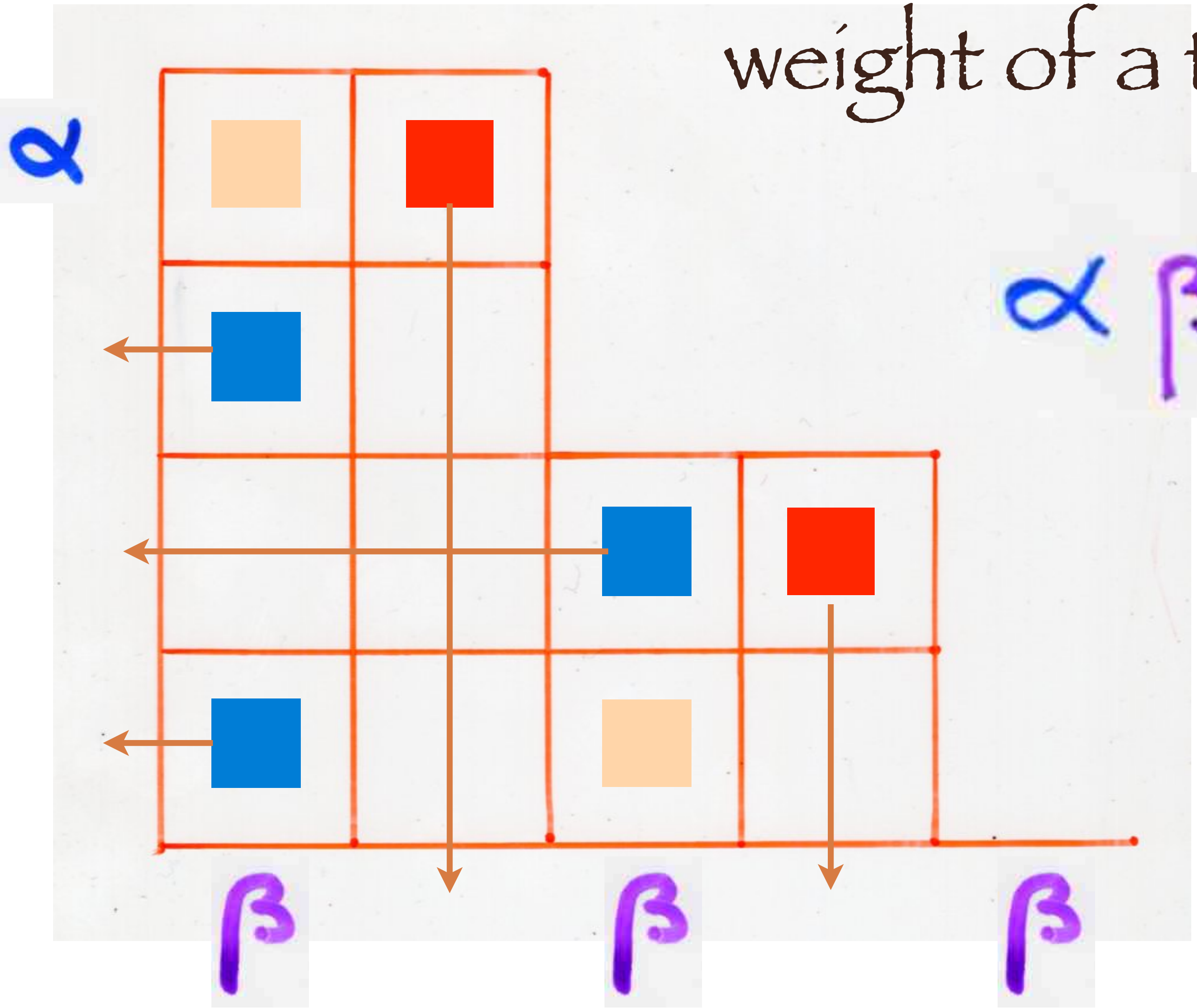
(ii)
- no coloured cell at the left of a **blue** cell
- no coloured cell below a **red** cell







weight of a tableau



$$\alpha \beta^2 q^2$$

q $k(T)$ = nb of cells 🟧

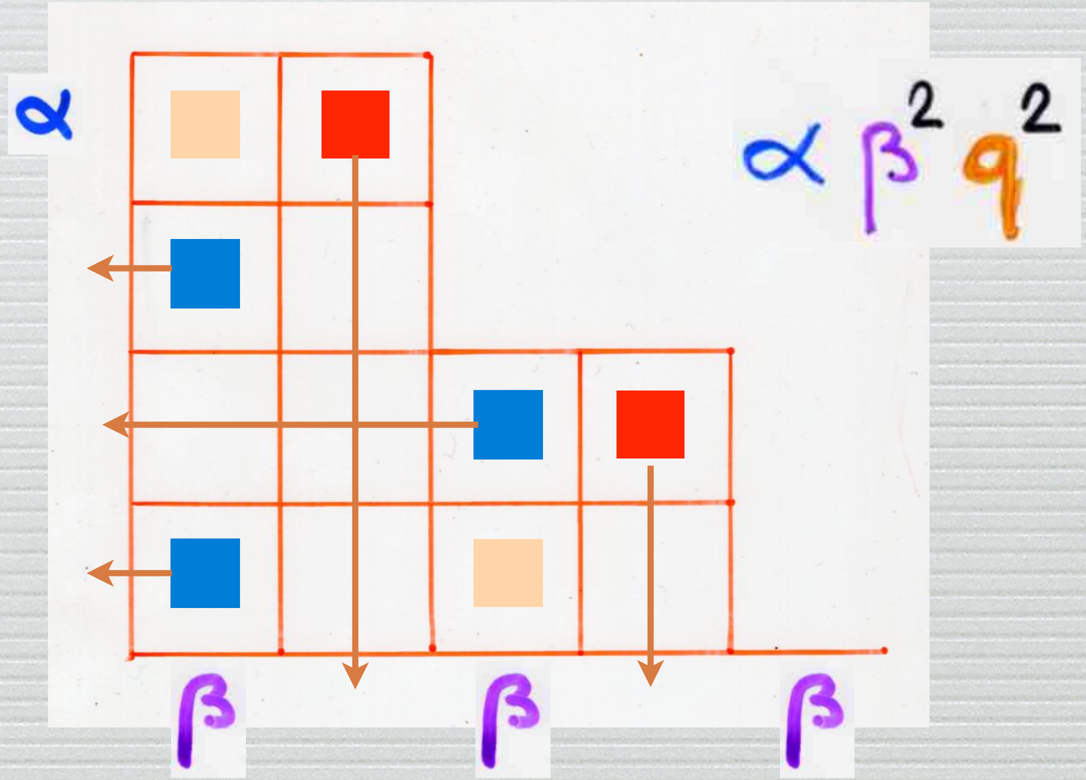
α $i(T)$ = nb of rows without 🔵

β $j(T)$ = nb of columns without 🔴

Partition function

Z_n

Sum of the weight of all tableaux of size n



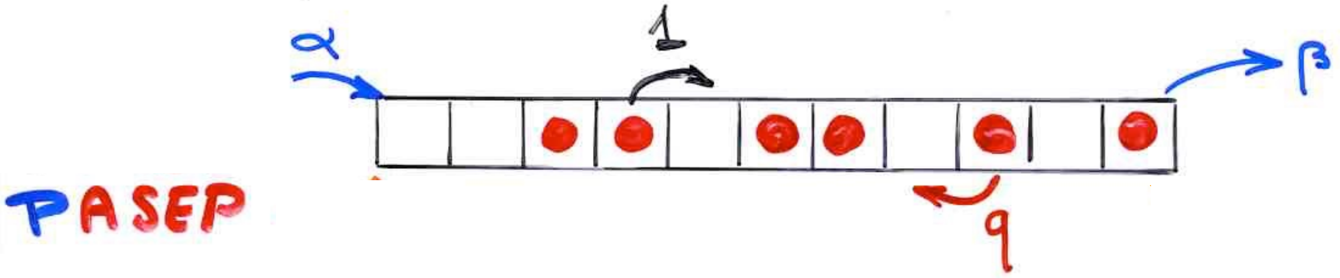
q $k(T)$ = nb of cells 🟧

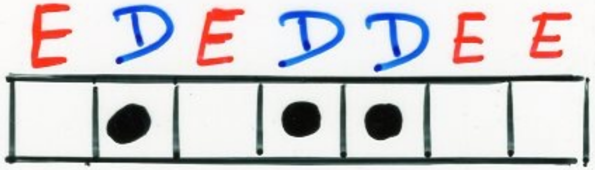
α $i(T)$ = nb of rows without 🔵

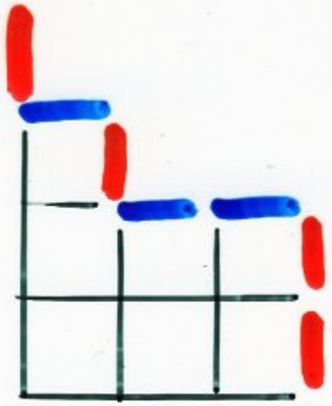
β $j(T)$ = nb of columns without 🔴

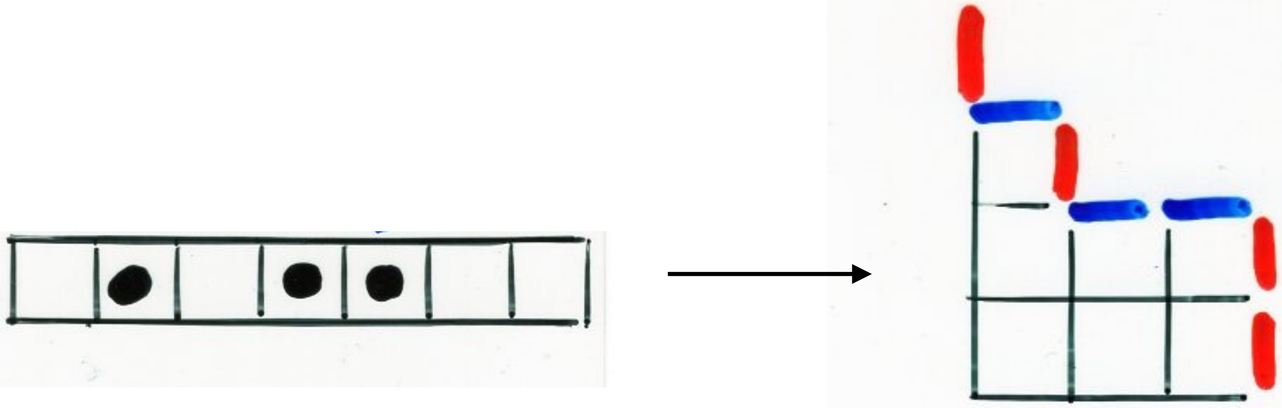
computation of the
"stationary probabilities"

PASEP with 3 parameters q, α, β









<u>Corollary.</u> The stationary probability associated to the state $\tau = (\tau_1, \dots, \tau_n)$ is

$$\text{proba}_\tau(q;\alpha,\beta) = \frac{1}{Z_n} \sum_{\substack{T \\ \text{alternative} \\ \text{tableaux} \\ \text{profile } \tau}} q^{k(T)} \alpha^{-i(T)} \beta^{-j(T)}$$

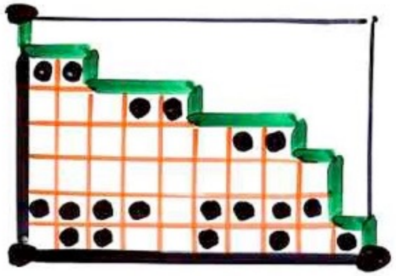
alternative
tableau
X.V. (2008)

permutation
tableau

S. Corteel, L. Williams
(2007, 2008, 2009)

permutation tableaux

Ferrers diagram $F \subseteq k \times (n-k)$
rectangle



filling of the cells with 0 and 1

(i) in each column : at least one 1

☐ = 0 ▣ = 1

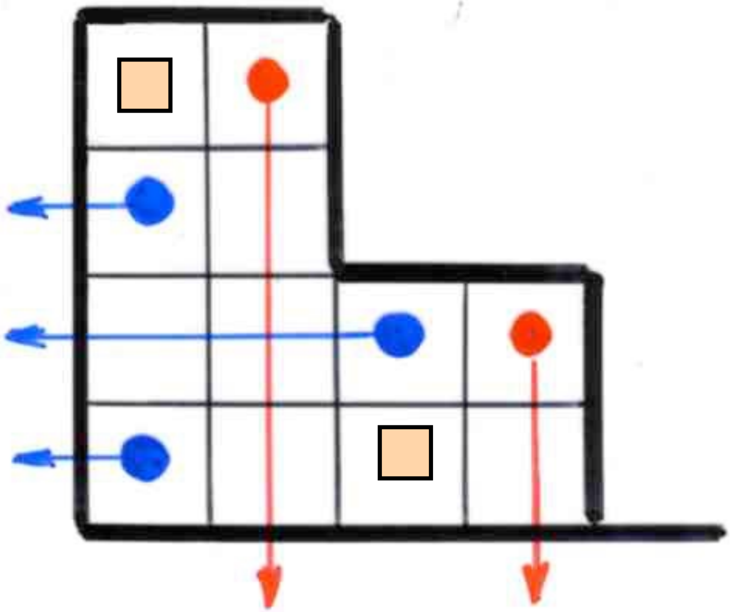
(ii)

forbidden

permutation tableau

A. Postnikov (2001, ...)
totally nonnegative part of the Grassmannian
E. Steingrímsson, L. Williams (2005)

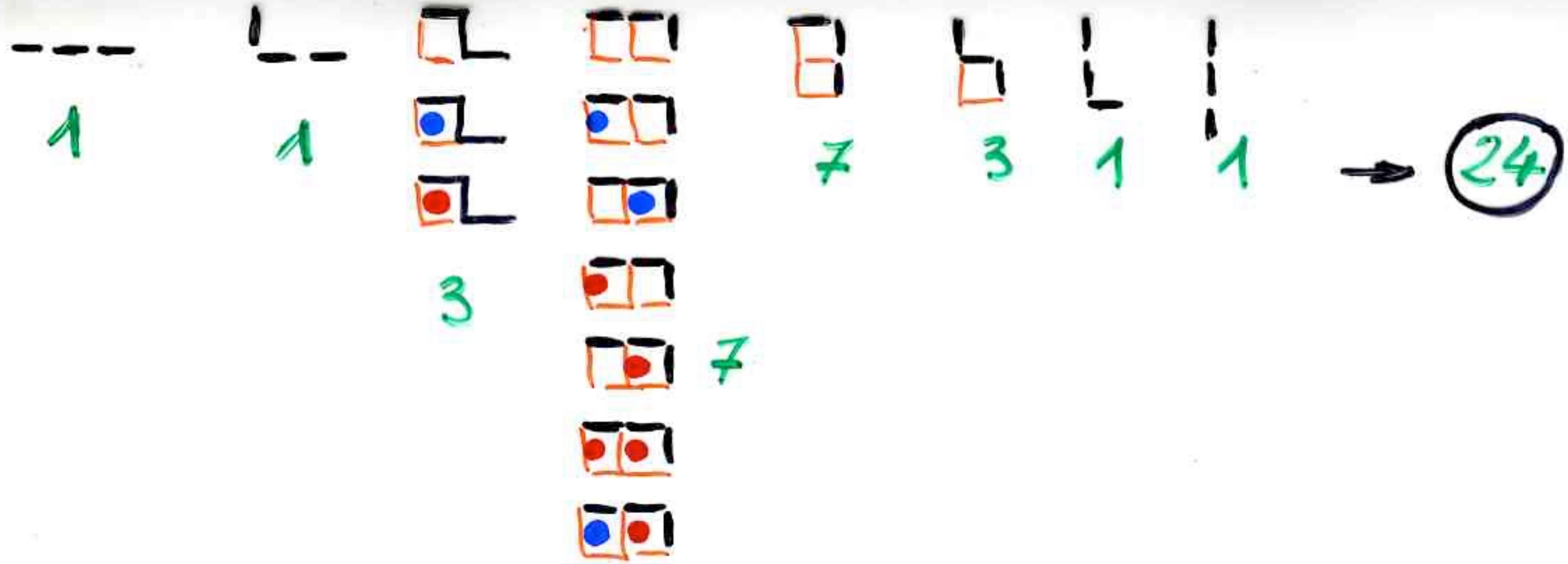
Enumeration of alternative tableaux



Prop. The number of alternative tableaux of size n is

(n+1)!

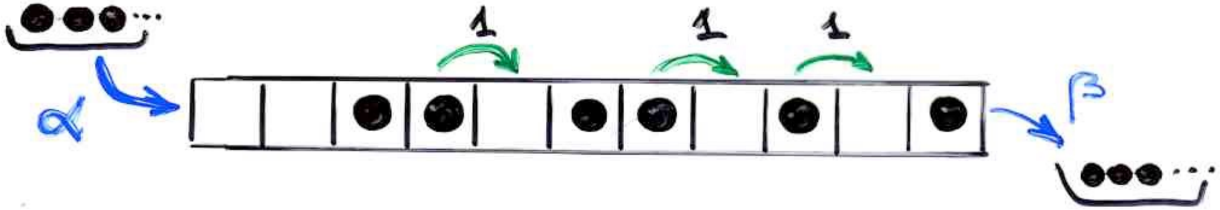
ex: n=2



$q=0$ TASEP (α,β)

TASEP

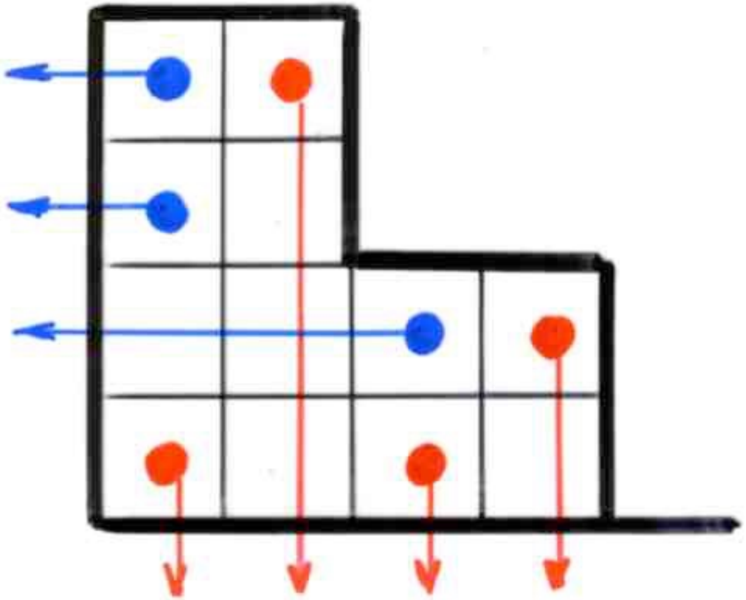
"totally asymmetric exclusion process"



Definition Catalan alternative tableau

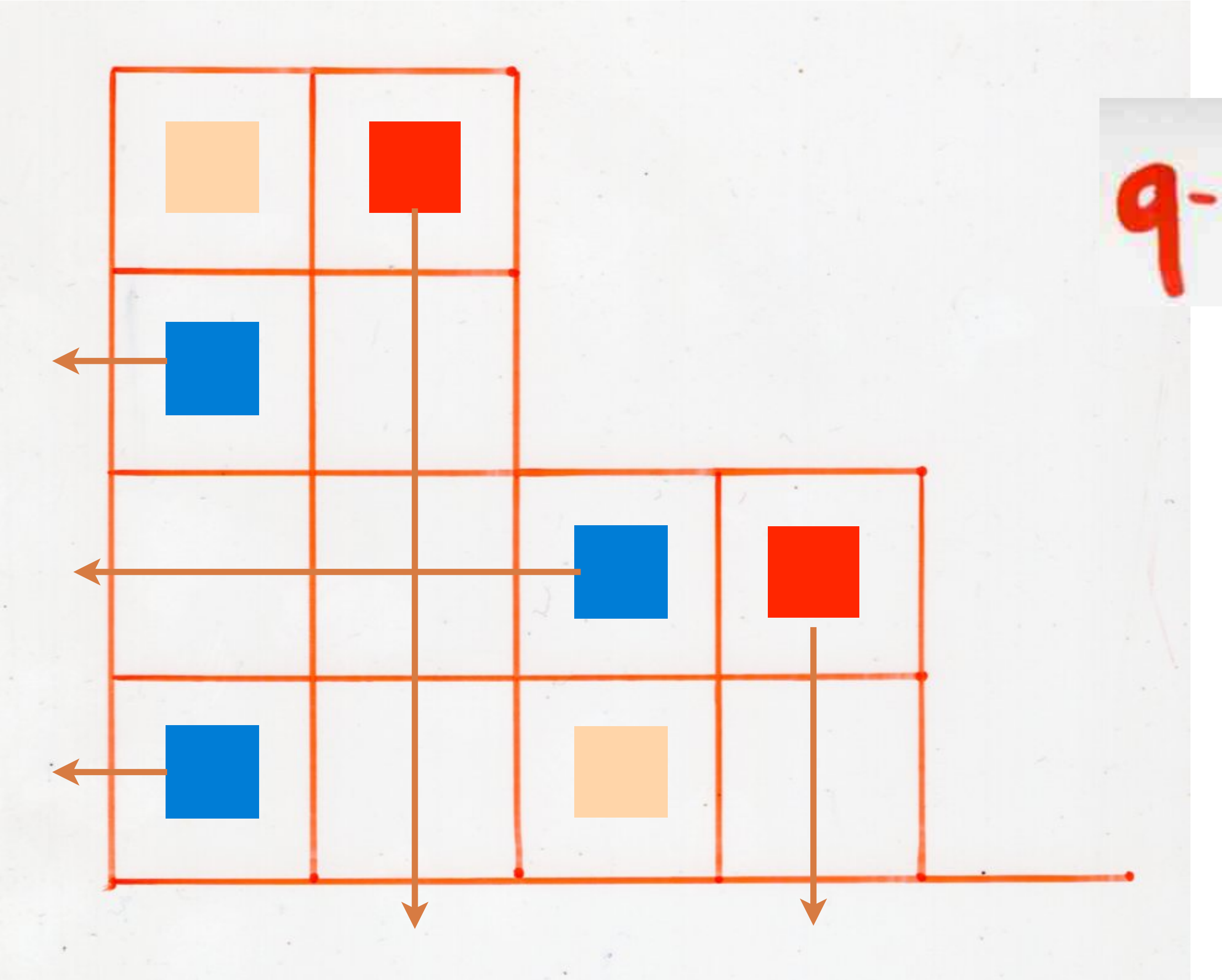
alternative tableau T without cells 🟧

i.e. every empty cell is below a red cell or on the left of a blue cell



$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

Catalan numbers



q- analog of n!

?

q- analog of n!

$$[k]_q = 1 + q + q^2 + \dots + q^{k-1}$$

Inv

number
of inversions

Maj

Major
index

Interpretation of the 3-parameters
Partition function

q, α, β

q-analogue of n! ?

First bijection: tableaux— permutations

Corteel, Williams (2007)

Steingrimsson, Williams (2007)

bijection

permutation tableaux → permutations

α, β

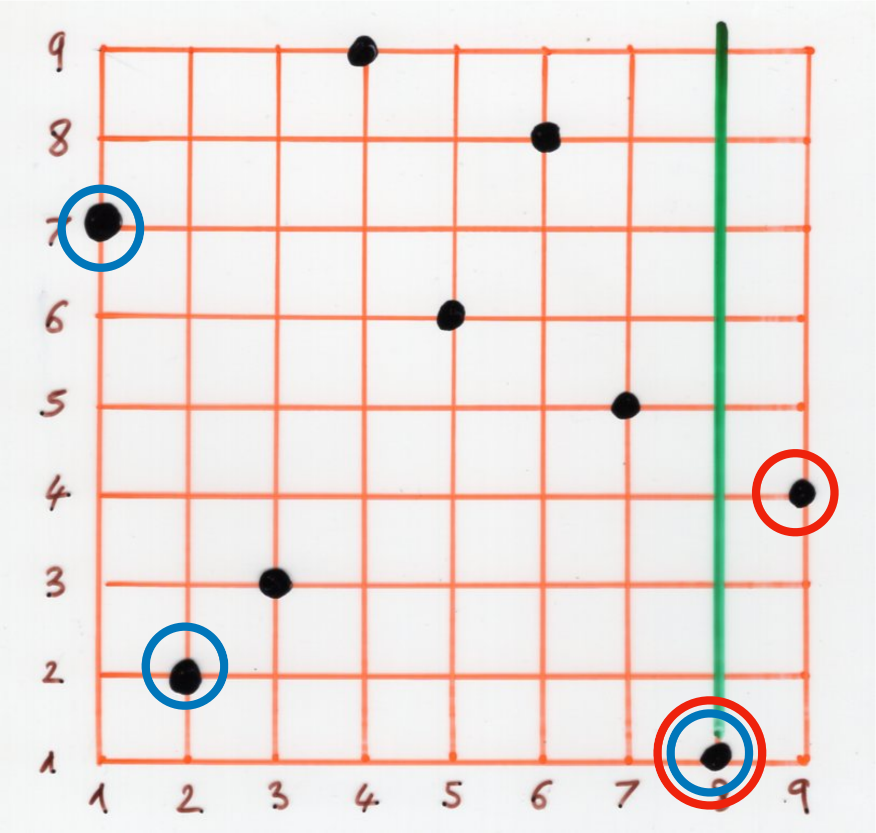
"special"

left-to-right

right-to-left

minimum

elements



"special"

$\sigma = 7\ 2\ 3\ 9\ 6\ 8\ 5\ 1\ 4$

permutation

word

left-to-right
right-to-left

minimum

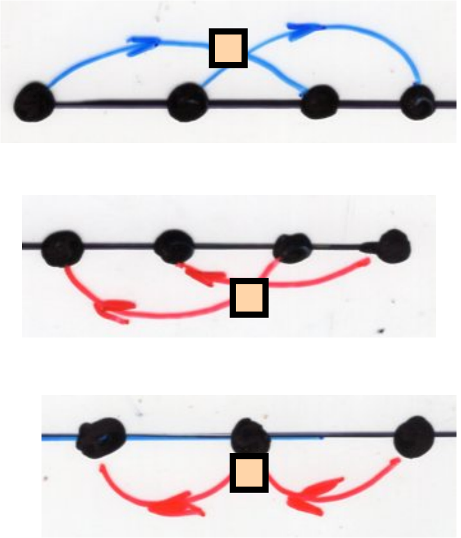
elements

9

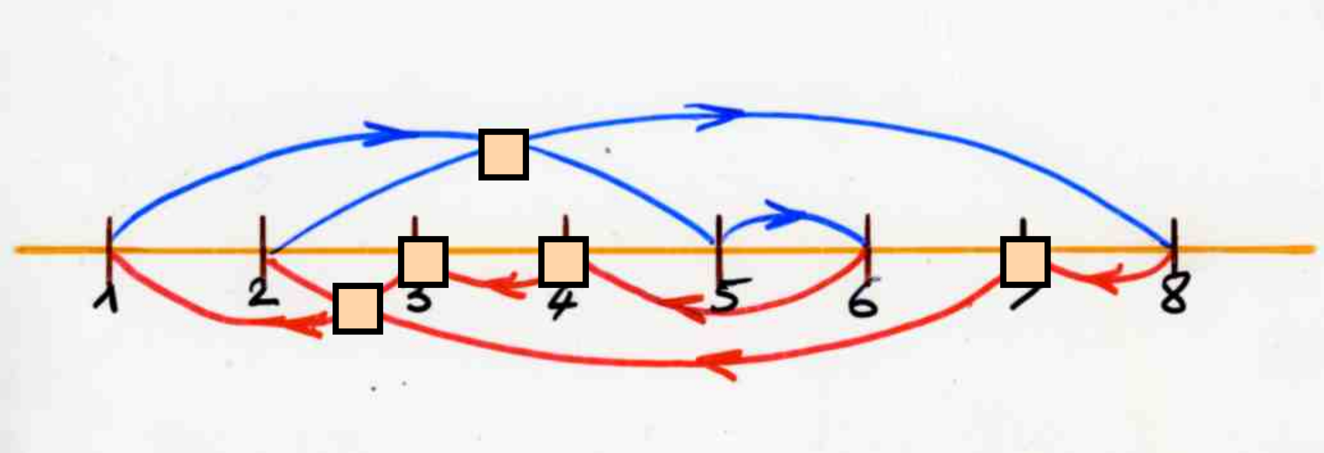
number of crossings of a permutation

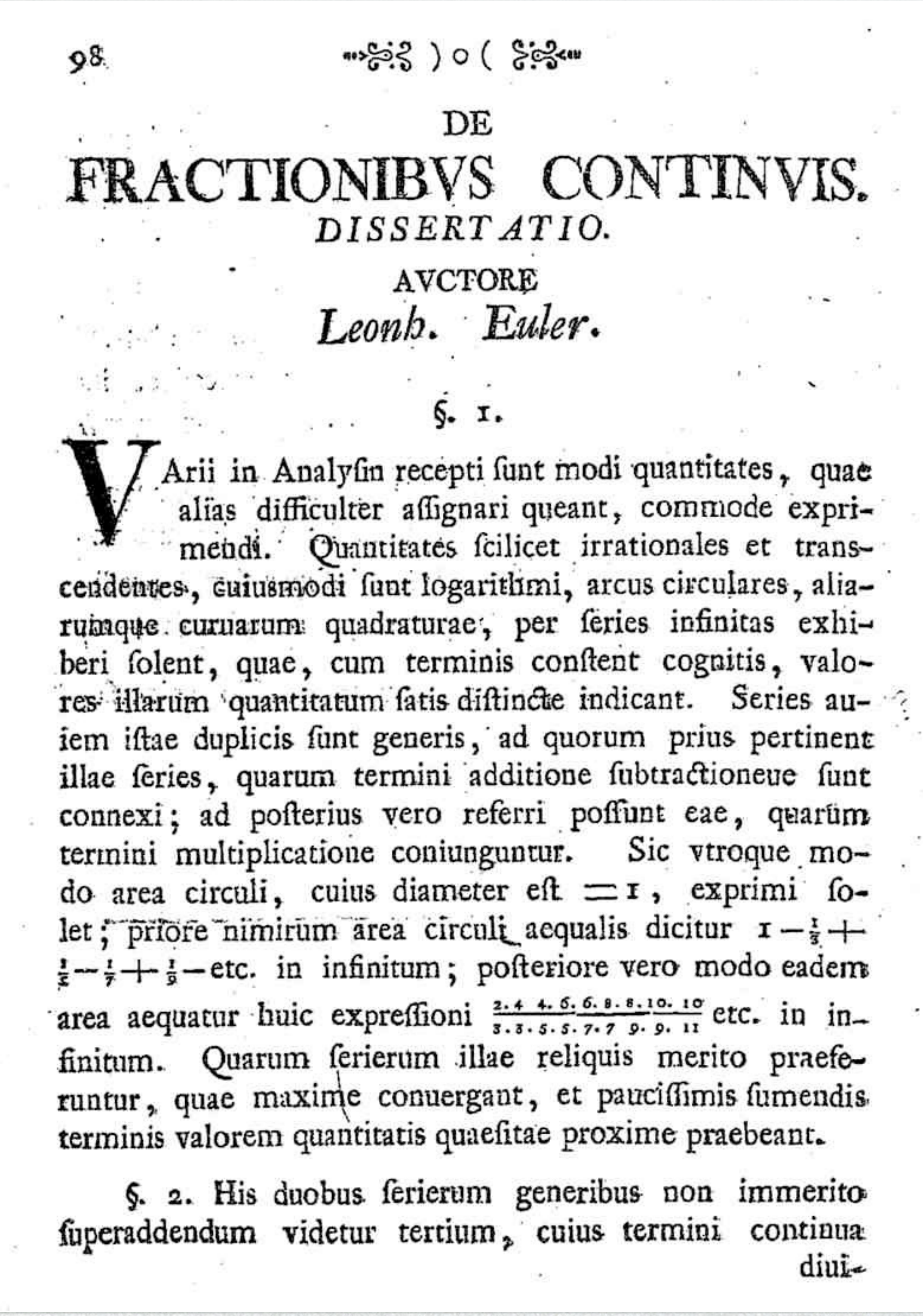
Corteel (2007)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 8 & 1 & 3 & 6 & 4 & 2 & 7 \end{pmatrix}$$



(strict) exceedances





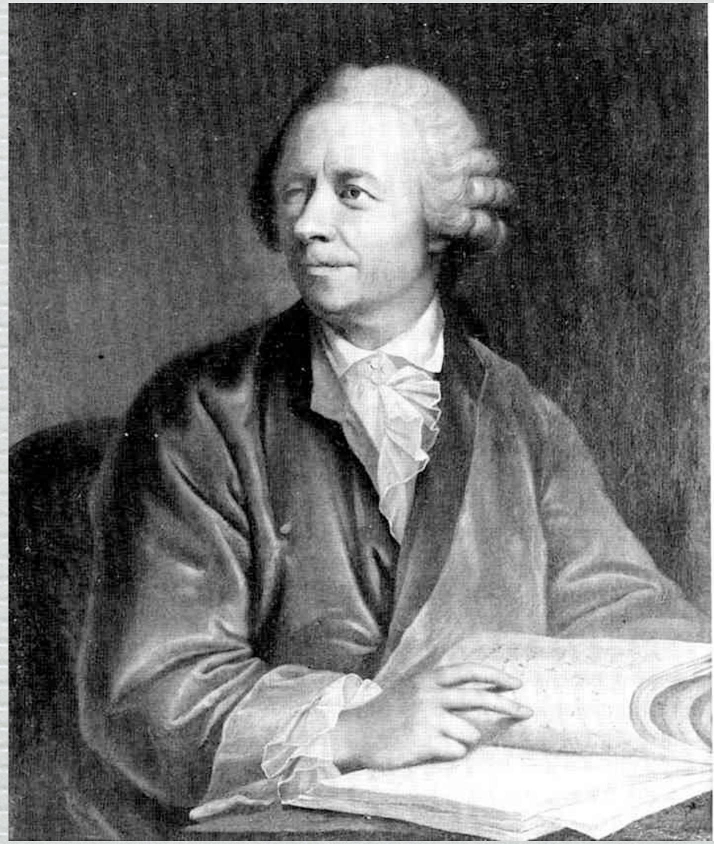
DE
FRACTIONIBVS CONTINVIS.
DISSERTATIO.

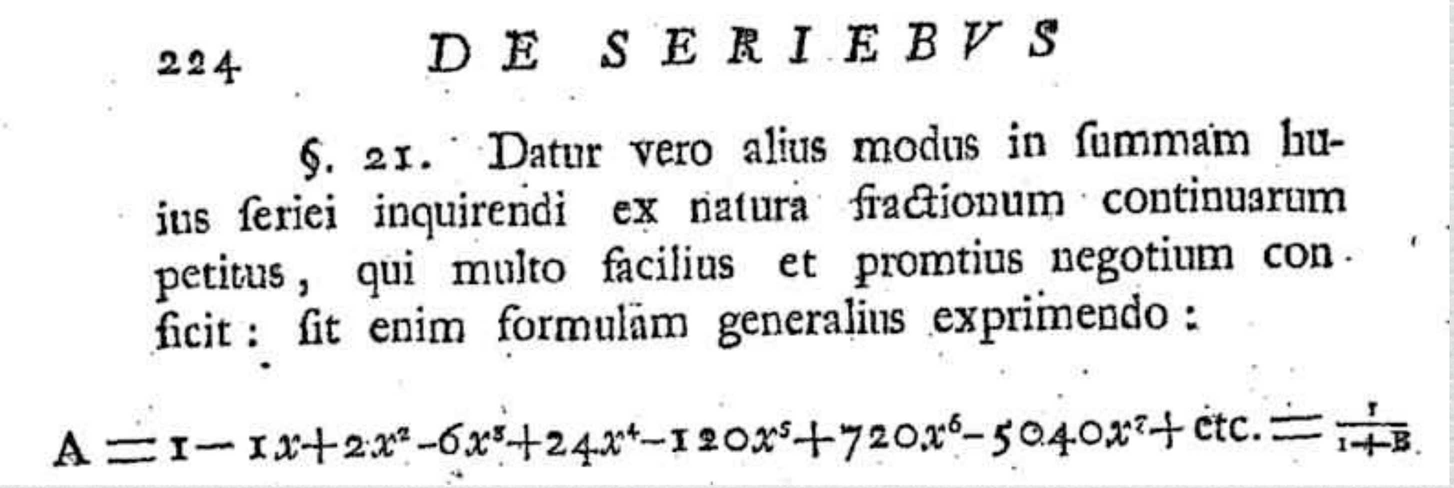
AVCTORE
Leonh. Euler.

§. 1.

Varii in Analyſin recepti ſunt modi quantitates, quae alias difficulter aſſignari queant, commode exprimendi. Quantitates ſcilicet irrationales et transcendentes, cuiusmodi ſunt logarithmi, arcus circulares, aliarumque curuarum quadraturae, per ſeries infinitas exhiberi ſolent, quae, cum terminis conſtent cognitis, valores illarum quantitatum ſatis diſtincte indicant. Series autem iſtae duplicis ſunt generis, ad quorum prius pertinent illae ſeries, quarum termini additione ſubtractioneue ſunt connexi; ad poſterius vero referri poſſunt eae, quarum termini multiplicatione coniunguntur. Sic vtroque modo area circuli, cuius diameter eſt $= 1$, exprimi ſolet; priore nimirum area circuli aequalis dicitur $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} -$ etc. in infinitum; poſteriore vero modo eadem area aequatur huic expreſſioni $\frac{2.4.4.6.6.8.8.10.10}{3.3.5.5.7.7.9.9.11}$ etc. in infinitum. Quarum ſerierum illae reliquis merito praeferuntur, quae maxime conuergant, et pauciſſimis ſumendis terminis valorem quantitatis quaeſitae proxime praebeant.

§. 2. His duobus ſerierum generibus non immerito ſuperaddendum videtur tertium, cuius termini continua diui-





§. 21. Datur vero alius modus in ſummam huius ſeriei inquirendi ex natura fractionum continuarum petitus, qui multo facilius et promtius negotium conficit: ſit enim formulam generalius exprimendo:

$$A = 1 - 1x + 2x^2 - 6x^3 + 24x^4 - 120x^5 + 720x^6 - 5040x^7 + \text{etc.} = \frac{1}{1+B}$$

$$A = \cfrac{1}{1+\cfrac{x}{1+\cfrac{x}{1+\cfrac{2x}{1+\cfrac{2x}{1+\cfrac{3x}{1+\cfrac{3x}{1+\cfrac{4x}{1+\cfrac{4x}{1+\cfrac{5x}{1+\cfrac{5x}{1+\cfrac{6x}{1+\cfrac{6x}{1+\cfrac{7x}{\text{etc.}}}}}}}}}}}}}}$$

§. 22. Quemadmodum autem huiusmodi fractio-

9

$$\lambda_k = \left\lceil \frac{k}{2} \right\rceil$$

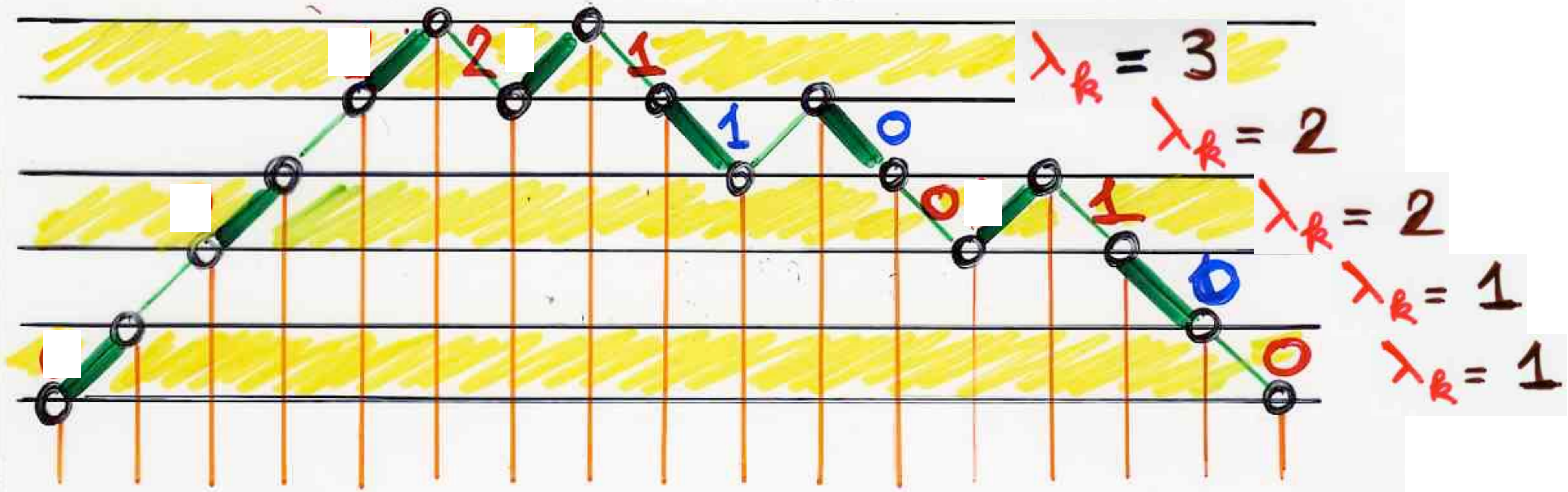
$$\sum_{n \geqslant 0} n!\, t^n = \cfrac{1}{1 - \cfrac{1t}{1 - \cfrac{1t}{1 - \cfrac{2t}{1 - \cfrac{2t}{1 - \cfrac{3t}{1 - \dots}}}}}}$$

$$\lambda_k = \left[\left\lceil \frac{k}{2} \right\rceil \right]_q$$

$$\sum_{n \geqslant 0} (n!)_q \, t^n = \cfrac{1}{1-(1)t \cfrac{}{1-(1)t \cfrac{}{1-(1+q)t \cfrac{}{1-(1+q)t \cfrac{}{1-(1+q+q^2)t \cfrac{}{1- \dots}}}}}}$$

subdivided
Laguerre
histories

$$\lambda_k = \left\lceil \frac{k}{2} \right\rceil$$



subdivided Laguerre history

bijection permutations
subdivided Laguerre histories

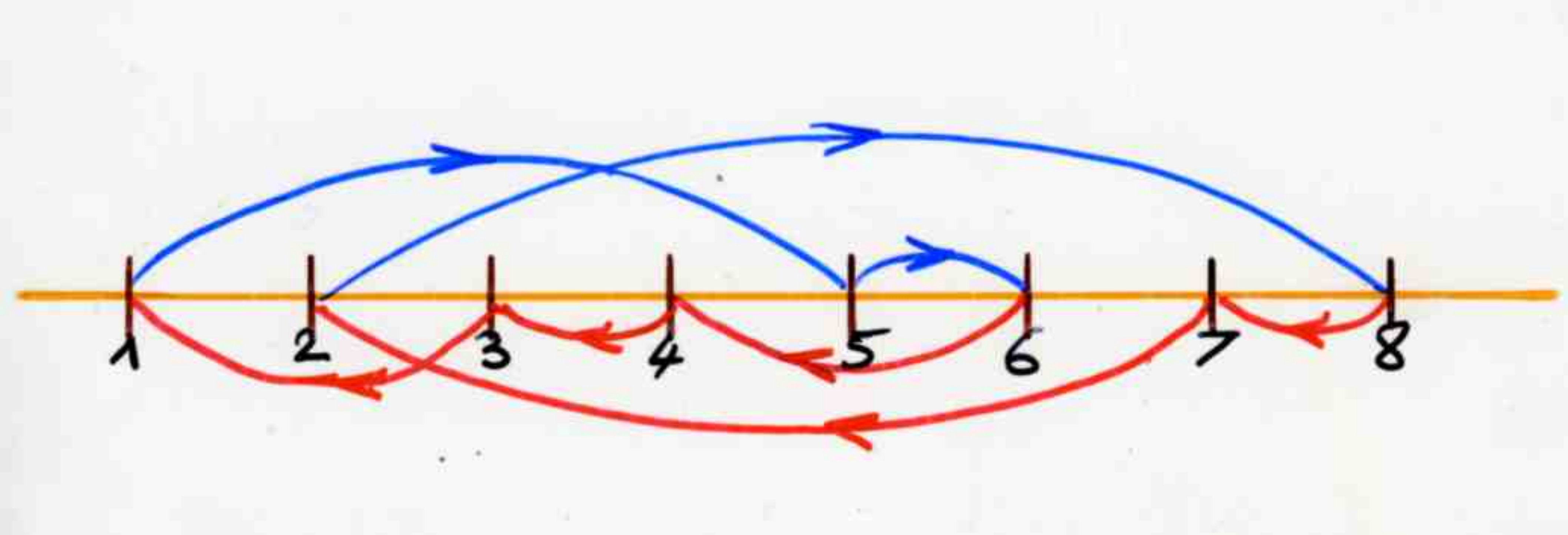
A. de Médicis, X.V.
(1994)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 8 & 1 & 3 & 6 & 4 & 2 & 7 \end{pmatrix}$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 8 & 1 & 3 & 6 & 4 & 2 & 7 \end{pmatrix}$$

(strict) exceedances

$$i < \sigma(i)$$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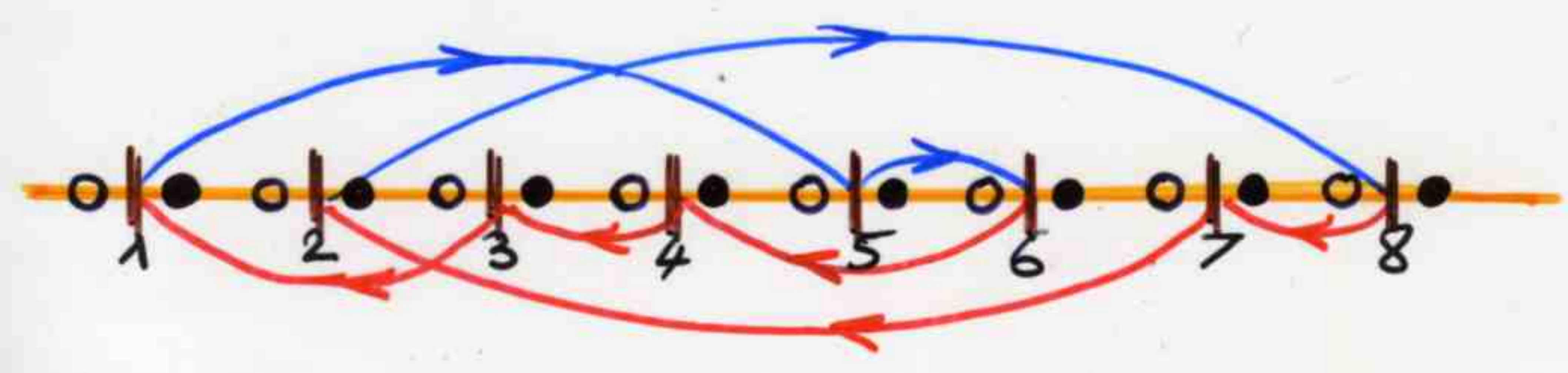
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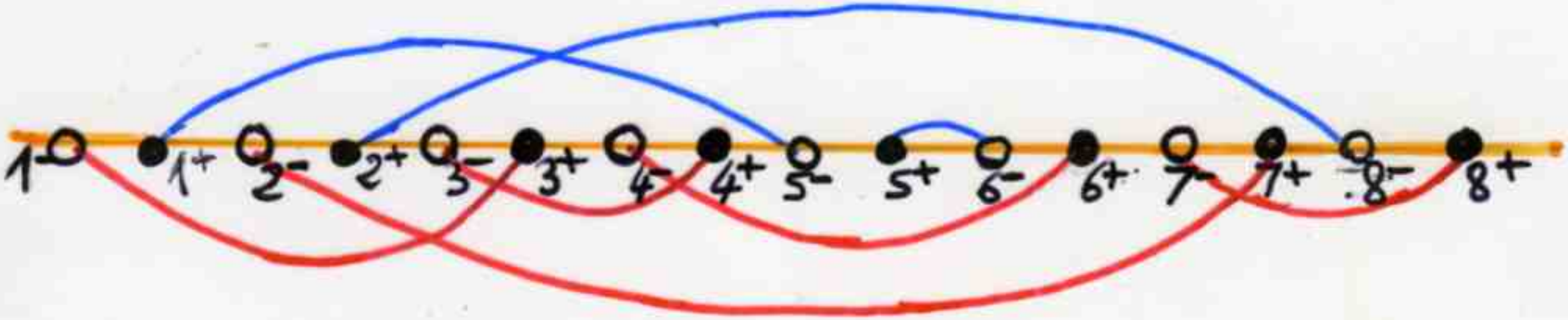
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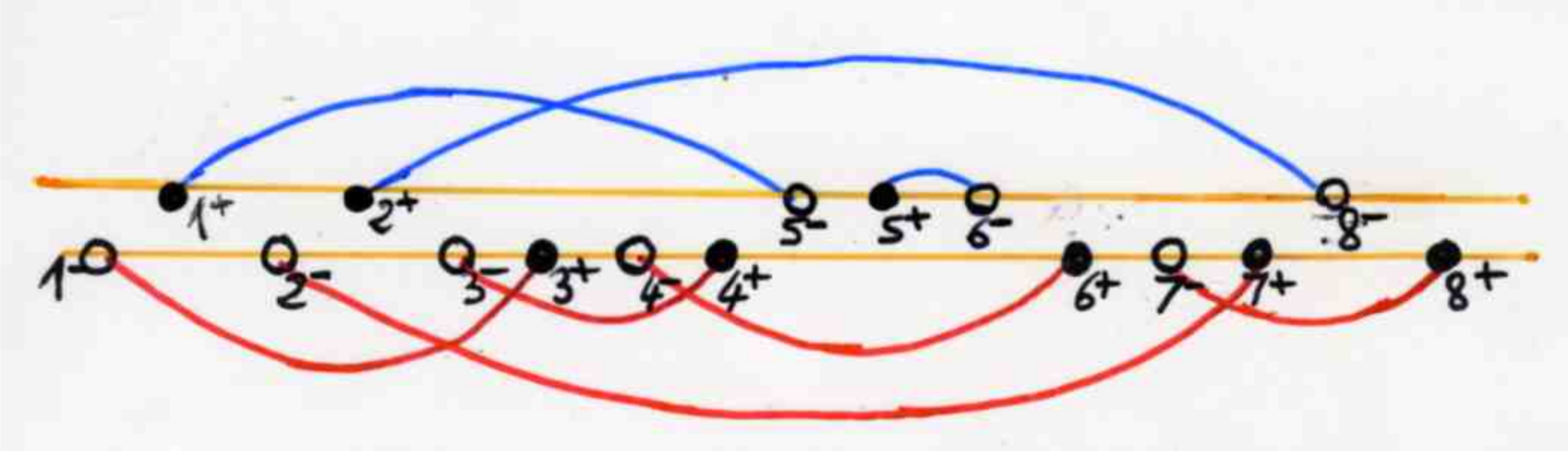
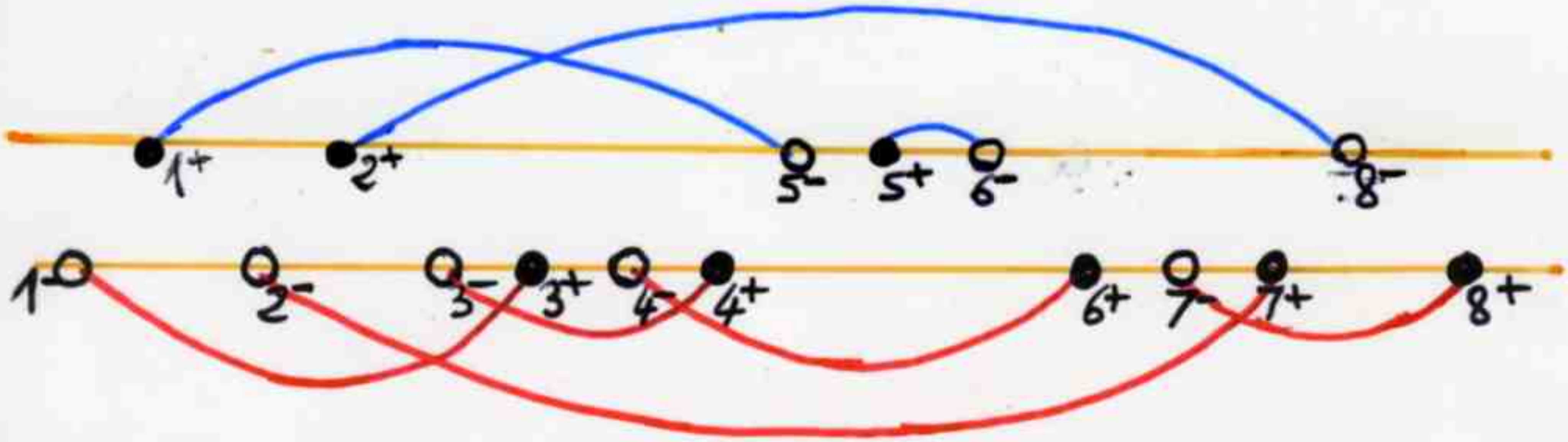
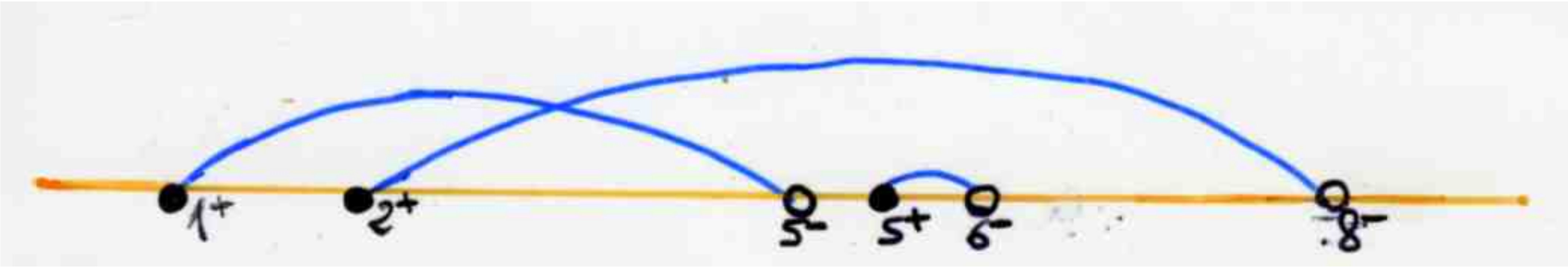
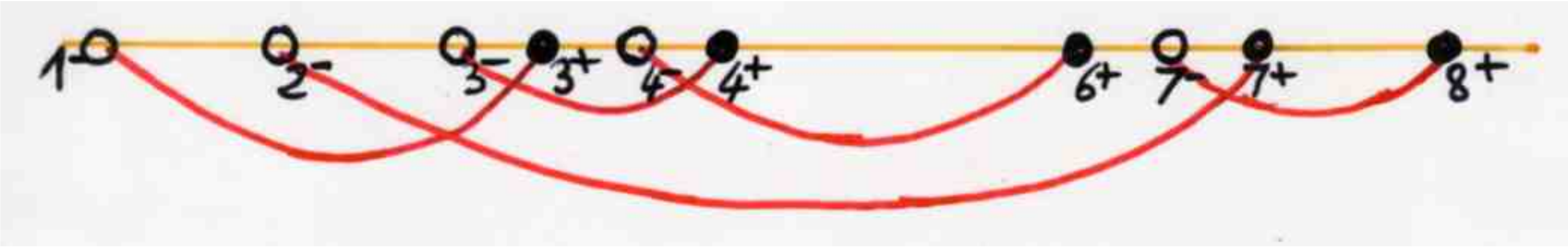
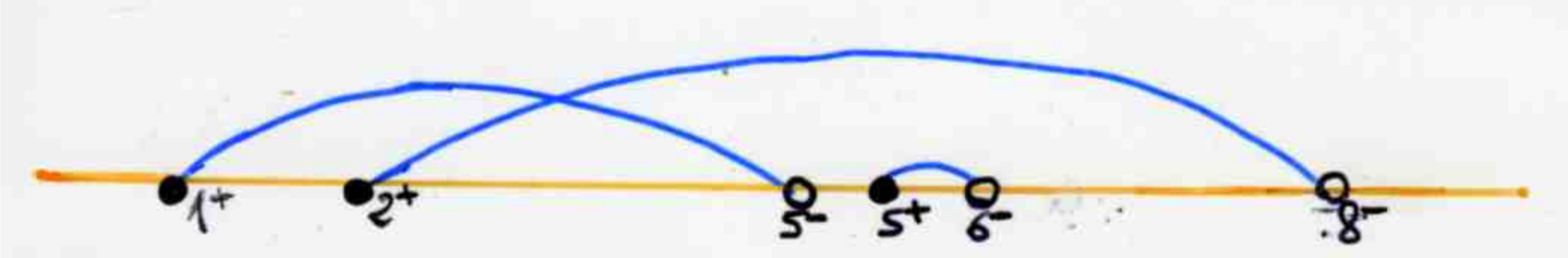
8−
8+

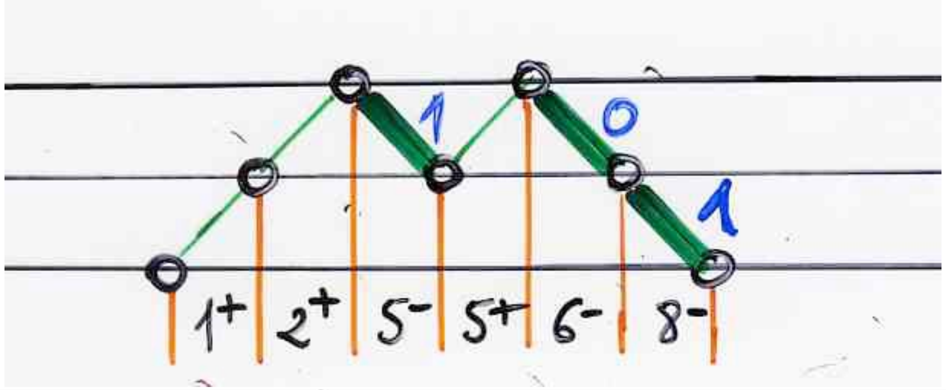

1+
2+
5-
5+
6-
8-
1-
2-
3-
3+
4-
4+
6+
7-
7+
8+

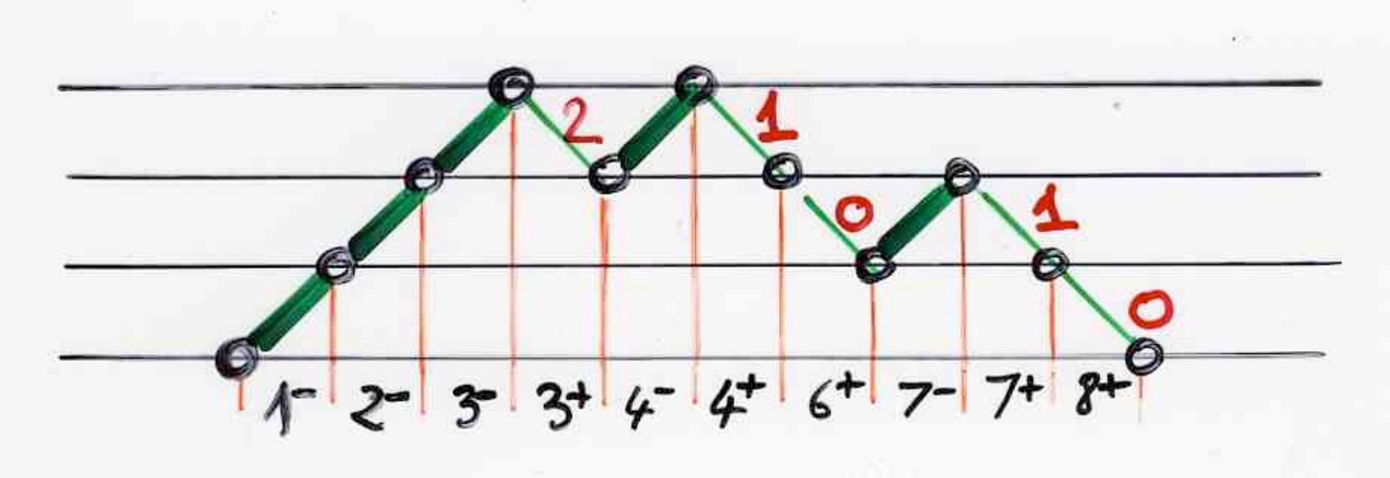

1+
2+
5-
5+
6-
8-
1-
2-
3-
3+
4-
4+
6+
7-
7+
8+

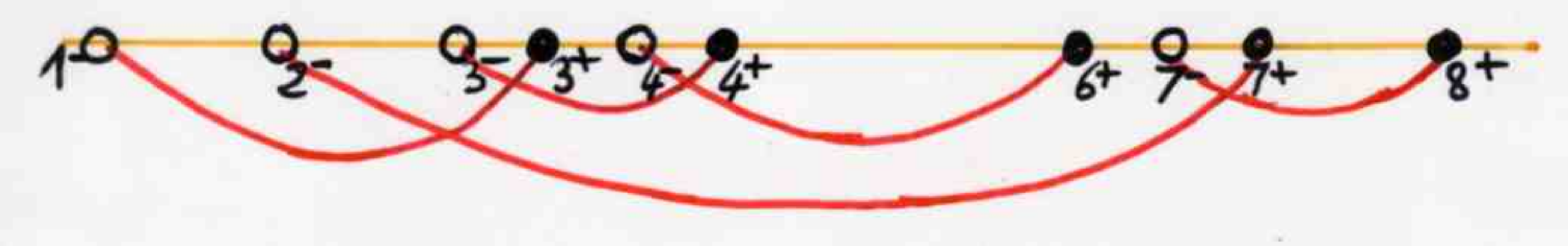

1+
2+
5-
5+
6-
8-


1-
2-
3-
3+
4-
4+
6+
7-
7+
8+


1+
2+
5-
5+
6-
8-


1
0
1
1+
2+
5-
5+
6-
8-


2
1
0
1
0
1-
2-
3-
3+
4-
4+
6+
7-
7+
8+


1-
2-
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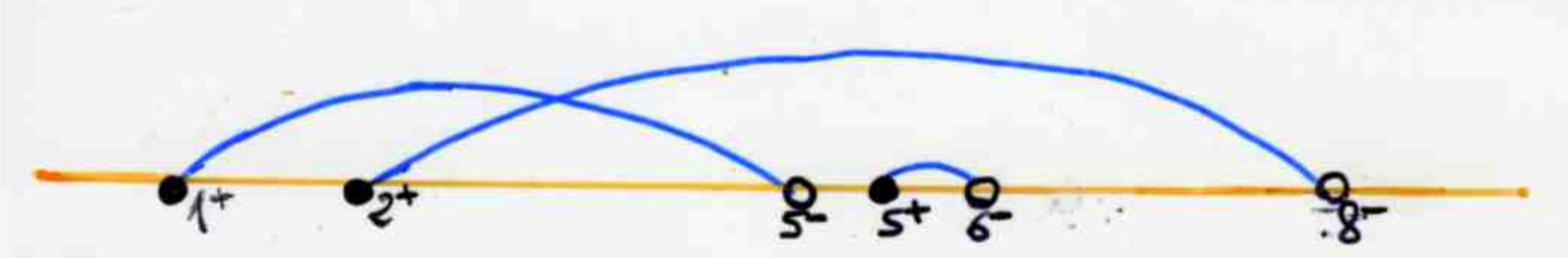
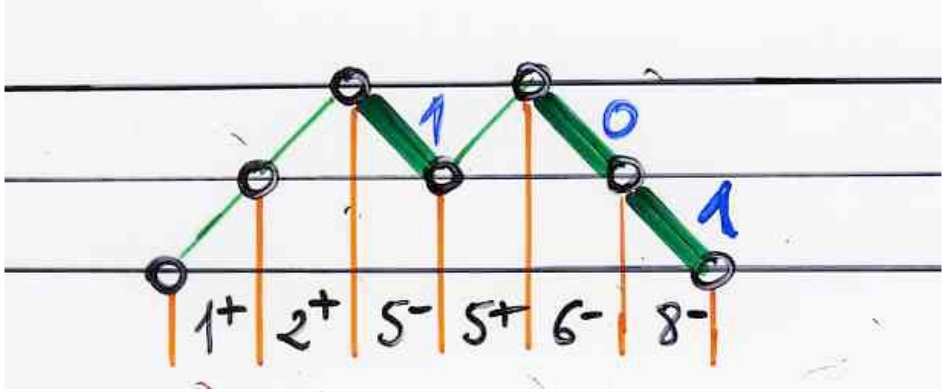
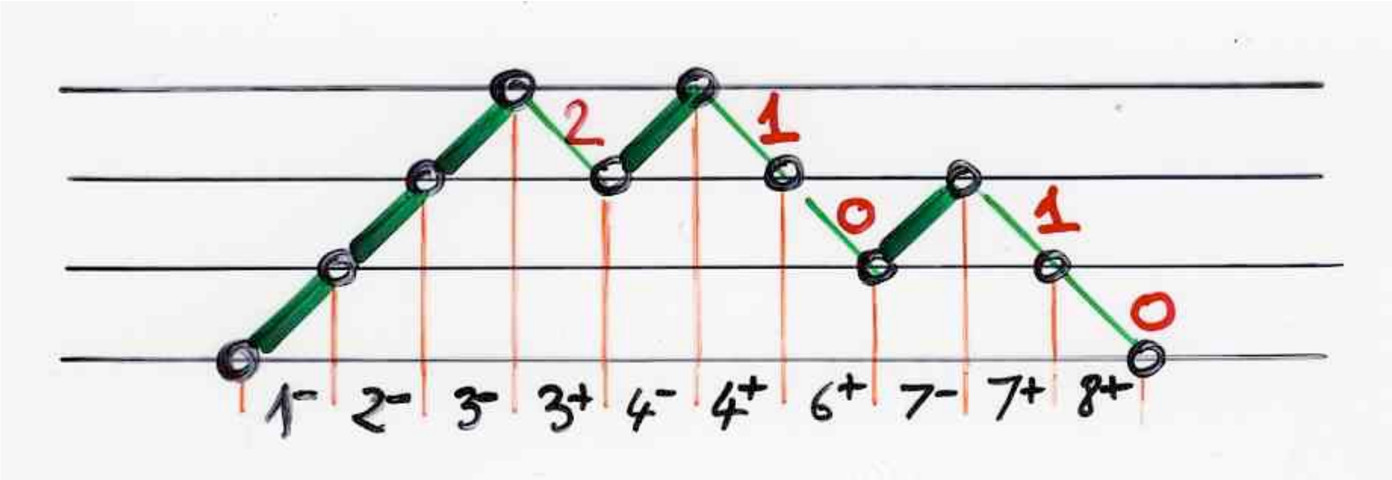
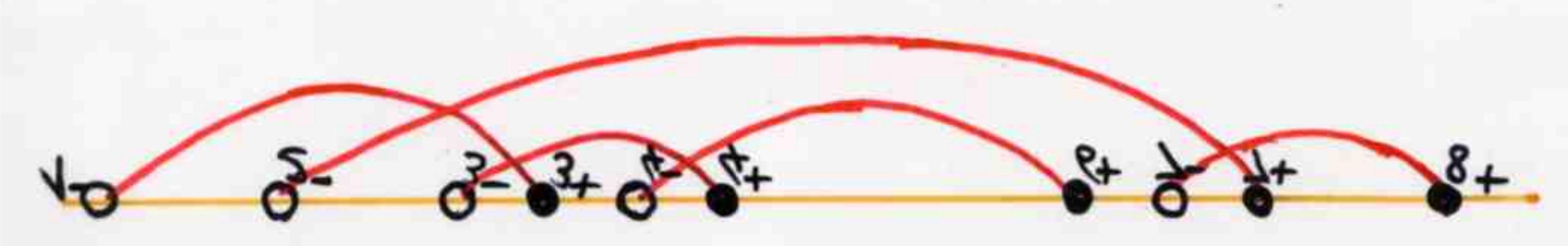
3+
4-
4+
6+
7-
7+
8+


1+
2+
5-
5+
6-
8-


1
0
1
1+
2+
5-
5+
6-
8-


2
1
0
1
0
1-
2-
3-
3+
4-
4+
6+
7-
7+
8+


1-
2-
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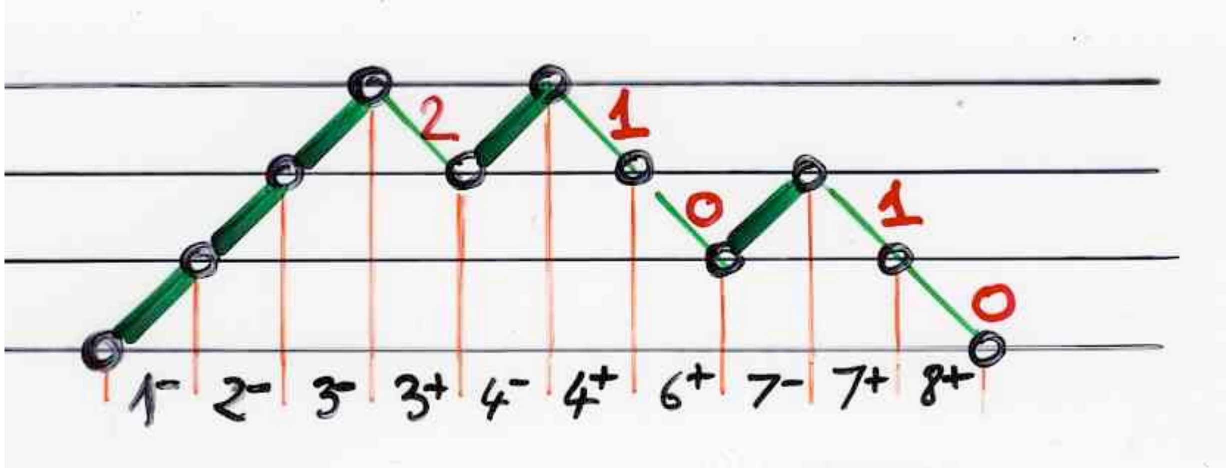
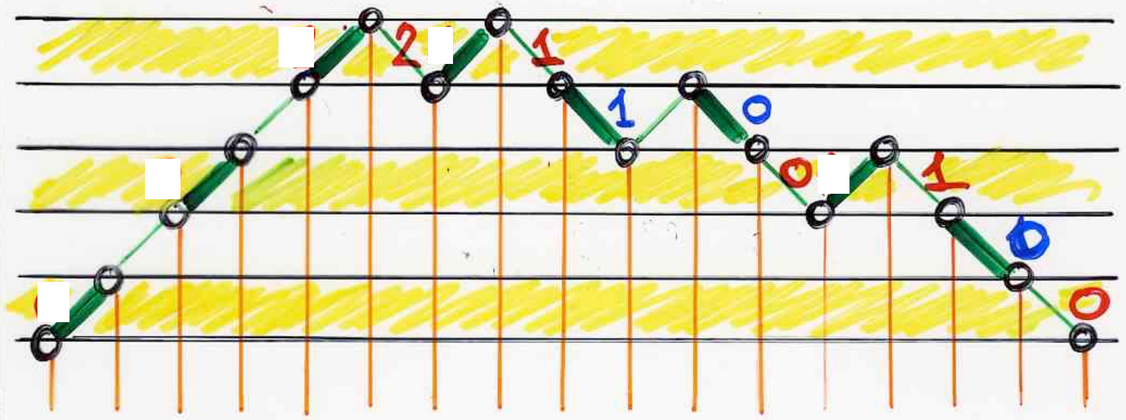
3+
4-
4+
6+
7-
7+
8+

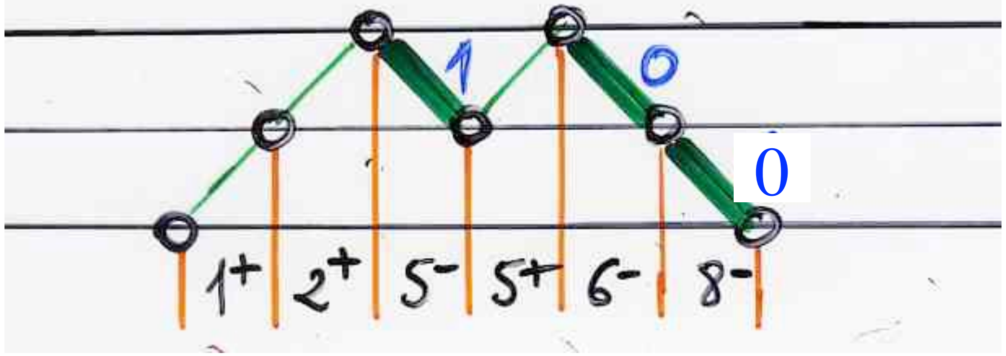


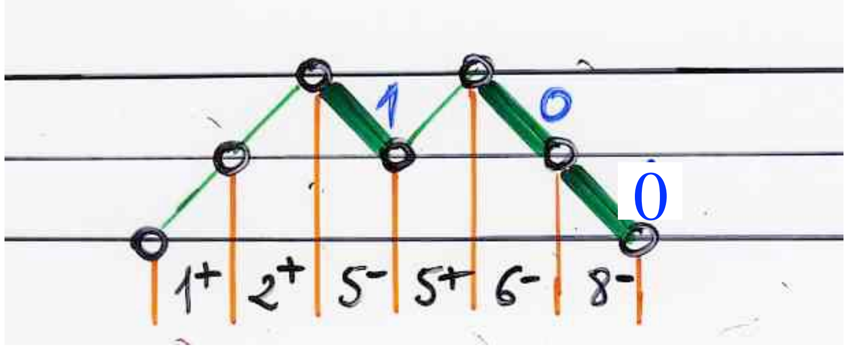
pair of two
Hermite histories
("shuffle")

=

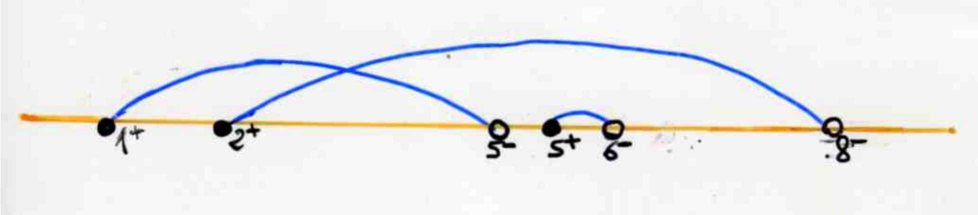


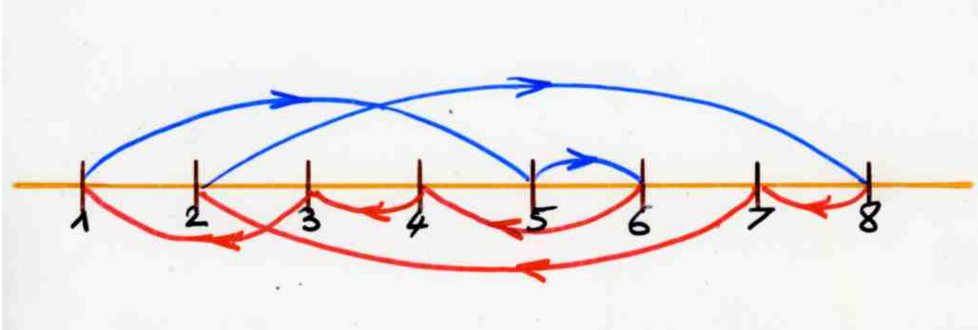
subdivided Laguerre history

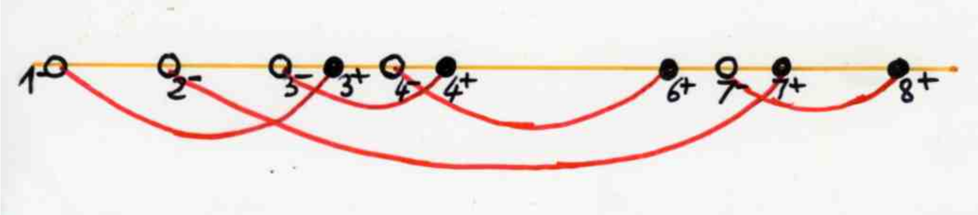


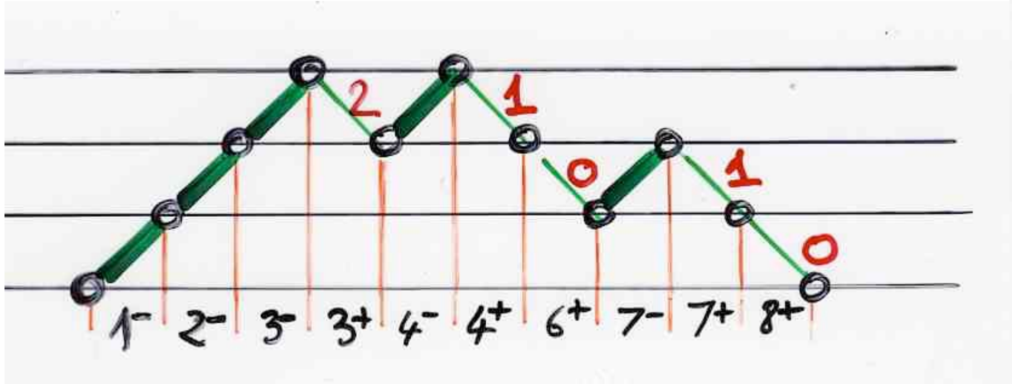


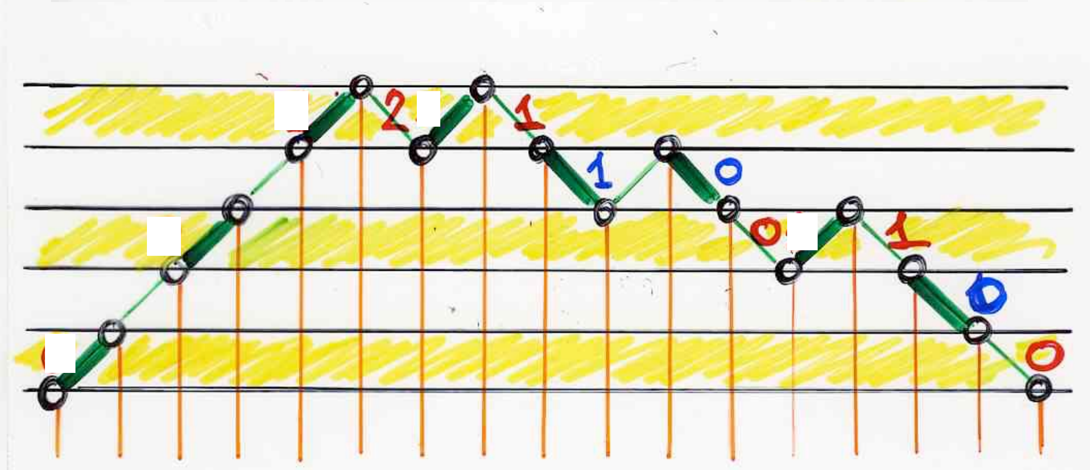
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 8 & 1 & 3 & 6 & 4 & 2 & 7 \end{pmatrix}$$



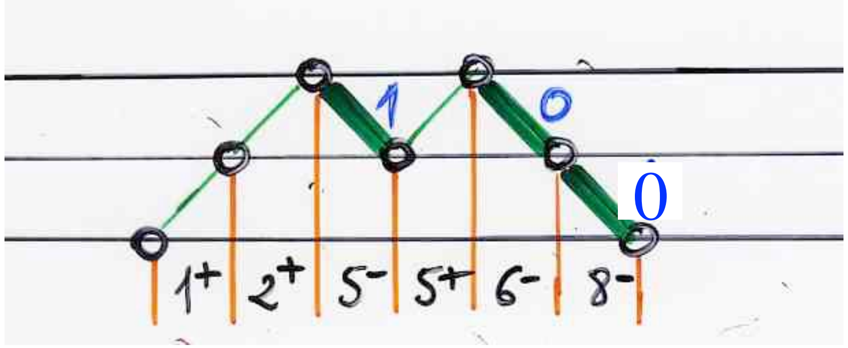




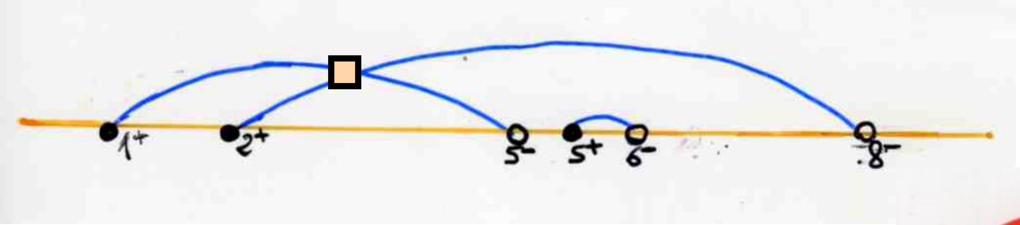


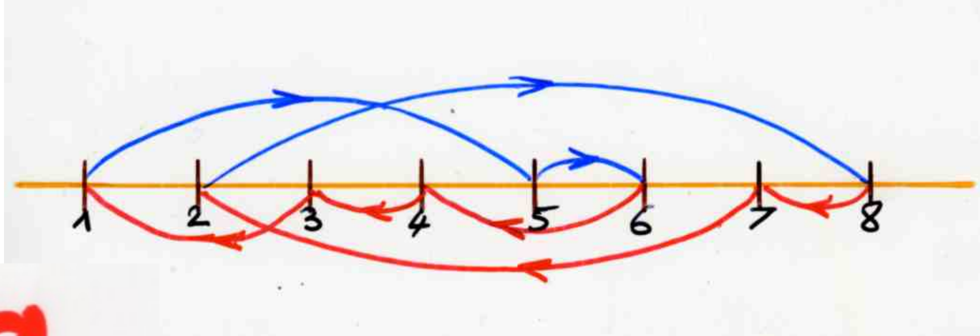


subdivided Laguerre history



$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 8 & 1 & 3 & 6 & 4 & 2 & 7 \end{pmatrix}$$

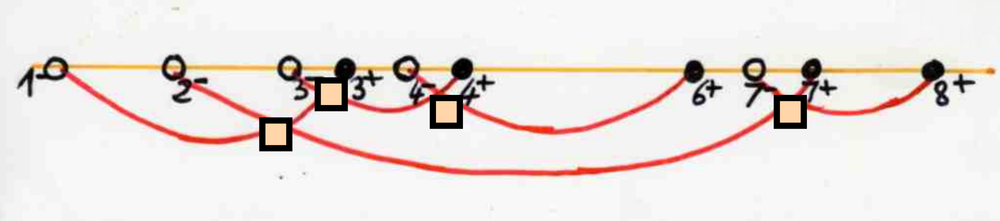


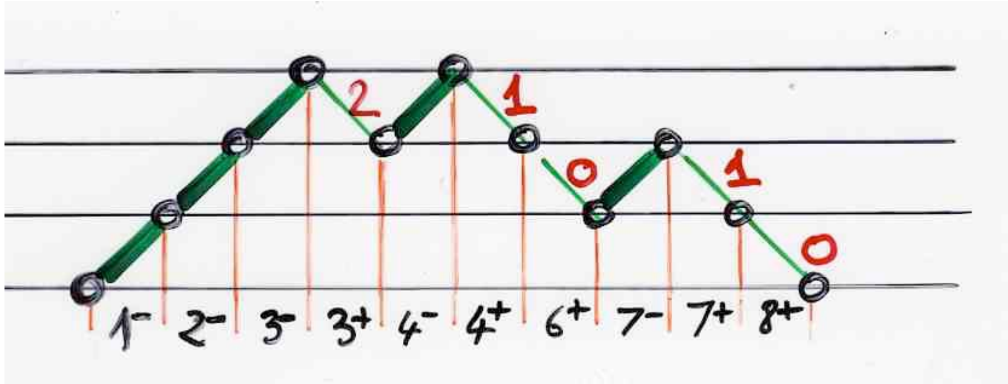


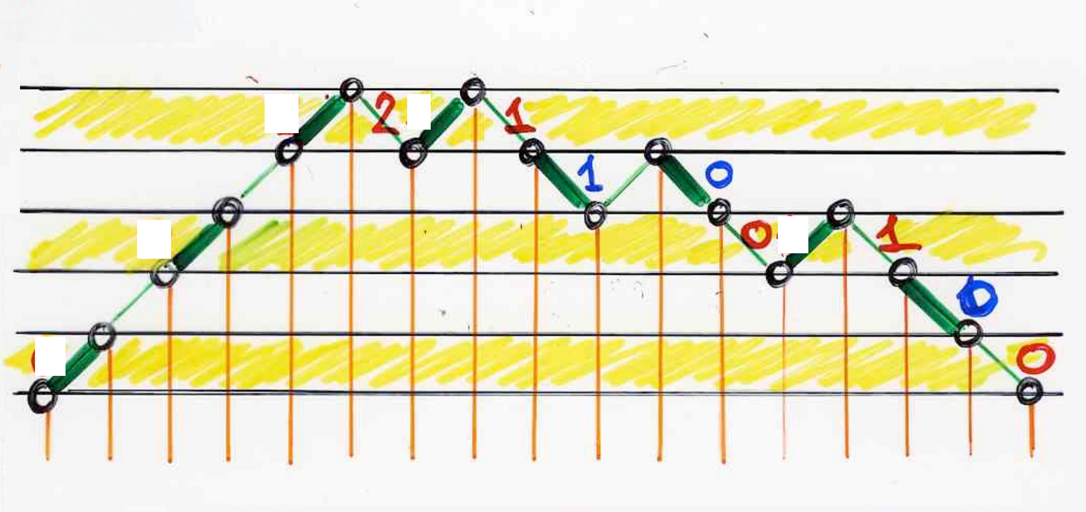
nb of crossings

9

nb of crossings







subdivided Laguerre history

$$\lambda_k = \left\lceil \frac{k}{2} \right\rceil$$

$$\sum_{n\geqslant 0} n!\, t^n = \cfrac{1}{1-\cfrac{1t}{1-\cfrac{1t}{1-\cfrac{2t}{1-\cfrac{2t}{1-\cfrac{3t}{1-\ldots}}}}}}$$

$$\lambda_k = \left[\left\lceil \frac{k}{2} \right\rceil \right]_q$$

$$\sum_{n \geq 0} (n!)_q \, t^n = \cfrac{1}{1-(1)t \cfrac{}{1-(1)t \cfrac{}{1-(1+q)t \cfrac{}{1-(1+q)t \cfrac{}{1-(1+q+q^2)t \cfrac{}{1-\ \ldots}}}}}}$$

Pairs of Hermite histories

permutations τ

exceedances

permutation tableaux

subdivided Laguerre histories

q

Interpretation of the 3-parameters
Partition function

q, α, β

Second bijection: tableaux— permutations

bijection
Corteel, Nadeau (2007)

equivalent to

"exchange-fusion"
"exchange-delete"
algorithm X.V. (2007)

~~"special"~~

q, α, β

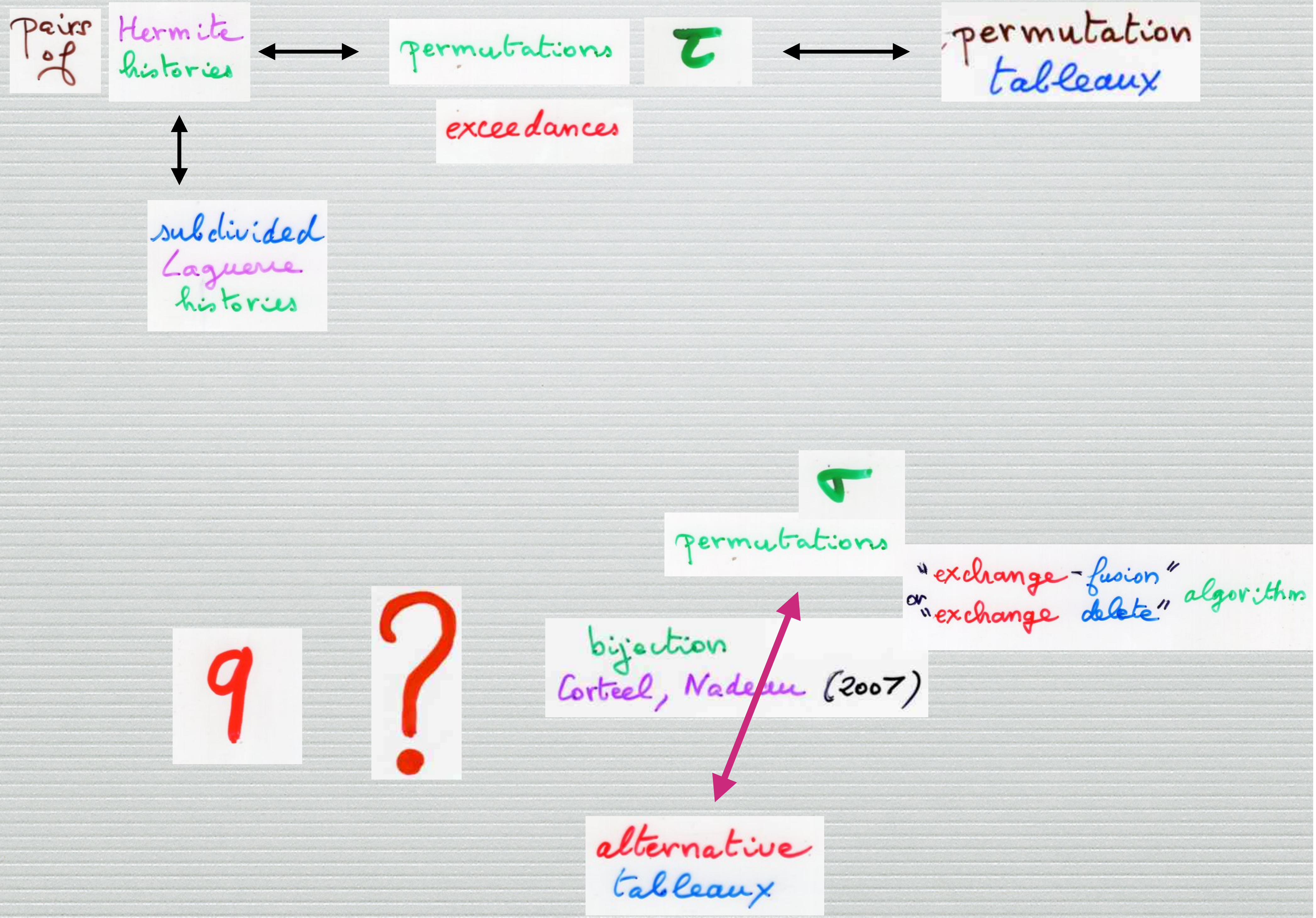
left-to-right
right-to-left

minimum

elements

generating polynomial:

$$xy(x+y)(x+1+y)\ldots\ldots(x+n-1+y)$$



Pairs of Hermite histories

permutations τ

permutation tableaux

exceedances

subdivided Laguerre histories

σ

permutations

"exchange-fusion" or "exchange delete" algorithm

bijection Corteel, Nadeau (2007)

alternative tableaux

q ?

q, α, β

Josuat-Vergès (2011)

$$\bar{Z}_N = Z_N(\alpha^{-1}, \beta^{-1}, q)$$

Josuat-Vergès (2011)

Proposition

$$\bar{Z}_N = \sum_{\sigma \in \mathfrak{S}_{N+1}} \alpha^{s(\sigma)-1} \beta^{t(\sigma)-1} q^{31\text{-}2(\sigma)}$$

$s(\sigma)$

$t(\sigma)$

$31\text{-}2(\sigma)$

$s(\sigma) =$ number right-to-left maxima

$t(\sigma) =$ number right-to-left minima

$31\text{-}2(\sigma) =$ number of patterns 31-2

$$\bar{Z}_N = Z_N(\alpha^{-1}, \beta^{-1}, q)$$

Josuat-Vergès (2011)

Proposition

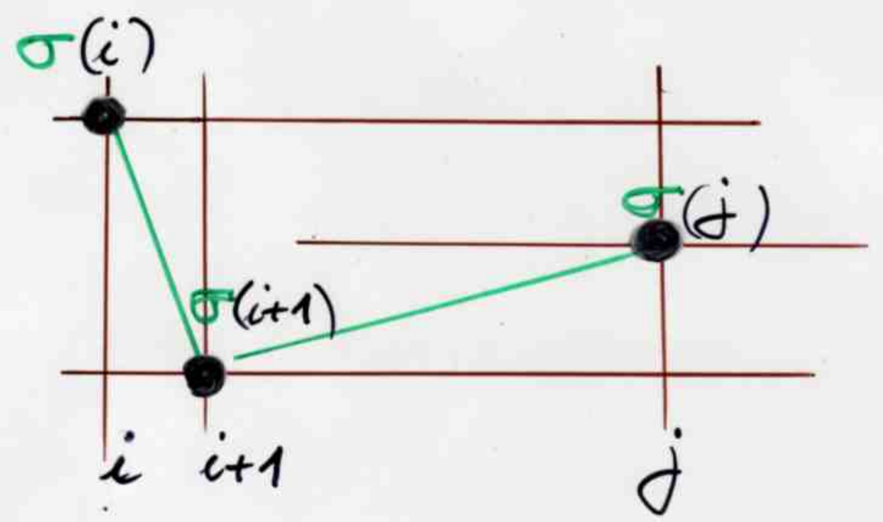
$$\bar{Z}_N = \sum_{\sigma \in \mathfrak{S}_{N+1}} \alpha^{s(\sigma)-1} \beta^{t(\sigma)-1} q^{31\text{-}2(\sigma)}$$

$s(\sigma)$

$t(\sigma)$

$31\text{-}2(\sigma)$

31-2



$$\bar{Z}_N = Z_N(\alpha^{-1}, \beta^{-1}, q)$$

Josuat-Vergès (2011)

Proposition

$$\bar{Z}_N = \sum_{\sigma \in \mathfrak{S}_{N+1}} \alpha^{s(\sigma)-1} \beta^{t(\sigma)-1} q^{31\text{-}2(\sigma)}$$

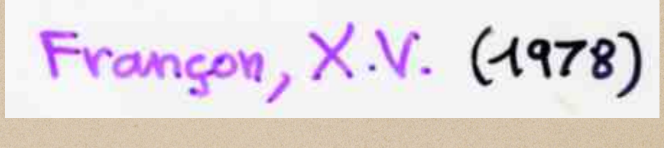
- Steingrímsson-Williams
- reverse - complement - inverse
- Foata-Zeilberger
- Françon-V.

Al-Salam-Chihara polynomials

$$2x\,Q_n(x) = Q_{n+1}(x) + (a+b)q^n Q_n(x) + (1-q^n)(1-abq^{n-1})Q_{n-1}(x)$$

Laguerre histories

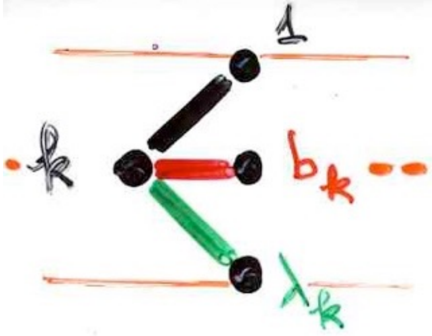
The FV bijection

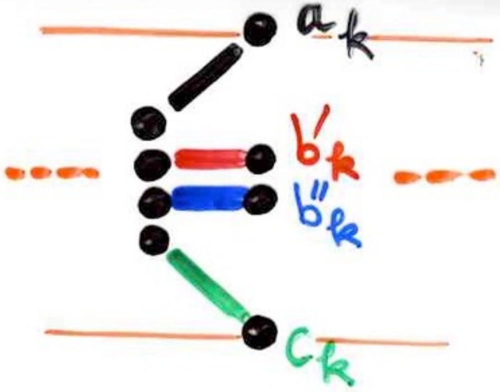

Françon, X.V. (1978)

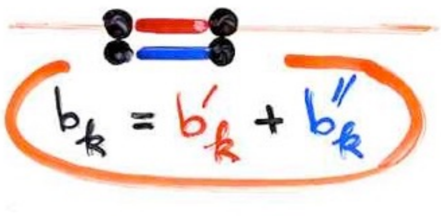
Laguerre polynomial

$$b_k = (2k+2)$$
$$\lambda_k = k(k+1)$$

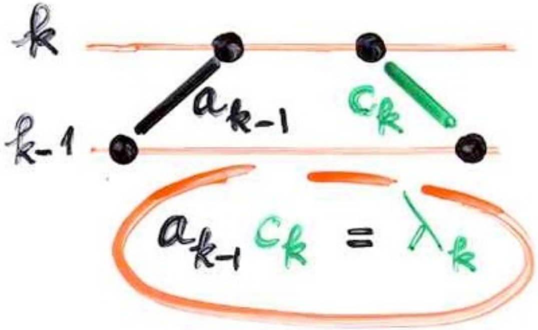
$$\mu_n = (n+1)!$$







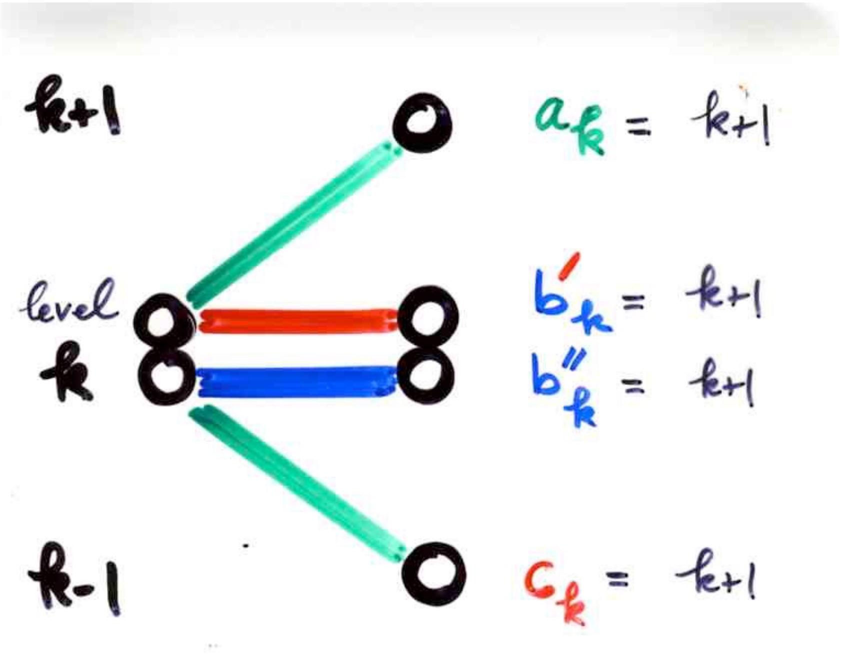
$$b_k = b'_k + b''_k$$

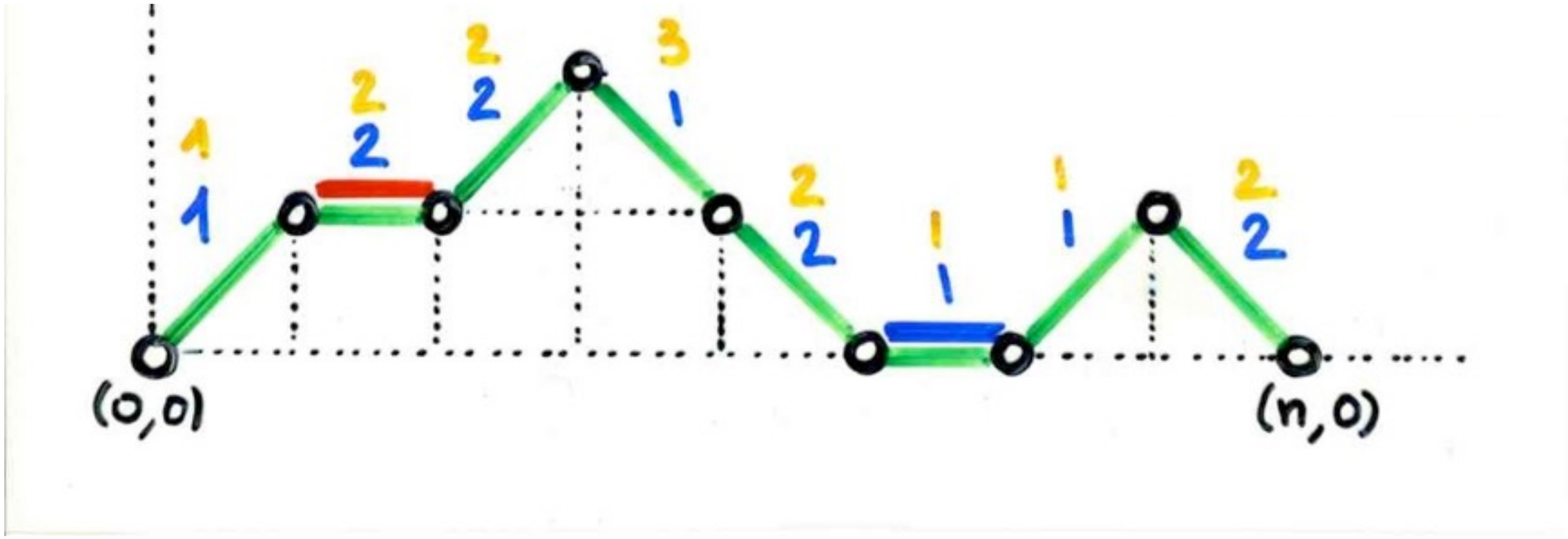


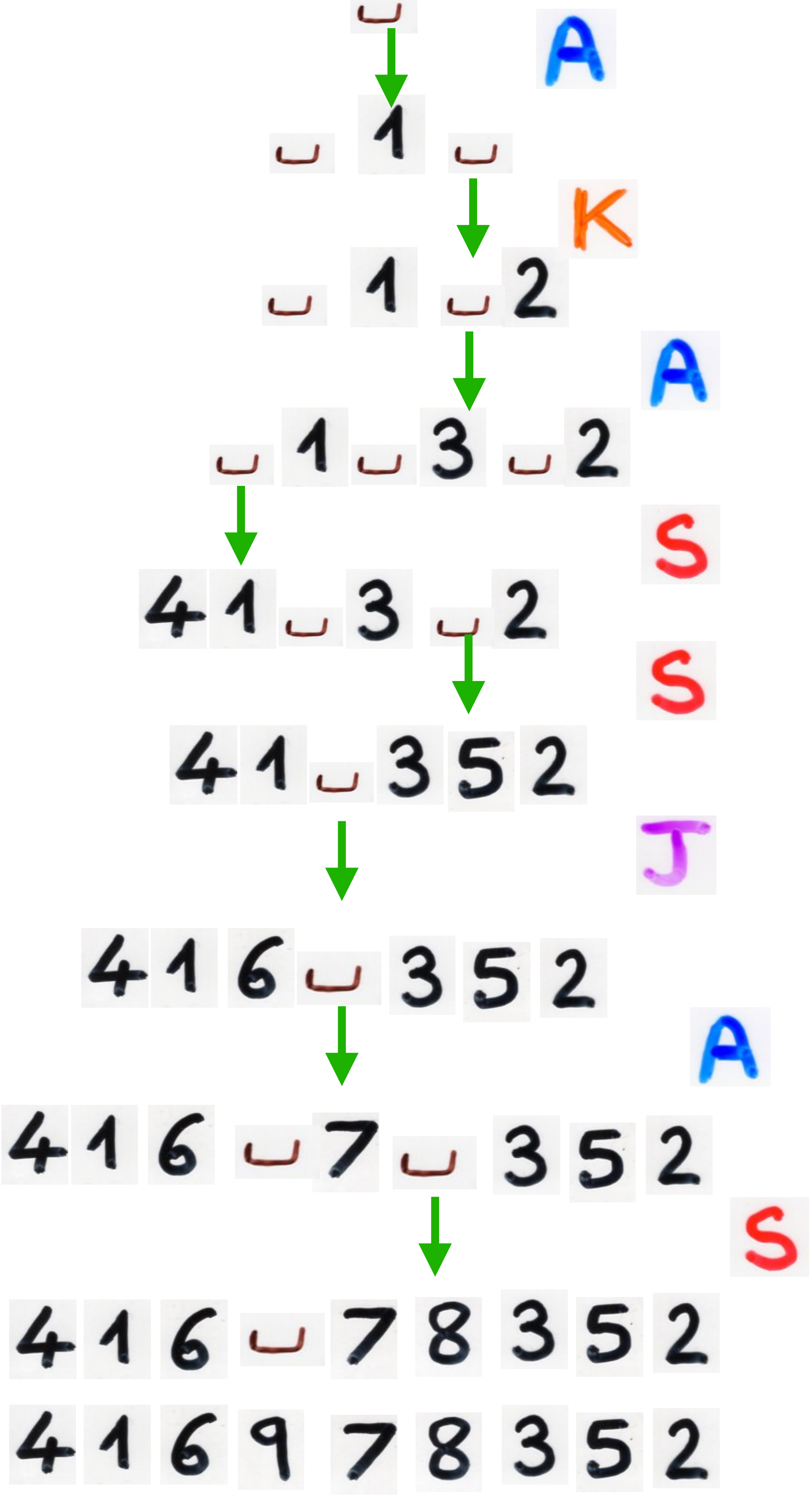
$$a_{k-1} c_k = \lambda_k$$

$$b_k = (2k+2)$$

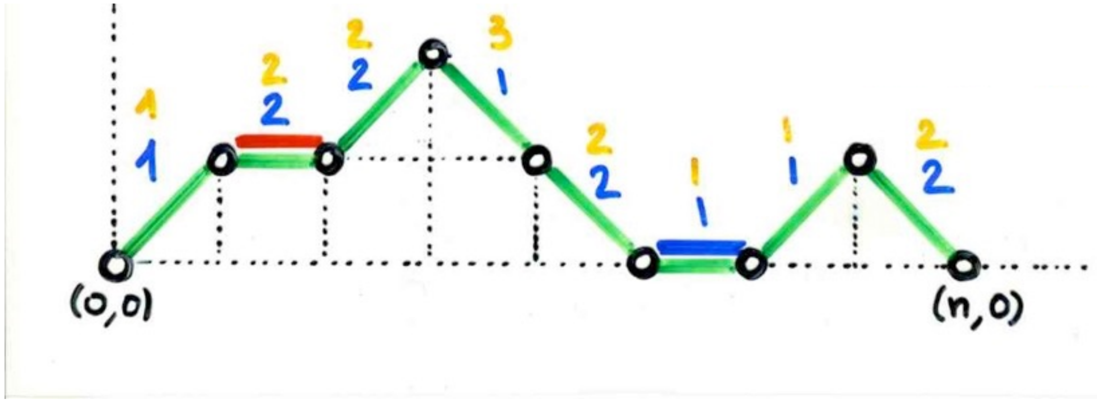
$$\lambda_k = k(k+1)$$


k+1
level
k
k-1
$a_k = k+1$
$b'_k = k+1$
$b''_k = k+1$
$c_k = k+1$


1
1
2
2
2
2
3
1
2
2
1
1
1
1
2
2
(0,0)
(n,0)



Laguerre histories

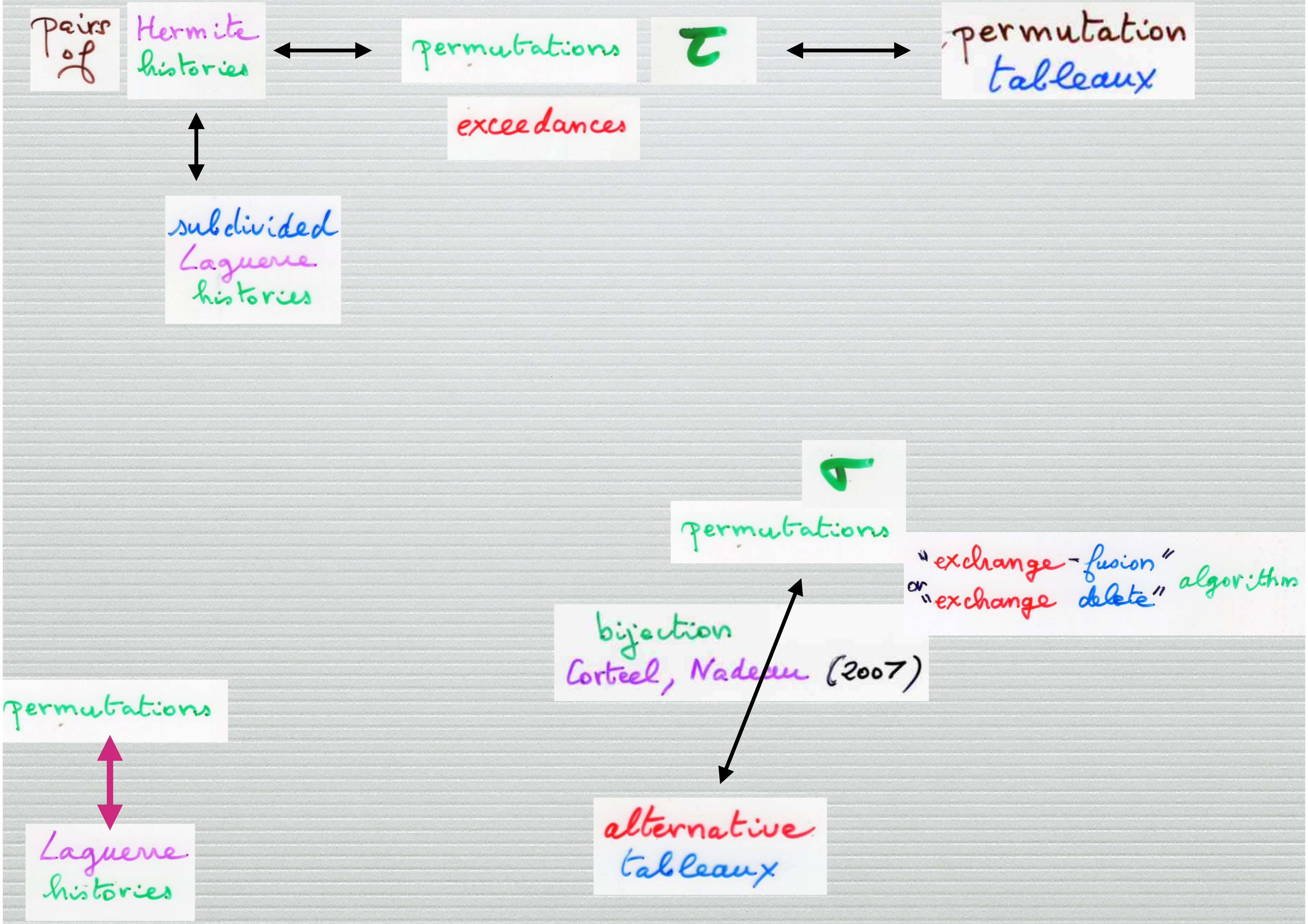


$$\langle k|\, A = (k+1)\, \langle (k+1)|$$

$$\langle k|\, K = (k+1)\, \langle k|$$

$$\langle k|\, J = (k+1)\, \langle k|$$

$$\langle k|\, S = (k+1)\, \langle (k-1)|$$



Pairs of Hermite histories ↔ permutations τ ↔ permutation tableaux

exceedances

subdivided Laguerre histories

σ permutations

“exchange-fusion” or “exchange delete” algorithm

bijection Corteel, Nadeau (2007)

alternative tableaux

permutations

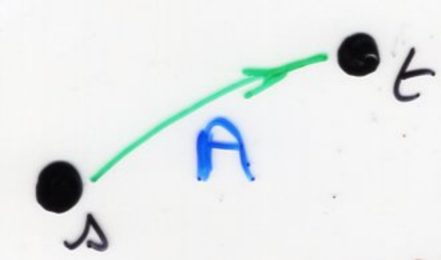
Laguerre histories

The essence of the parameter 31-2

the philosophy of « histories »

and its q-analogues

S states



operators

history

weight

$v_A(s,t)$ = number of possibilities to apply A

$$H = h_1 h_2 \cdots h_n$$

sequence of operators

initial state s_0

$$P = (p_1, p_2, \ldots, p_n)$$

$$s_i \xrightarrow{h_i} s_{i+1}$$

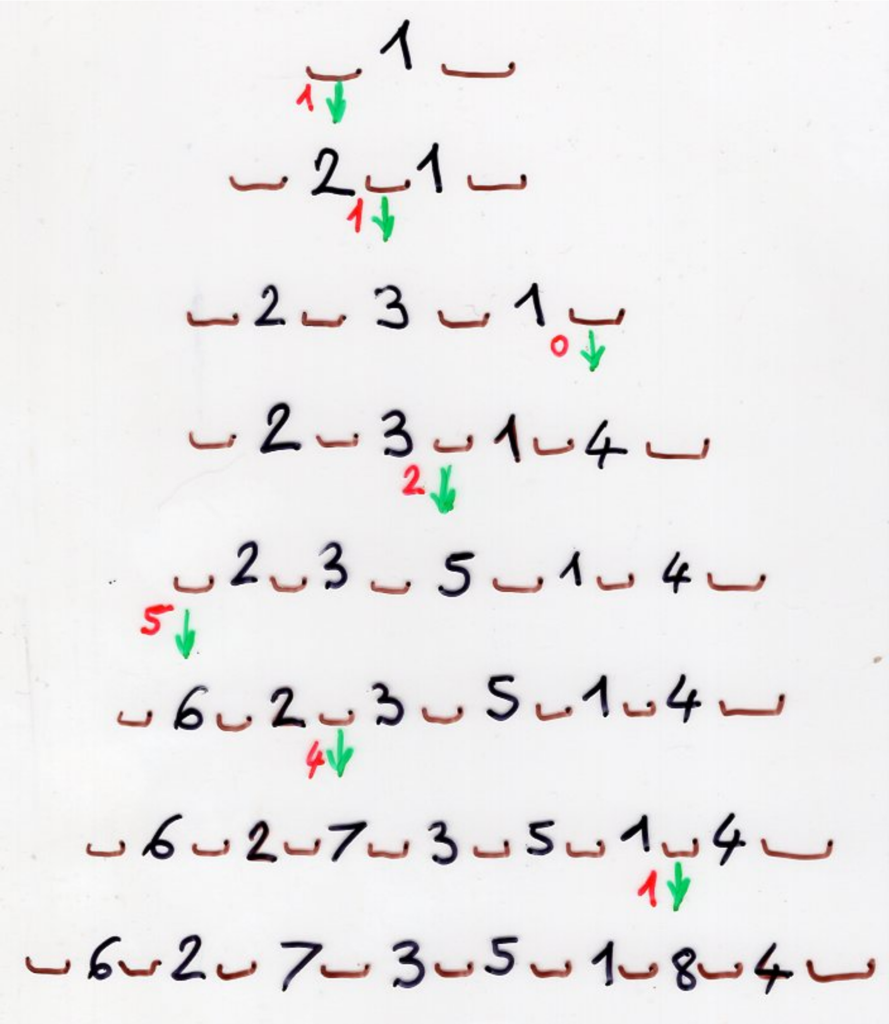
$$0 \leq p_i < v_{h_i}(s_i, s_{i+1})$$

q- weight

$$v_q(H) = q^{\left(\sum_{i=1}^{n} p_i\right)}$$

Inv

number of inversions



Maj

Major index

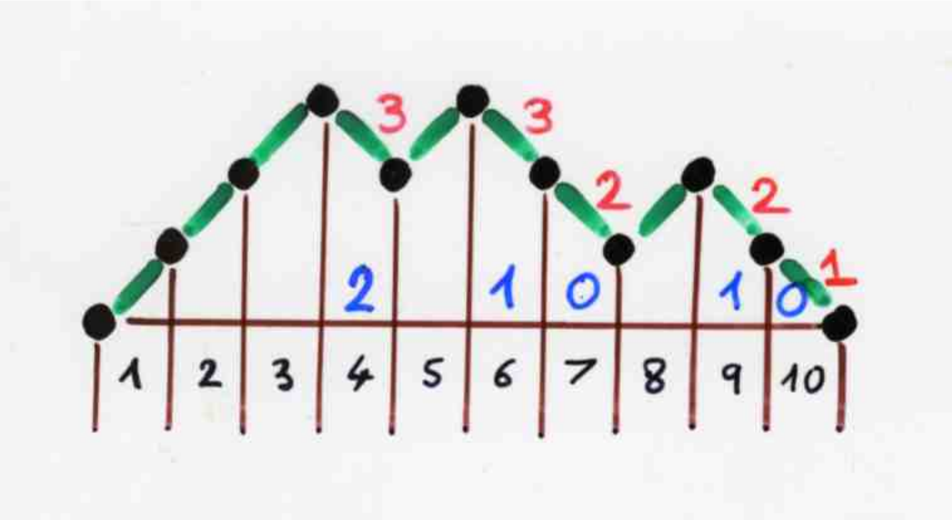
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$

$$\mathrm{maj}(\sigma) = \sum_{\substack{i \\ \sigma(i) > \sigma(i+1)}} i$$

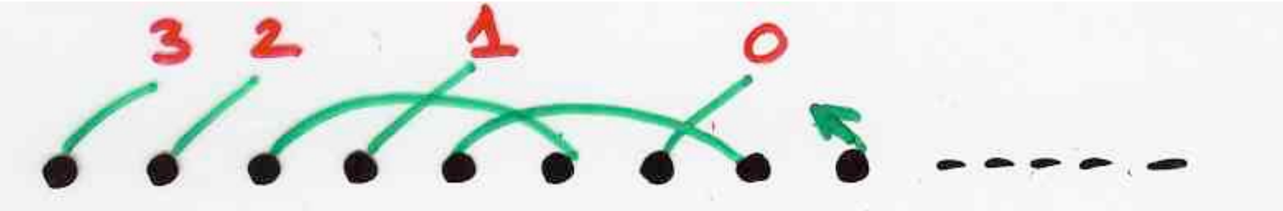
$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)}$$

Mahonian
distribution

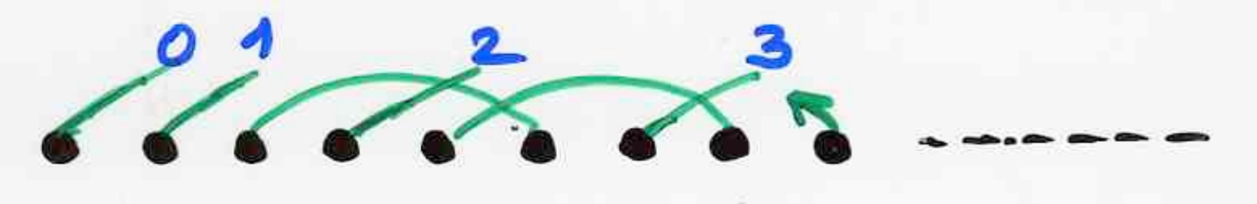
Hermite history

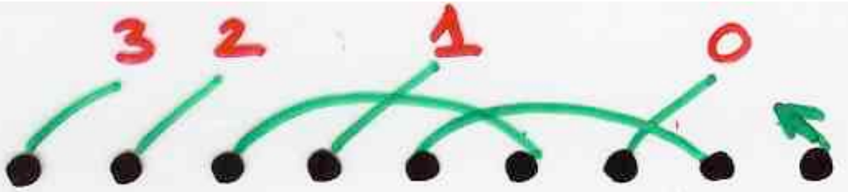


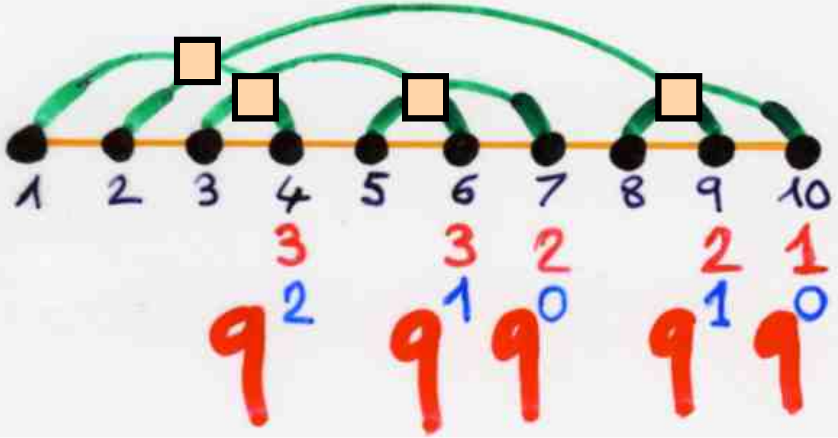
crossing

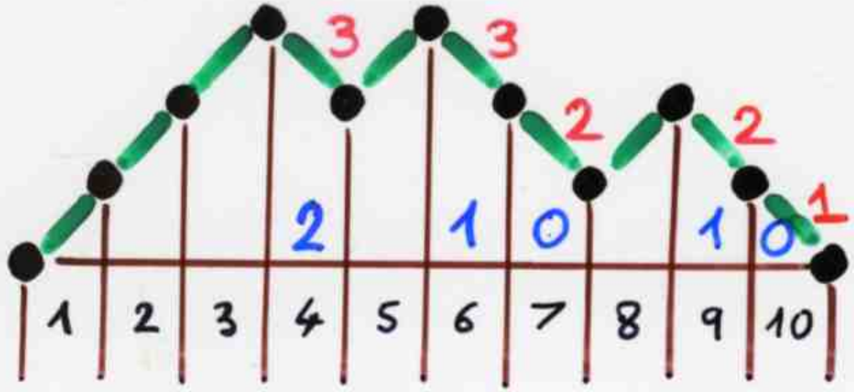


nesting









crossing

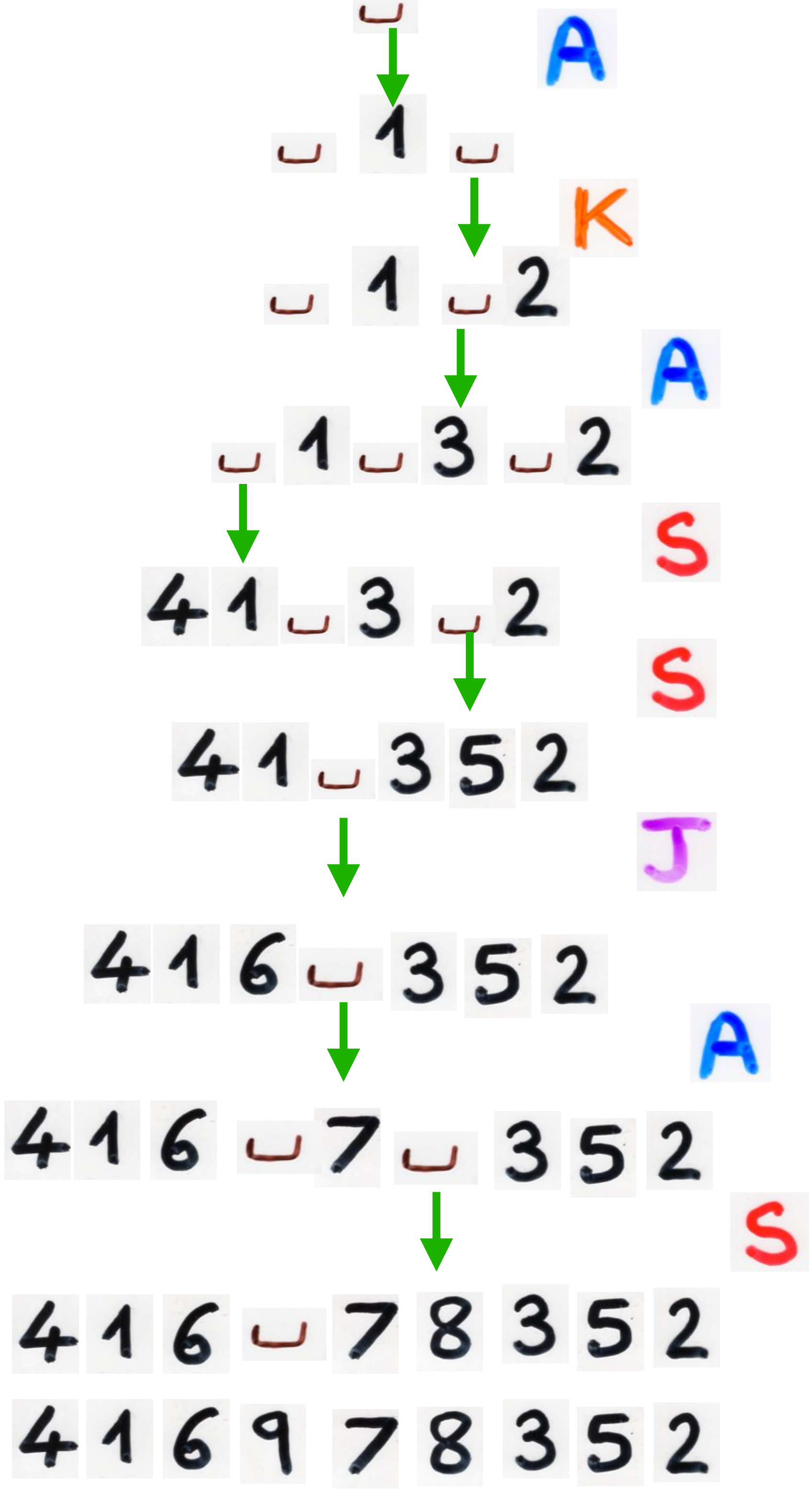
$= q^4$

nesting

q-Laguerre polynomials

$$\begin{cases} b_k = [k+1]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k+1]_q \end{cases}$$

$$\begin{cases} b'_k = [k+1]_q \\ b''_k = [k+1]_q \\ a_k = [k+1]_q \\ c_k = [k+1]_q \end{cases}$$



"q-analogue" of Laguerre histories

choice function

$i=$	1	2	3	4	5	6	7	8
$p_i=$	1	2	2	1	2	1	1	2
$p_i-1=$	0	1	1	0	1	0	0	1

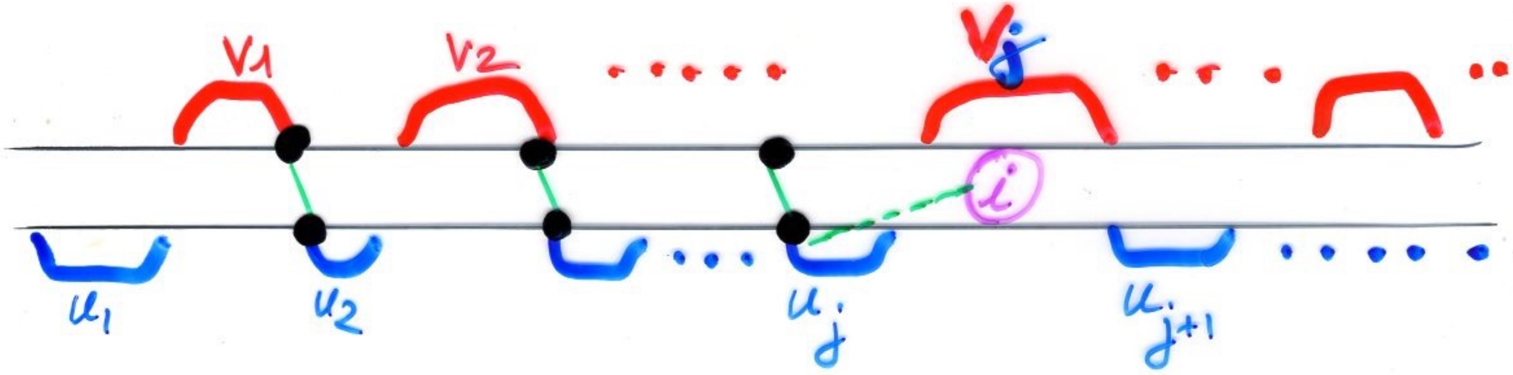
weighted q-Laguerre histories

q^4

weighted
q- Laguerre
histories

$$q^{\left[\sum_{i=1}^{n}(p_i-1)\right]}$$

this is also q^m where m is the number of subsequences (a, b, c) of σ having the pattern (31-2)



q-Laguerre polynomials

$$\begin{cases} b_k = [k+1]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k+1]_q \end{cases}$$

$$\mu_n = (n+1)!$$

$$\begin{cases} b_k = [k]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k]_q \end{cases}$$

q-Laguerre restricted histories

$$\mu_n = n!$$

q- Laguerre II

if $\mu_n = [n!]_q$

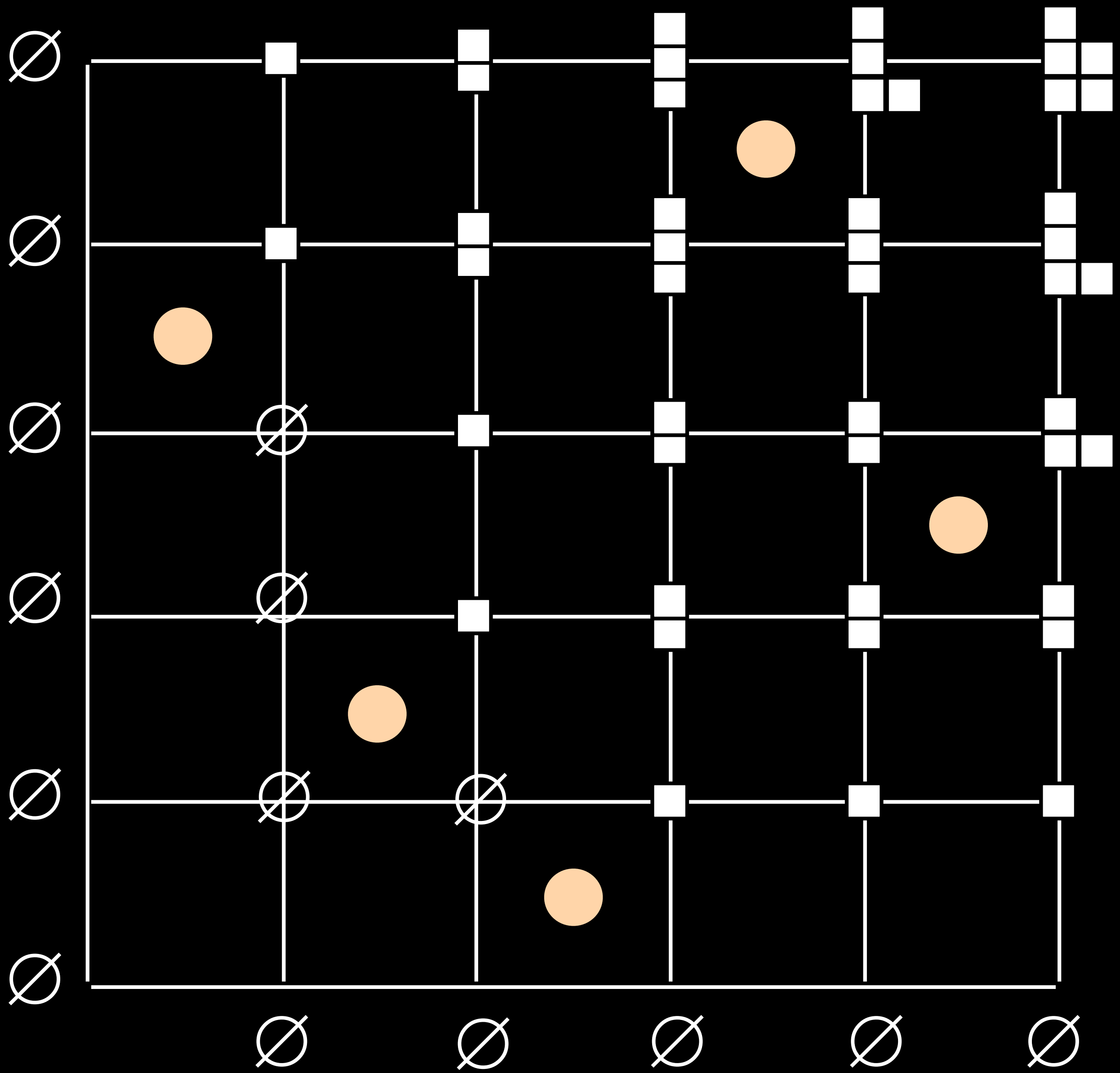
then $$\begin{cases} b_k = q^k \left([k]_q + [k+1]_q \right) \\ \lambda_k = q^{2k-1} [k]_q \times [k]_q \end{cases}$$

The essence of bijections

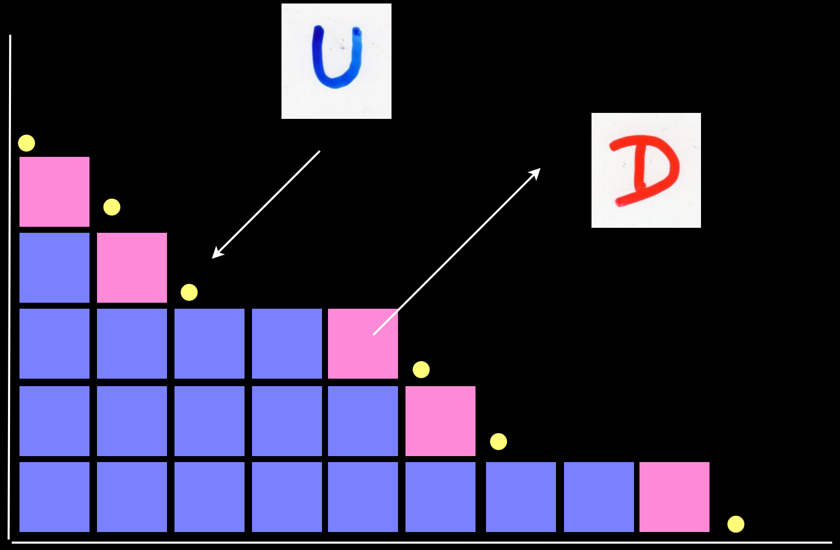
growth diagrams and the RS bijection

S. Fomin, 1986, 1994

Сергей Владимирович Фомин

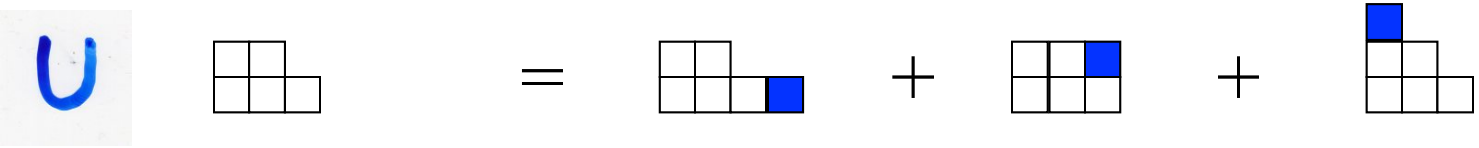


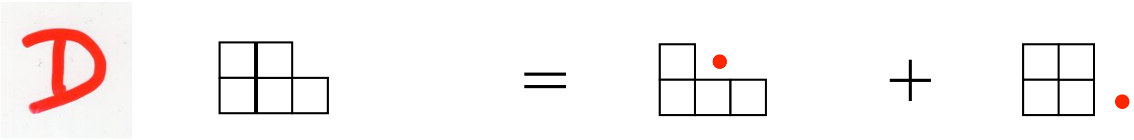
operators U and D



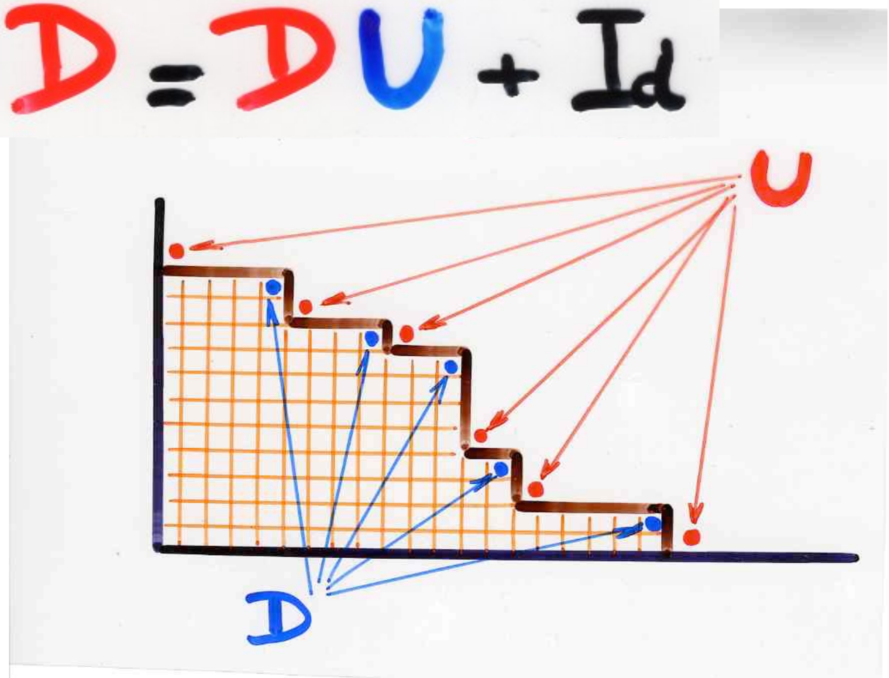
Young lattice

U adding
D deleting
a cell in a Ferrers diagram





$UD = DU + Id$



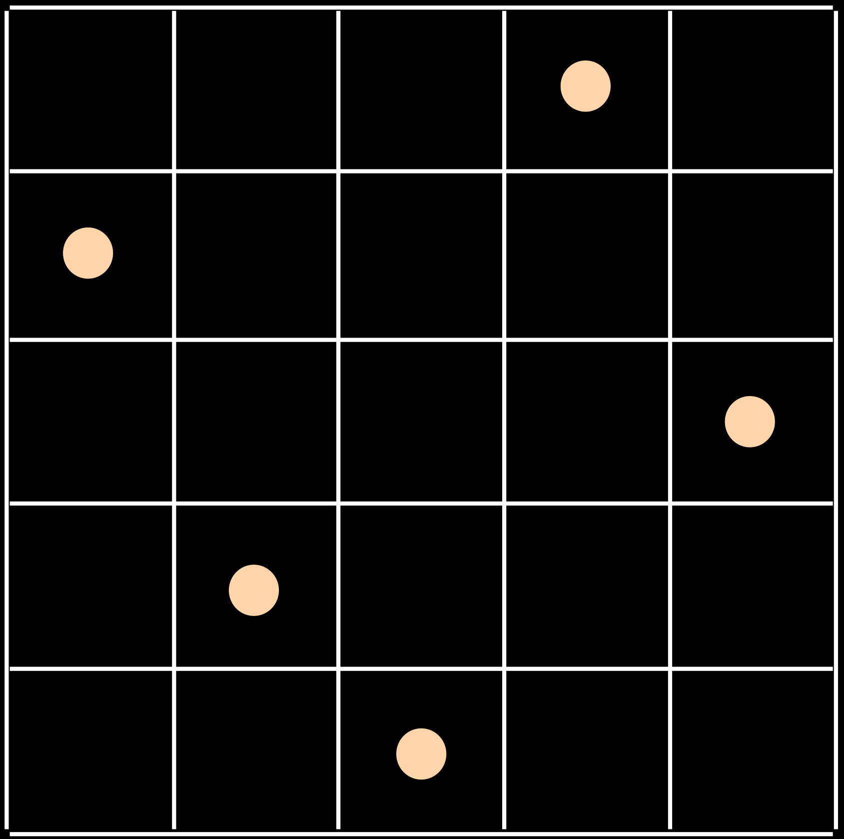
$$UD = DU + I_d$$

normal ordering
in physics

Lemma Every word w with letters U and D can be written in a unique way

$$w = \sum_{i,j \geq 0} c_{ij}(w)\, D^i U^j$$

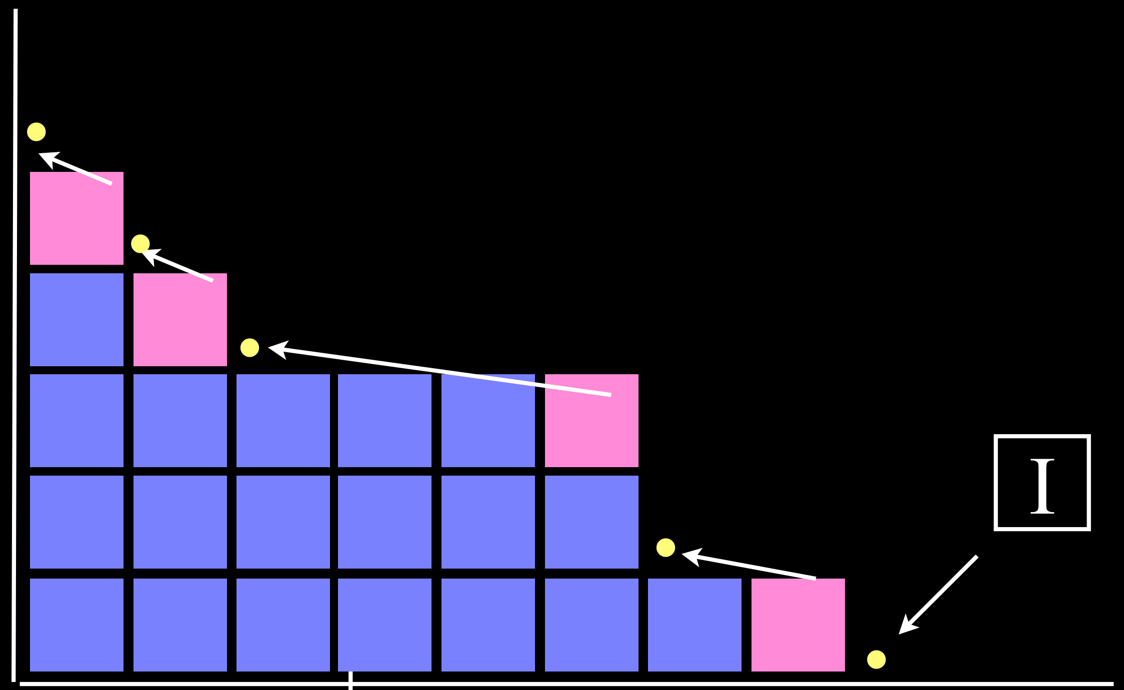
$$UD = qDU + I$$

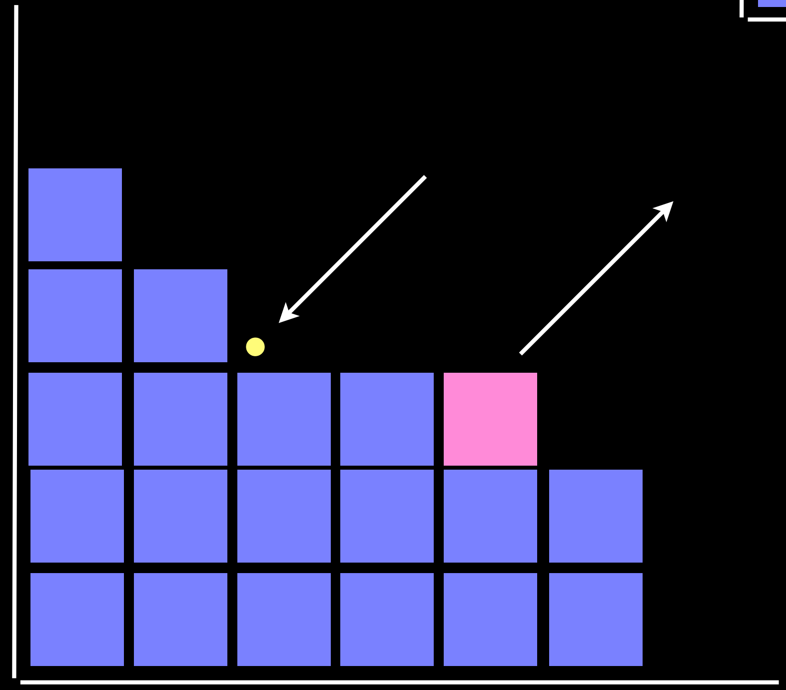


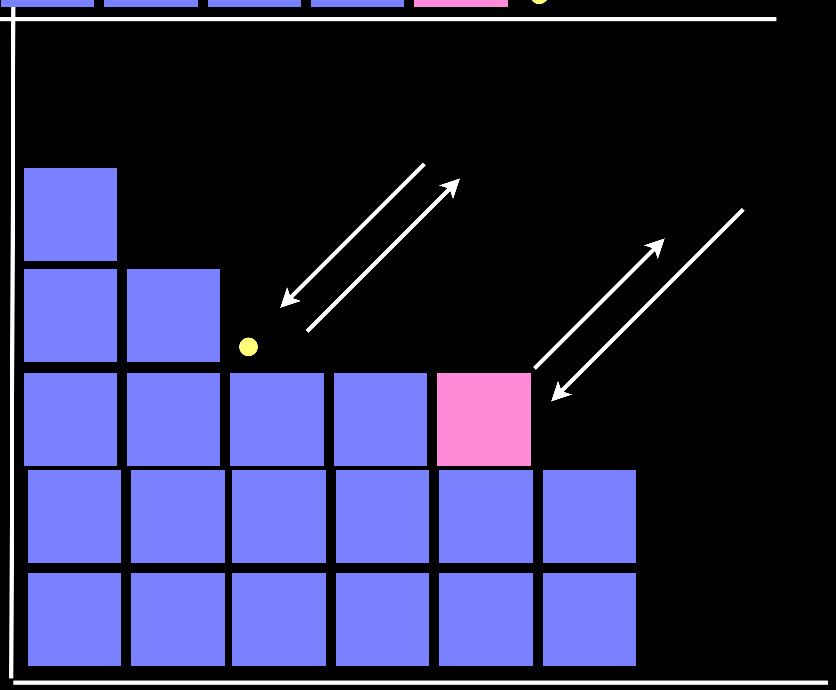
normal ordering

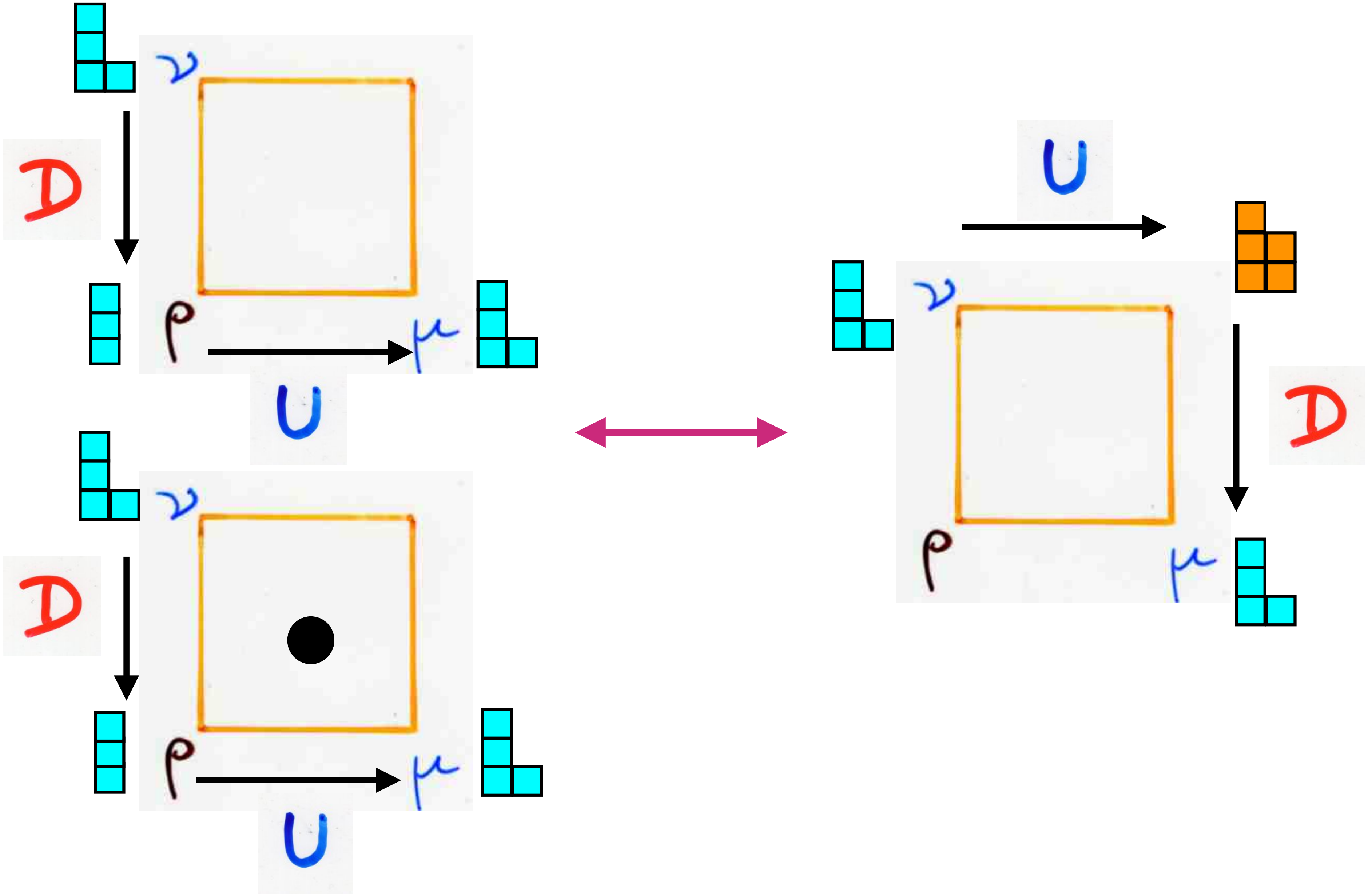
rook placement

$$\sigma = 4, 2, 1, 5, 3$$


I



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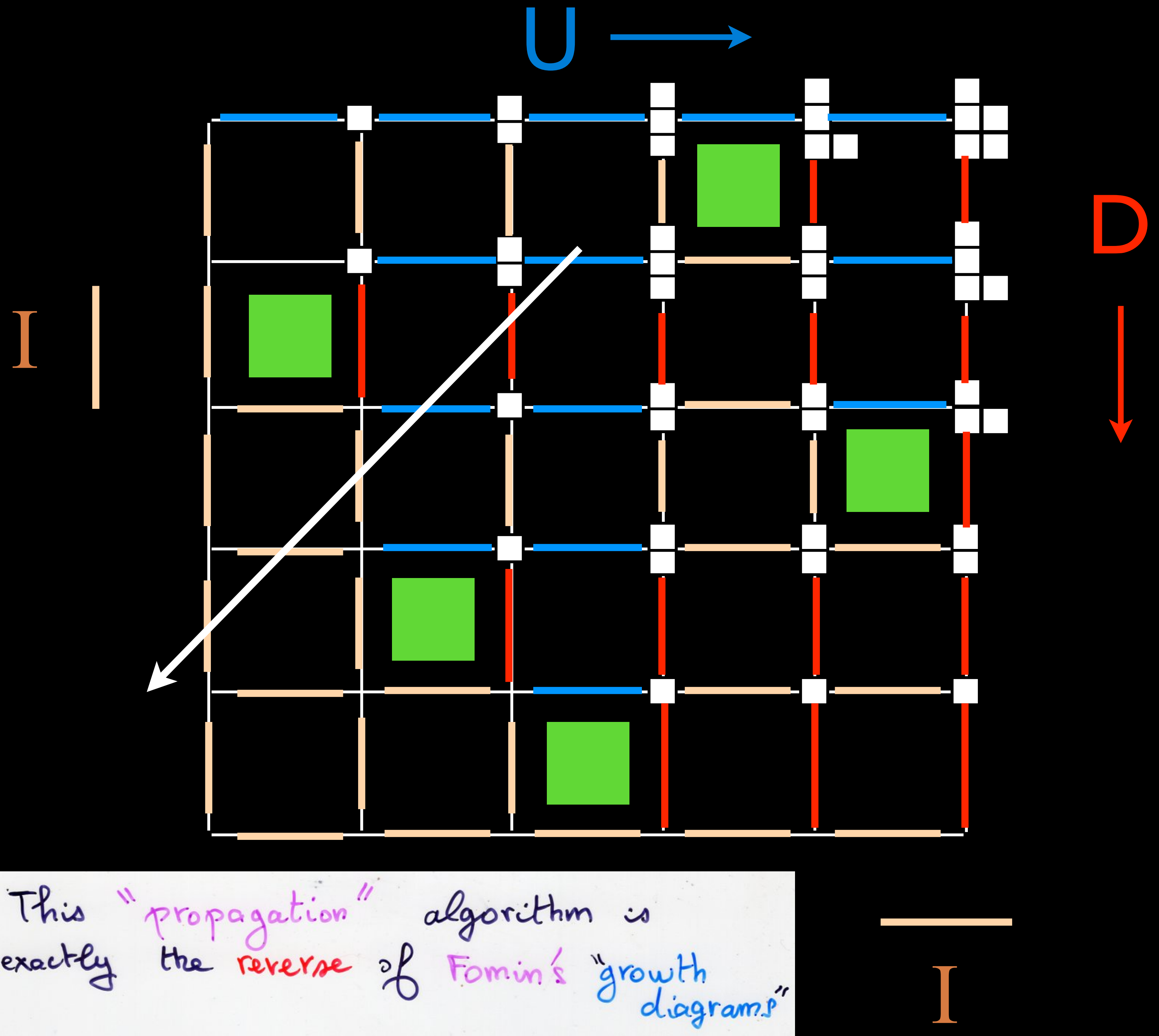


This "propagation" algorithm is exactly the reverse of Fomin's "growth diagrams"

The PASEP algebra

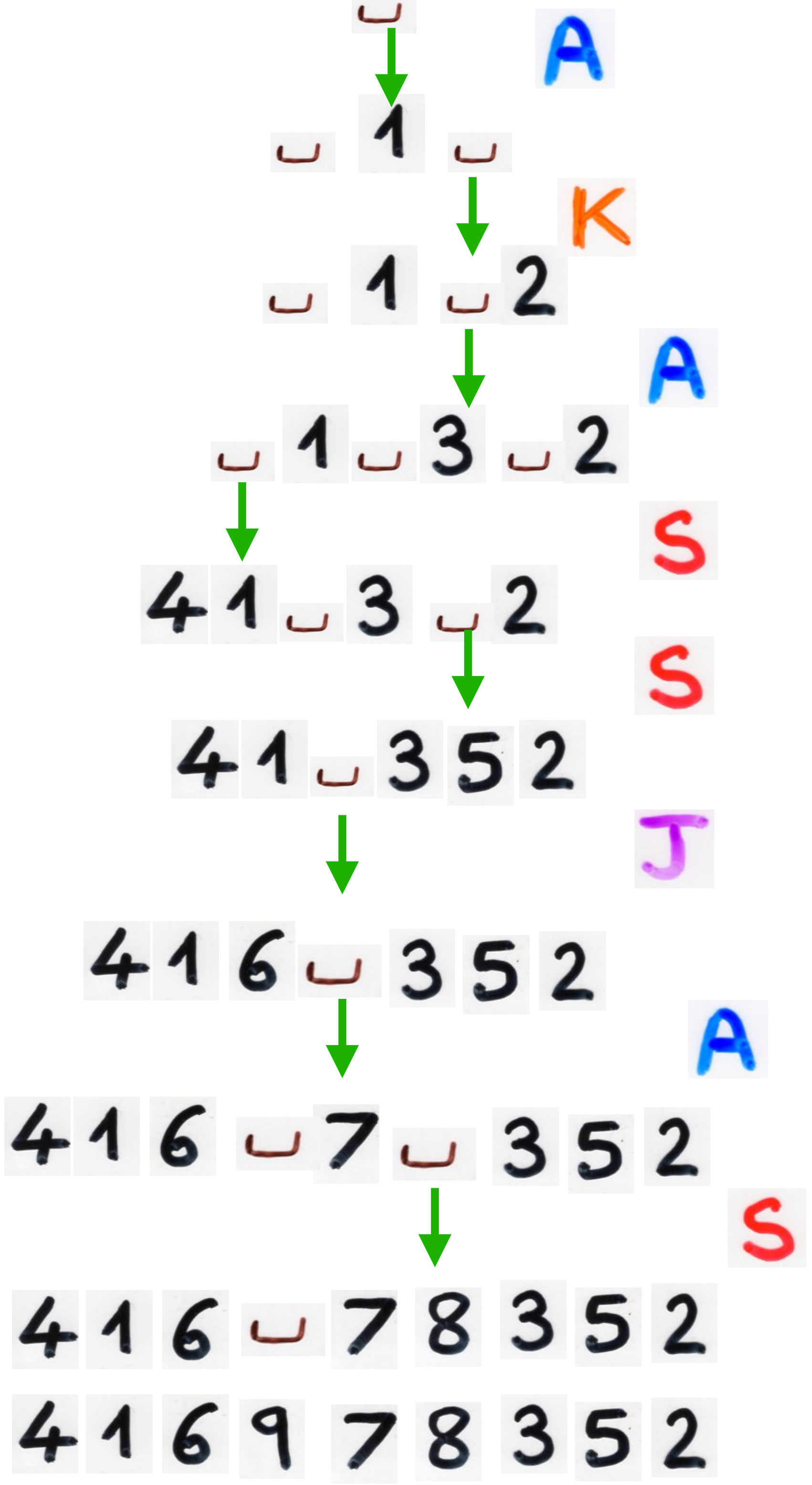
$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

word

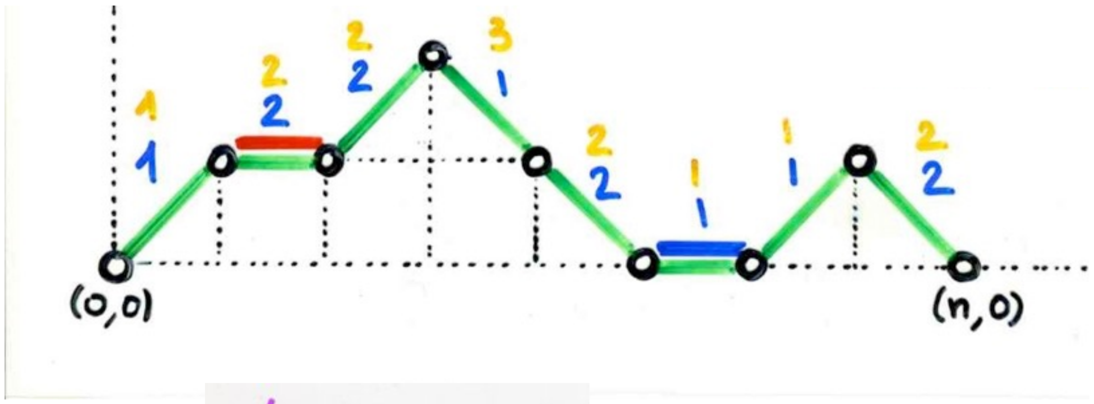
tableau

unique

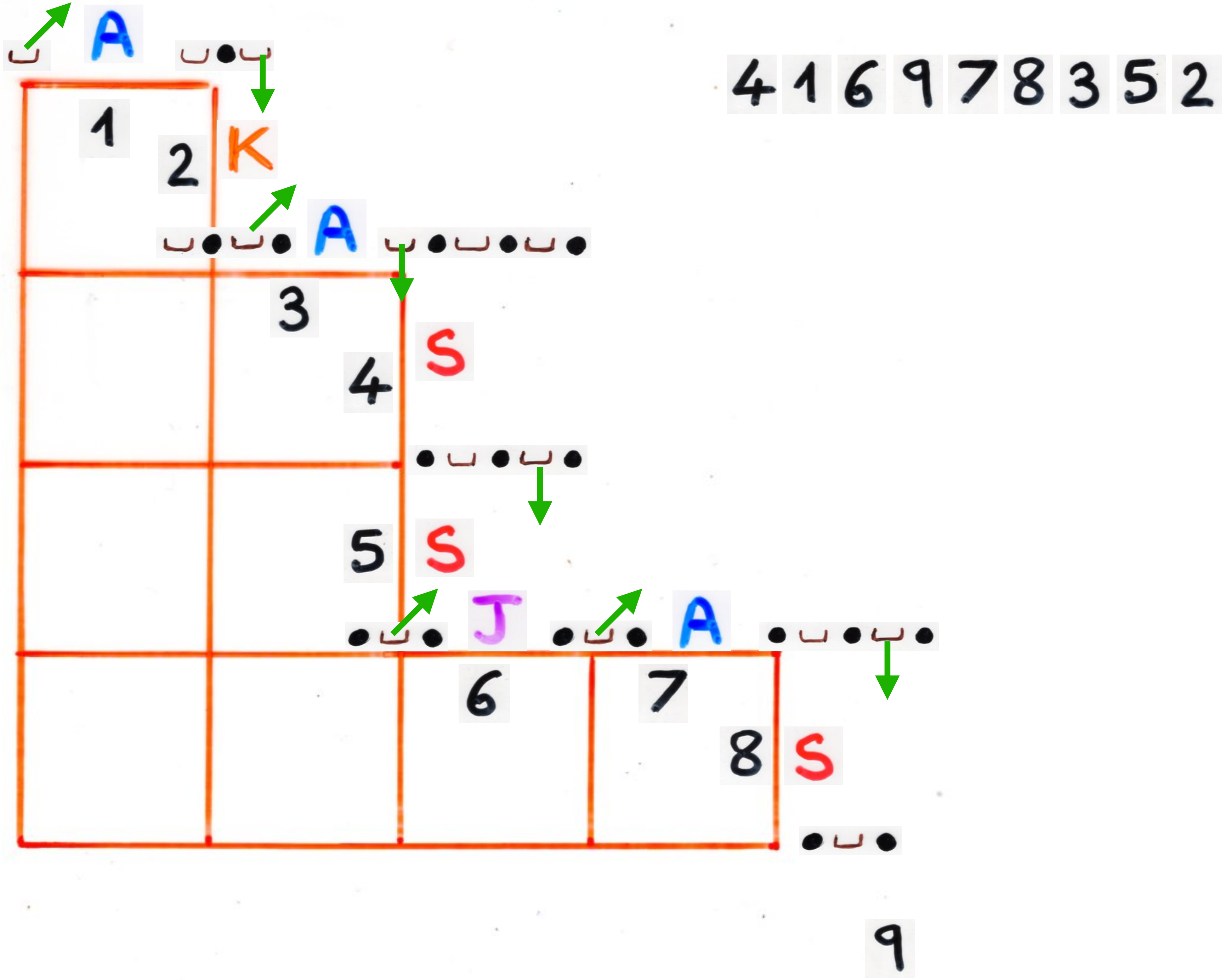


$$D = A + K$$
$$E = S + J$$

$$DE = ED + E + D$$



Laguerre
histories


A
K
A
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5
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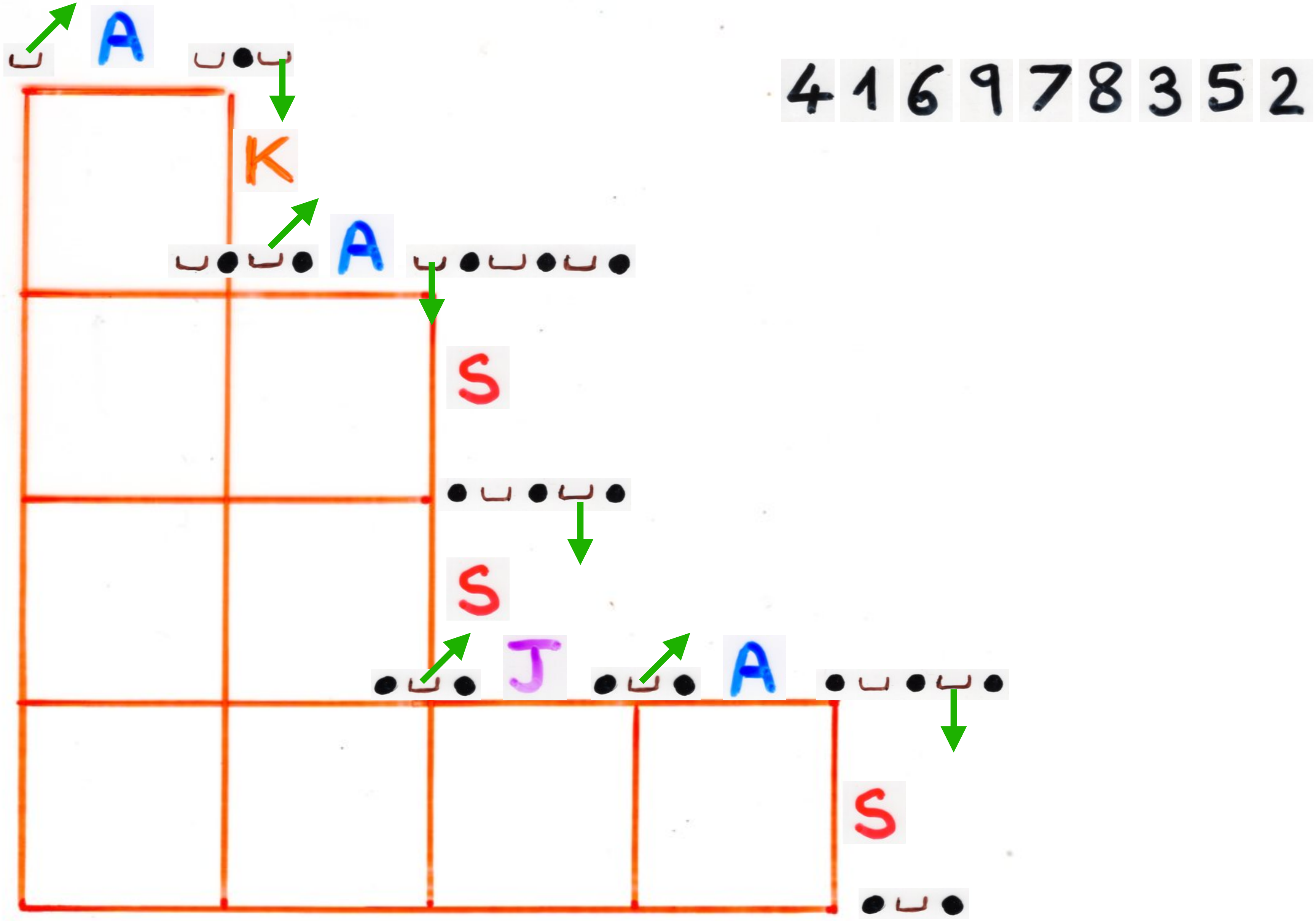
9
4 1 6 9 7 8 3 5 2


␣
A
␣•␣
K
␣•␣•
A
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S
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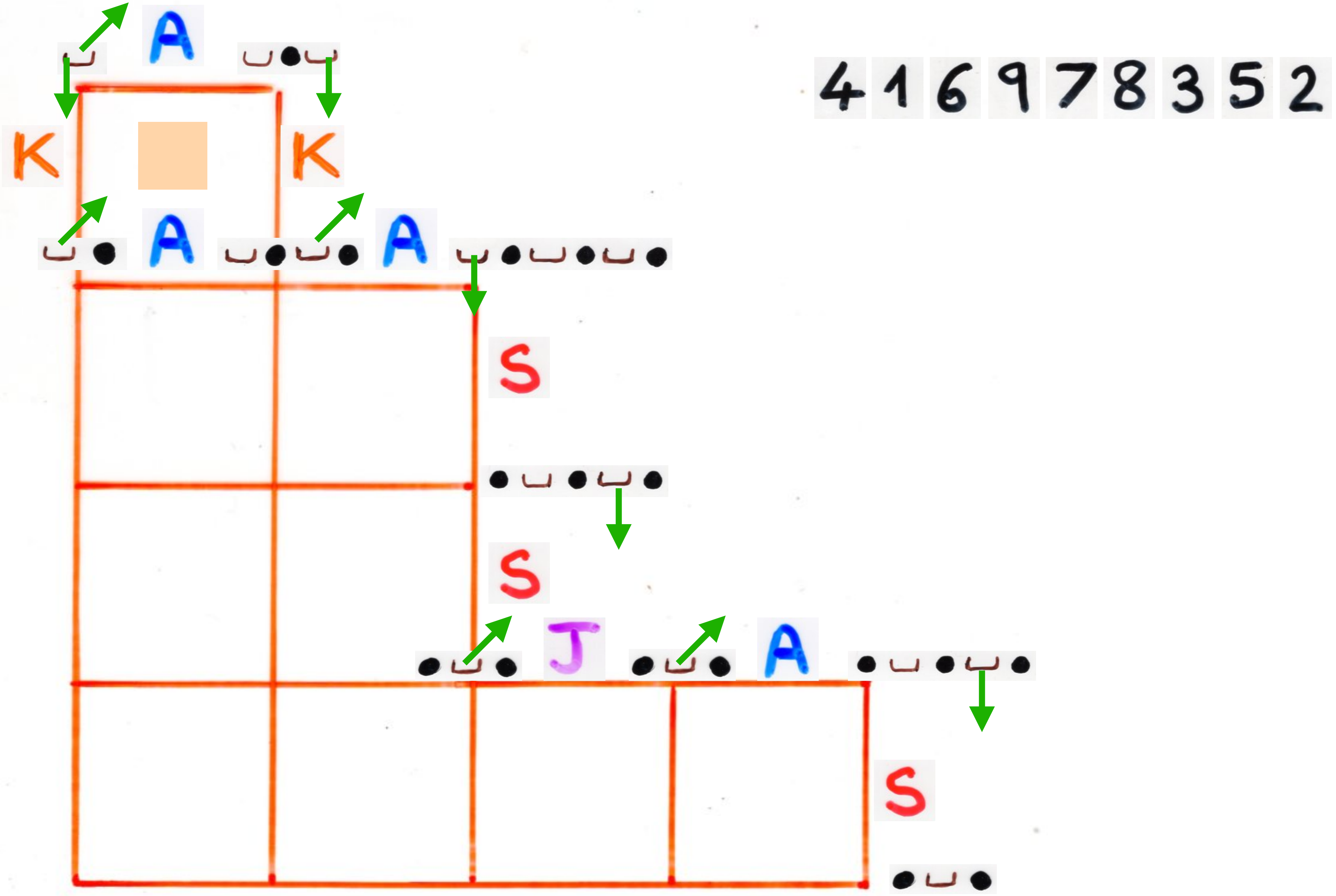
4 1 6 9 7 8 3 5 2


4 1 6 9 7 8 3 5 2
␣
A
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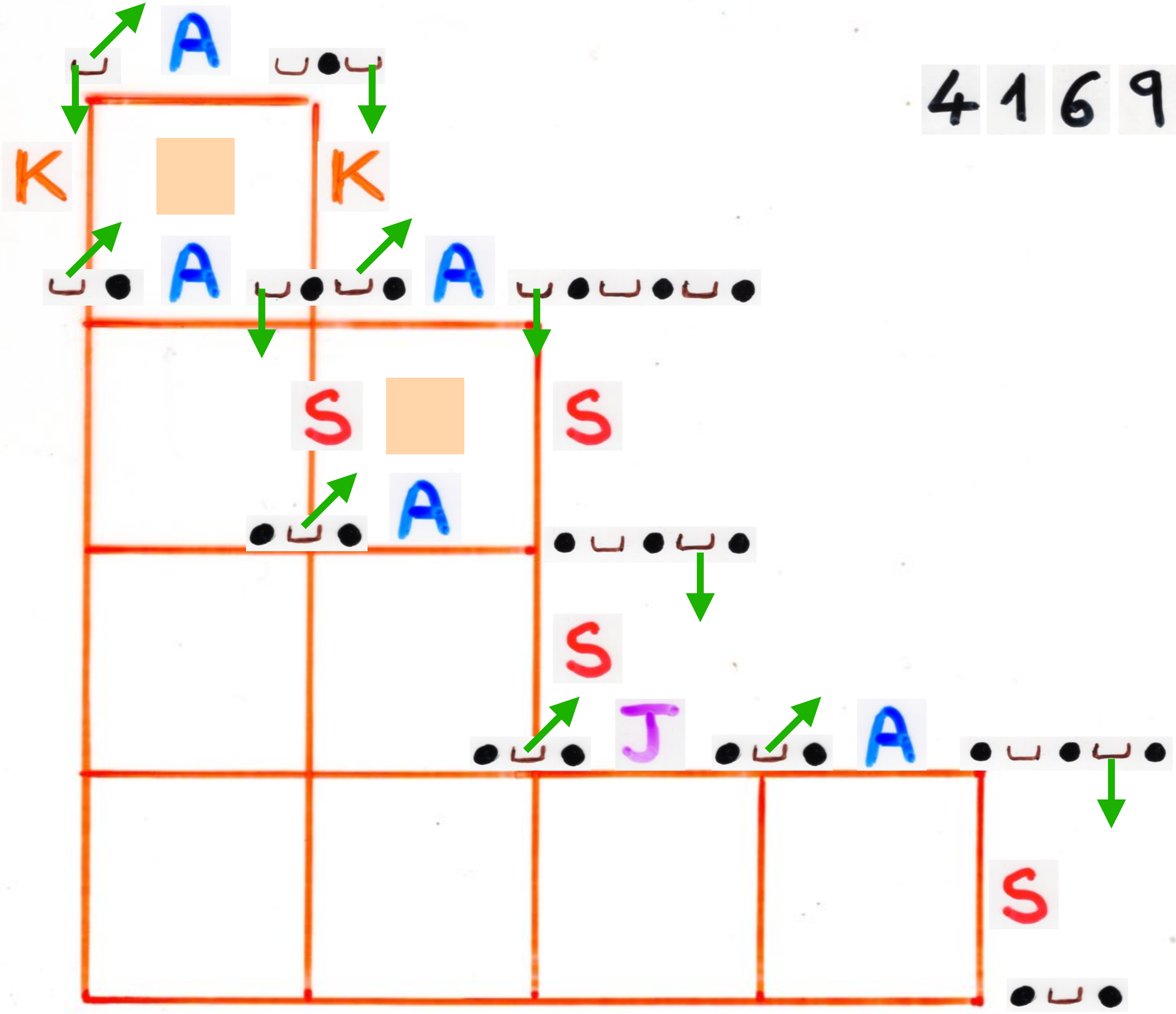
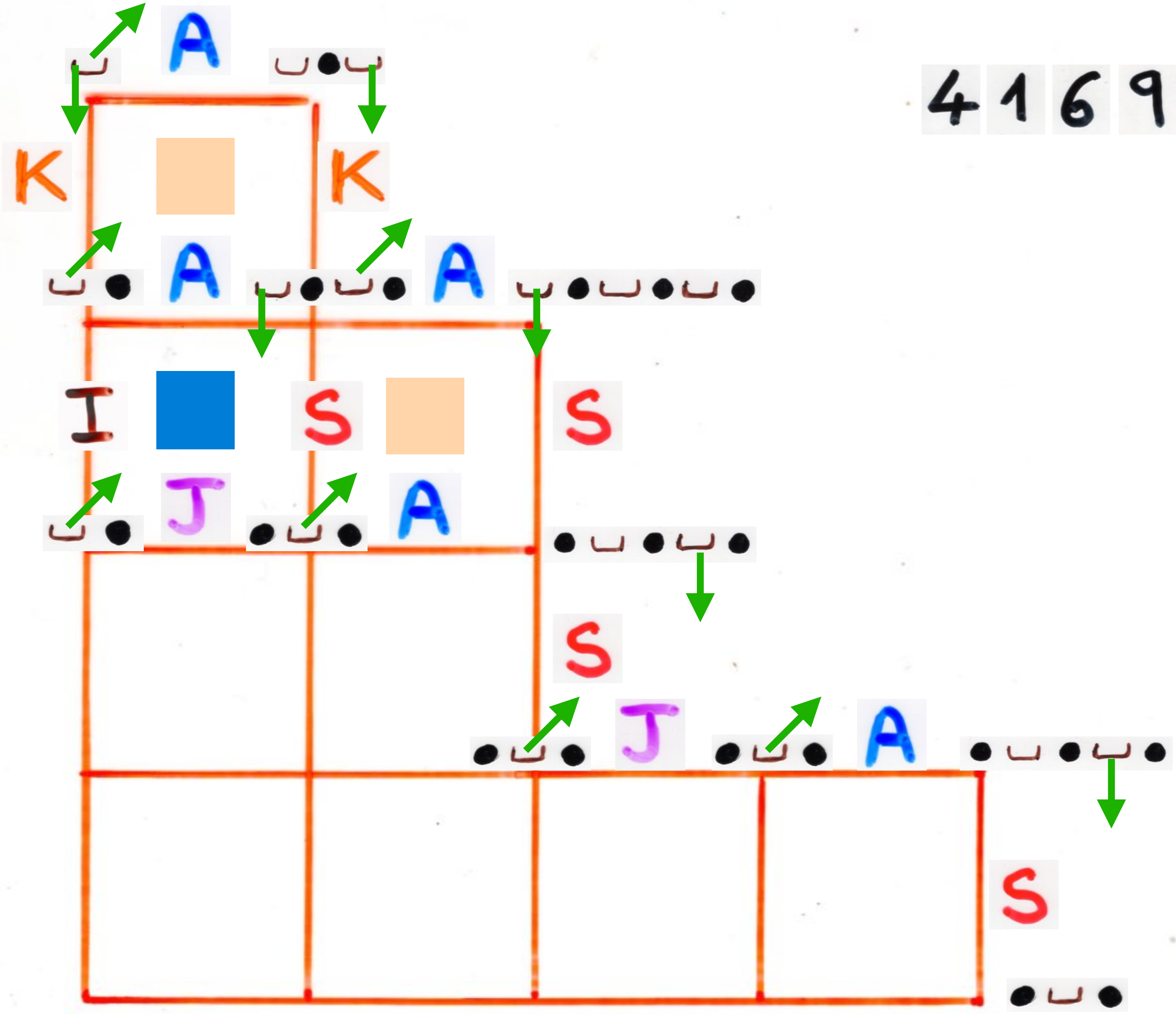


4 1 6 9 7 8 3 5 2



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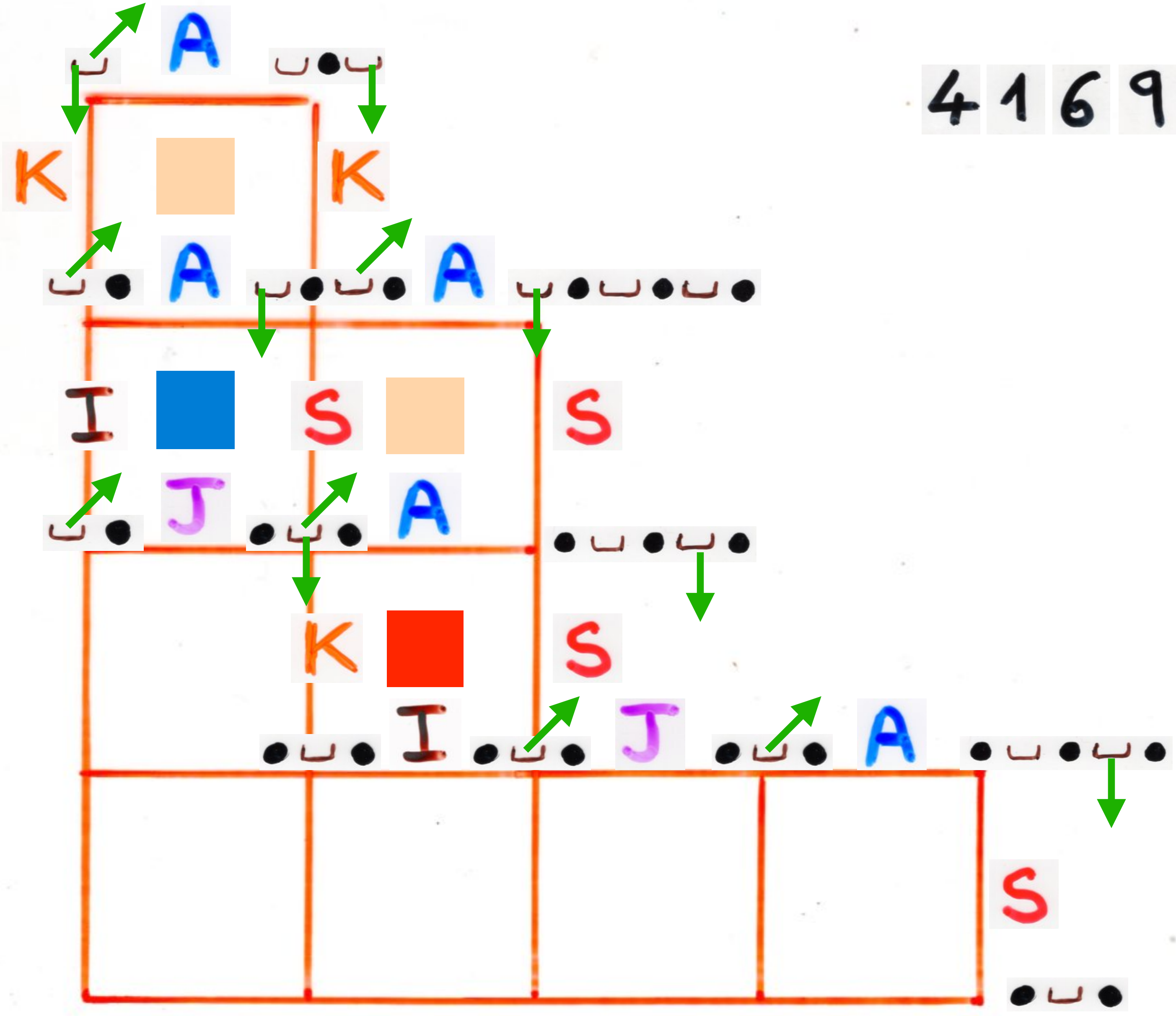
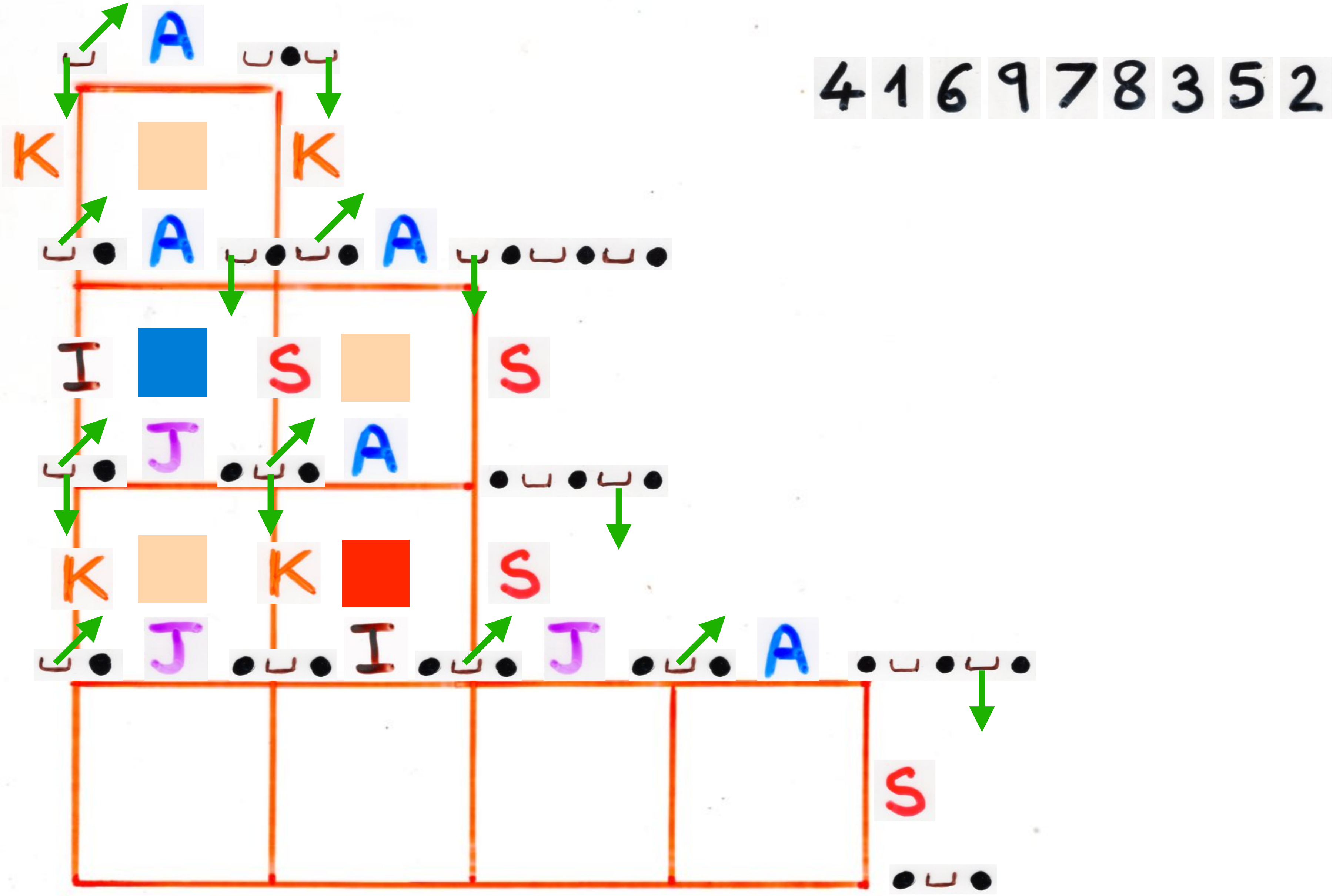


4 1 6 9 7 8 3 5 2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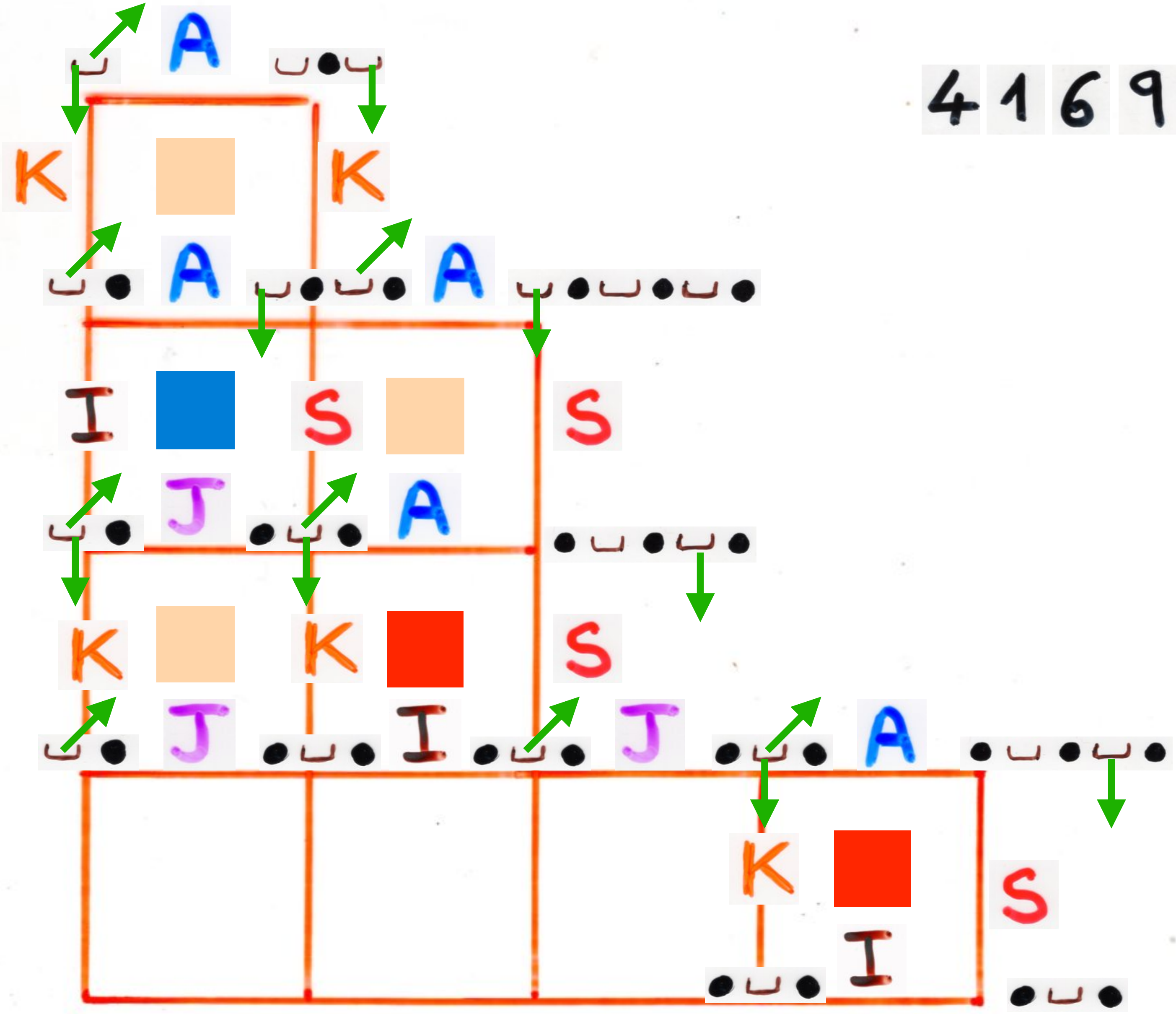

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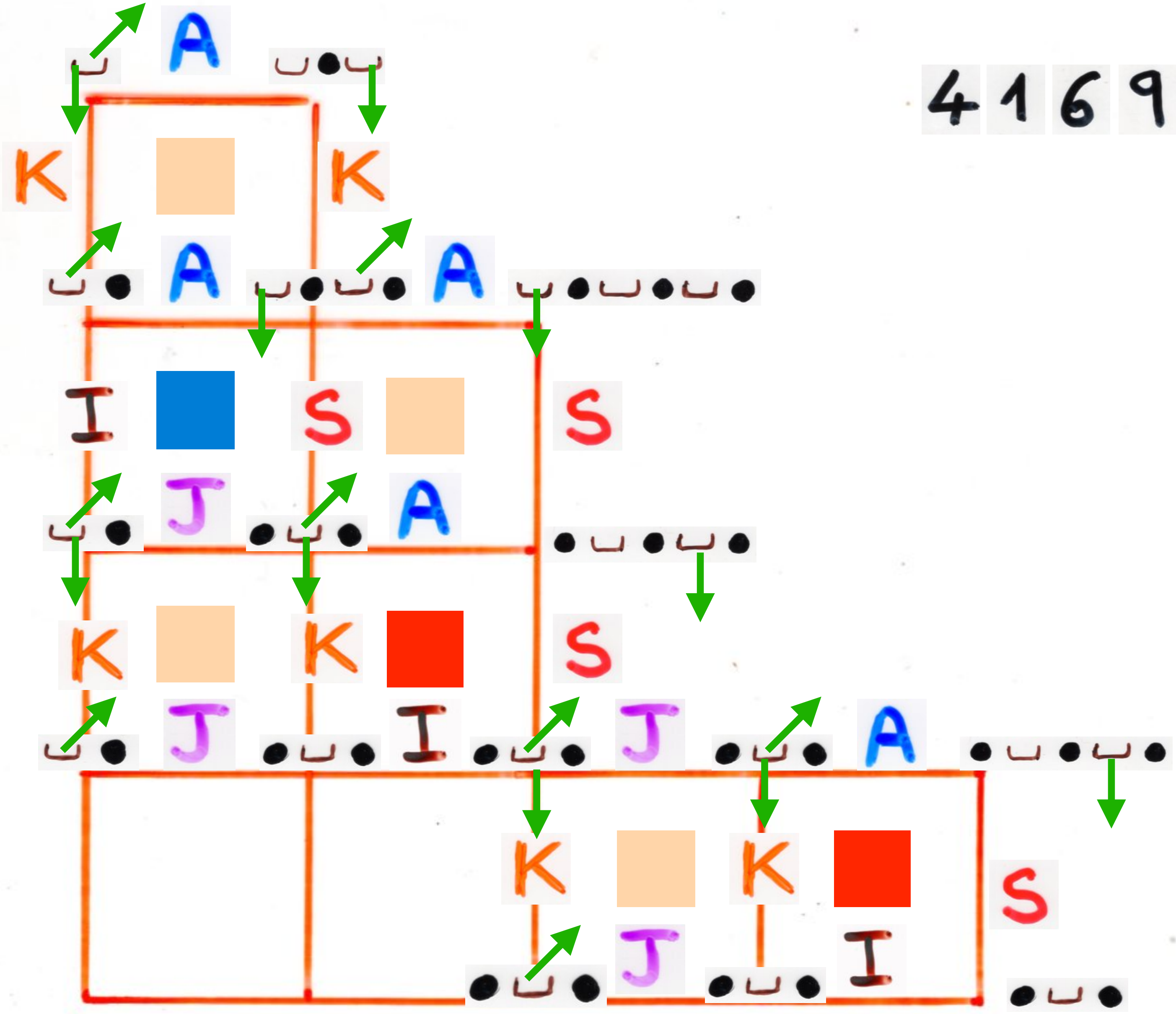
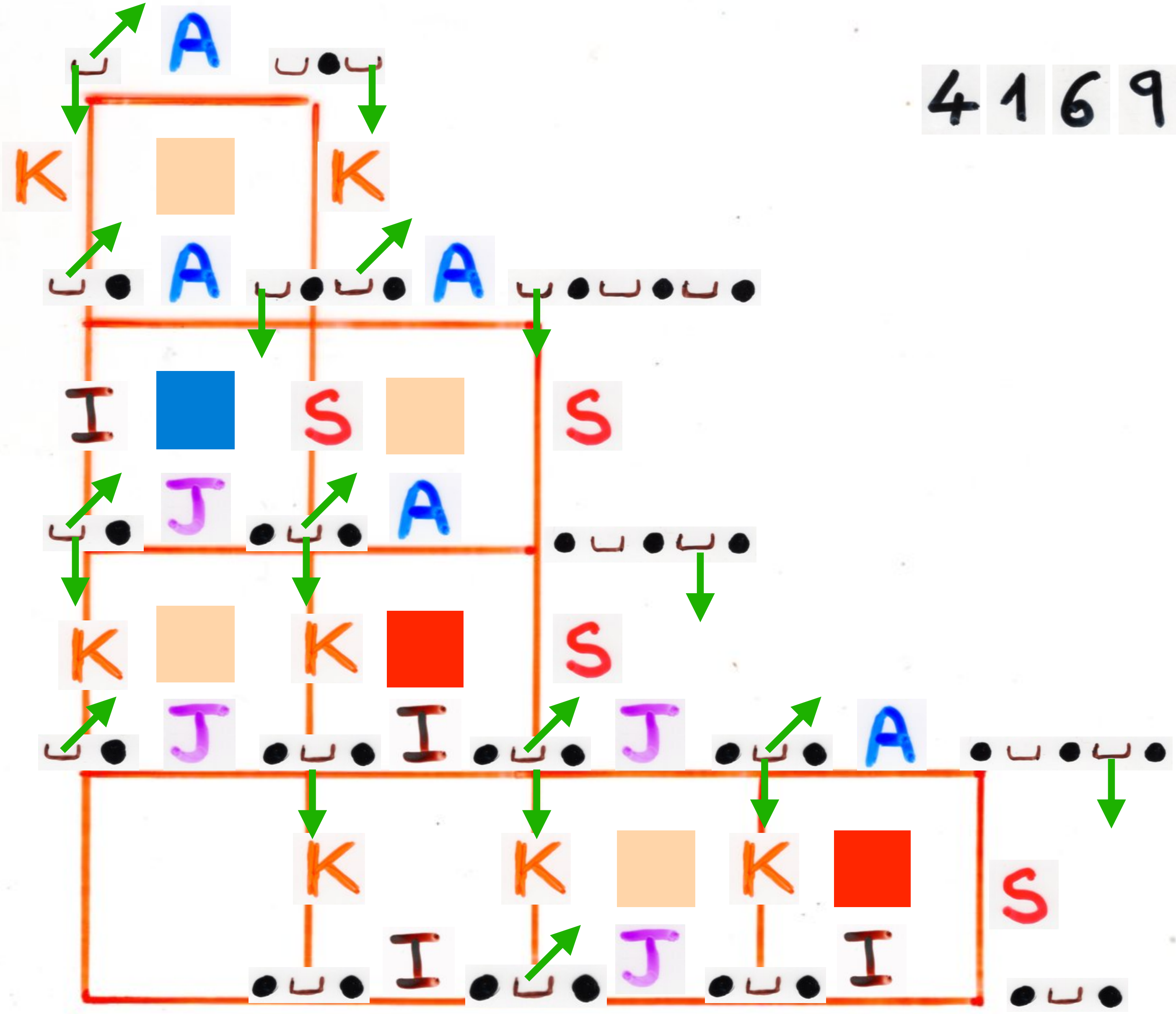
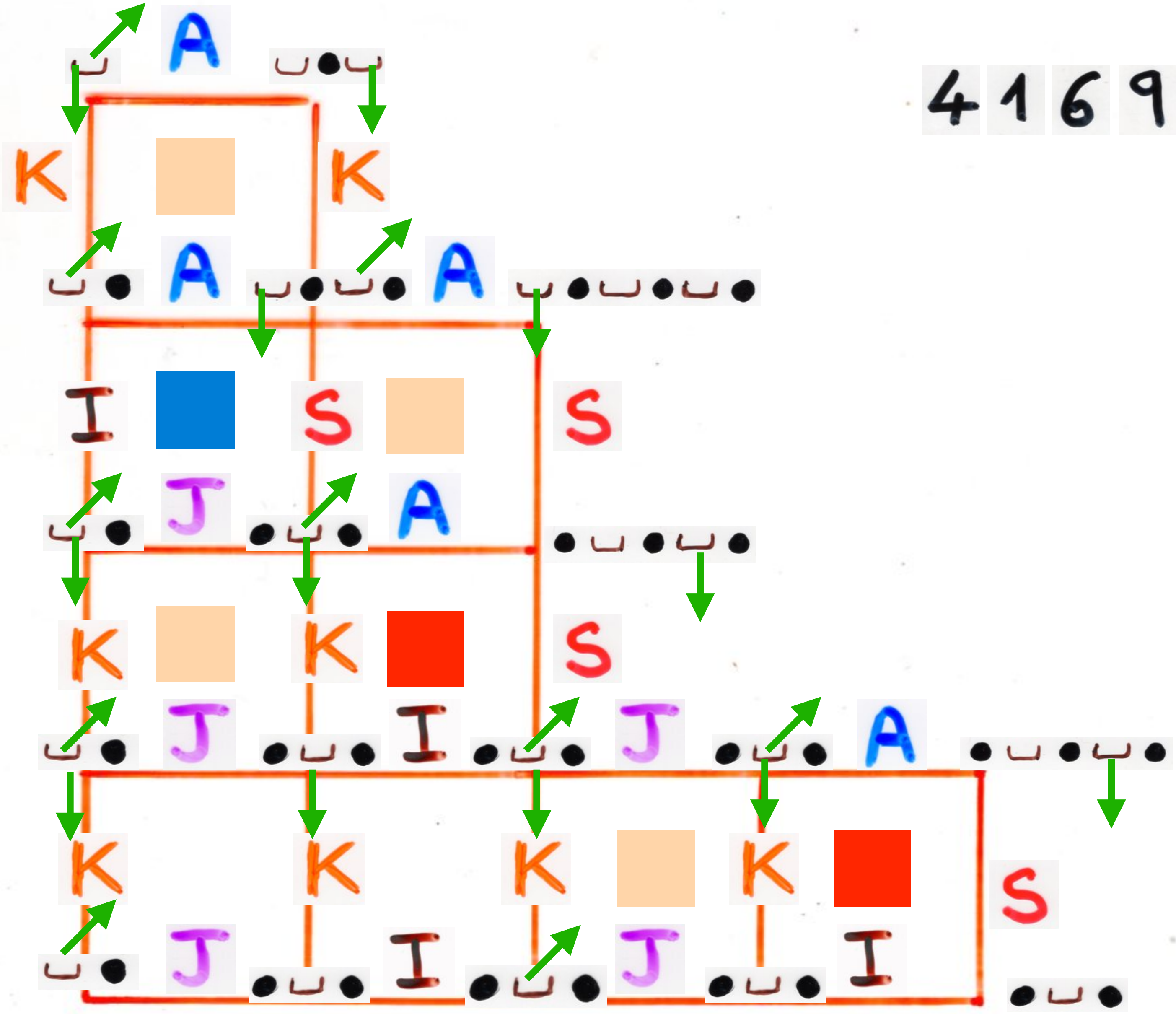


4 1 6 9 7 8 3 5 2

4 1 6 9 7 8 3 5 2





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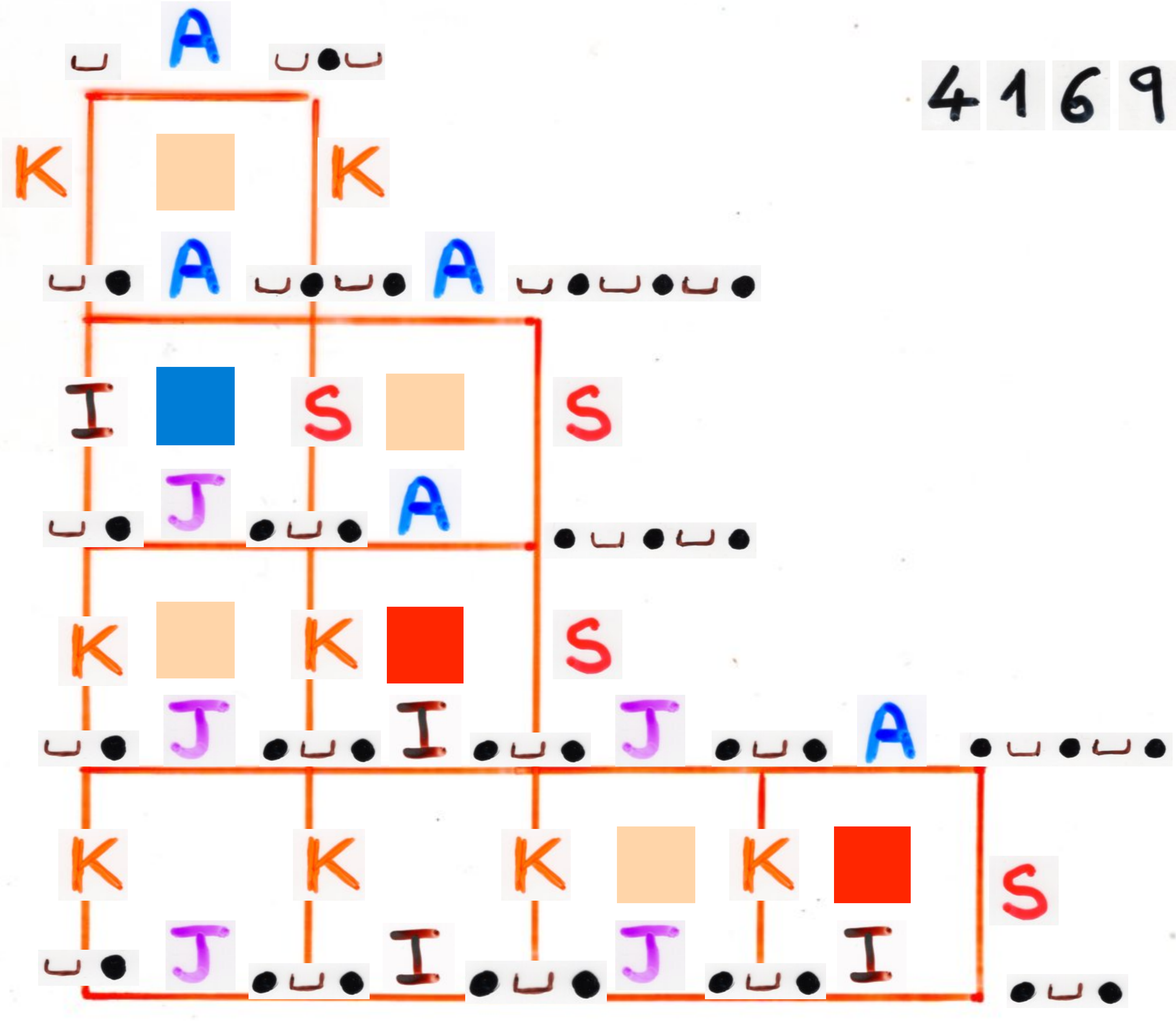

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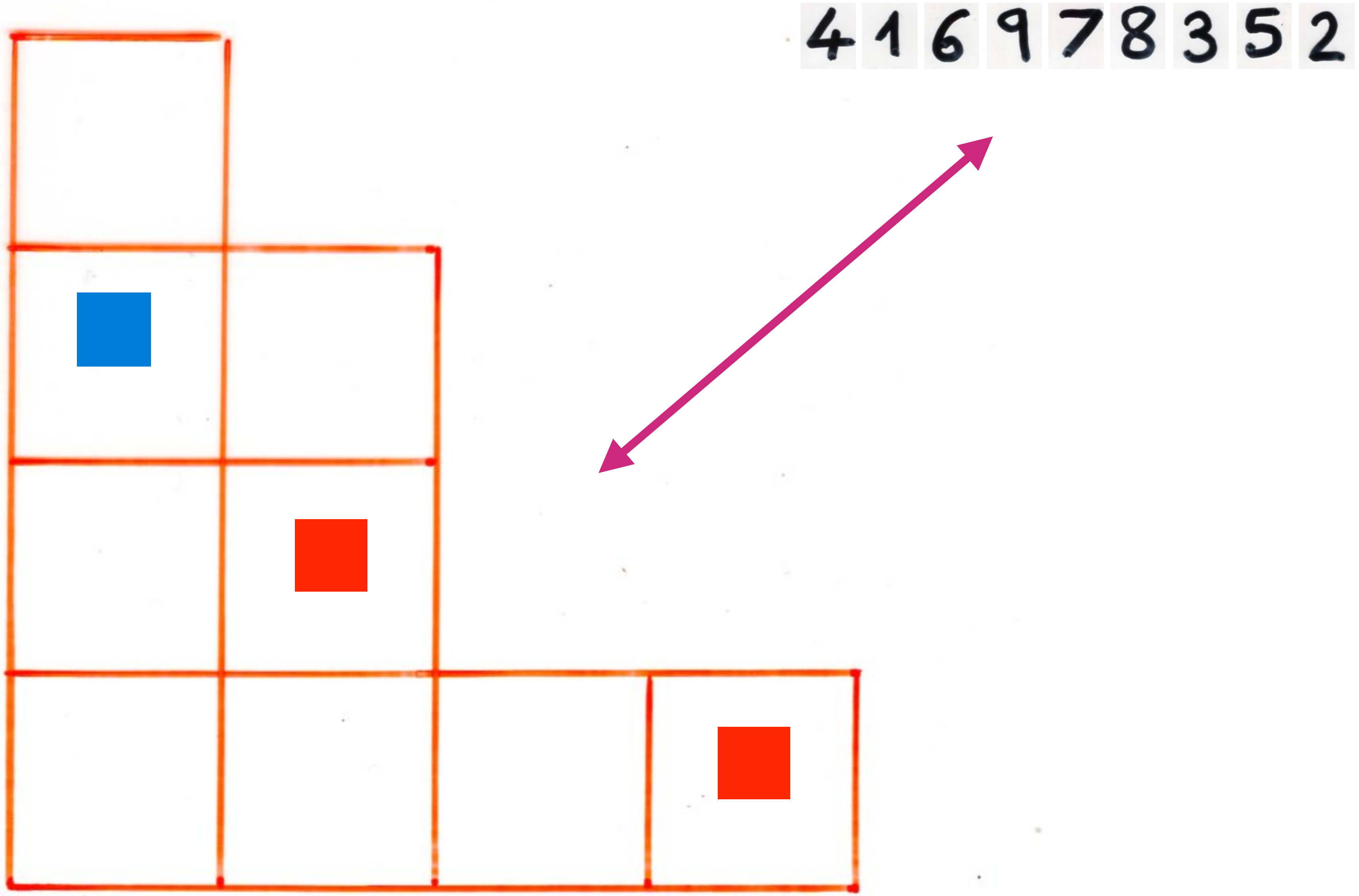
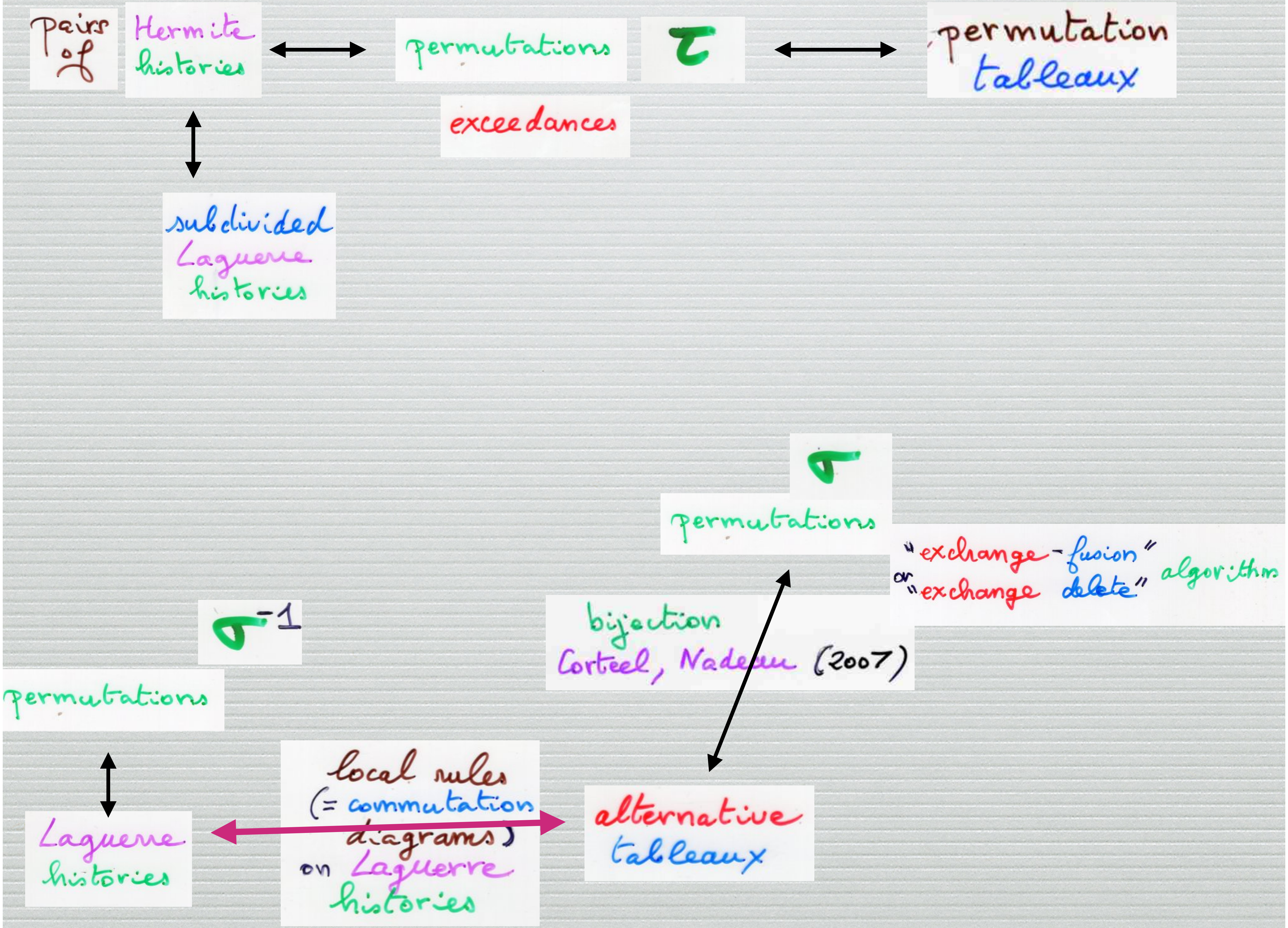

4 1 6 9 7 8 3 5 2



Pairs of Hermite histories

permutations τ

exceedances

permutation tableaux

subdivided Laguerre histories

σ

permutations

"exchange-fusion" or "exchange delete" algorithm

bijection Corteel, Nadeau (2007)

σ^{-1}

permutations

Laguerre histories

local rules (= commutation diagrams) on Laguerre histories

alternative tableaux

q, α, β

?

Josuat-Vergès (2011)

Third bijection

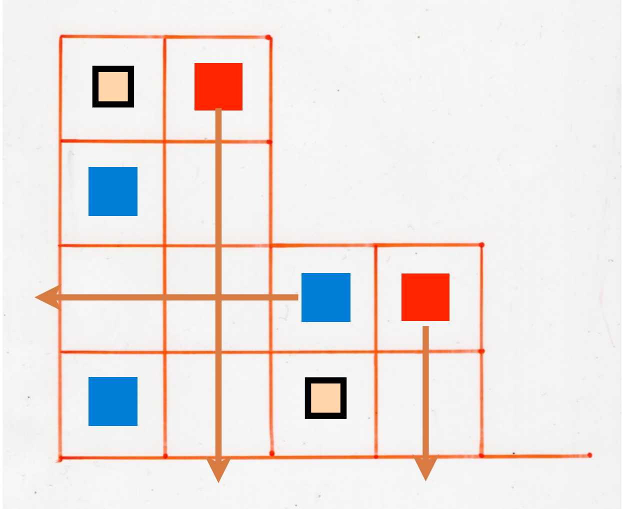
Tableaux — permutations

- direct bijection (with tree-like tableaux)

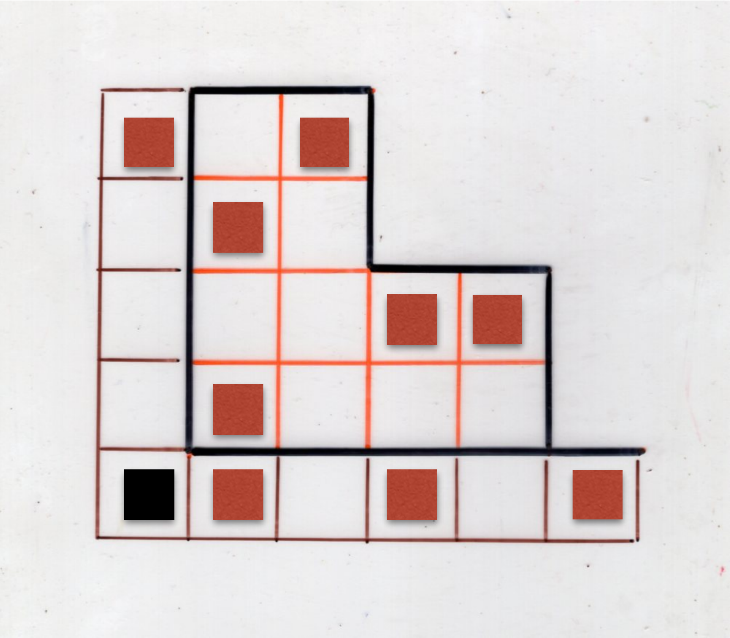
Aval, Boussicault, Nadeau (2011)

tableaux size $(n+1)$ $\longleftrightarrow$ (tableaux size n , $1 \leq i \leq n+1$)

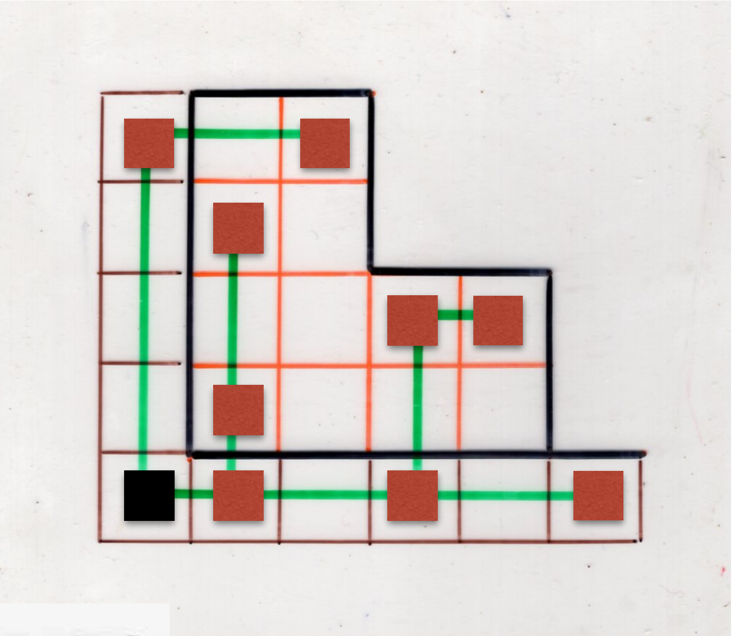
$(n+1)!$



alternative tableaux



tree-like tableaux



Aval, Boussicault, Nadeau (2013)

$q = 0$

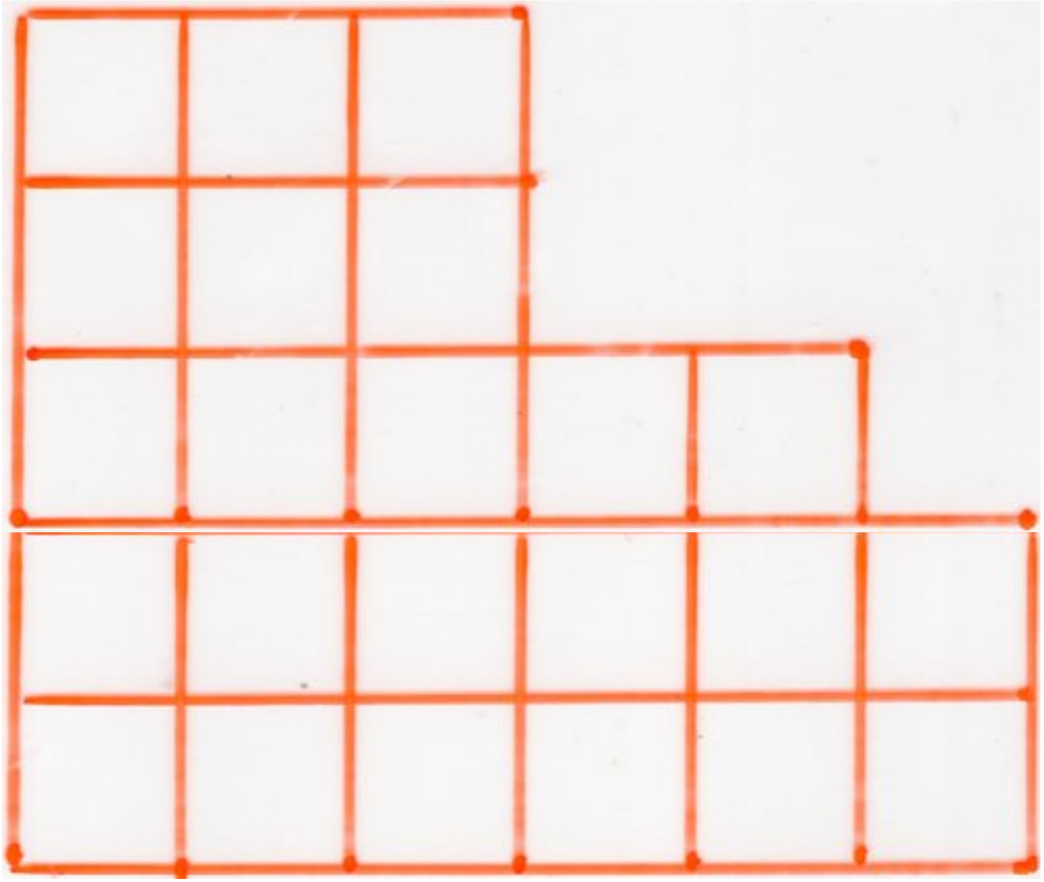
ν-trees

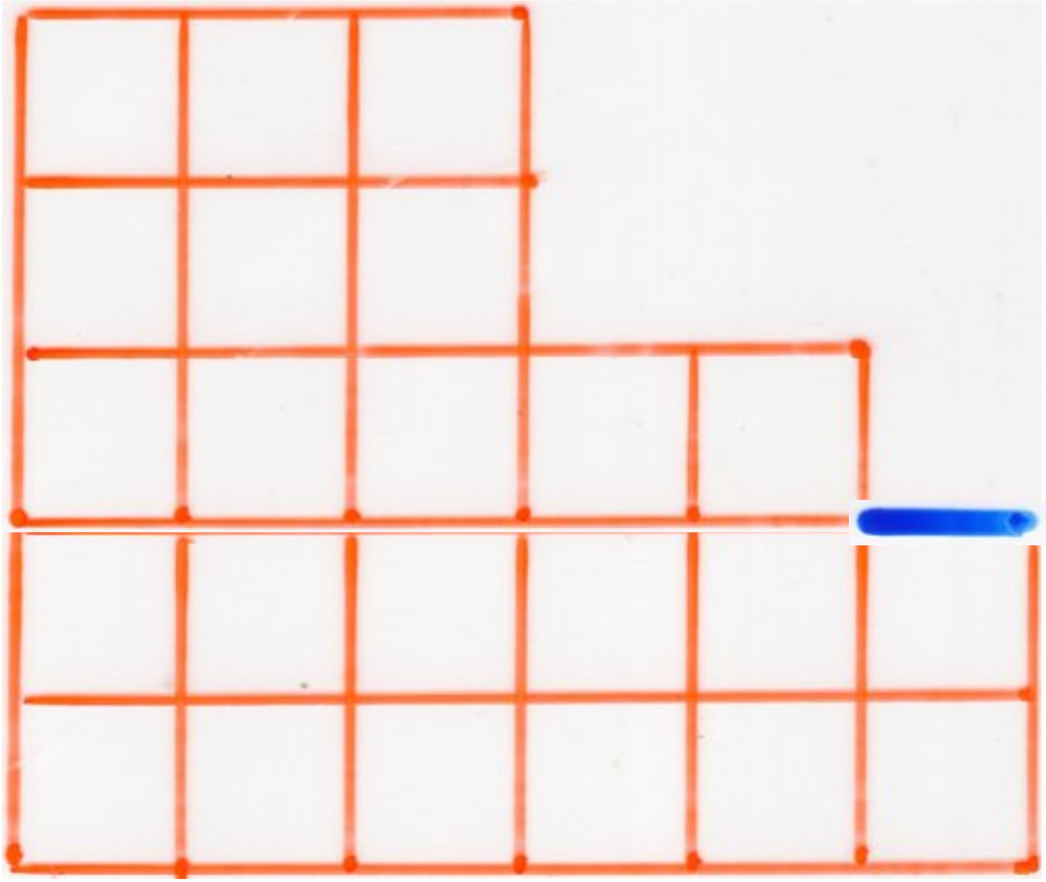
Ceballos, Padrol, Sarmiento
(2016) (2018)

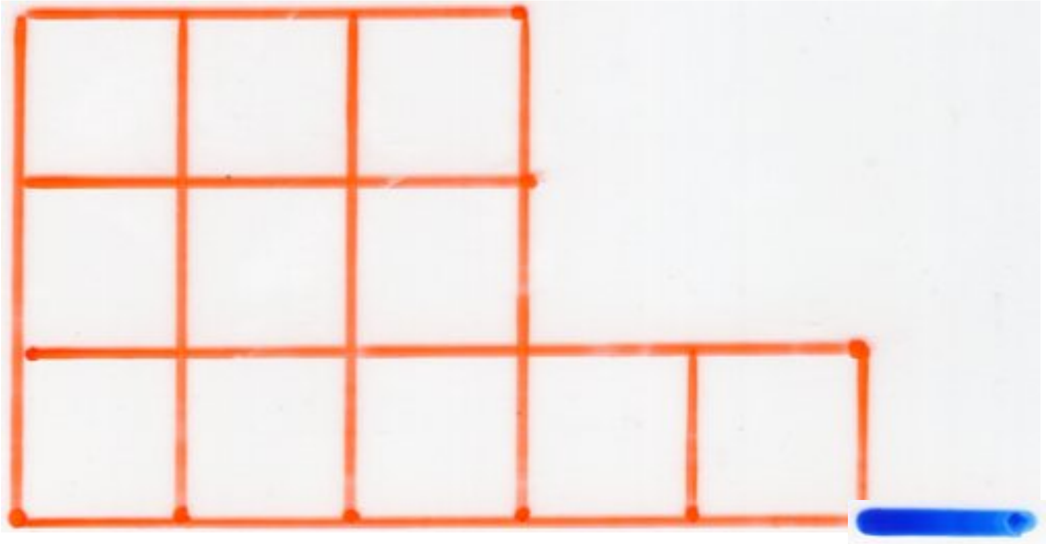
Catalan

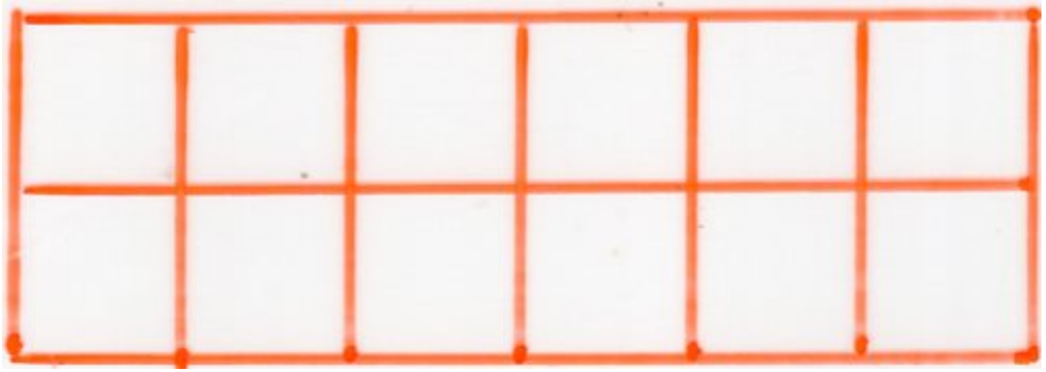
tree-like
tableaux

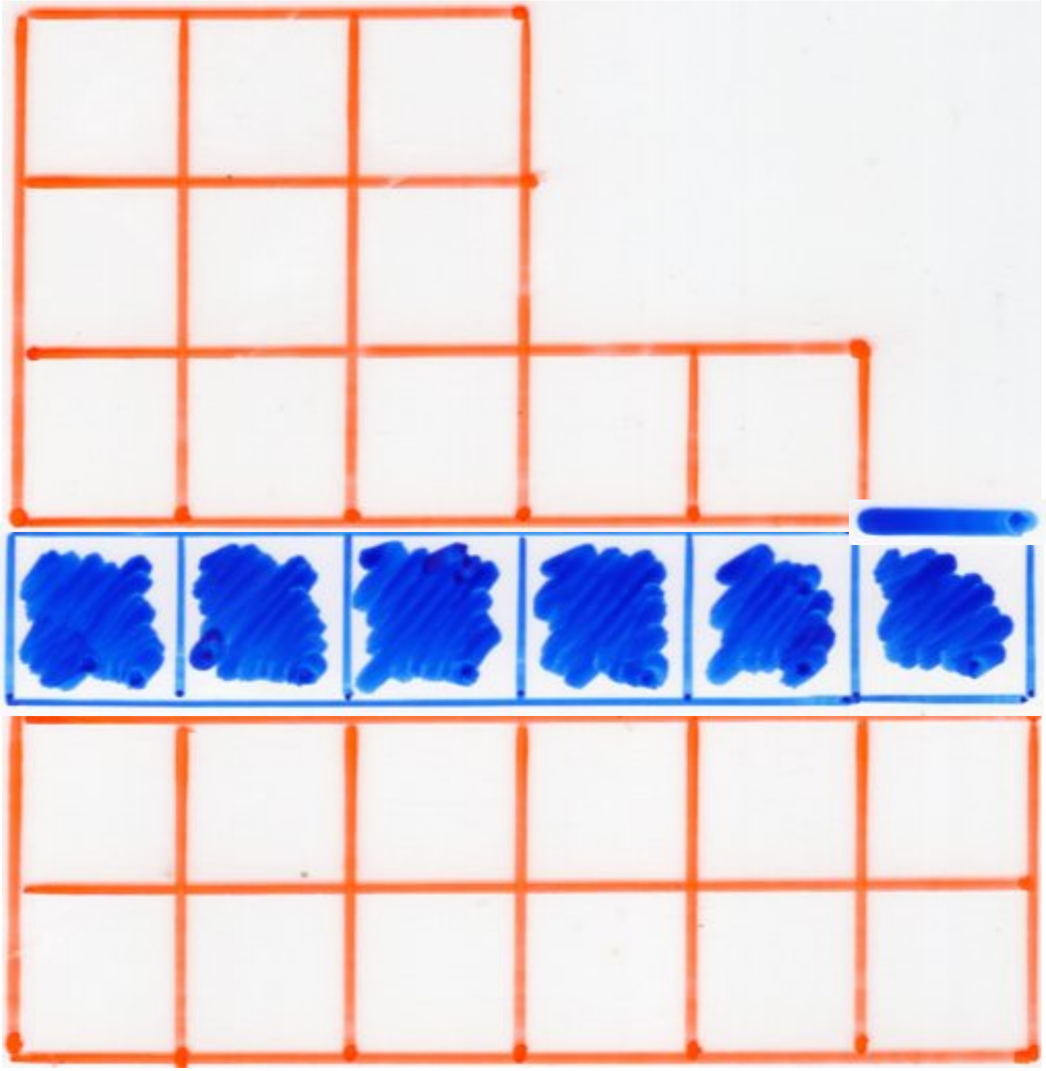
extensions of
Tamari
lattice

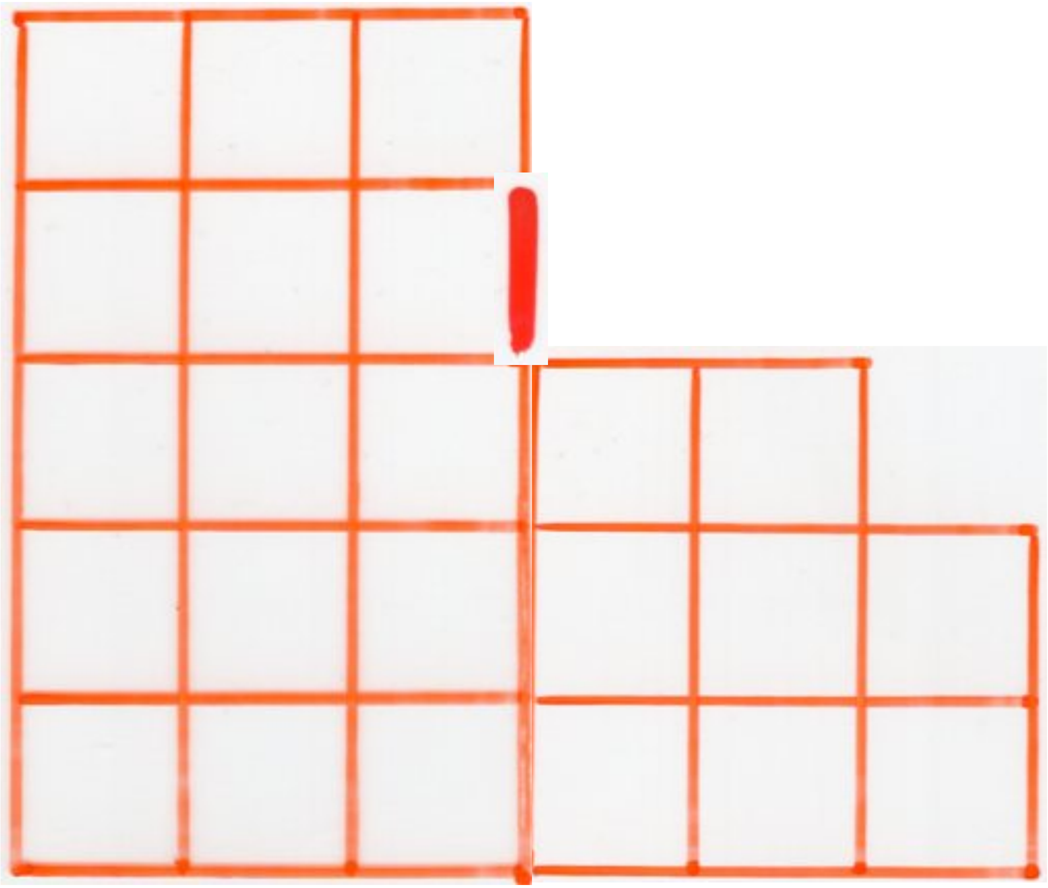


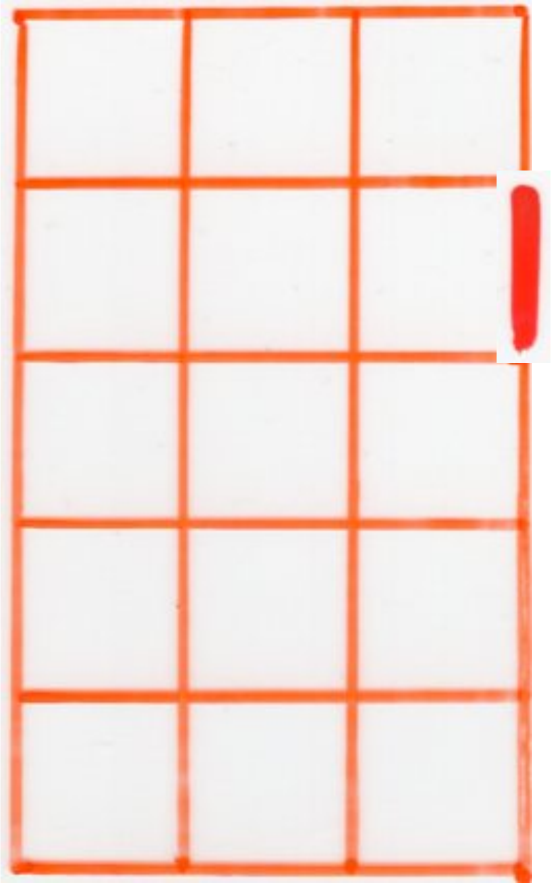


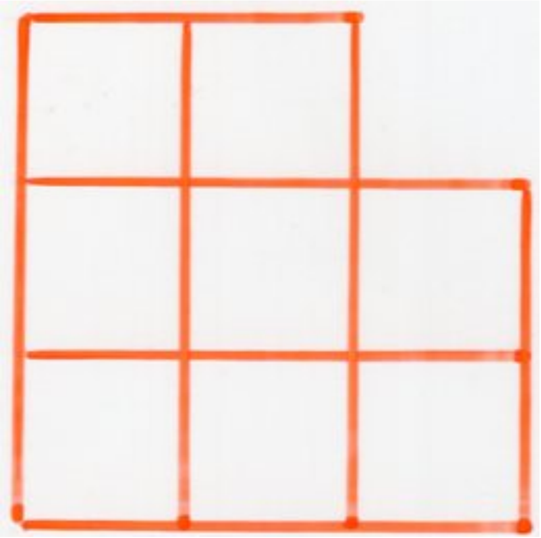


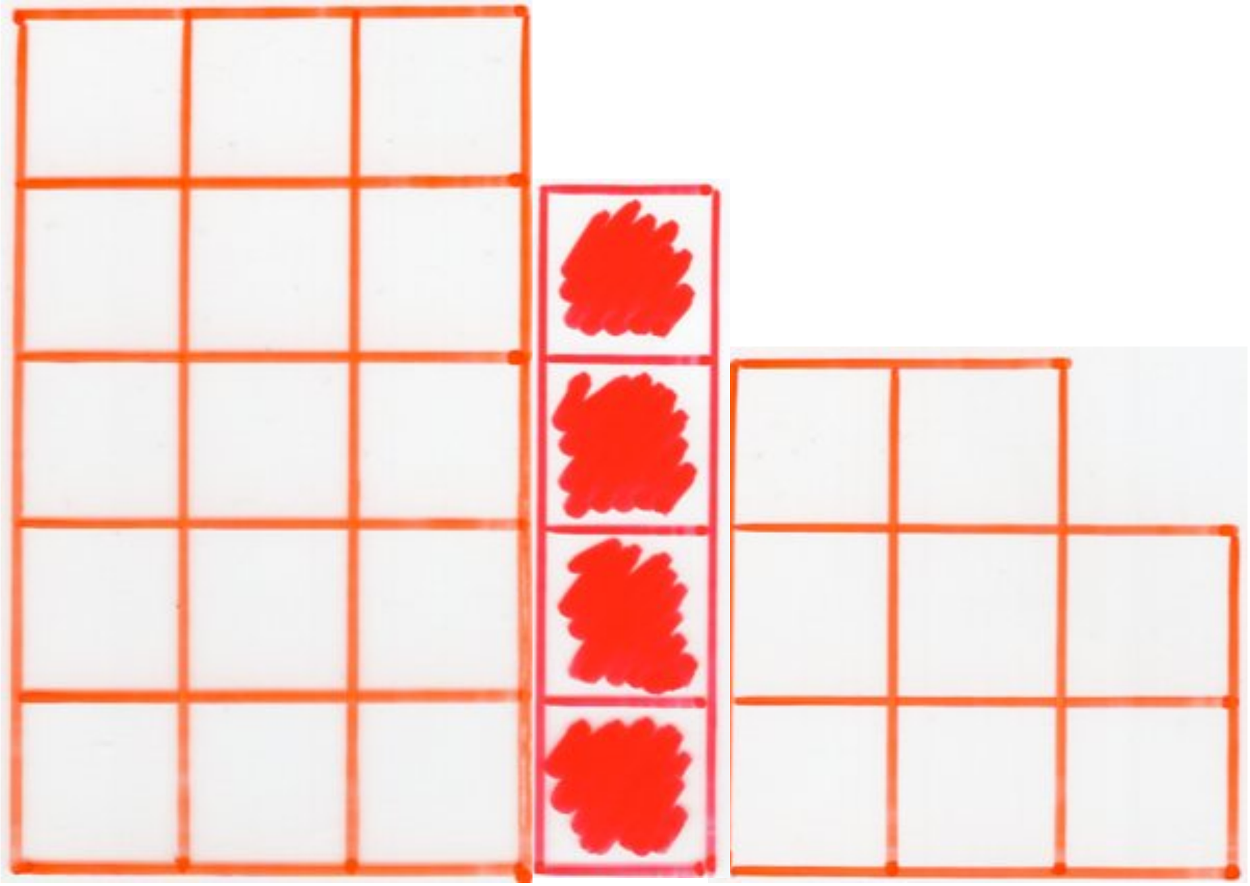


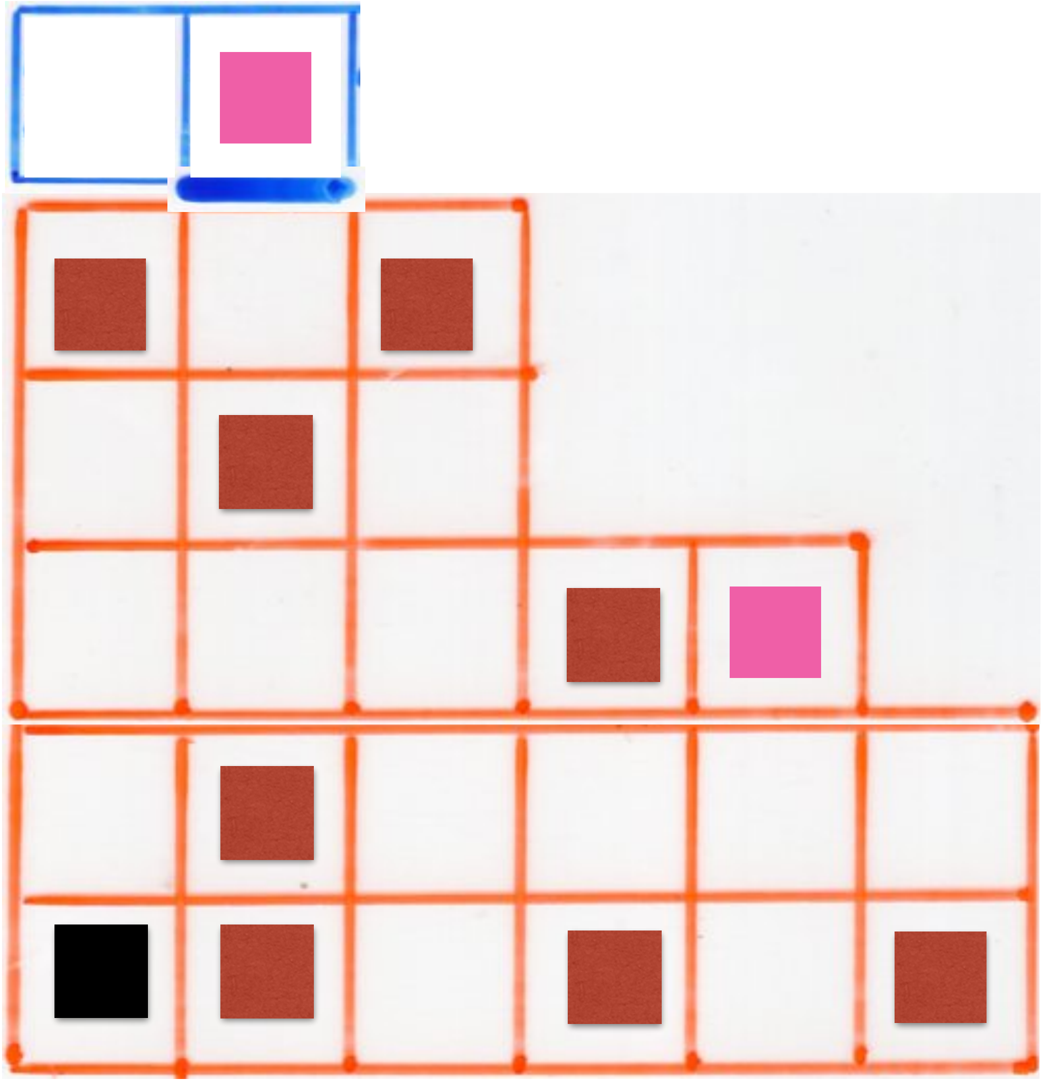


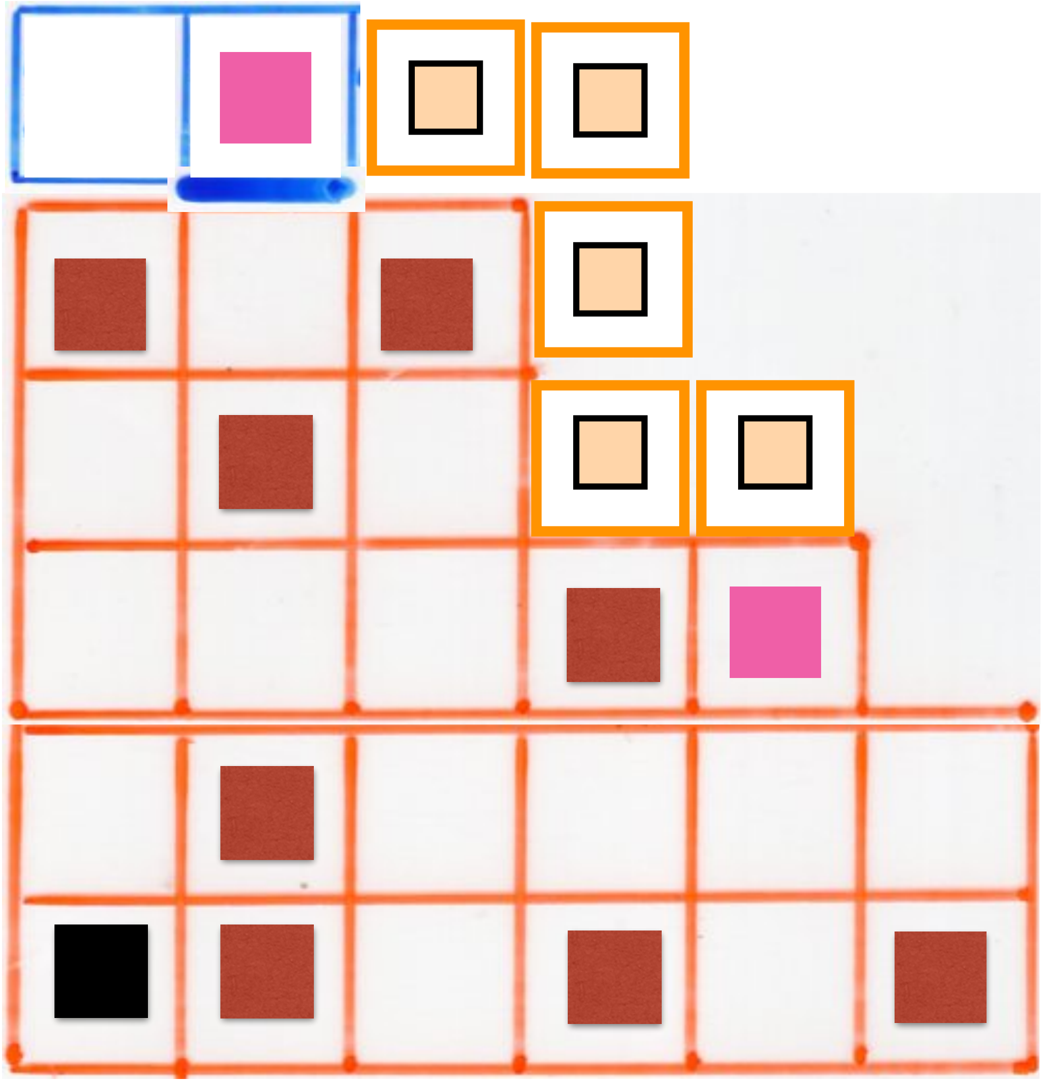


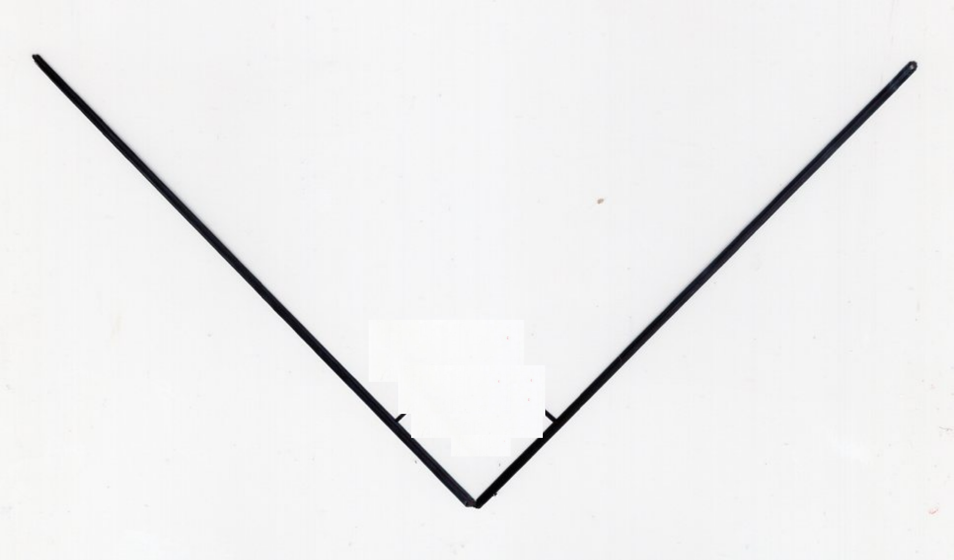






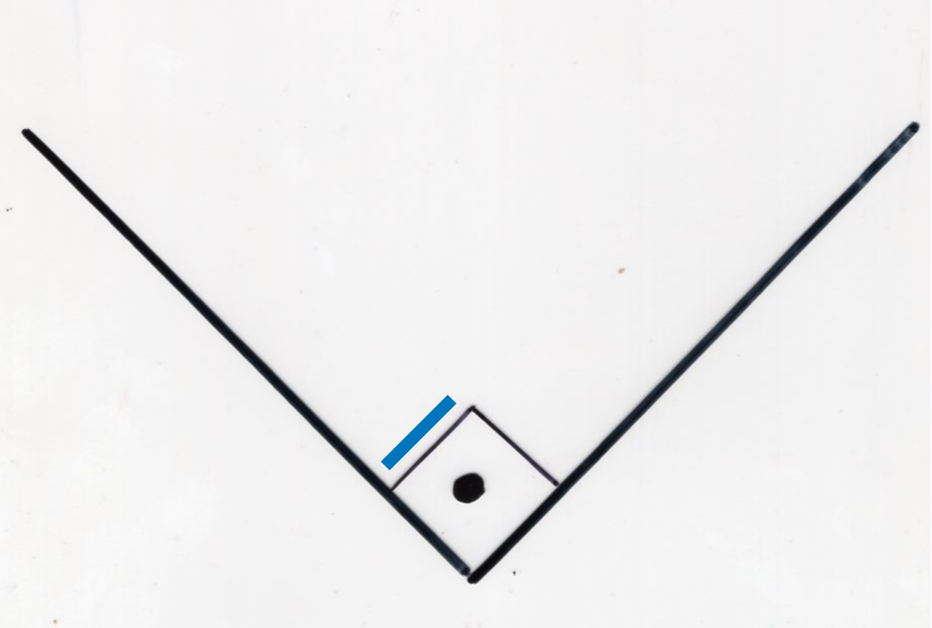






$\sigma = 6\ \ 2\ \ 7\ \ 3\ \ 5\ \ 1\ \ 8\ \ 4$

$\underbrace{\quad}_{1 \downarrow} \; 1 \; \underbrace{\quad}$



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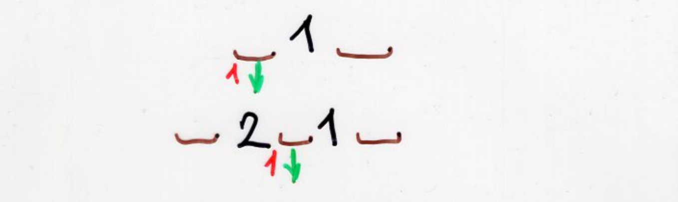

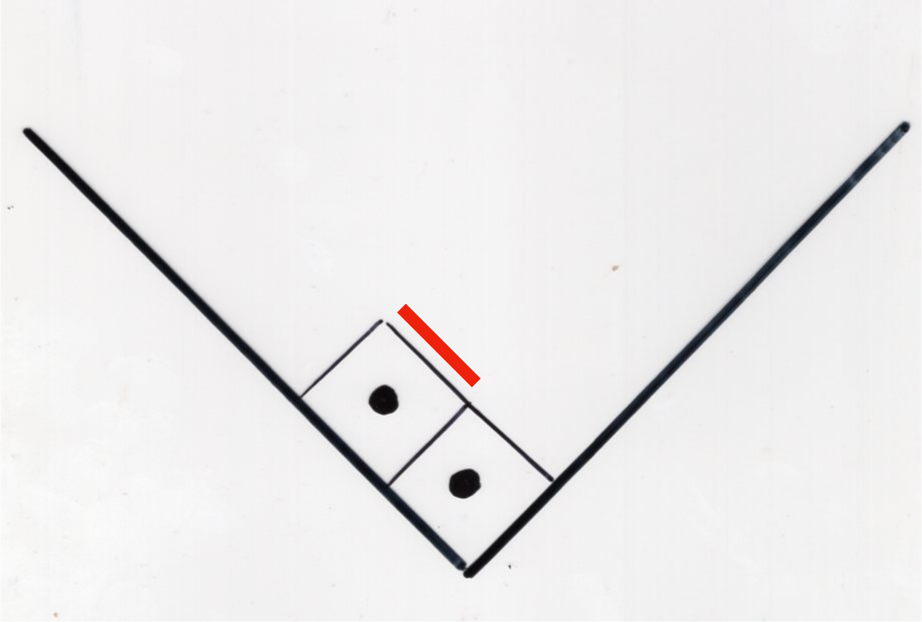

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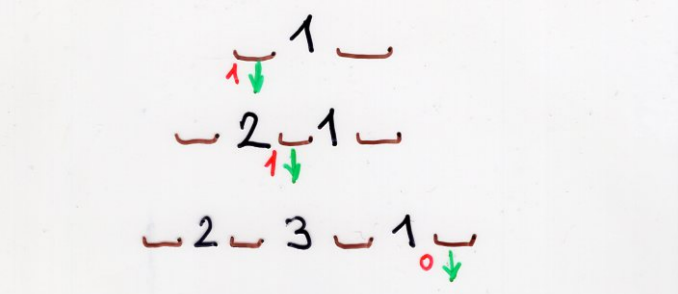
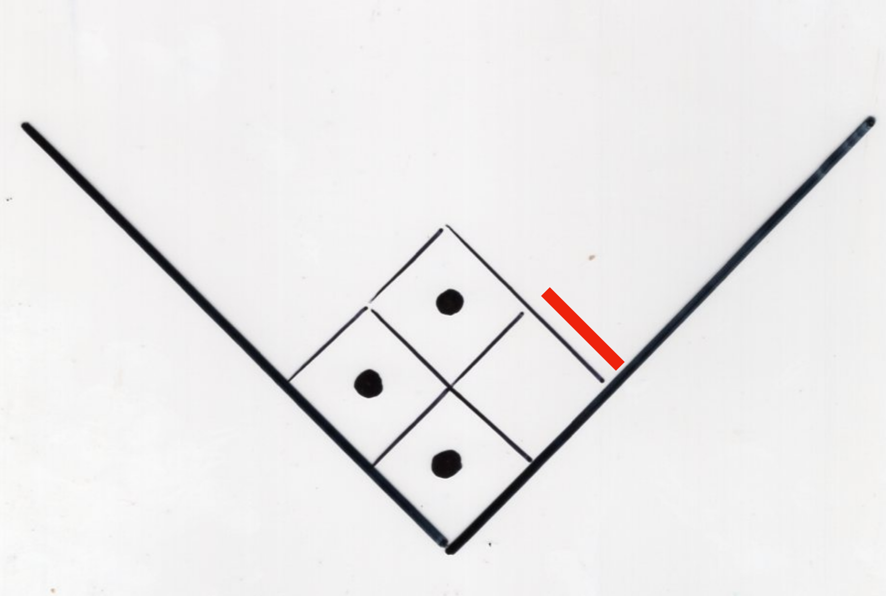
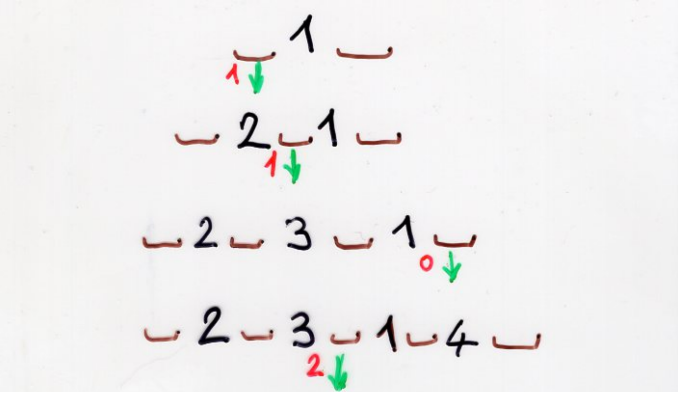
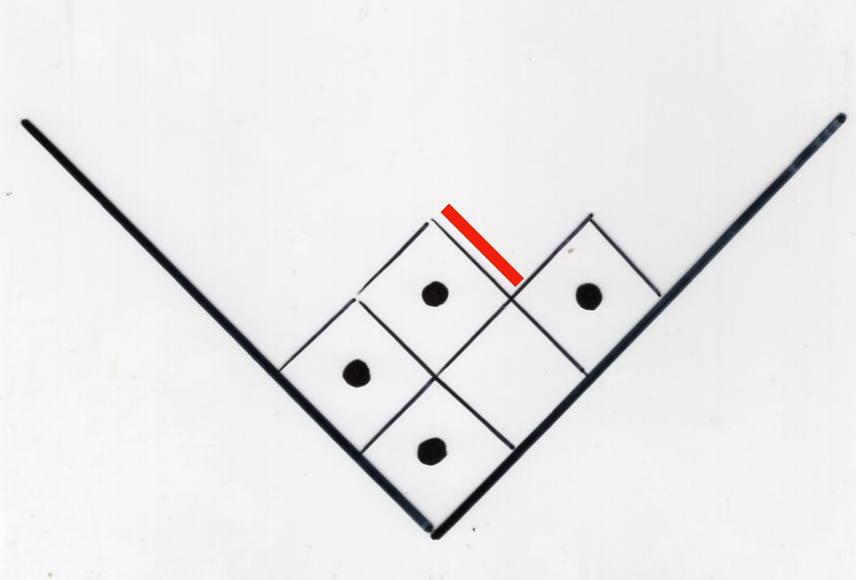


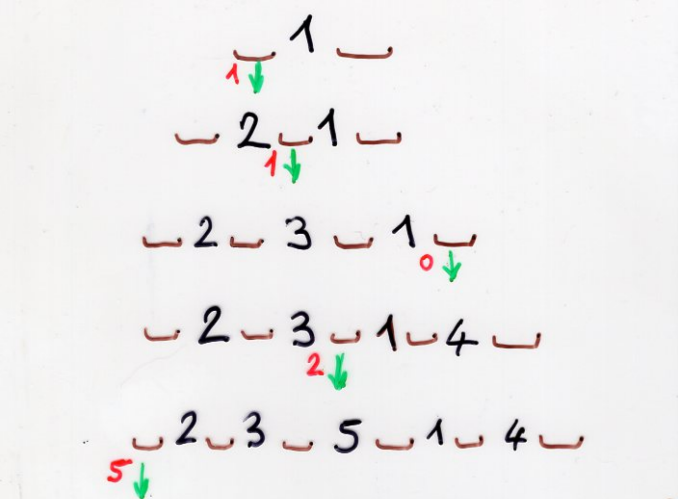


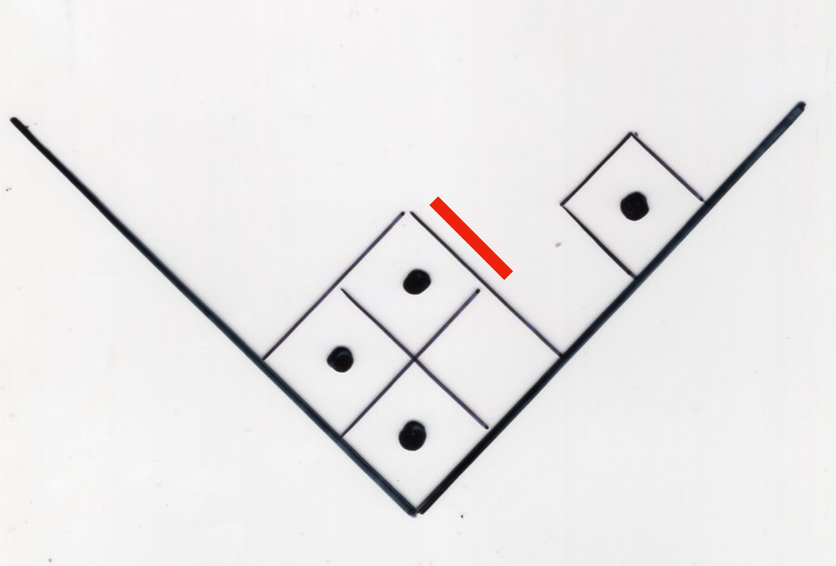
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$



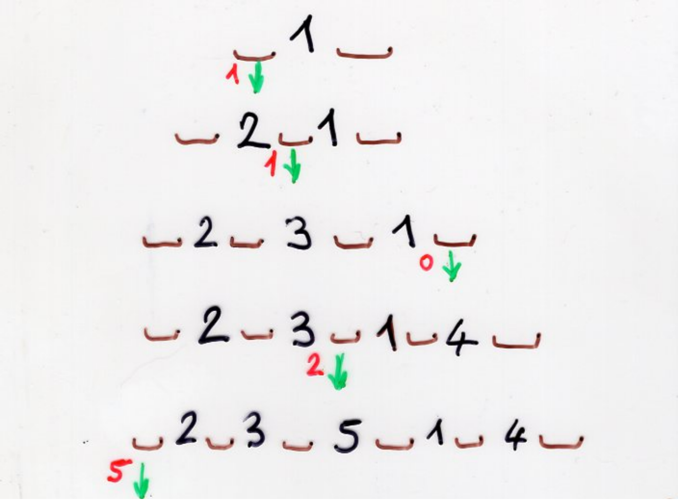


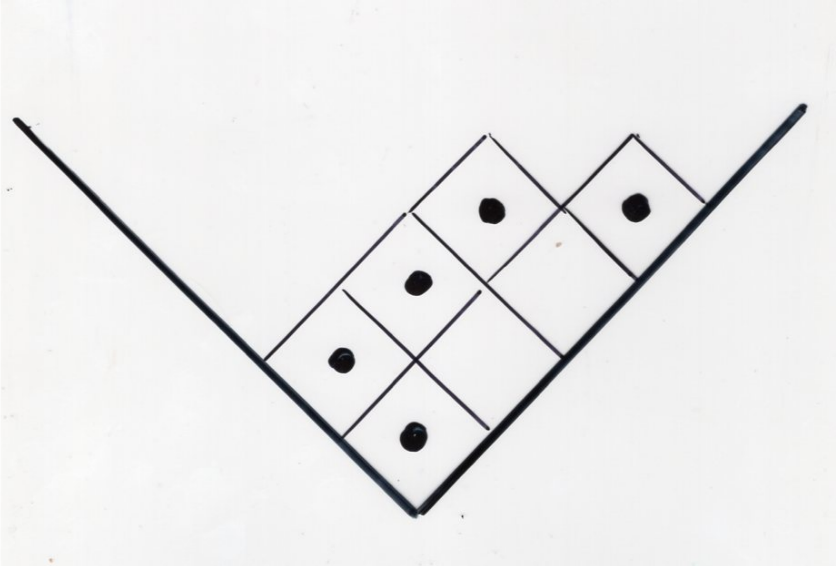
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$





$\sigma = 6\ \ 2\ \ 7\ \ 3\ \ 5\ \ 1\ \ 8\ \ 4$





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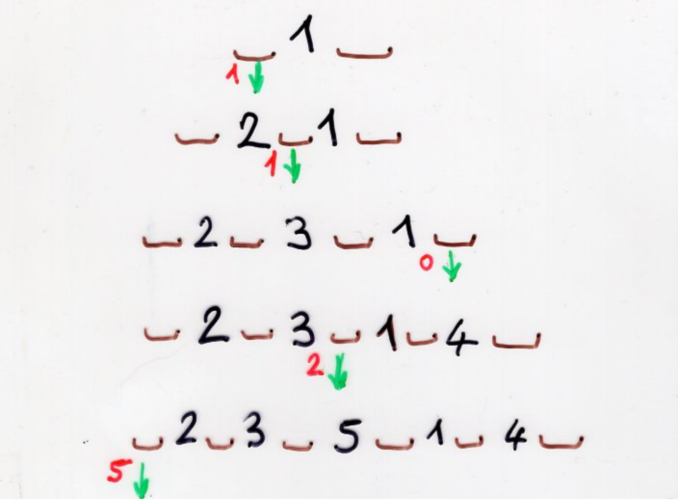
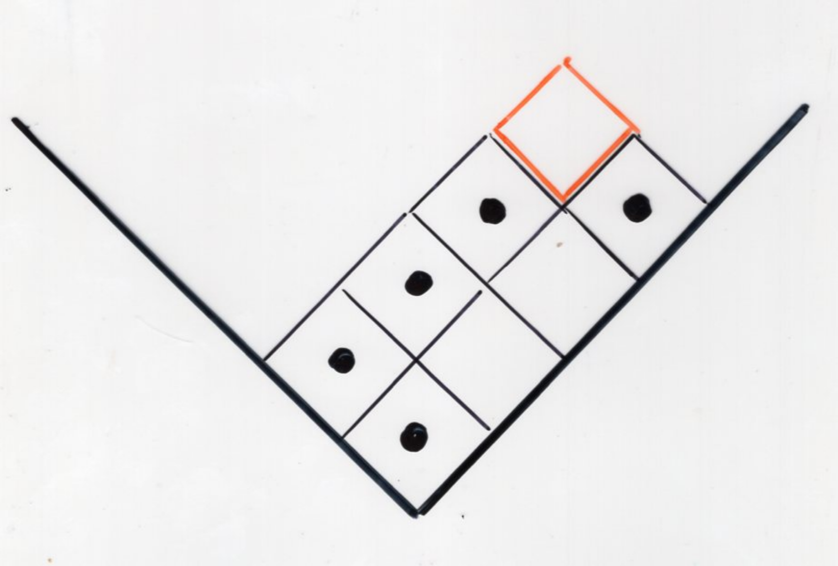
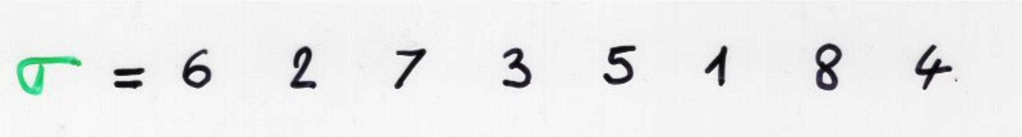
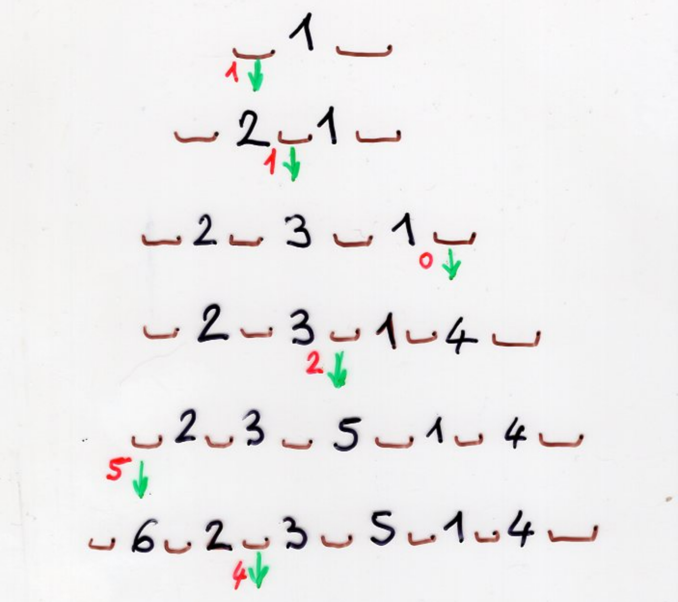
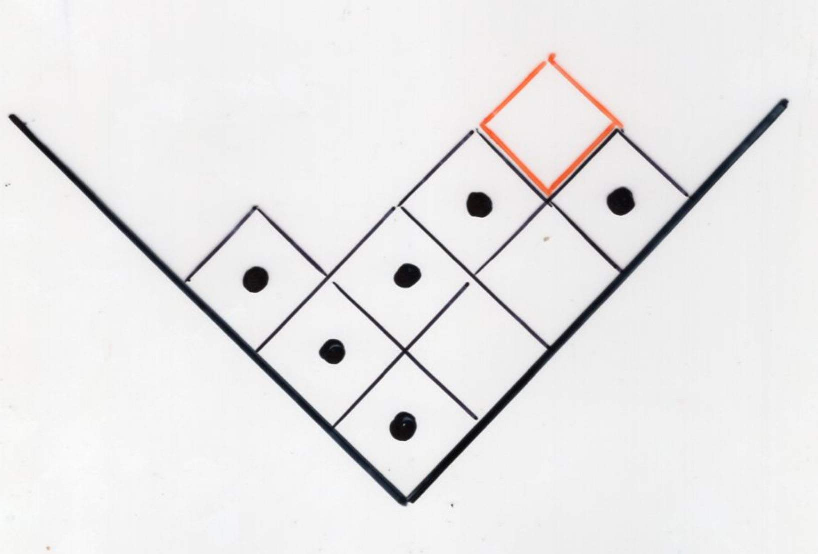


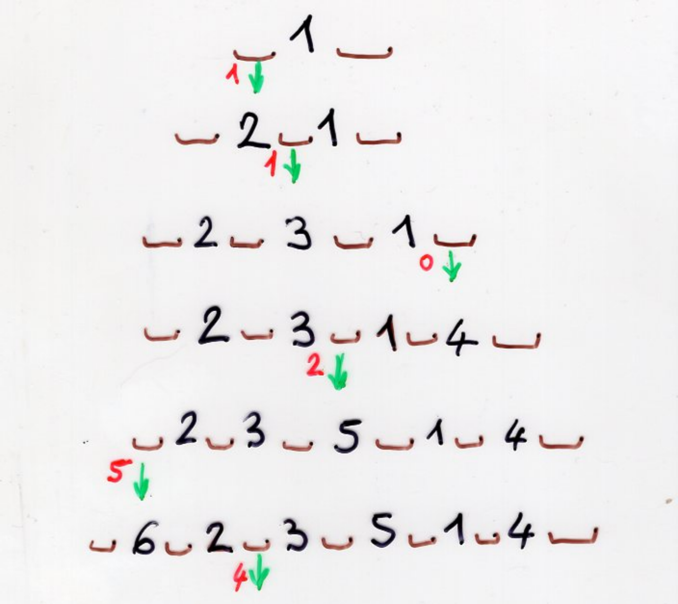


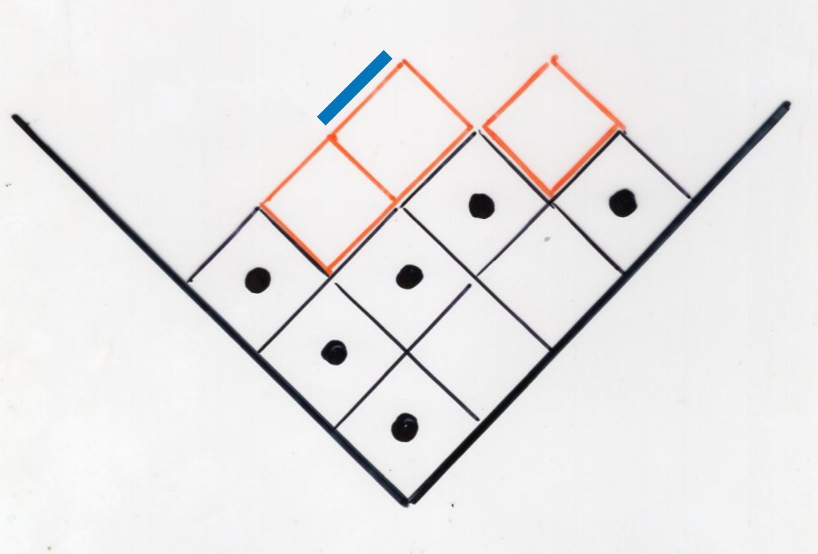

$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$

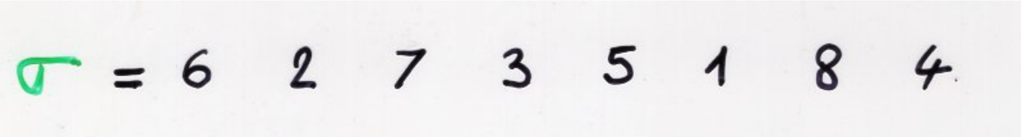




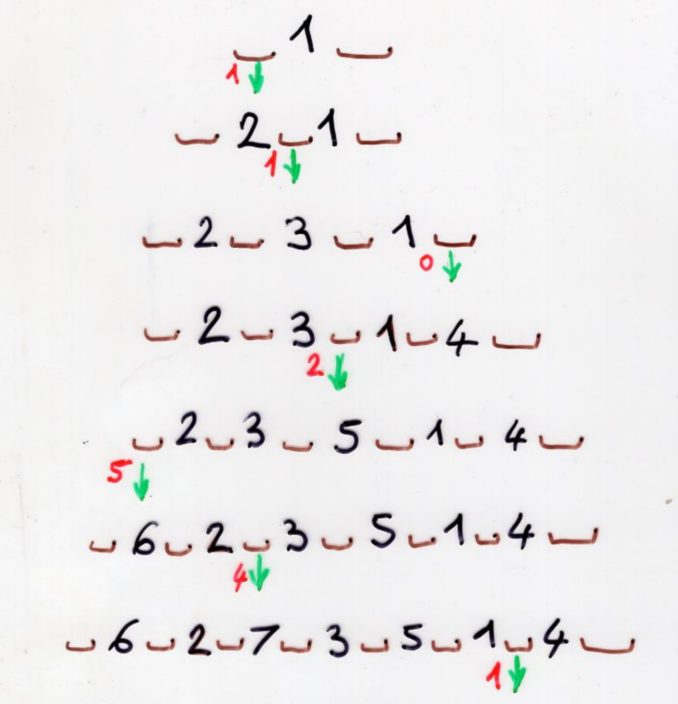
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$

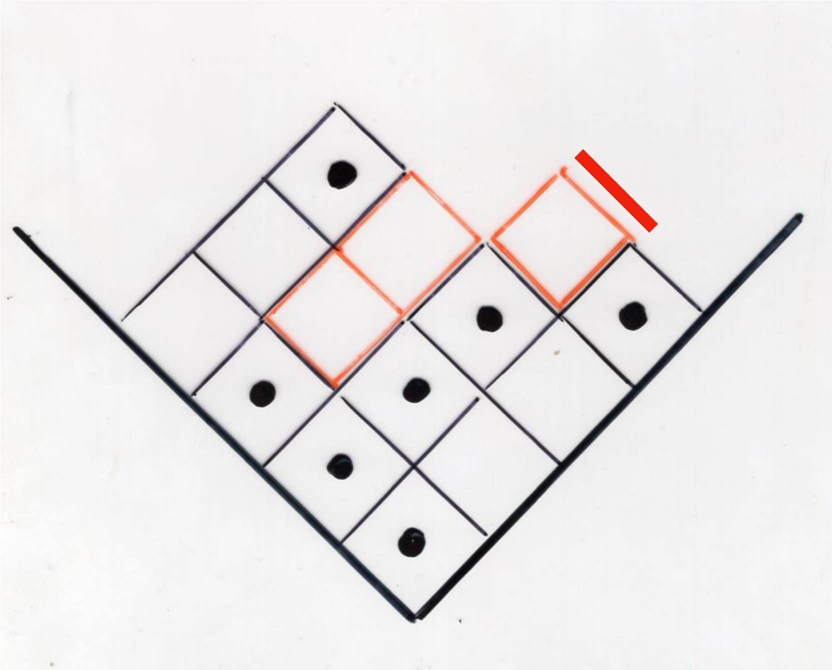




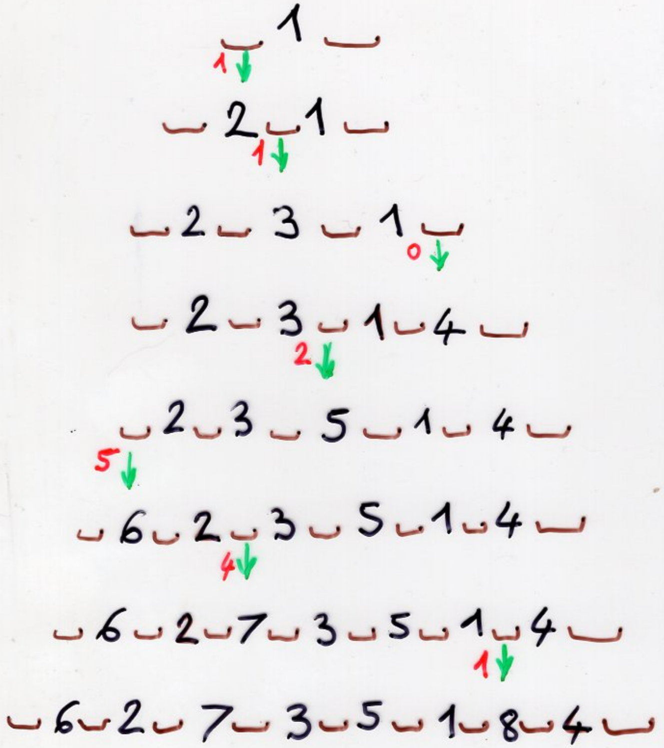


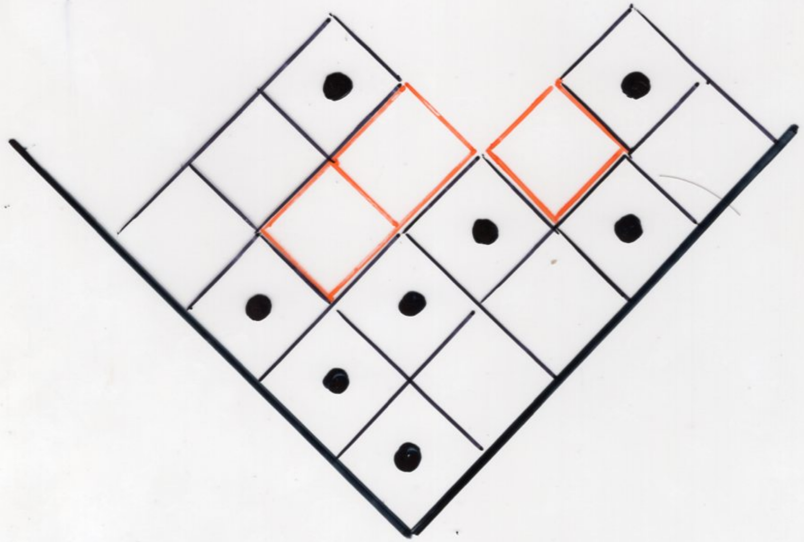
$\sigma = 6\ 2\ 7\ 3\ 5\ 1\ 8\ 4$



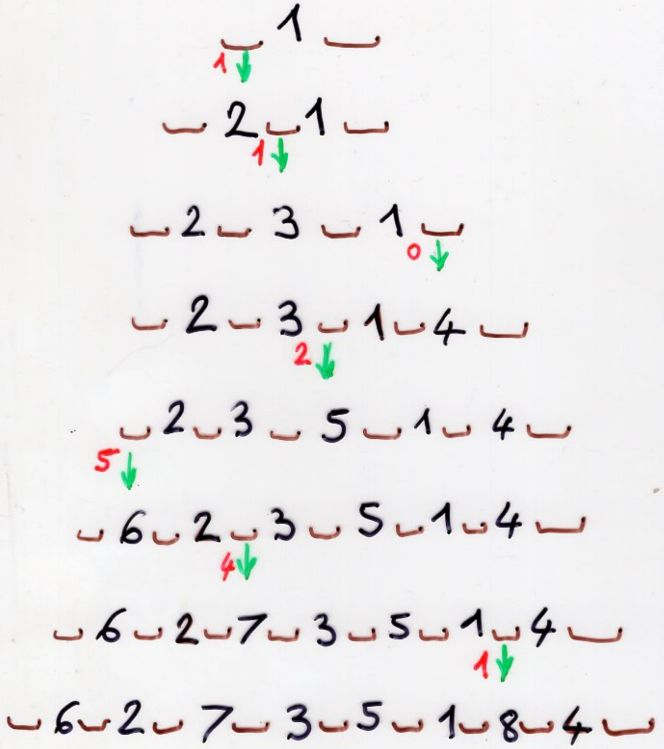


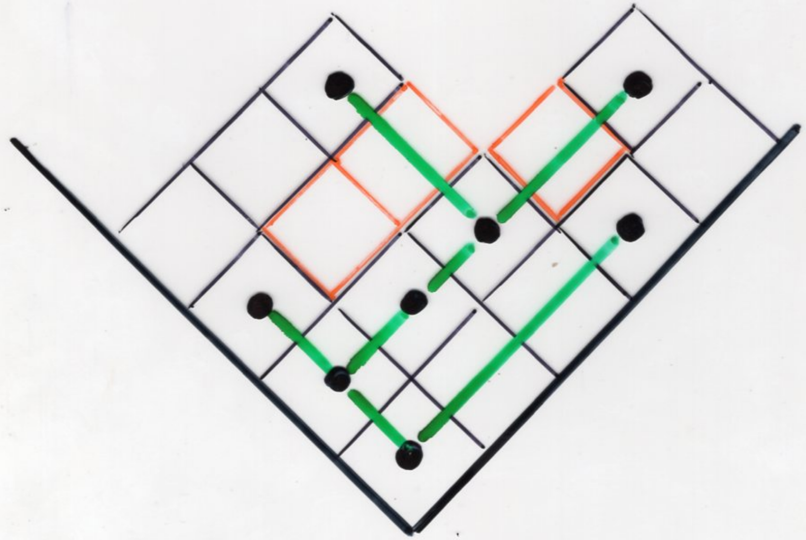
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$



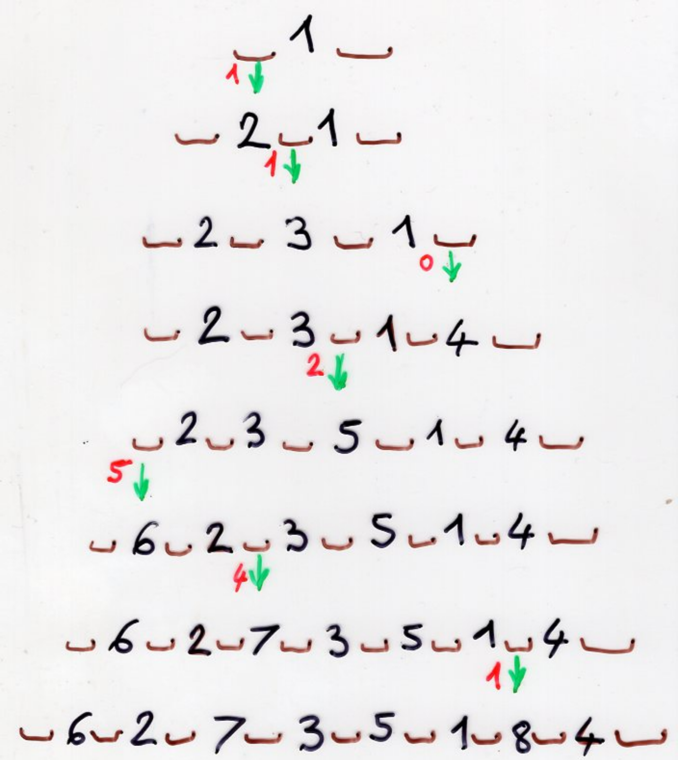


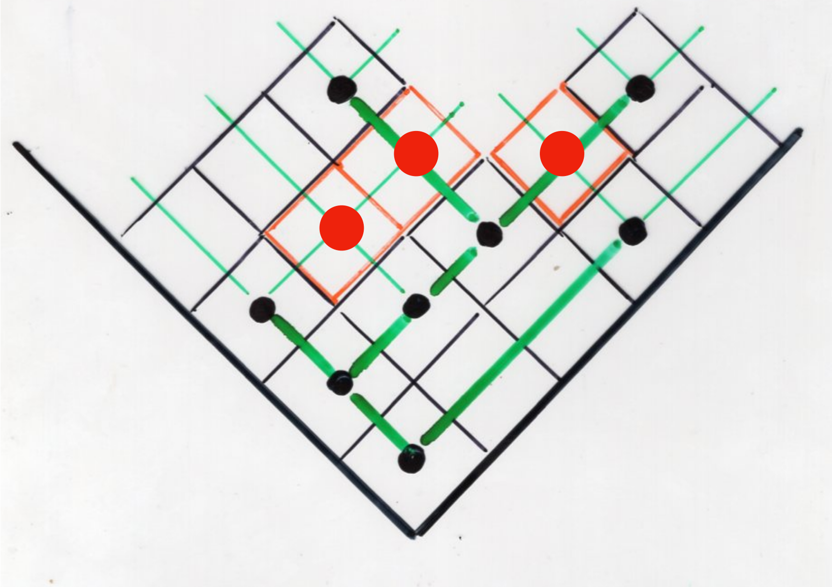
$\sigma = 6\ \ 2\ \ 7\ \ 3\ \ 5\ \ 1\ \ 8\ \ 4$



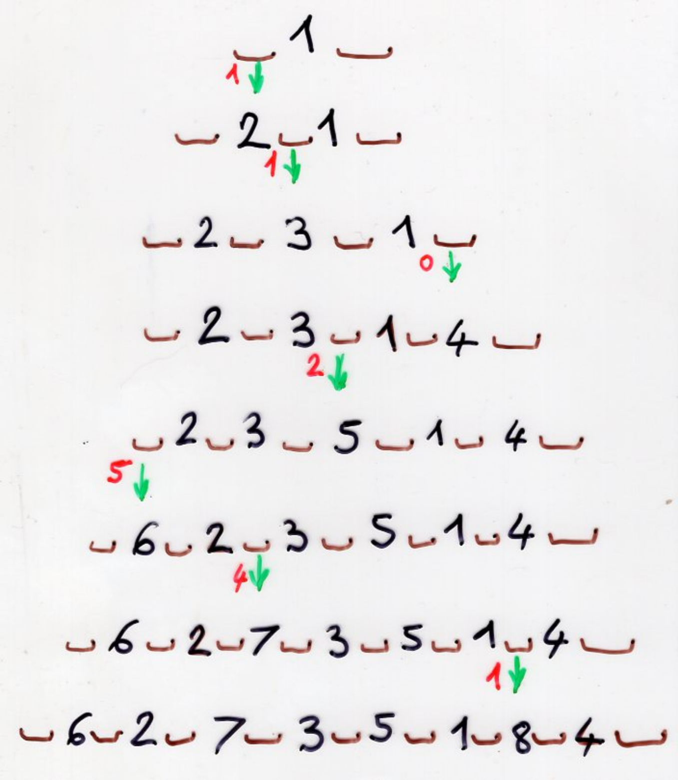


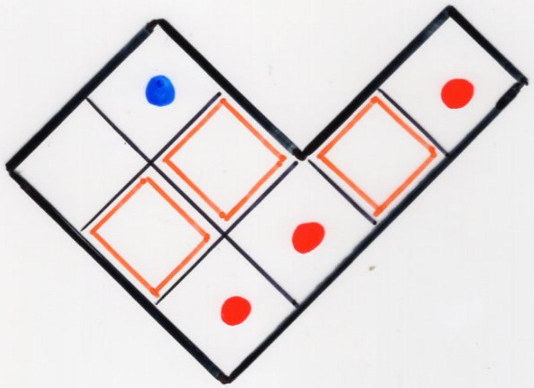
$\sigma = 6\quad 2\quad 7\quad 3\quad 5\quad 1\quad 8\quad 4$





$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$



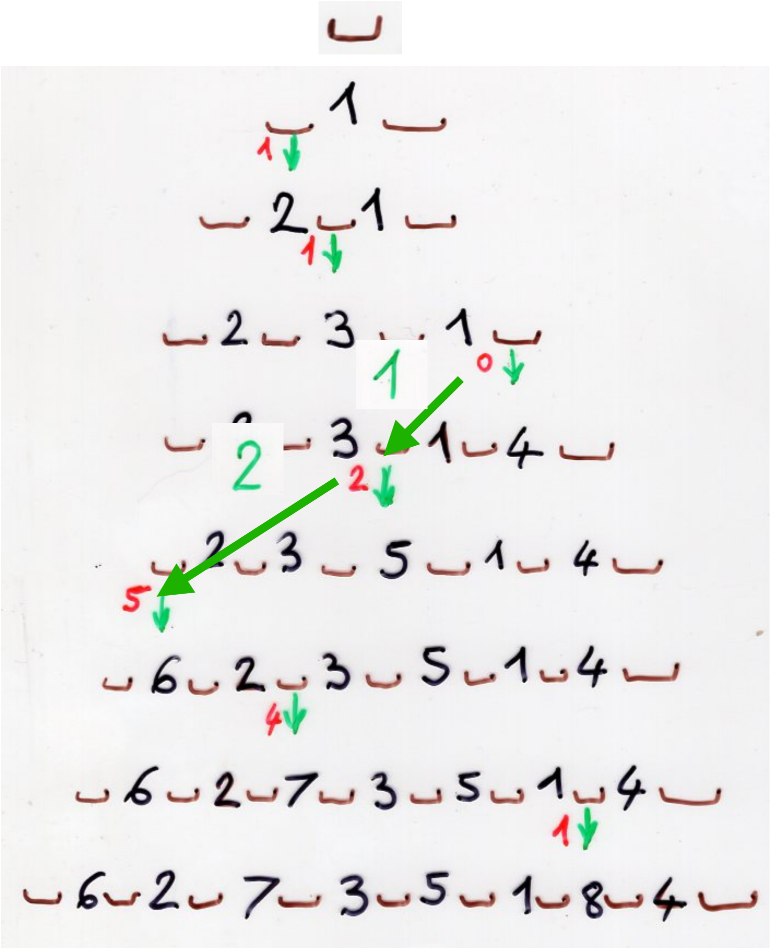


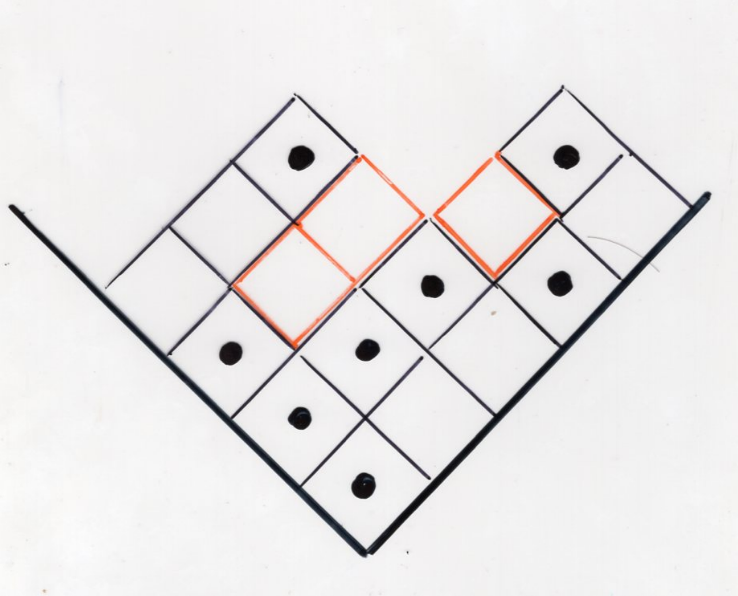
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$

bijection

$f \longrightarrow T$

number of crossings $cr(T)$

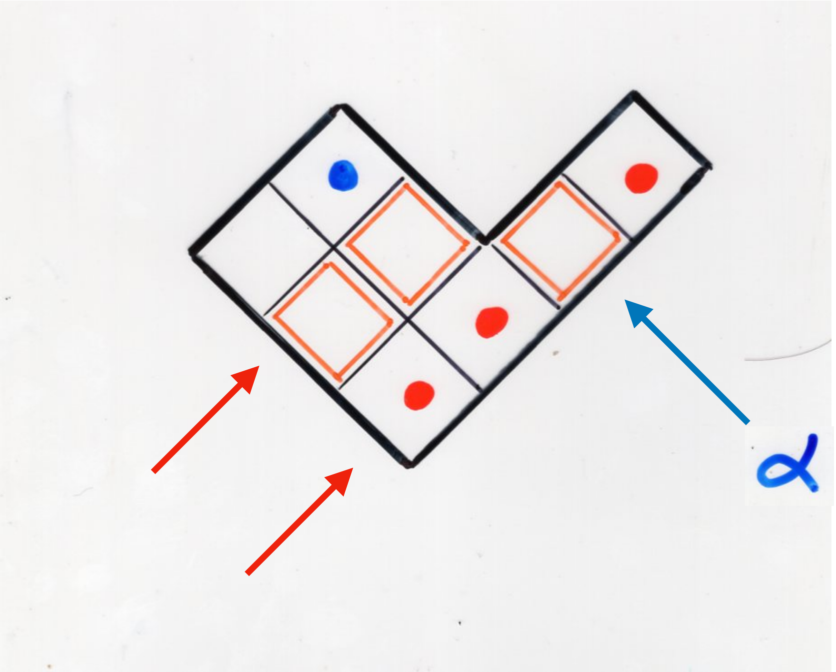


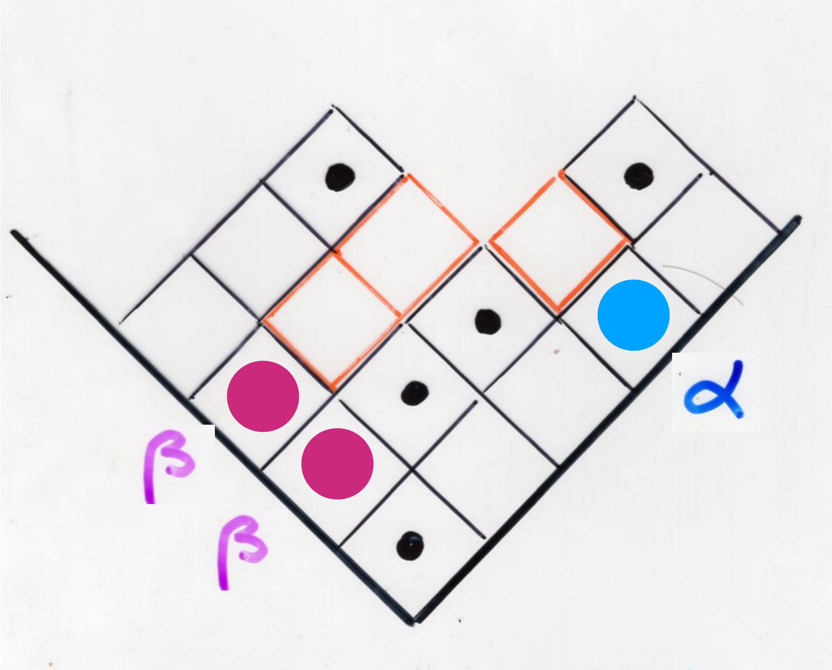


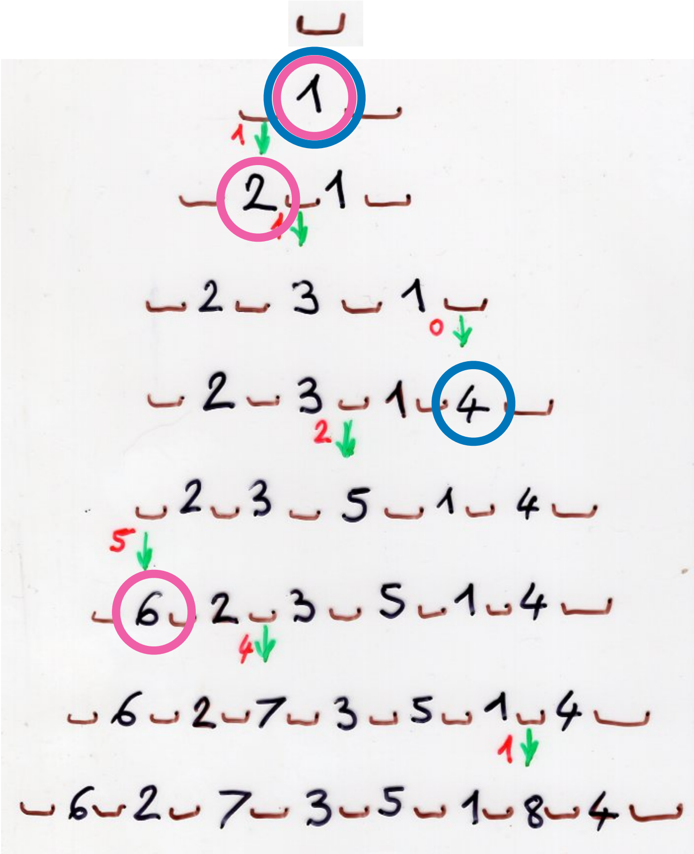
= sum of the length of all rim-hooks added in the insertion algorithm

$$= \sum_{1 \leqslant i \leqslant (n-1)} \max\Big[\big(f(i+1) - f(i)\big), 0\Big]$$

-1


α


α
β
β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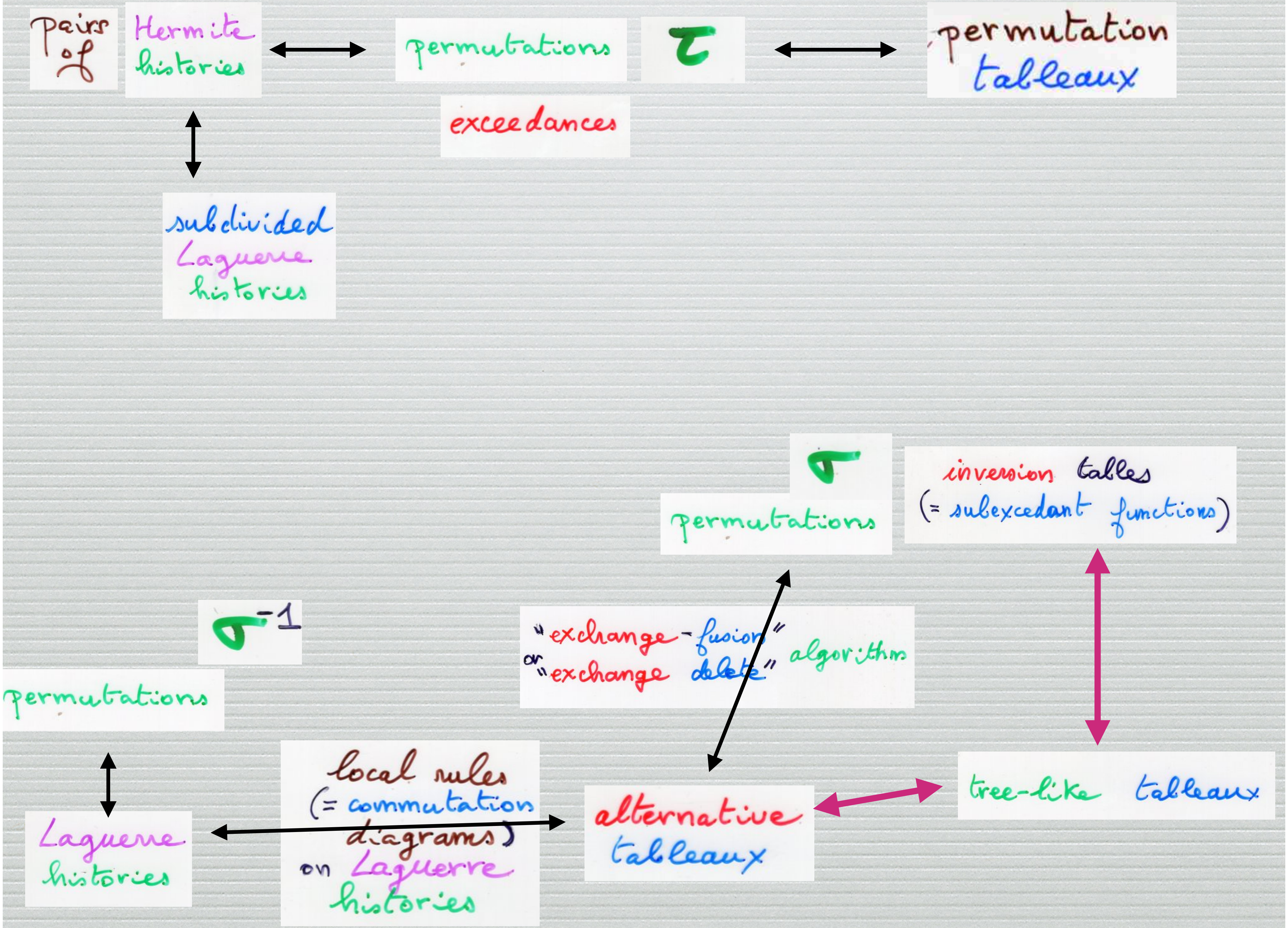
2 3 5 1 4
5
6 2 3 5 1 4
4
6 2 7 3 5 1 4
1
6 2 7 3 5 1 8 4



Pairs of Hermite histories

permutations τ

exceedances

permutation tableaux

subdivided Laguerre histories

σ
permutations

inversion tables
(= subexcedant functions)

"exchange-fusion" or "exchange delete" algorithm

σ^{-1}
permutations

Laguerre histories

local rules
(= commutation diagrams)
on Laguerre histories

alternative tableaux

tree-like tableaux

Laguerre heaps of segments

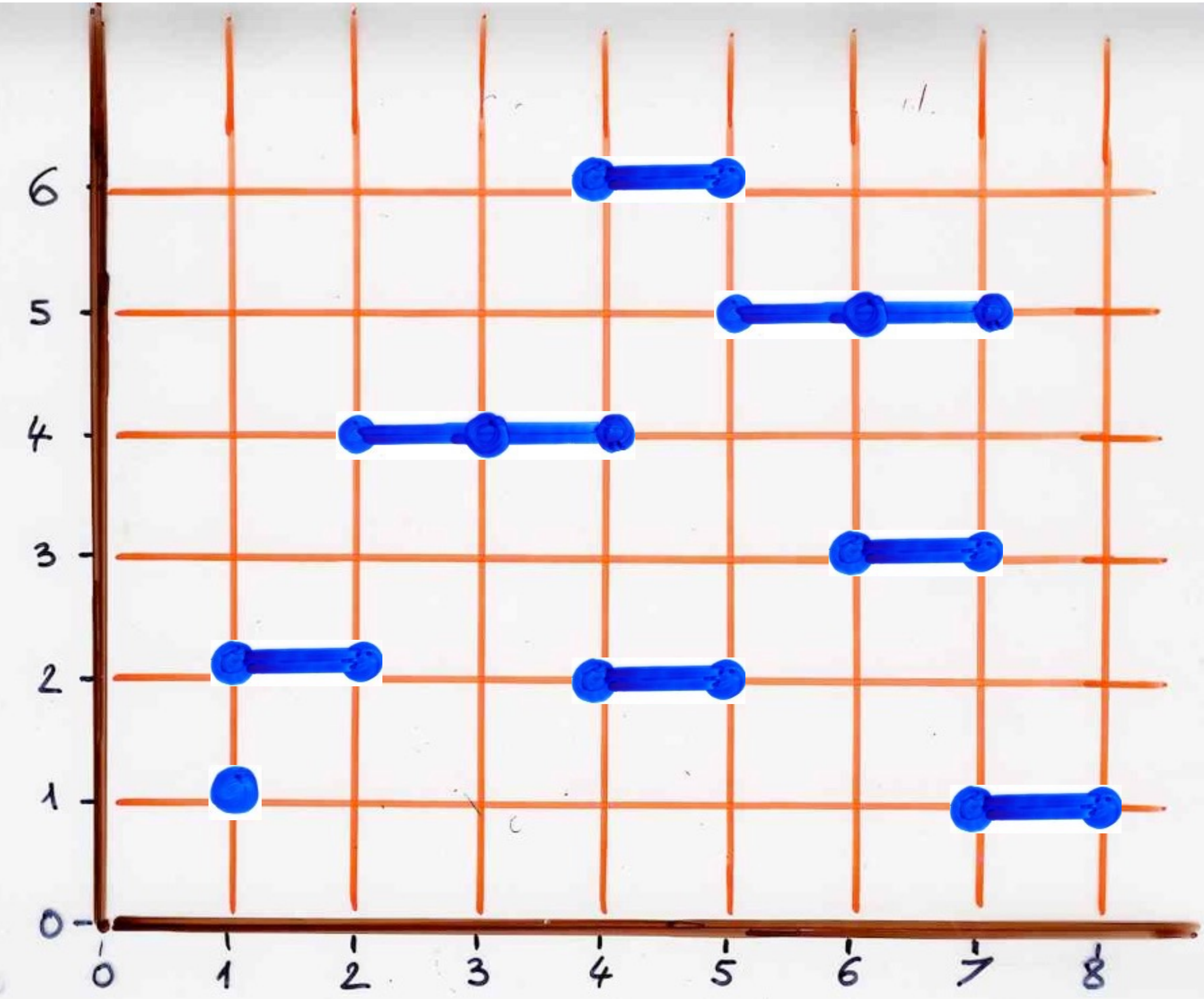
non-ambiguous
trees

tree-like
tableaux
rectangular shape

complete
non-ambiguous
trees

E. Jin (2014)

Bessel functions
heaps
logarithmic lemma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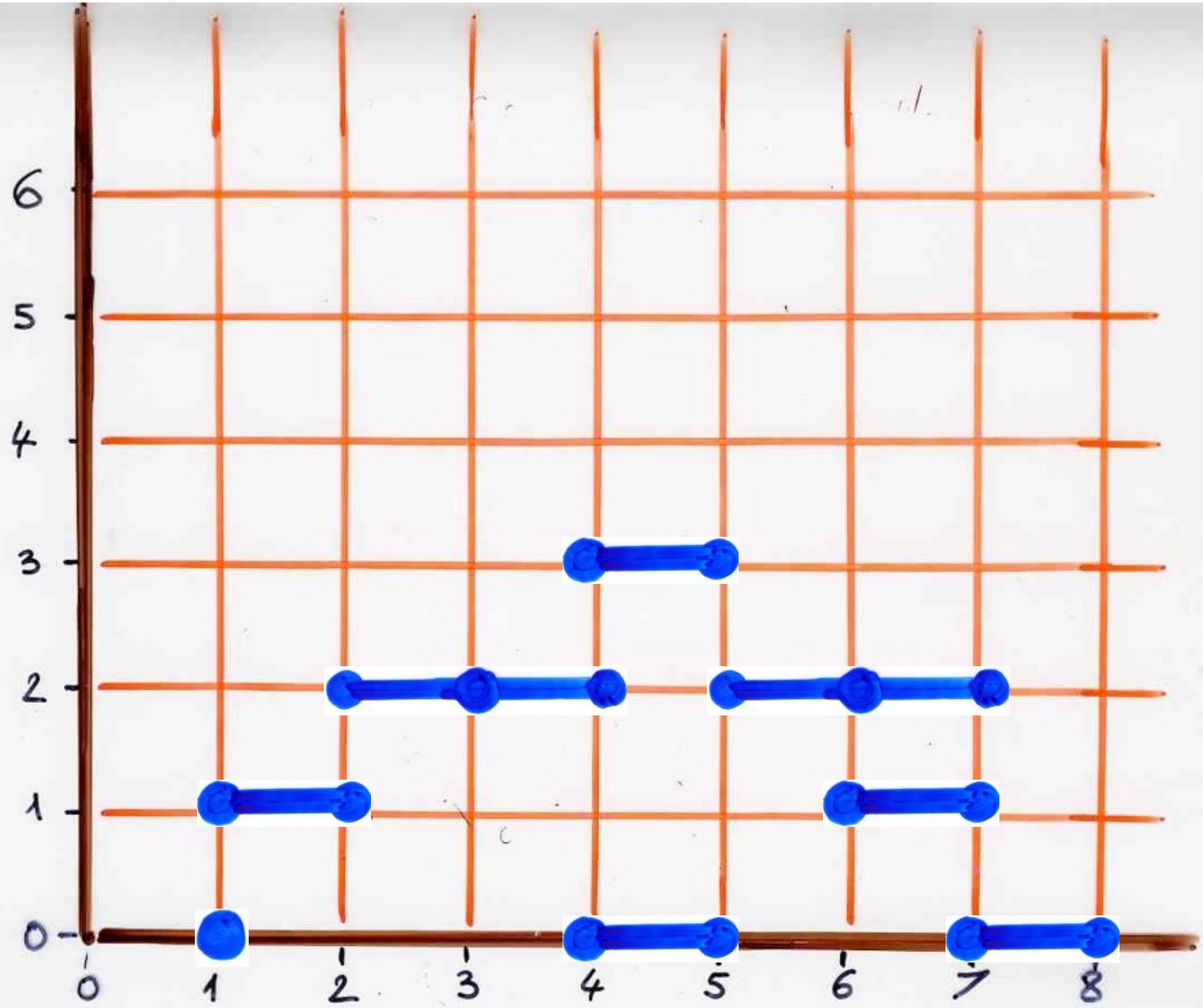
7
8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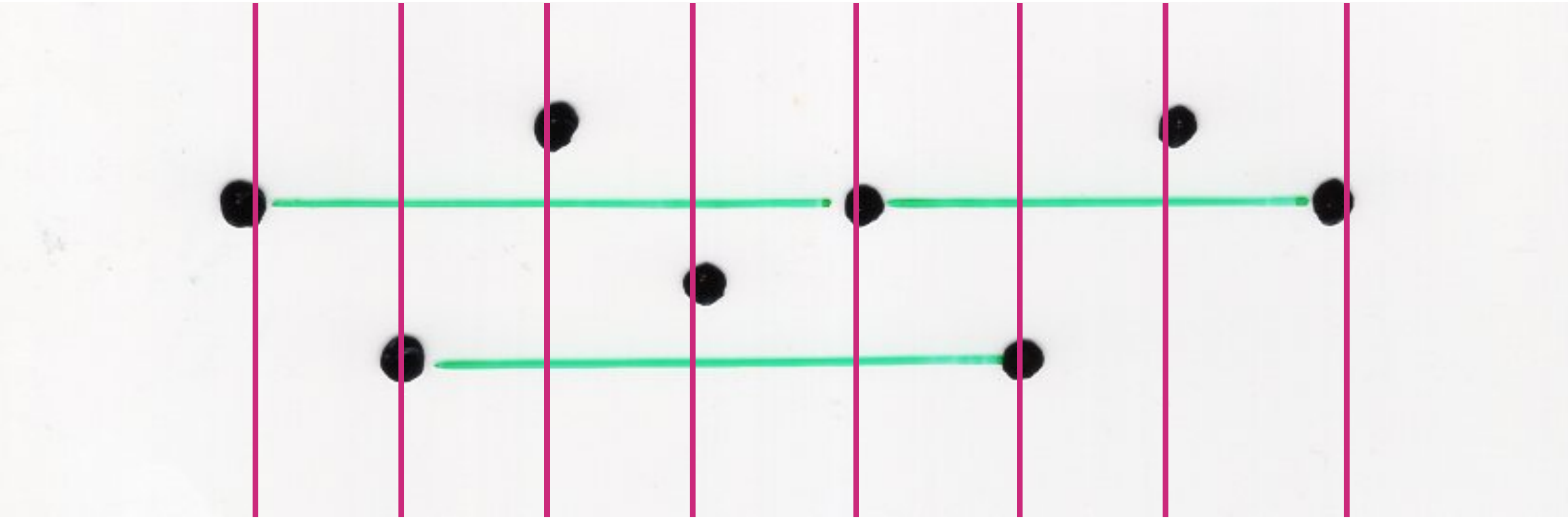


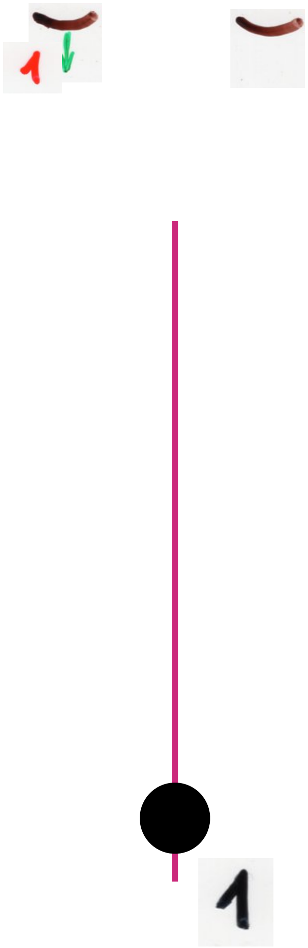
Laguerre heaps of segments

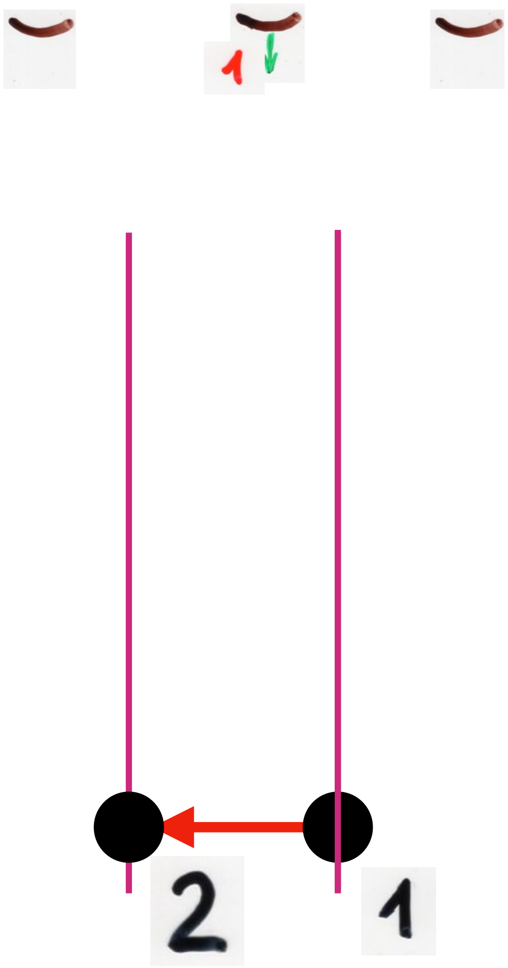
Bijection

Permutions

↕

Laguerre heaps of segments


1
1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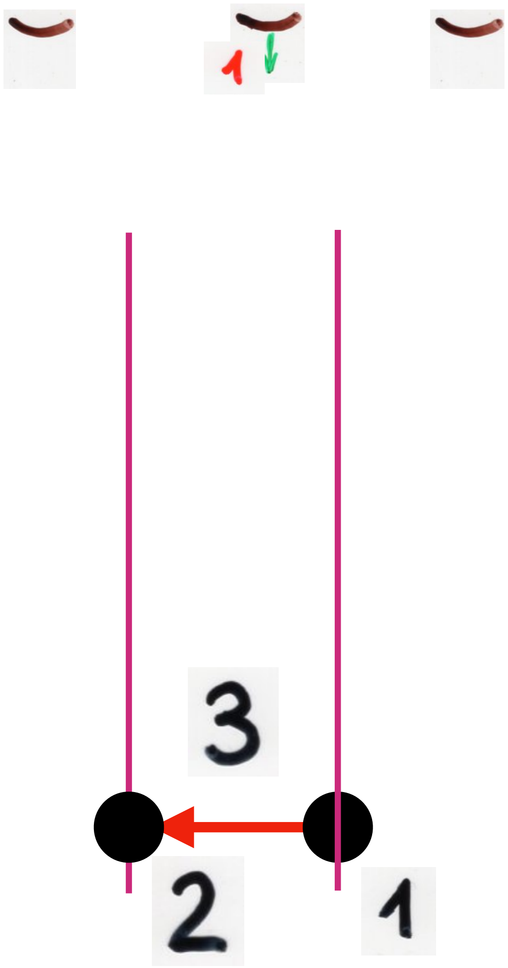
1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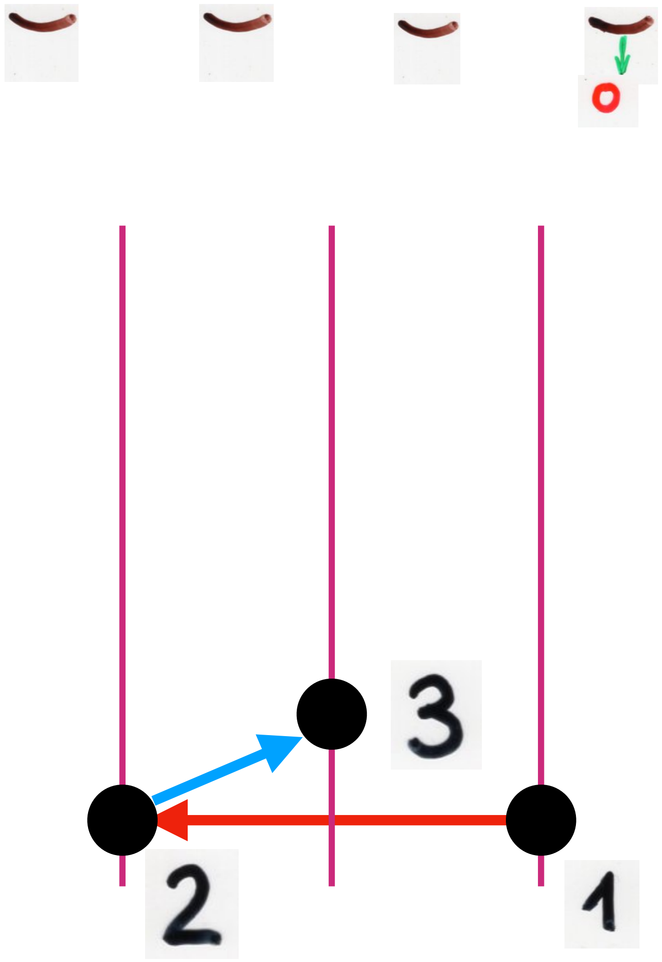
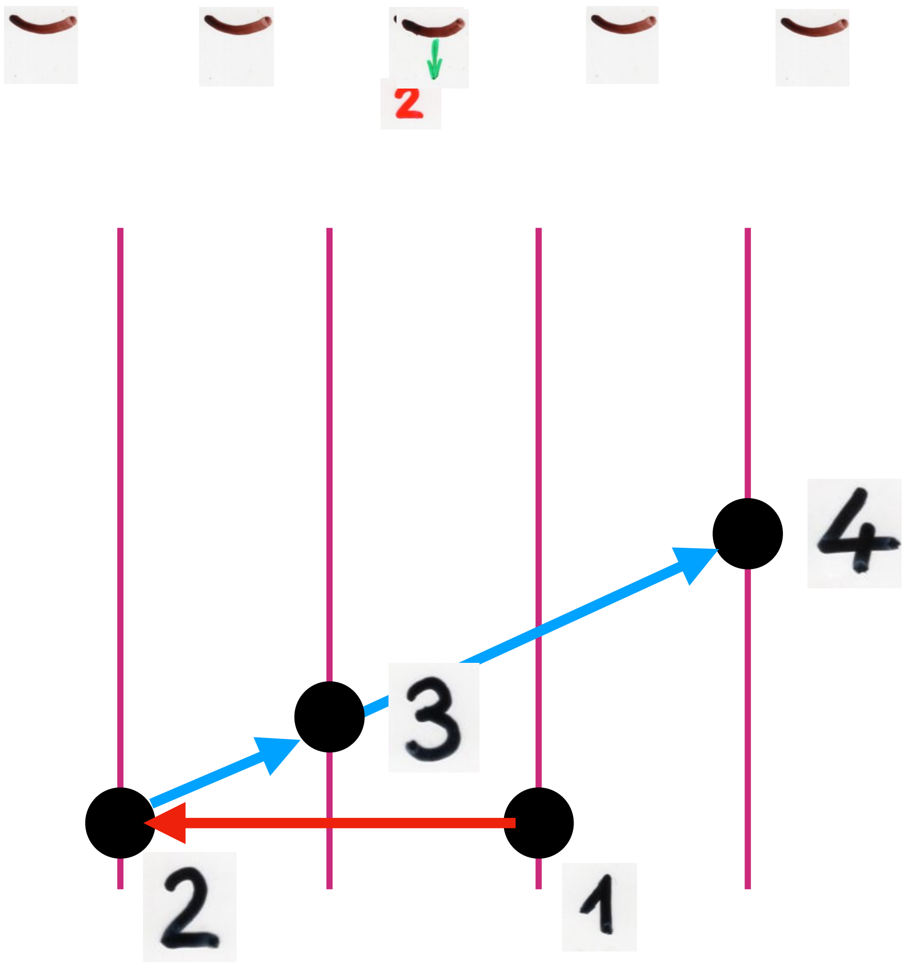
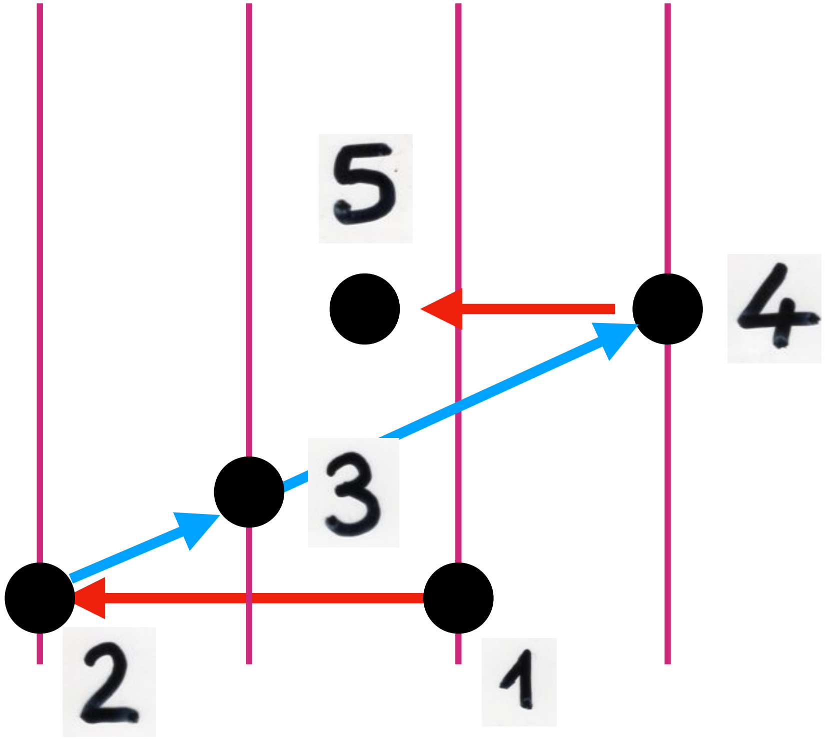
3
2
1


o
3
2
1


2
4
3
2
1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4
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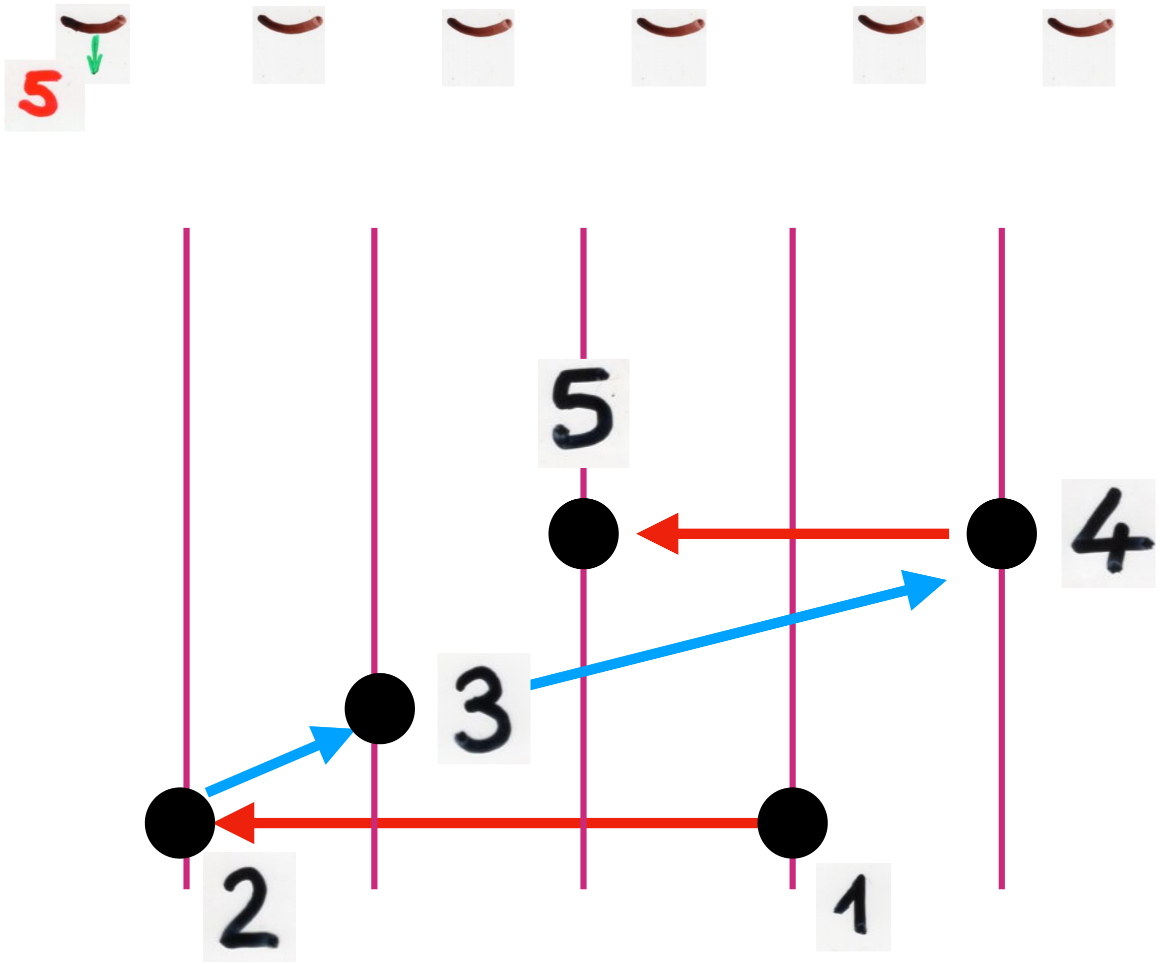
2
1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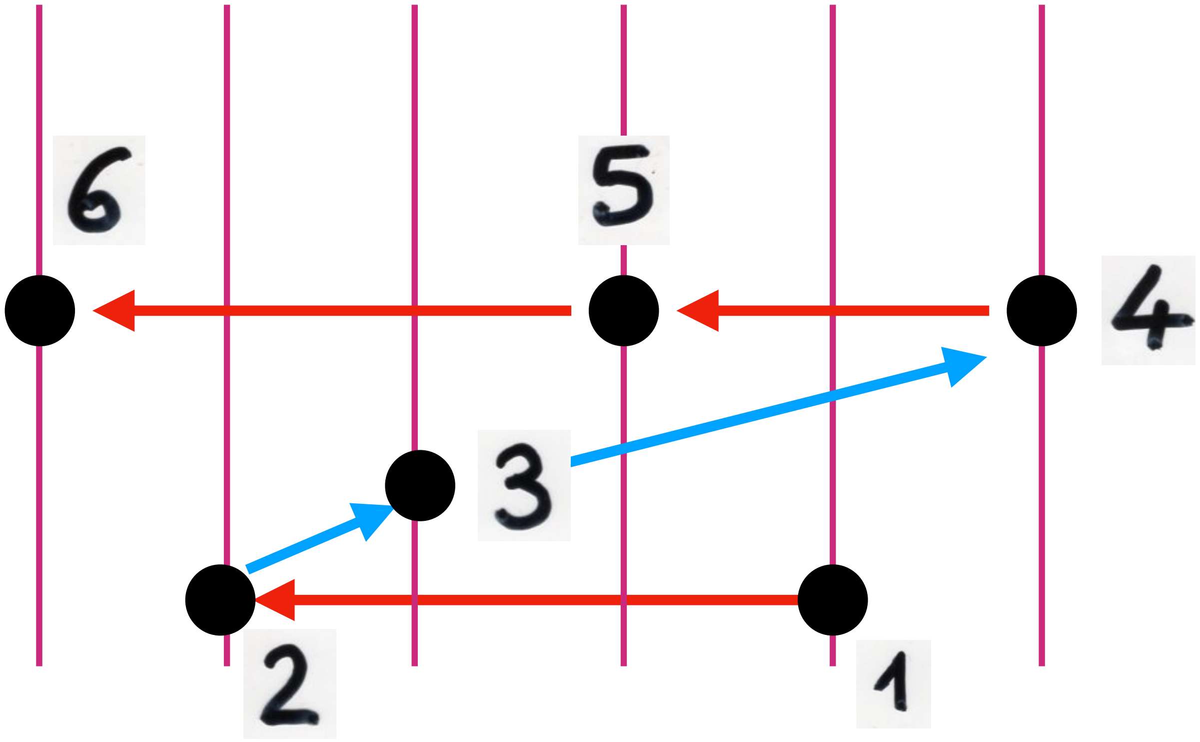
5
4
3
2
1


4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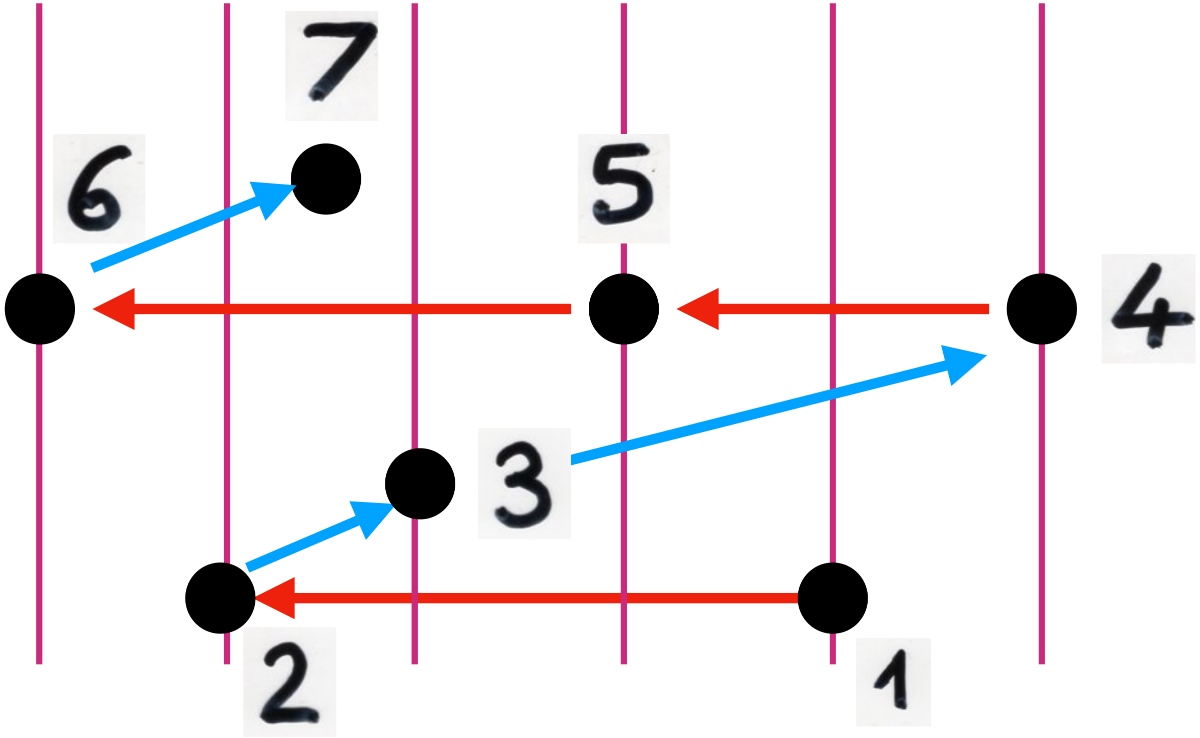
1


4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2
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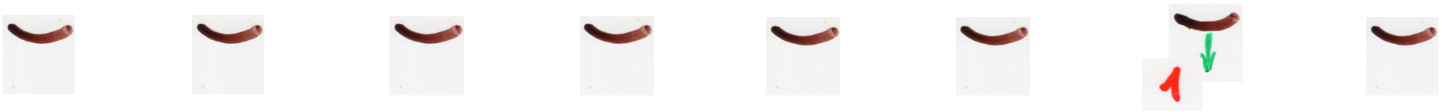
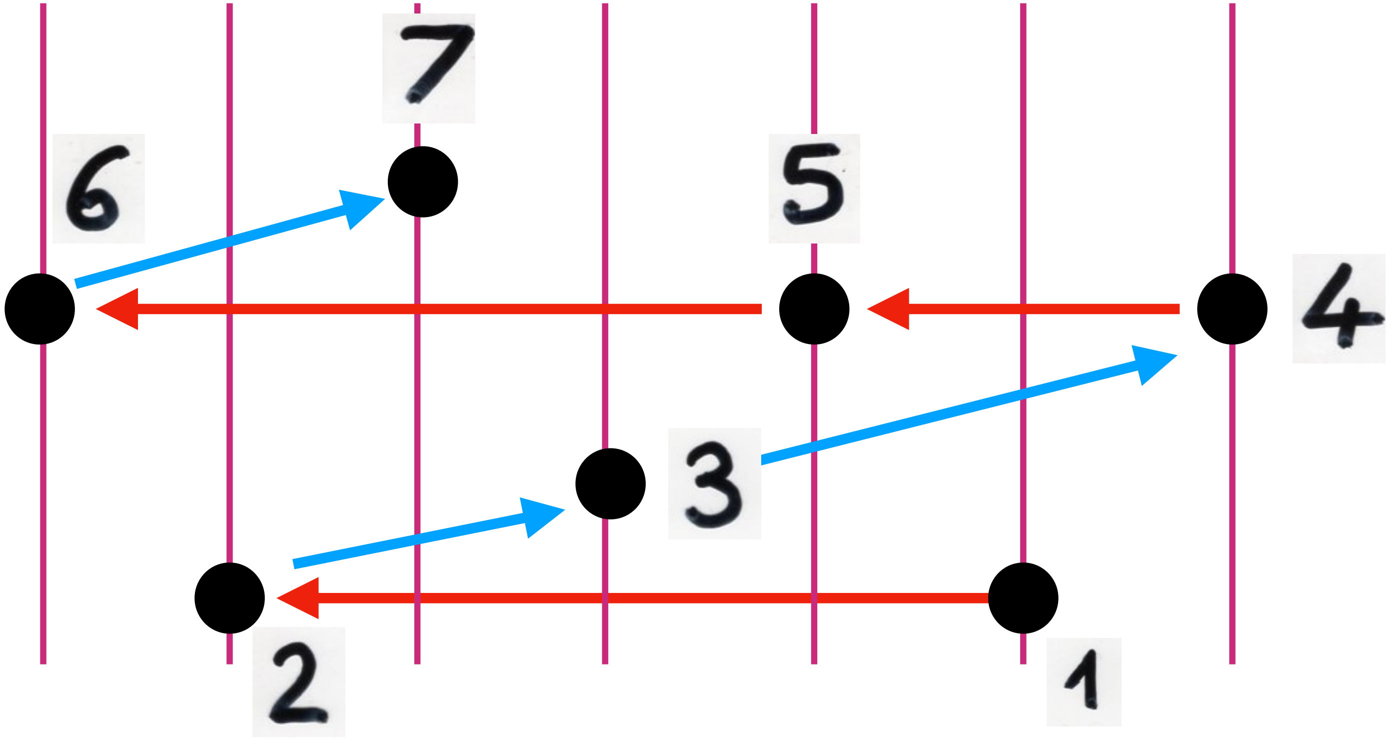

1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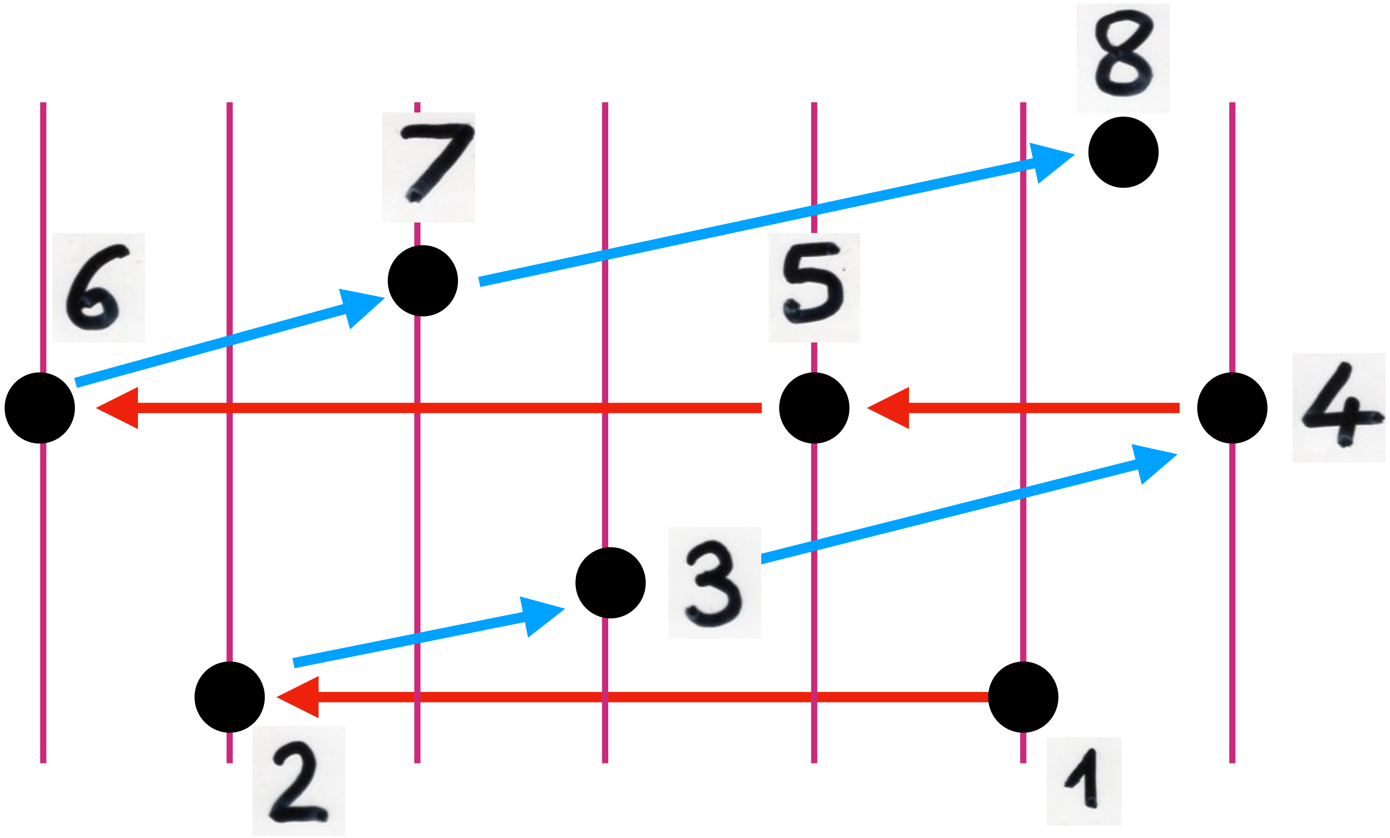
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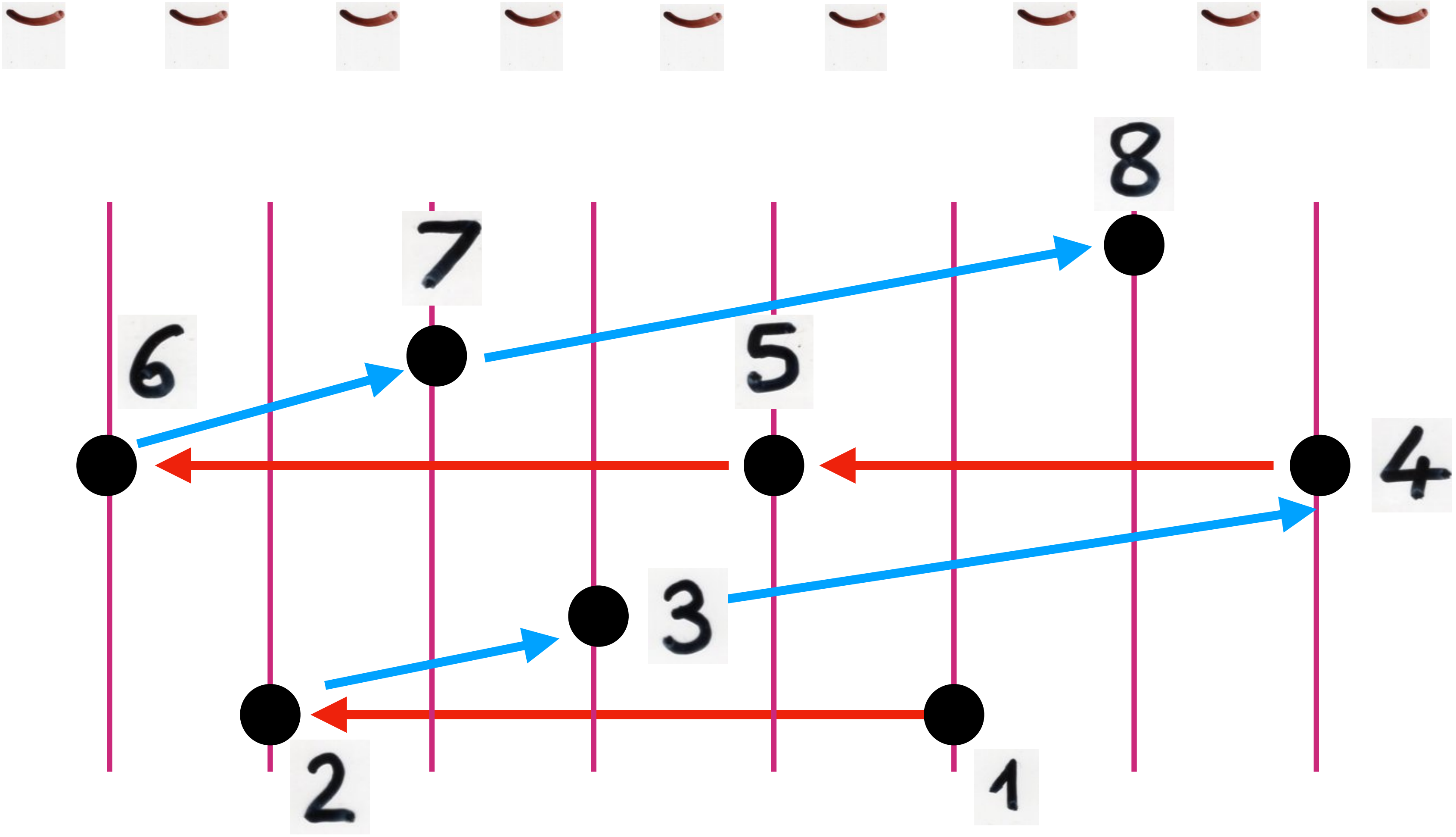
2
1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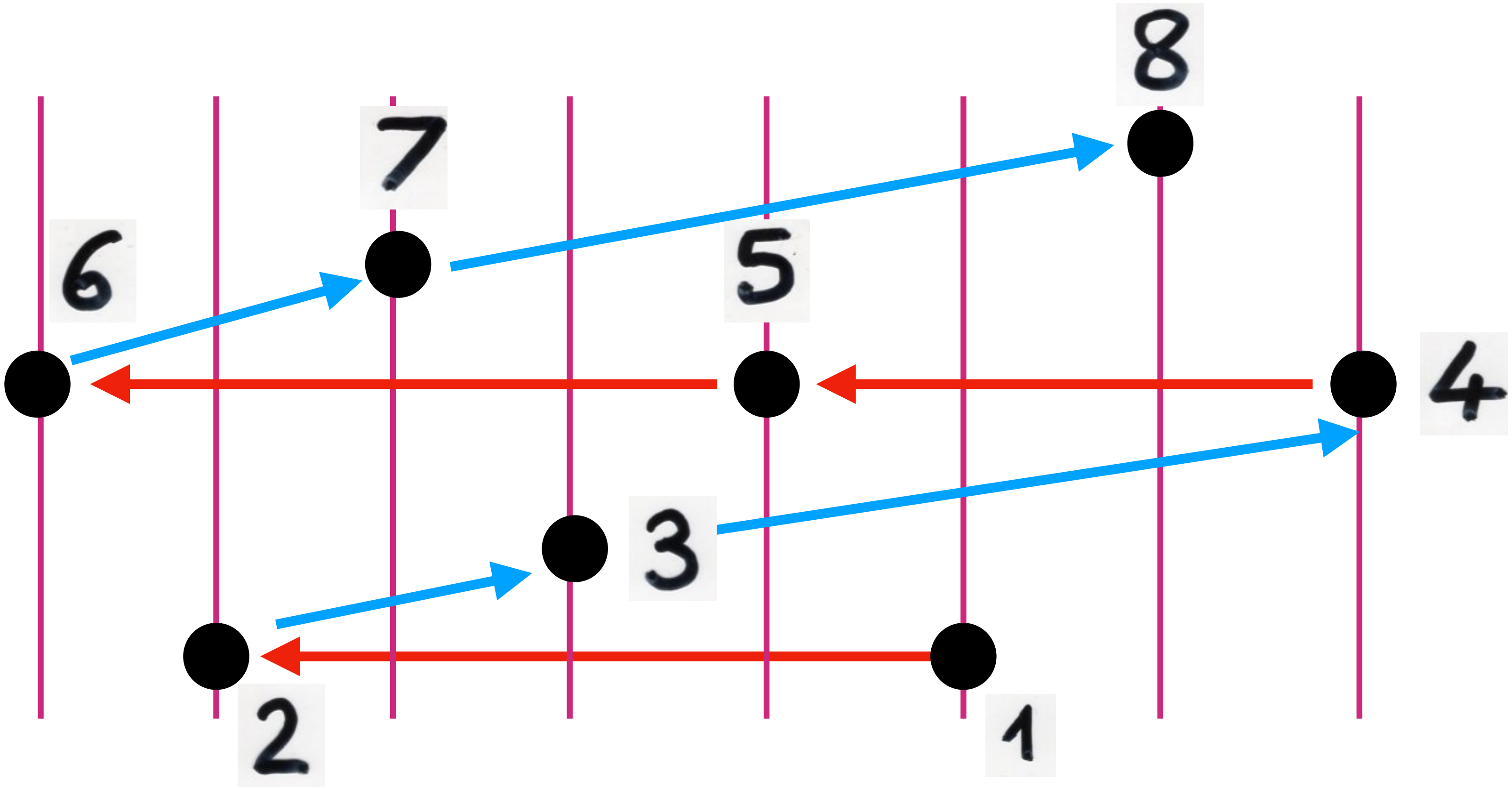
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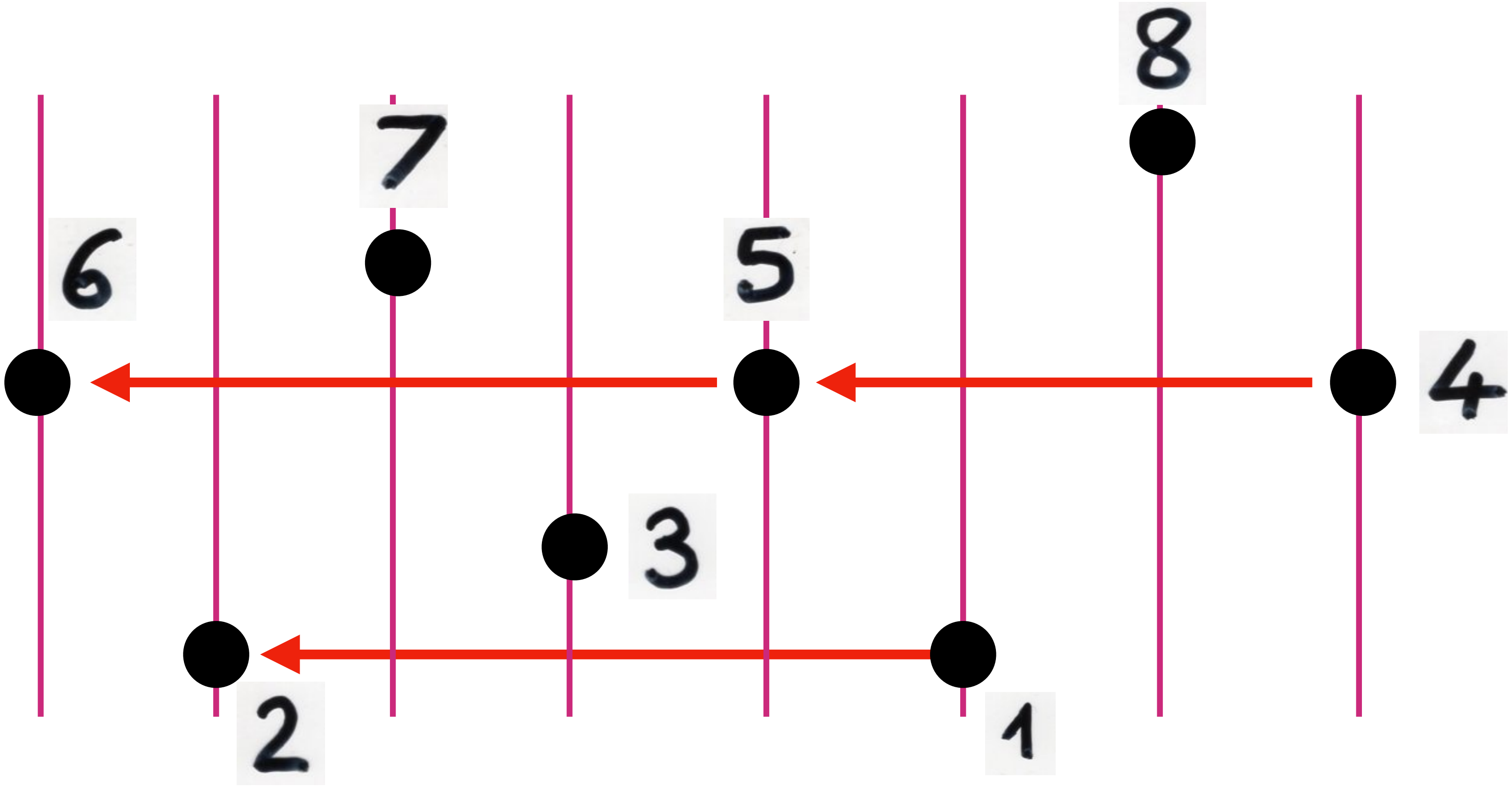
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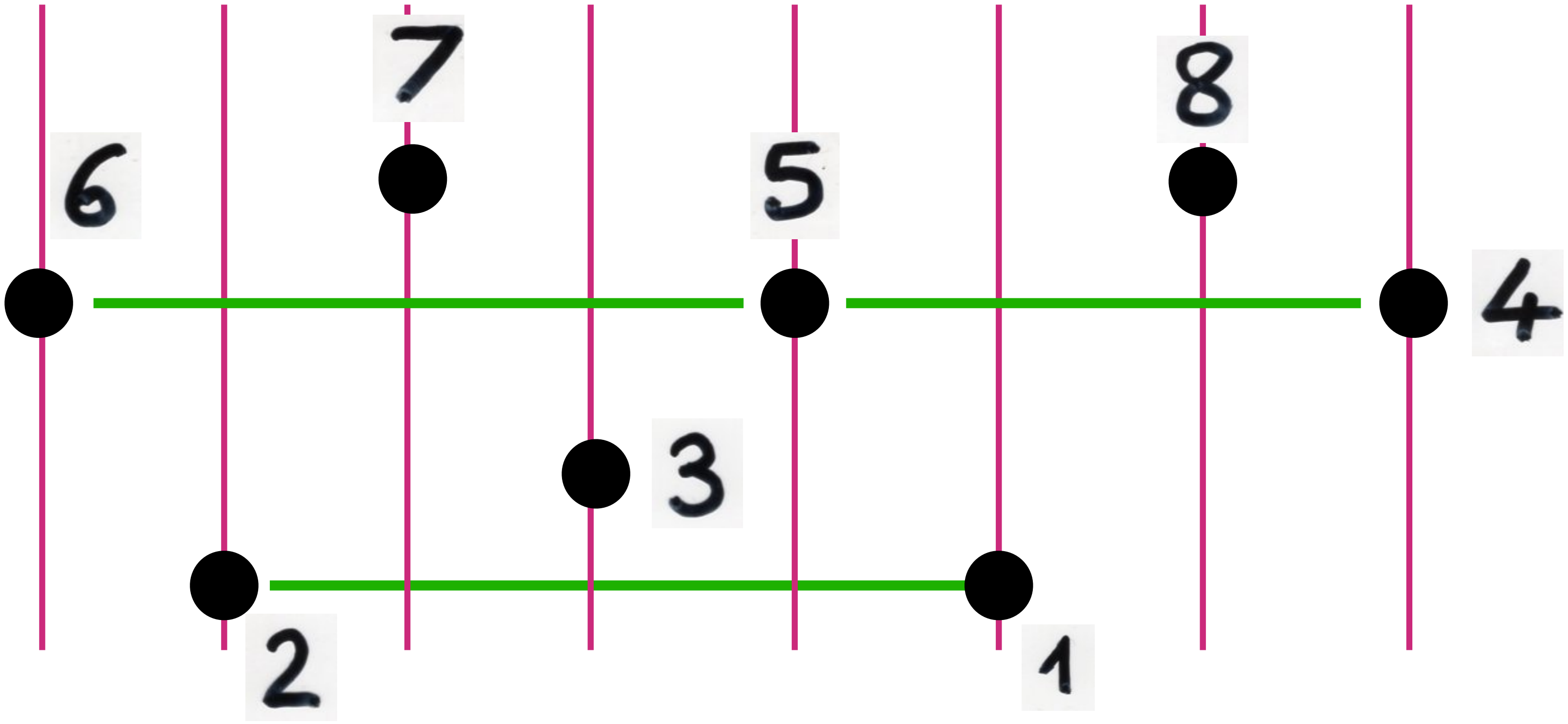
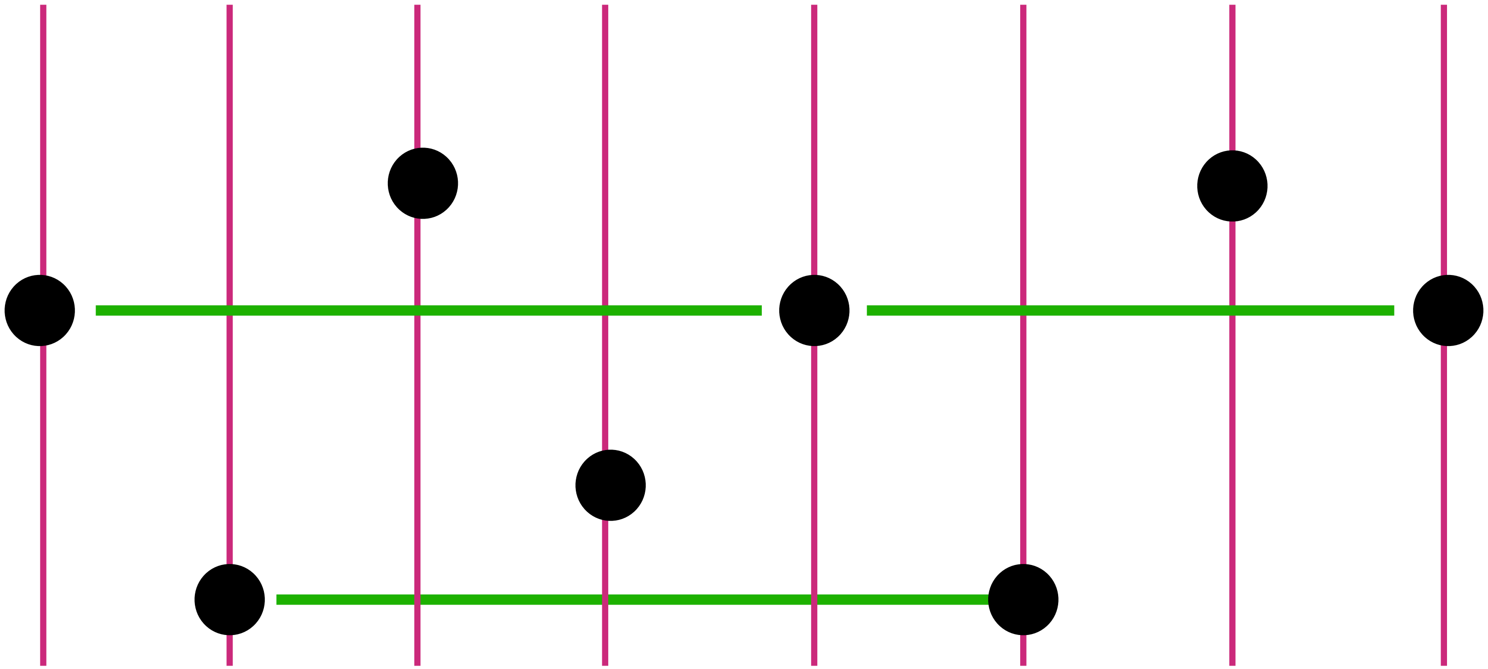
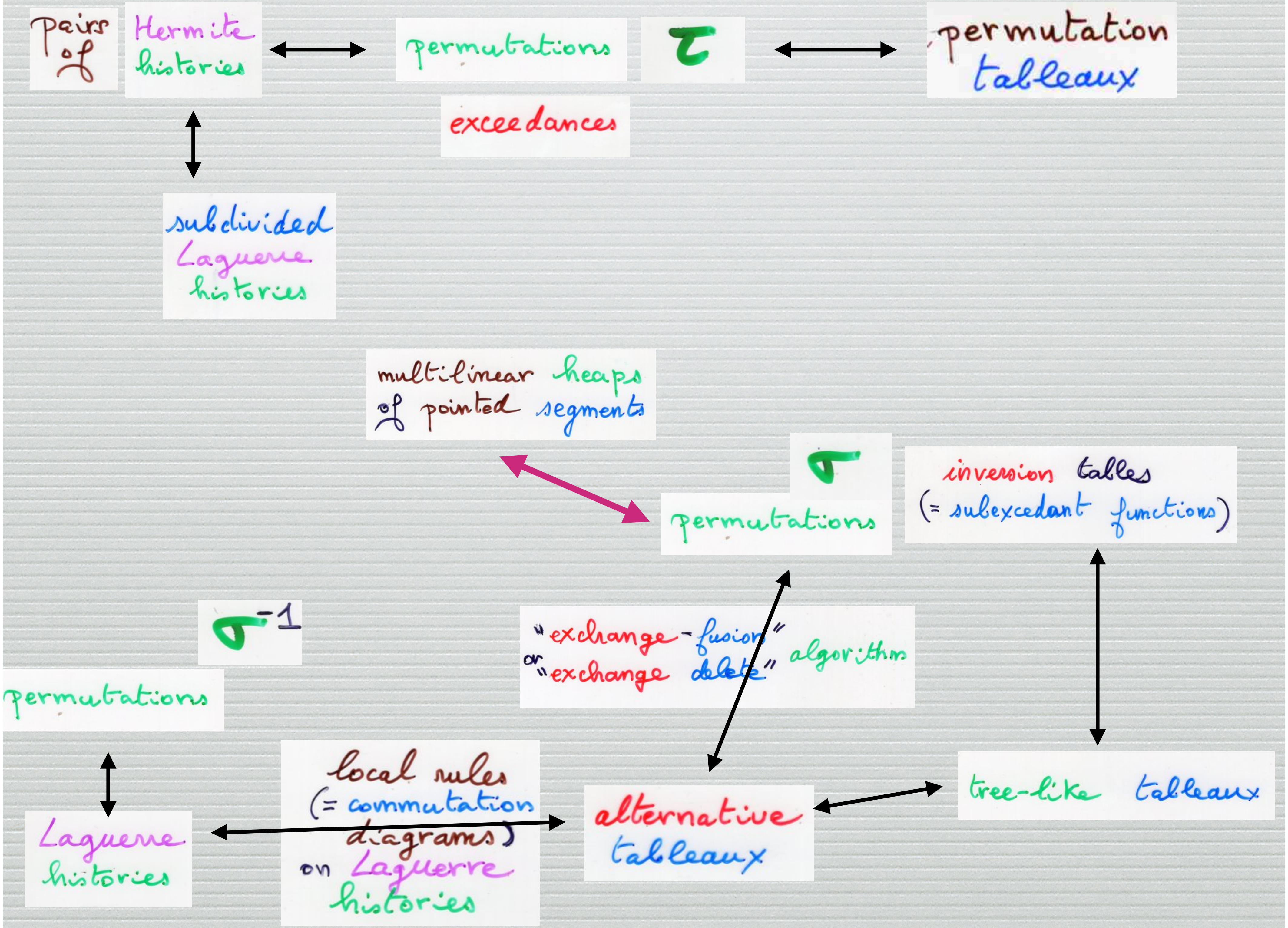

7
8
6
5
4
3
2
1

$\sigma = 6\ 2\ 7\ 3\ 5\ 1\ 8\ 4$





Pairs of Hermite histories

permutations τ

exceedances

permutation tableaux

subdivided Laguerre histories

multilinear heaps of pointed segments

σ permutations

inversion tables (= subexcedant functions)

"exchange-fusion" or "exchange delete" algorithm

σ^{-1}

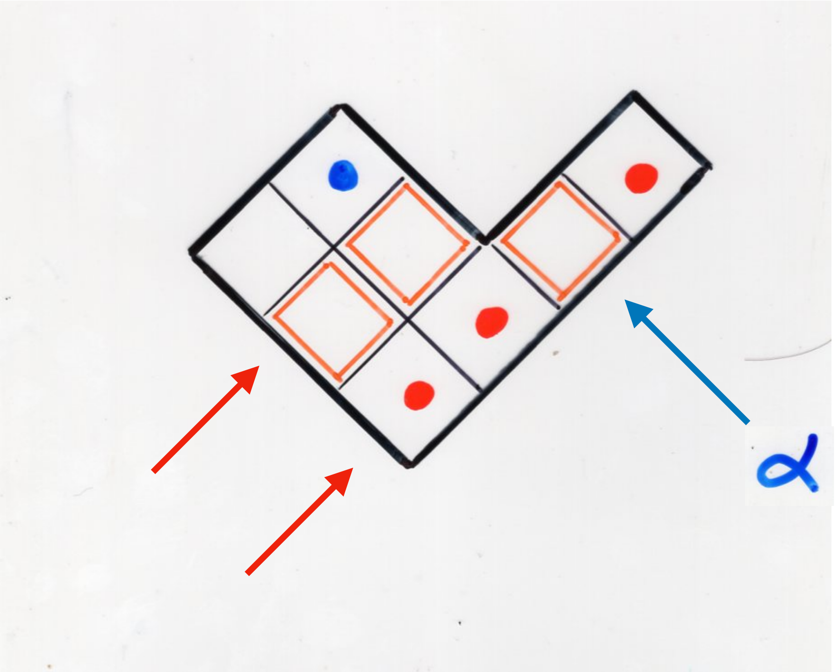
permutations

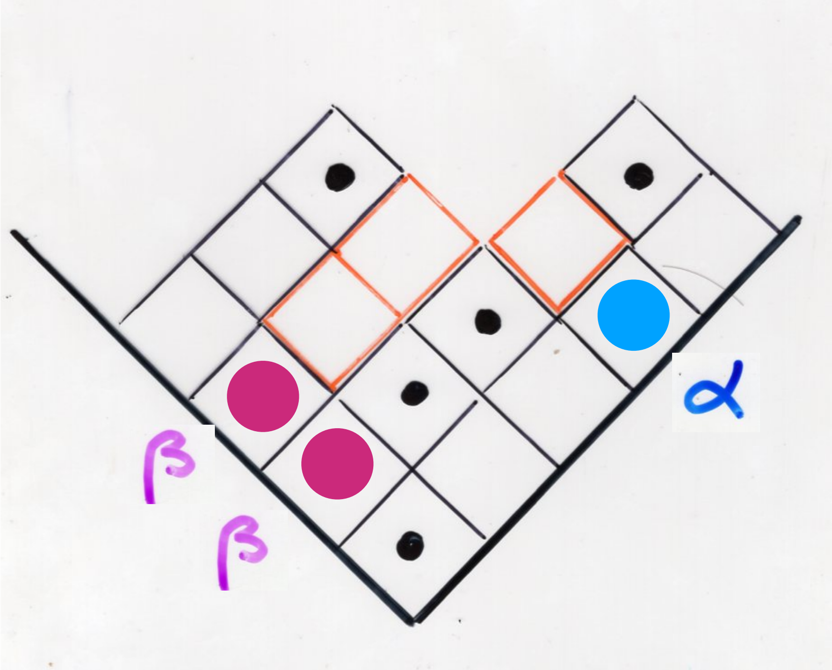
Laguerre histories

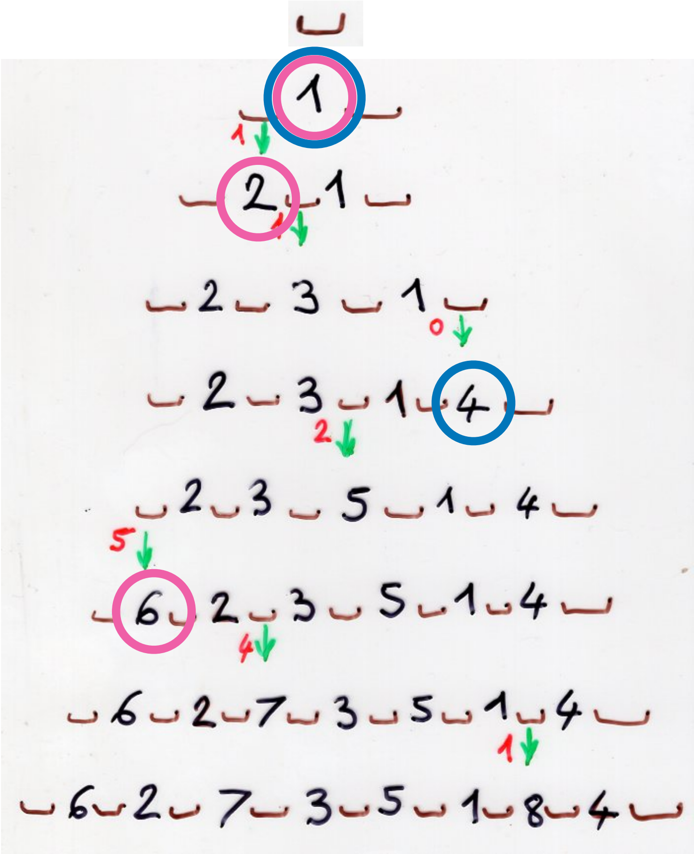
local rules (= commutation diagrams) on Laguerre histories

alternative tableaux

tree-like tableaux


α


β
β
α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1
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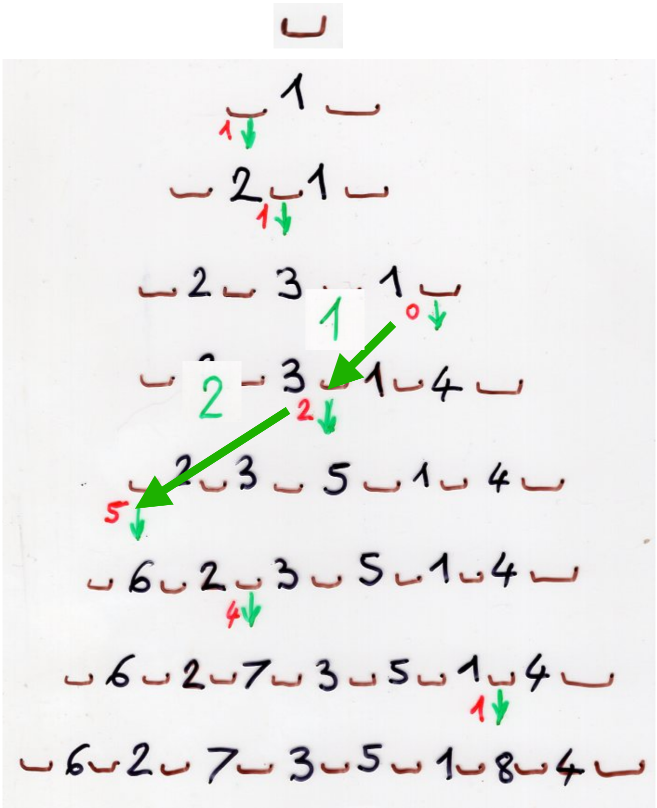
2
2 3 5 1 4
5
6 2 3 5 1 4
4
6 2 7 3 5 1 4
1
6 2 7 3 5 1 8 4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2 3 1
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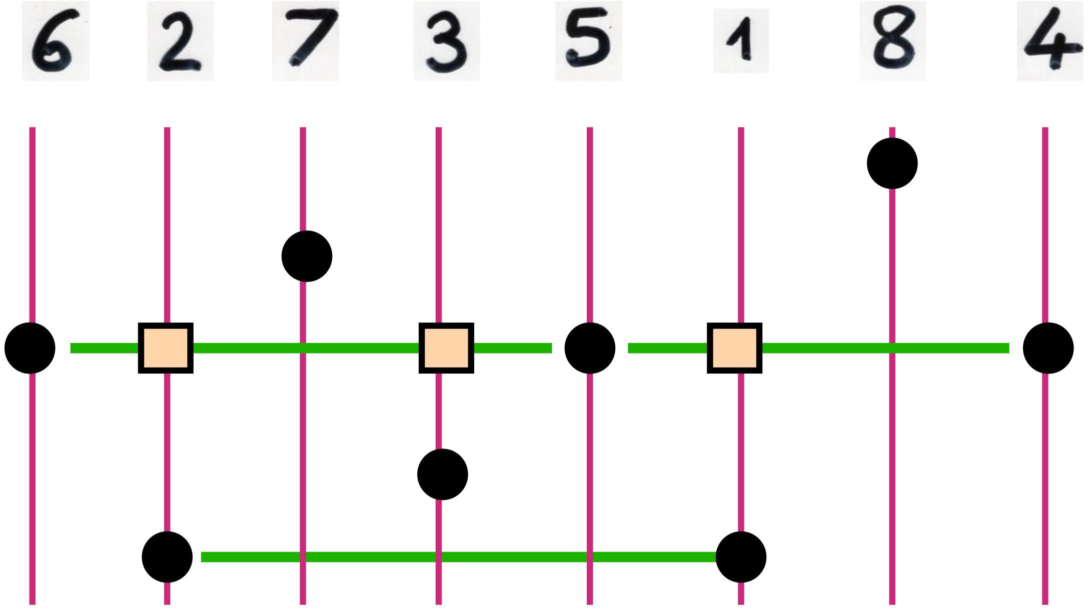
6 2 7 3 5 1 8 4


6
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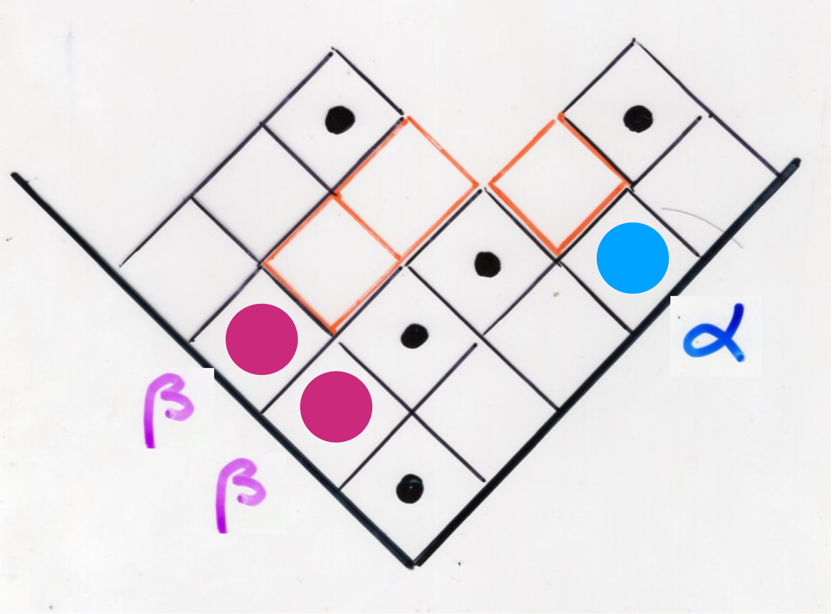
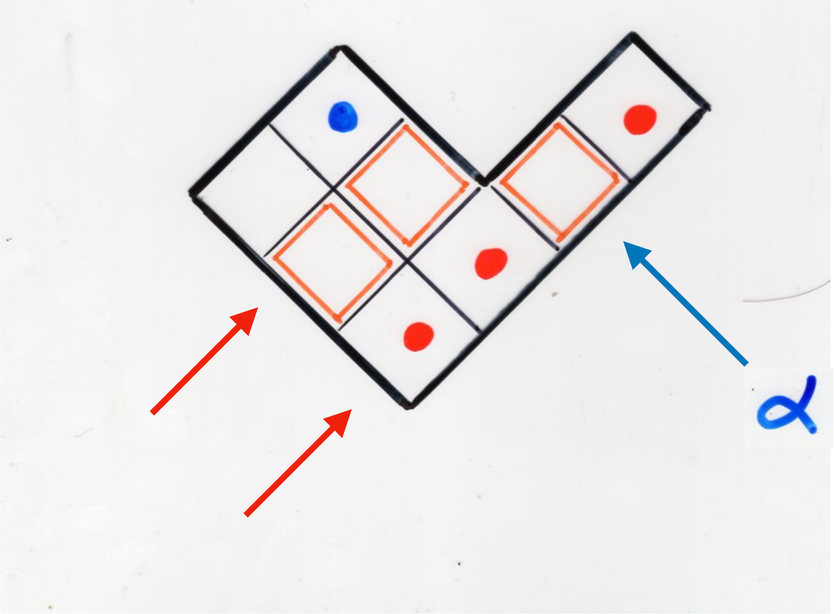
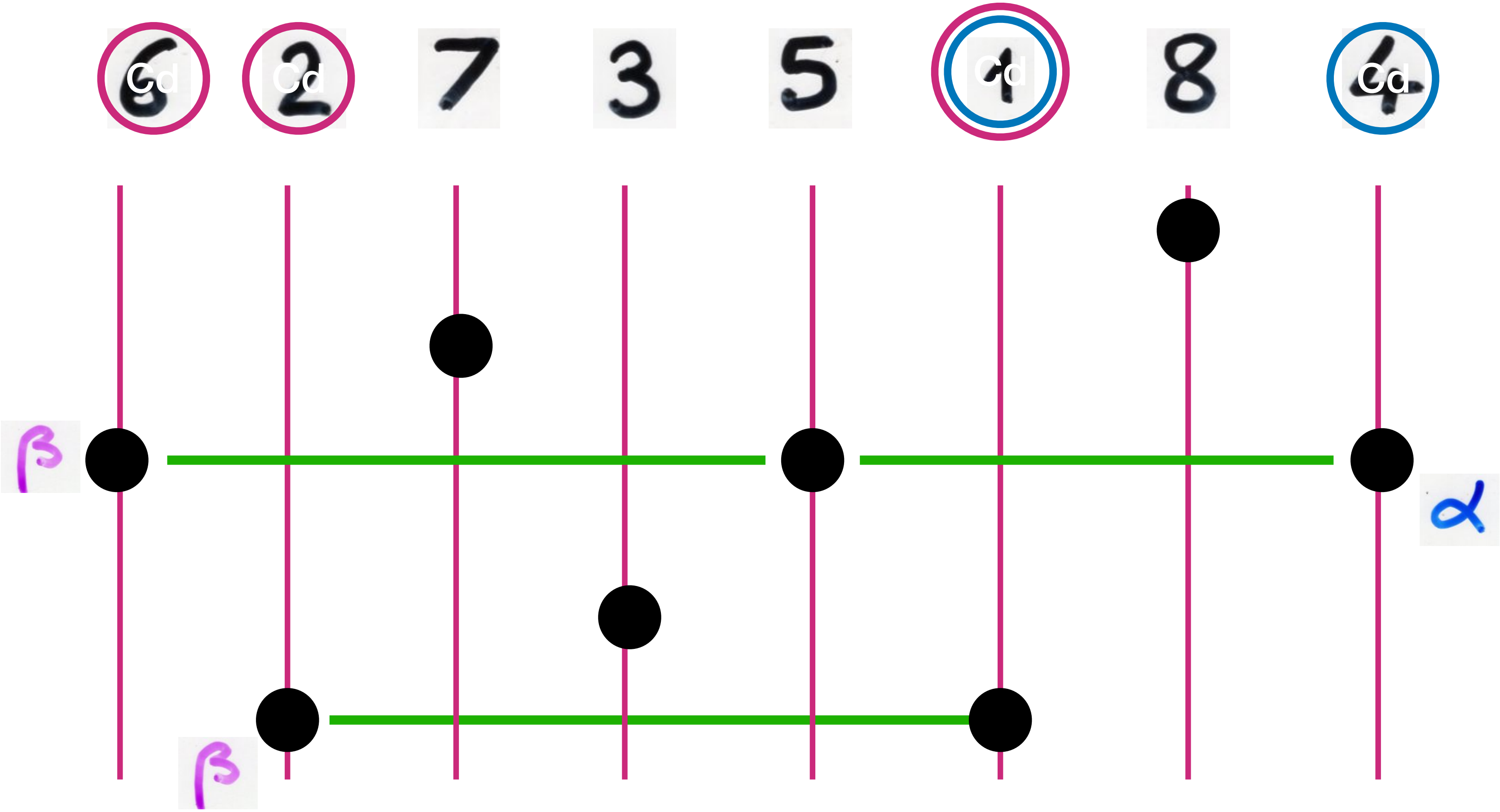
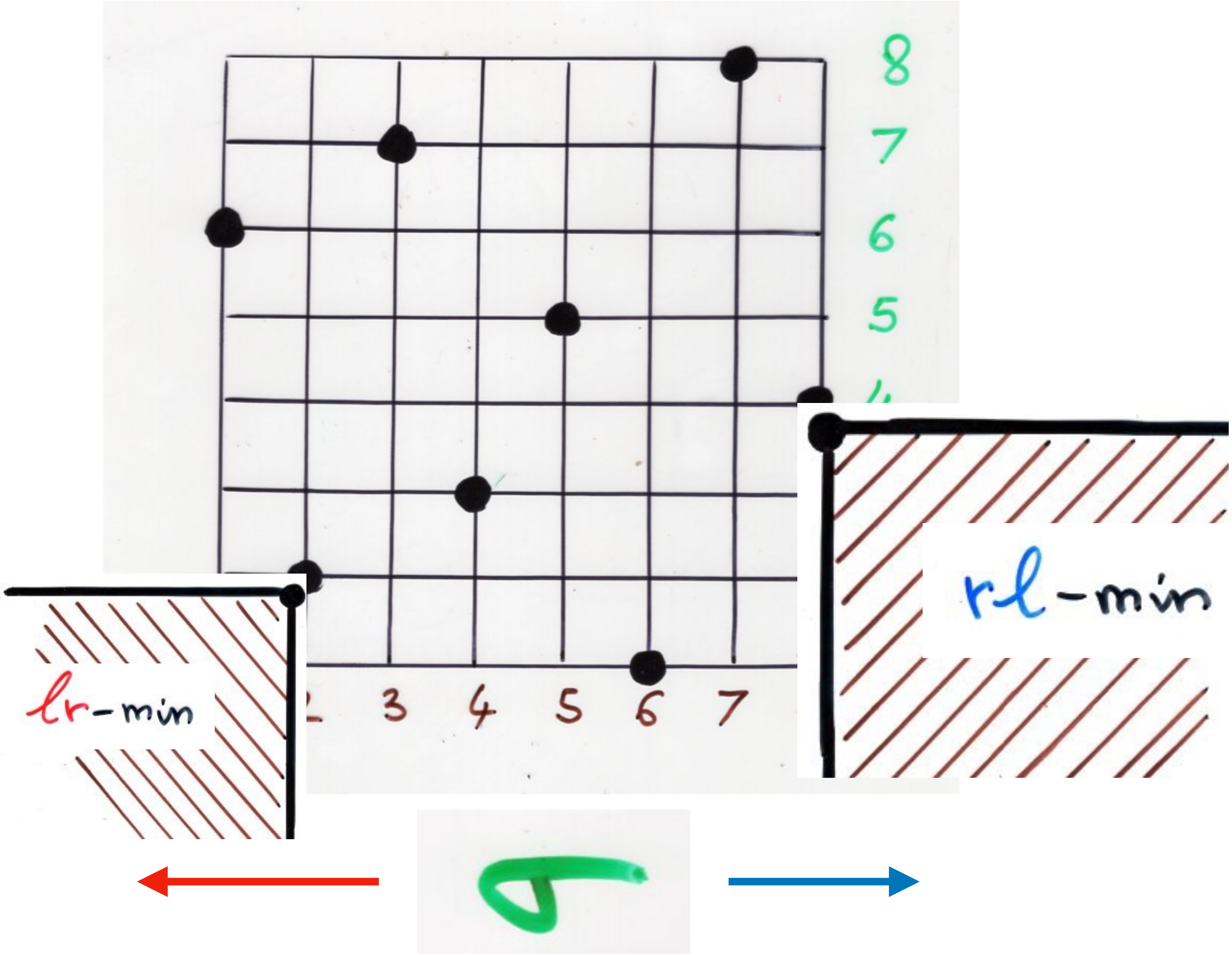
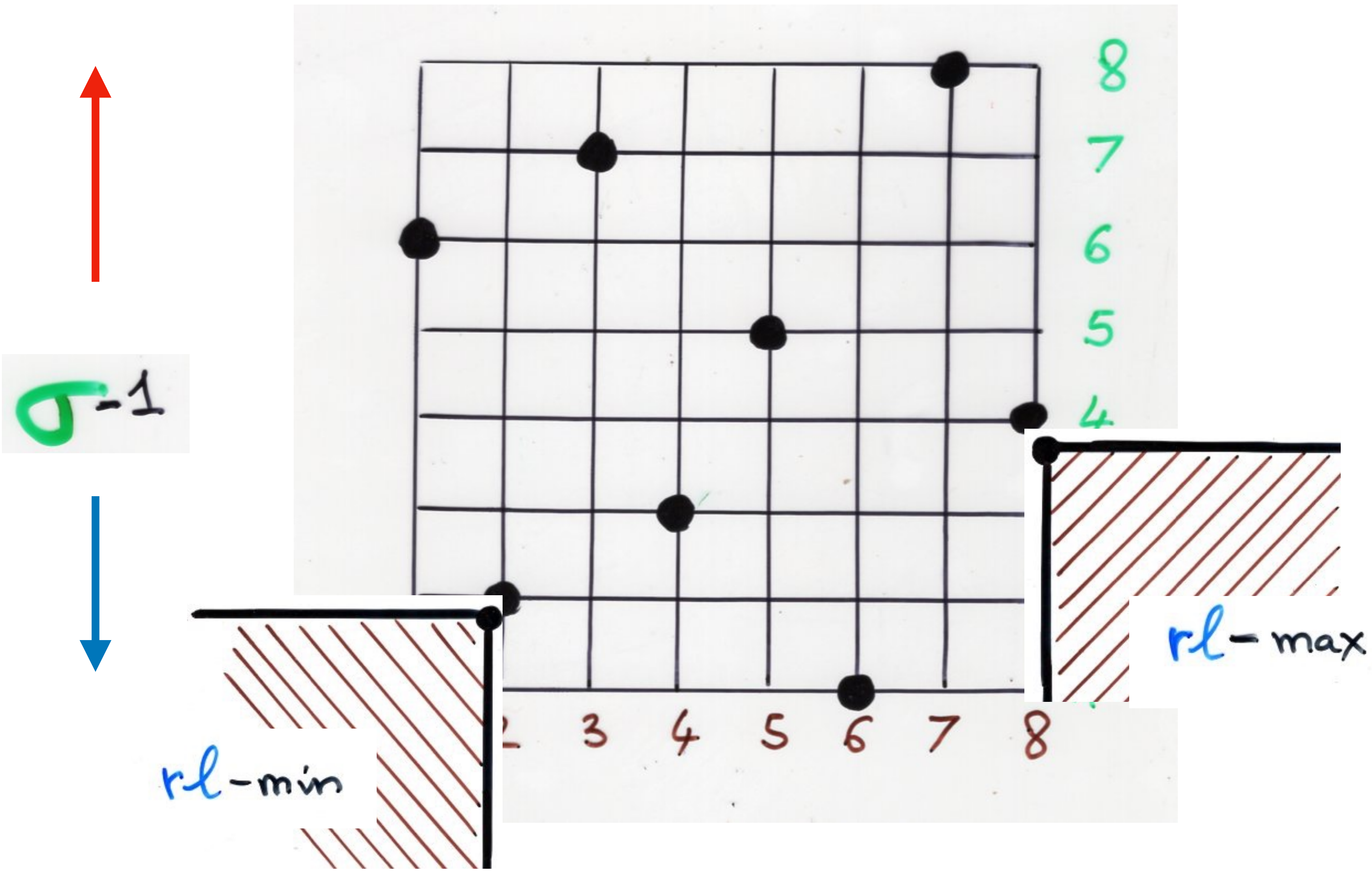
7
3
5
1
8
4

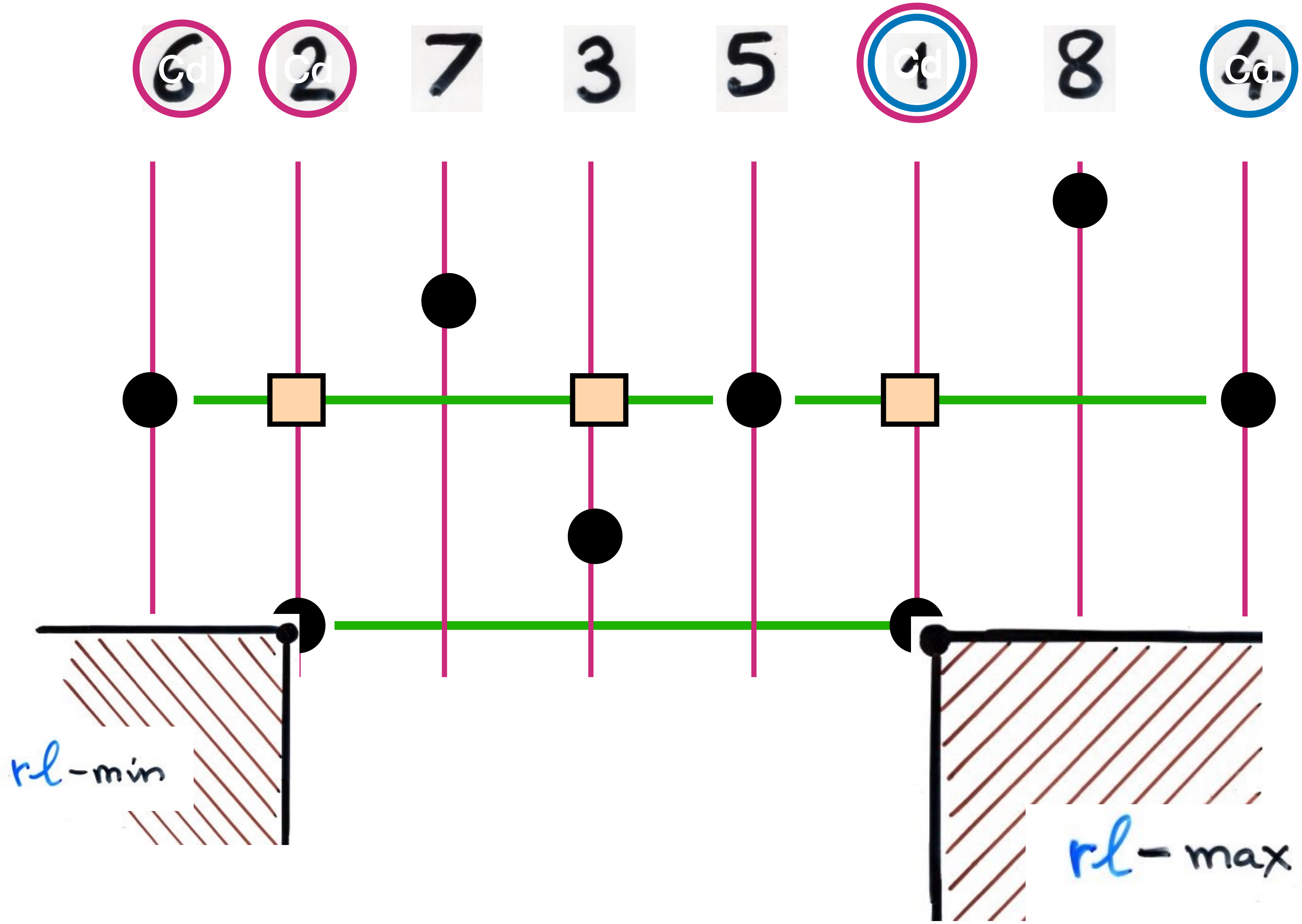

β
β
α


α


6
2
7
3
5
1
8
4
Cd
Cd
Cd
Cd
β
α
β


8
7
6
5
4
lr-min
2
3
4
5
6
7
rl-min
σ


σ-1
8
7
6
5
4
2
3
4
5
6
7
8
rl-min
rl-max


6
Cd
2
Cd
7
3
5
1
Cd
8
4
Cd
rl-min
rl-max

$$\bar{Z}_N = Z_N(\alpha^{-1}, \beta^{-1}, q)$$

Josuat-Vergès (2011)

Proposition

$$\bar{Z}_N = \sum_{\sigma \in \mathfrak{S}_{N+1}} \alpha^{s(\sigma)-1} \beta^{t(\sigma)-1} q^{31\text{-}2(\sigma)}$$

$s(\sigma)$

$t(\sigma)$

$31\text{-}2(\sigma)$

- Steingrímsson-Williams
- reverse - complement - inverse
- Foata-Zeilberger
- Françon-V.

Bijection (restricted) Laguerre histories

(of the inverse permutation)

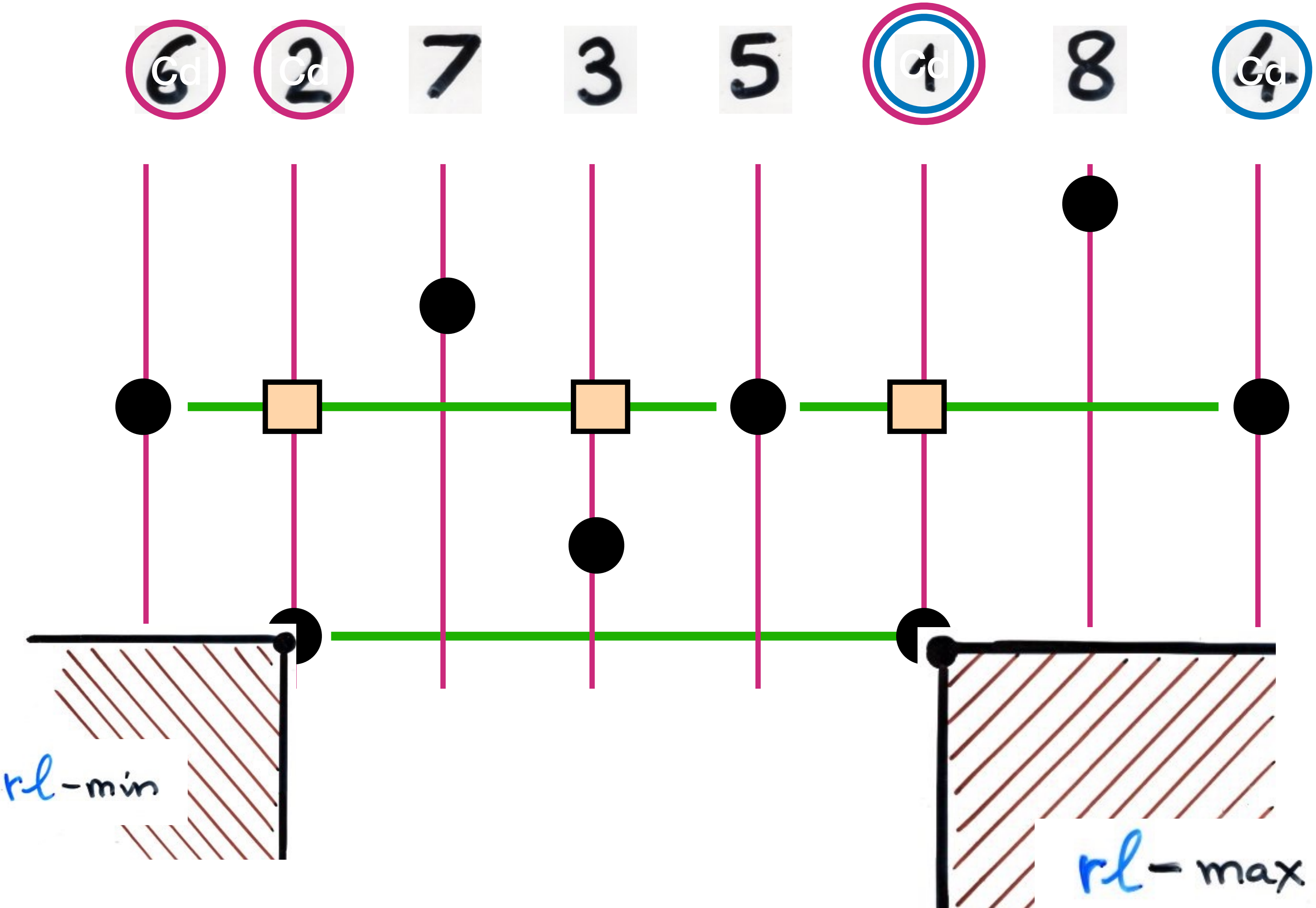
σ^{-1}

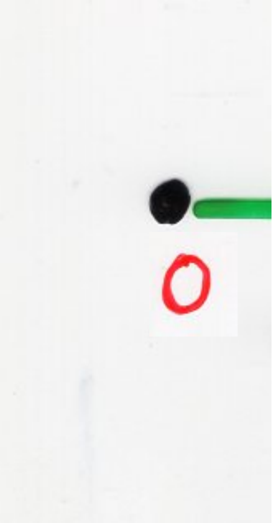
and Laguerre heaps of segments

Josuat-Vergès (2011)

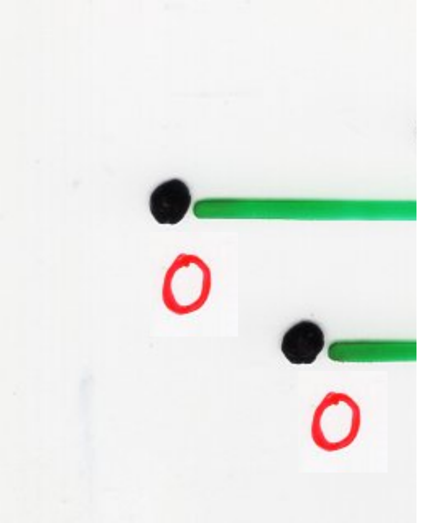
Proposition

$$\bar{Z}_N = \sum_{\sigma \in \mathfrak{S}_{N+1}} \alpha^{s(\sigma)-1} \beta^{t(\sigma)-1} q^{31-2(\sigma)}$$




O


A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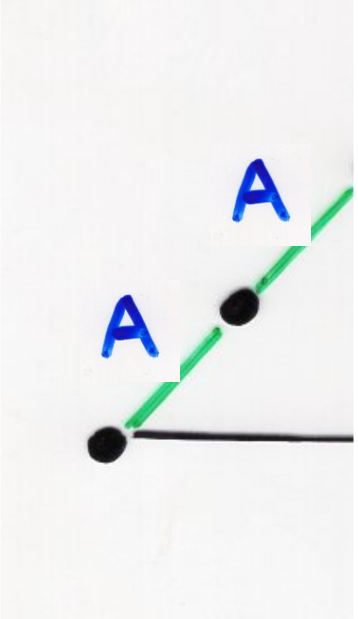
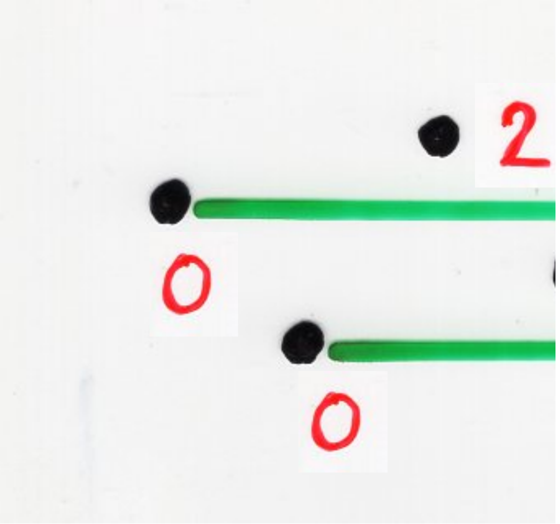
O


A
A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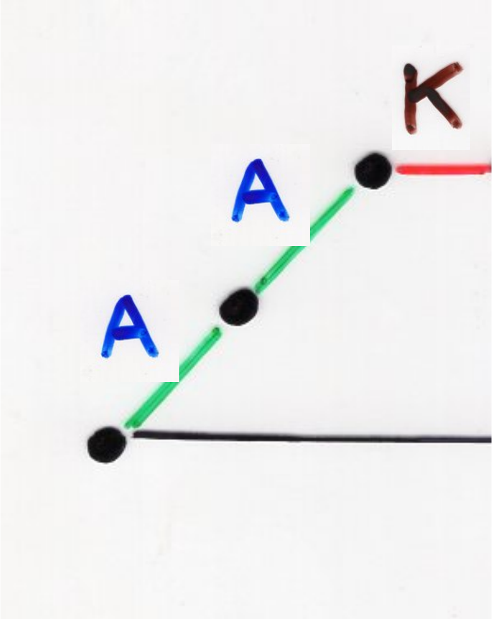
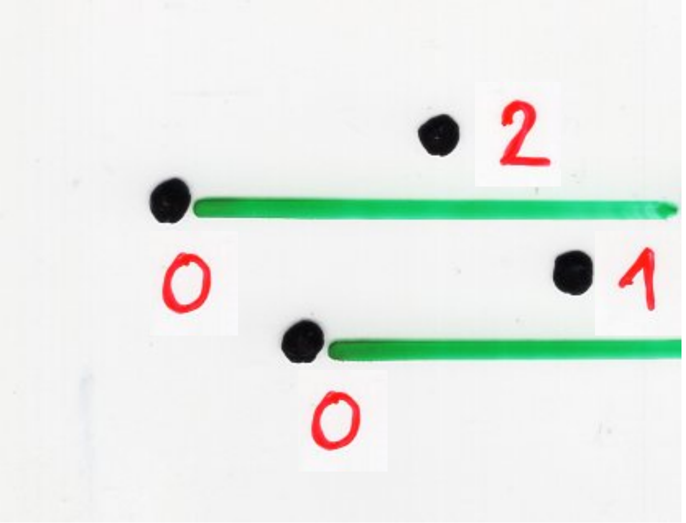
0


K
A
A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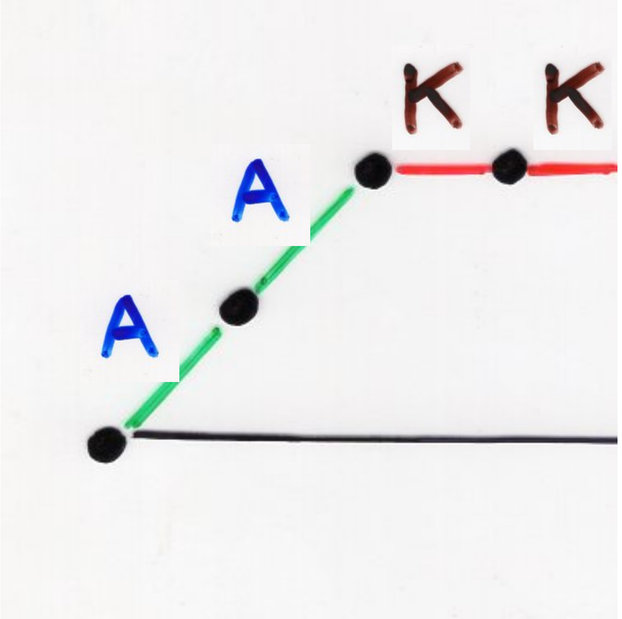
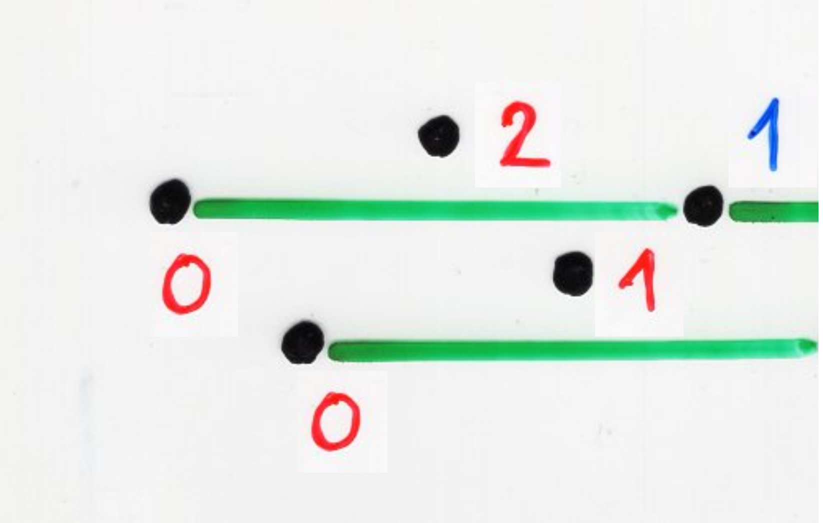
1
0


K
K
A
A


2
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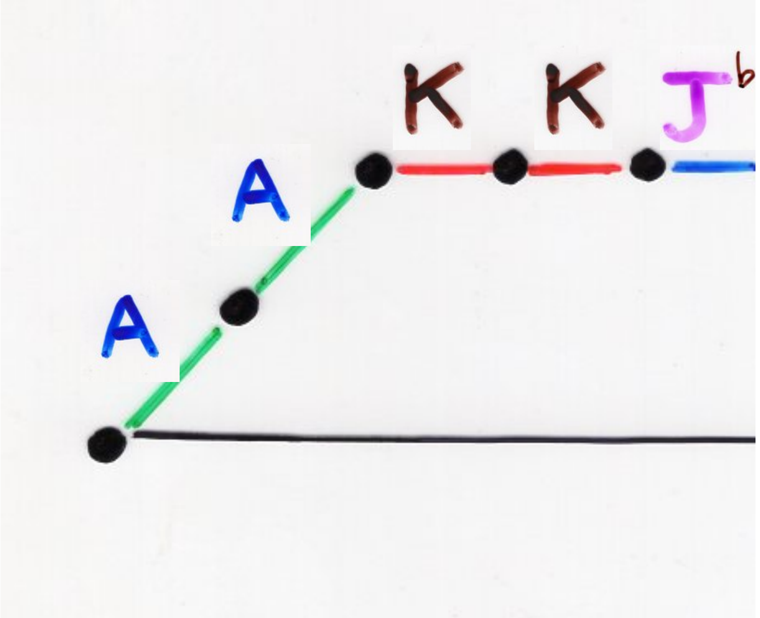
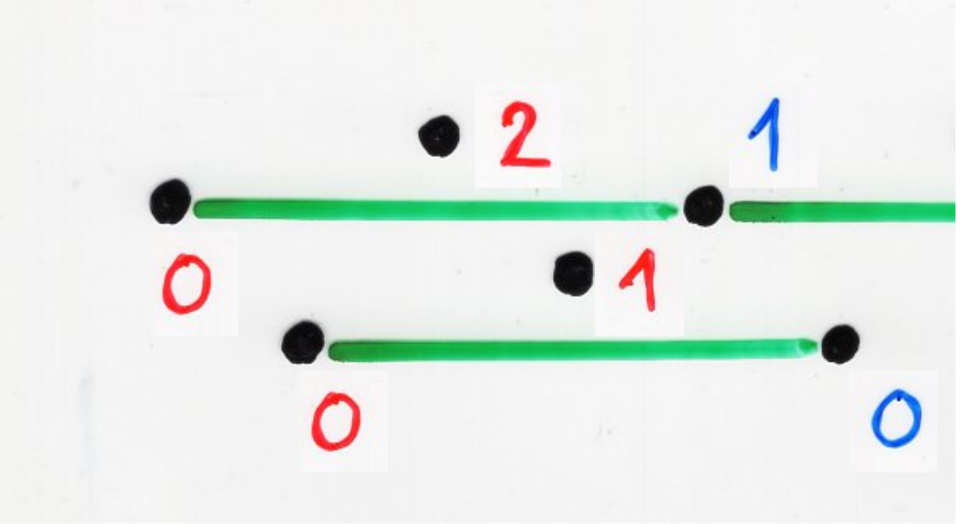
1
0


K
K
J^b
A
A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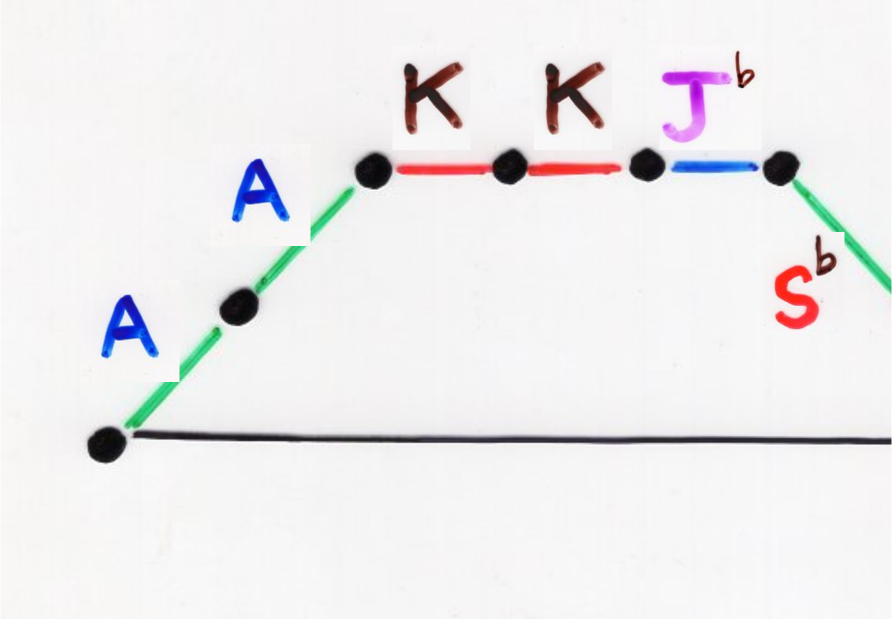
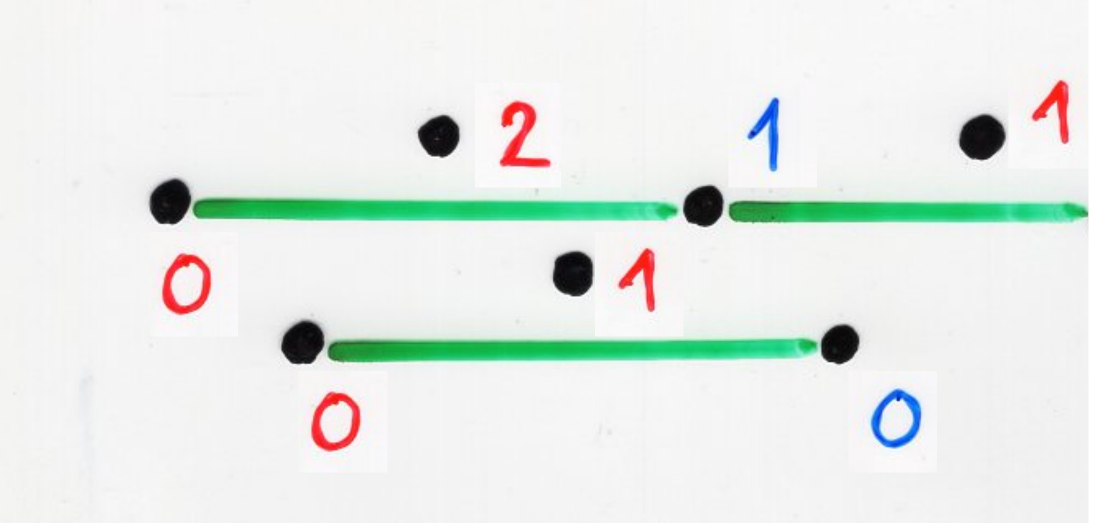
0
0


K
K
J^b
A
A
S^b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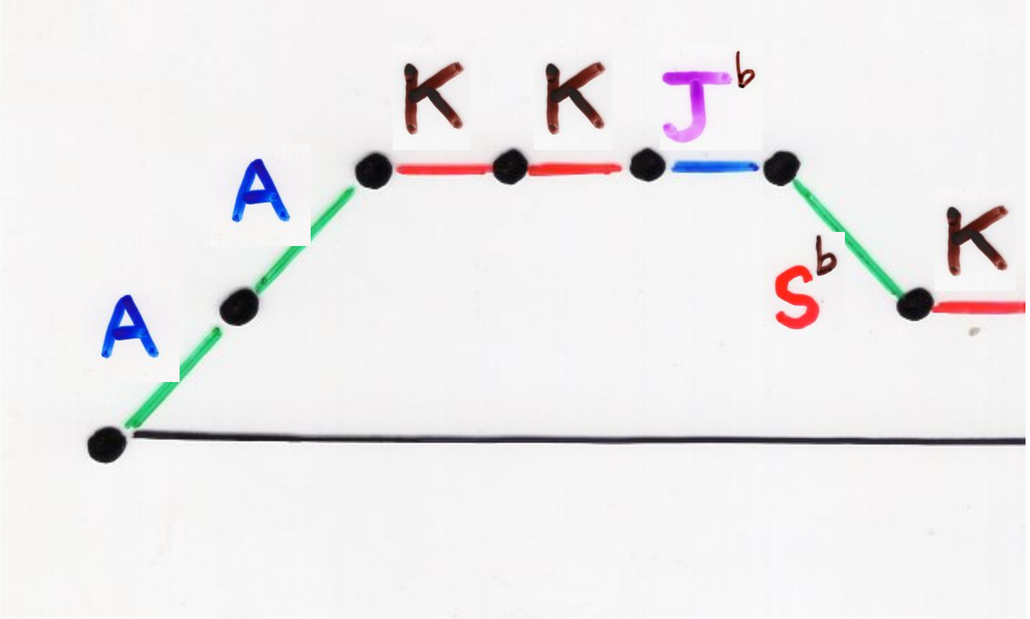
1
0
0


K
K
J^b
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K
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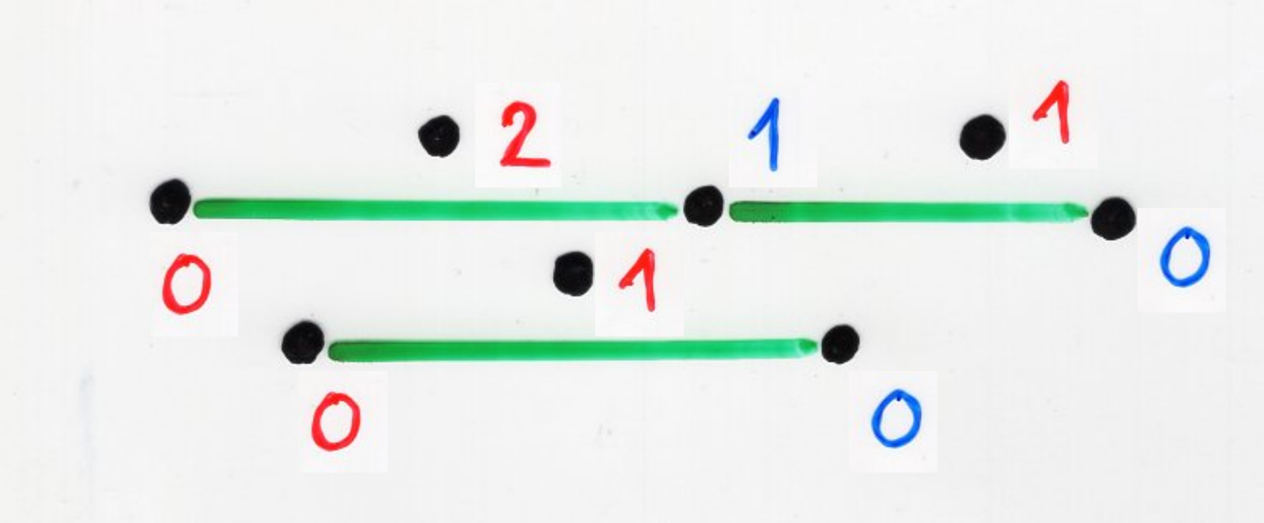
S^b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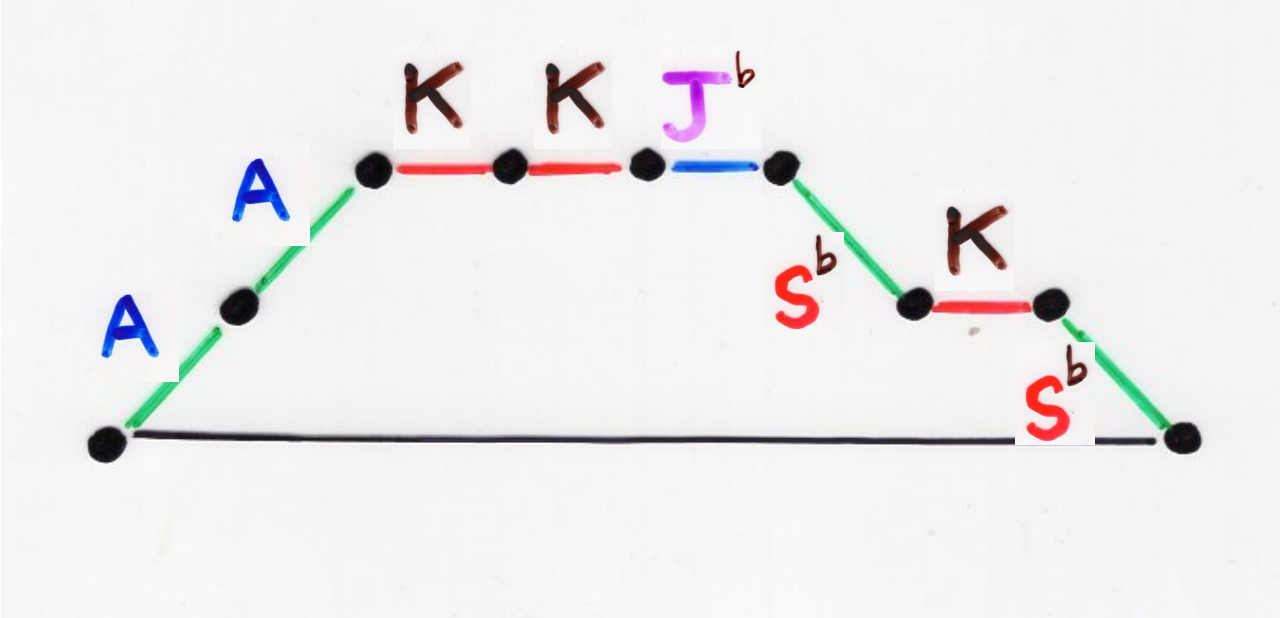
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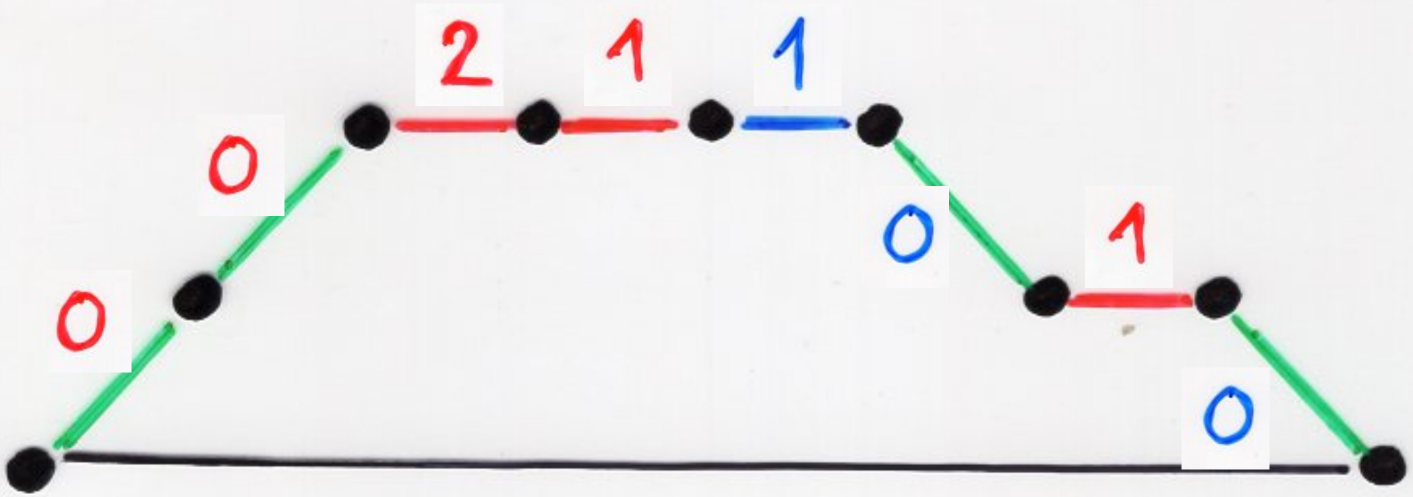
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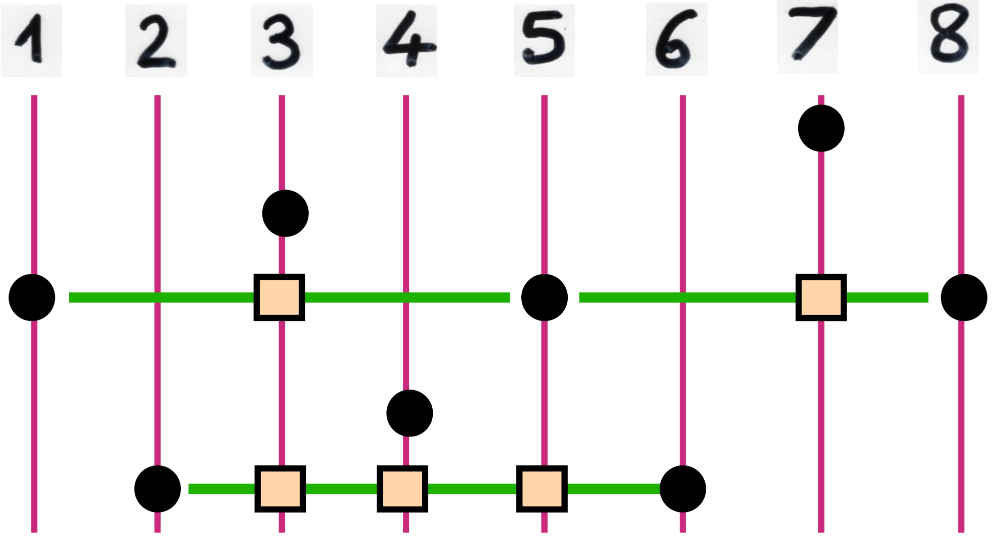
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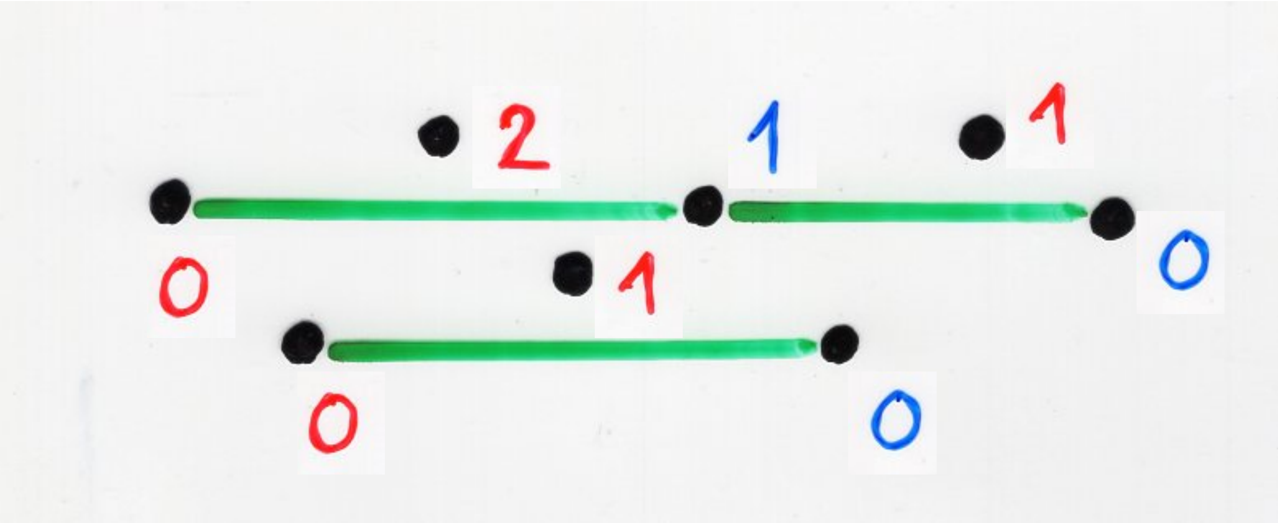
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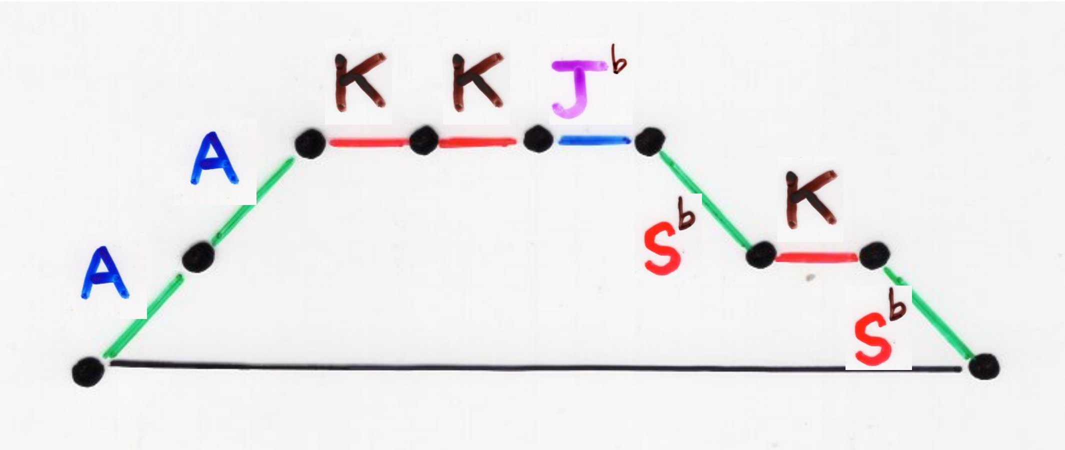
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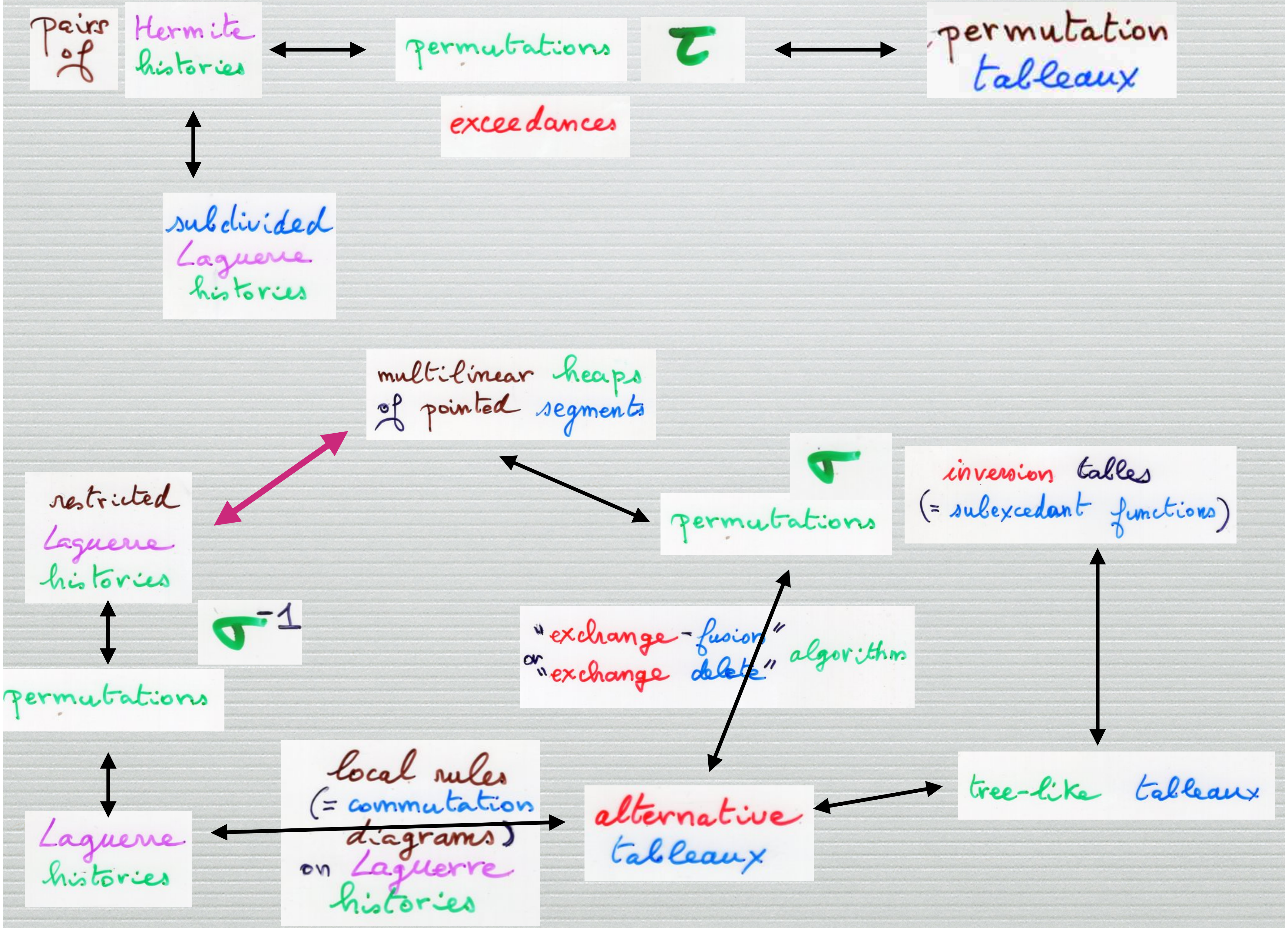
S^b



Pairs of Hermite histories

permutations τ

permutation tableaux

exceedances

subdivided Laguerre histories

multilinear heaps of pointed segments

restricted Laguerre histories

σ permutations

inversion tables (= subexcedant functions)

σ^{-1}

permutations

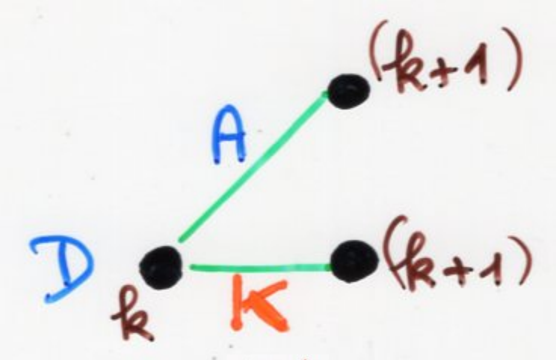
"exchange-fusion" or "exchange delete" algorithms

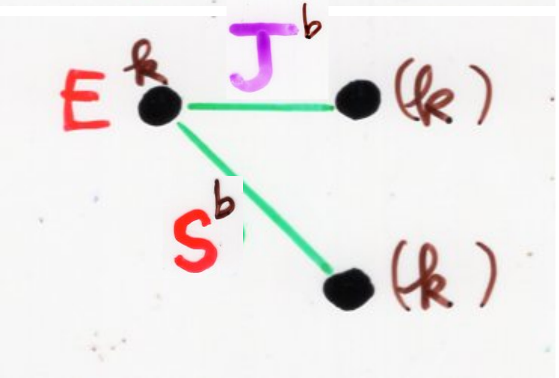
local rules (= commutation diagrams) on Laguerre histories

Laguerre histories

alternative tableaux

tree-like tableaux





D, E "restricted"

$$\begin{cases} D = A + K \\ E = S^b + J^b \end{cases}$$

$$DE = ED + E + D$$

$$\mu_n = n!$$

$$b_k = (2k+1)$$

$$\lambda_k = k^2$$

restricted
Laguerre
histories

$$\sigma(1) = (n+1)$$

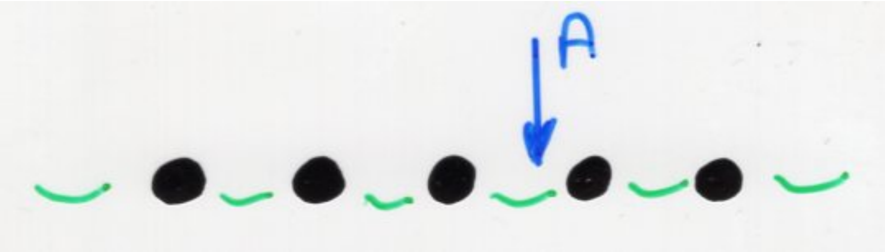
dictionnary data structure

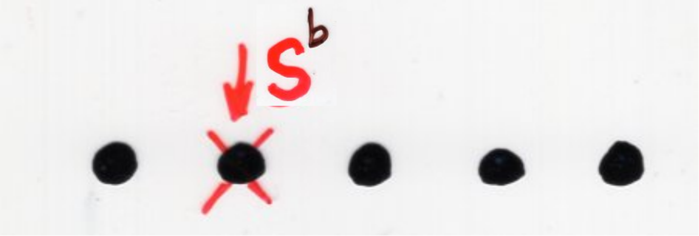
add or delete any element

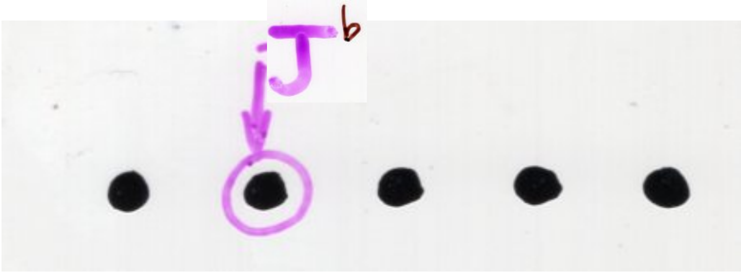
ask questions

J^b positive

K negative







$$\begin{cases} D = A + K \\ E = S^b + J^b \end{cases}$$

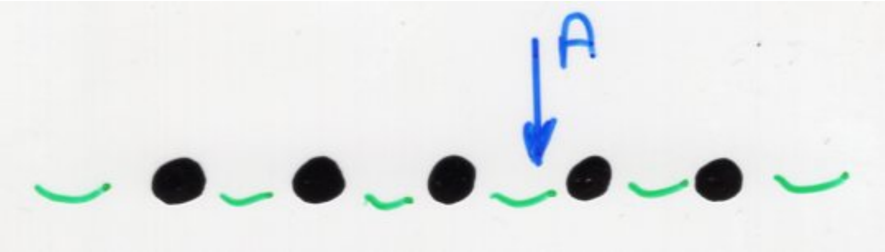
$$DE = ED + E + D$$

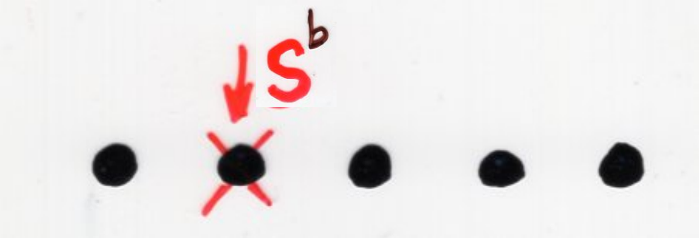
$$A\,|\,k\rangle = (k+1)\,|\,(k+1)\rangle$$

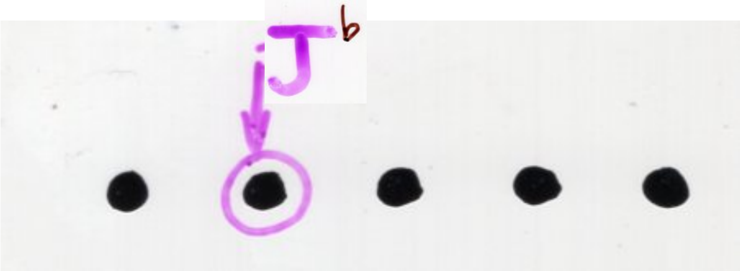
$$J^b\,|\,k\rangle = k\,|\,k\rangle$$

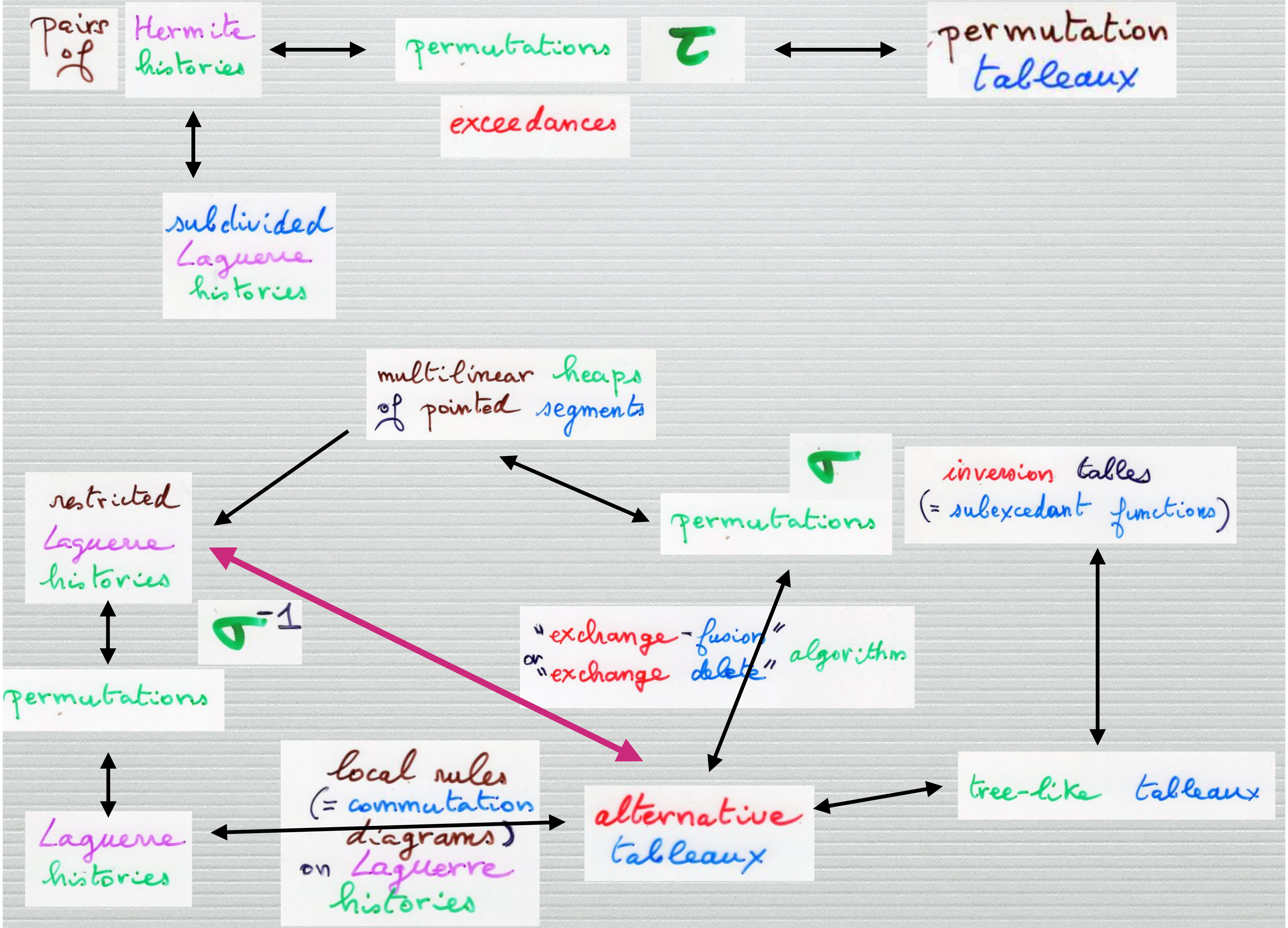
$$K\,|\,k\rangle = (k+1)\,|\,k\rangle$$

$$S^b\,|\,k\rangle = k\,|\,(k-1)\rangle$$









Pairs of Hermite histories

permutations τ

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subdivided Laguerre histories

multilinear heaps of pointed segments

σ

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tree-like tableaux

Laguerre histories

What about

the approach by physicists ?

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q- Hermite polynomial

α, β, q $\gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$

$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^{+}$$

$$\hat{a}\,\hat{a}^{+} - q\,\hat{a}^{+}\hat{a} = 1$$

$$UD = qDU + I$$

$$DE = qED + E + D$$

$$\tilde{\alpha} = (1-q)\frac{1}{\alpha} - 1$$

$$\tilde{\beta} = (1-q)\frac{1}{\beta} - 1$$

$$D = (D_{i,j})_{i,j \in \mathbb{N}}$$

$$E = (E_{i,j})_{i,j \in \mathbb{N}}$$

$$(1-q)\,D_{i,i} = 1 + \tilde{\beta} q^{i}$$

$$(1-q)\,D_{i,i+1} = 1 - \tilde{\alpha}\tilde{\beta} q^{i}$$

$$(1-q)\,E_{i,i} = 1 + \tilde{\alpha} q^{i}$$

$$(1-q)\,E_{i+1,i} = 1 - q^{i+1}$$

$$Z_N = \frac{1}{(1-q)^N} \sum_{n=0}^{N} R_{N,n}(\;, q)\, B_n(\tilde{\alpha}, \tilde{\beta}, y, q)$$

$$R_{N,n}(\; q) = \sum_{i=0}^{\left\lfloor \frac{N-n}{2} \right\rfloor} (-1)^i \left(\binom{2N}{N-n-2i} - \binom{2N}{N-n-2i-2} \right)$$

$$B_n(\tilde{\alpha}, \tilde{\beta}, \; q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \tilde{\alpha}^k (\; \tilde{\beta})^{n-k}$$

$$Z_N = \frac{1}{(1-q)^N} \sum_{n=0}^{N} R_{N,n}(y,q)\, B_n(\tilde{\alpha}, \tilde{\beta}, y, q)$$

$$R_{N,n}(1,q) = \sum_{i=0}^{\lfloor \frac{N-n}{2} \rfloor} (-1)^i \left(\binom{2N}{N-n-2i} - \binom{2N}{N-n-2i-2} \right)$$

$$B_n(\tilde{\alpha}, \tilde{\beta}, y, q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \tilde{\alpha}^k (y\, \tilde{\beta})^{n-k}$$

Josuat-Vergès (2011)

$$R_{N,n}(y,q) = \sum_{i=0}^{\lfloor \frac{N-n}{2} \rfloor} (-y)^i q^{\binom{i+1}{2}} \begin{bmatrix} n+i \\ i \end{bmatrix}_q \sum_{j=0}^{N-n-2i} y^j \left(\binom{N}{j}\binom{N}{n+2i+j} - \binom{N}{j-1}\binom{N}{n+2i+j+1} \right)$$

$$\bar{Z}_N = Z_N(\alpha^{-1}, \beta^{-1}, q)$$

Josuat-Vergès (2011)

Proposition

$$\bar{Z}_N = \sum_{\sigma \in \mathfrak{S}_{N+1}} \alpha^{s(\sigma)-1} \beta^{t(\sigma)-1} q^{31-2(\sigma)}$$

Al-Salam-Chihara polynomials

$$2x\, Q_n(x) = Q_{n+1}(x) + (a+b)q^n Q_n(x) + (1-q^n)(1-abq^{n-1}) Q_{n-1}(x)$$

PASEP
with 5 parameters

→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta, \quad q$

Askey-Wilson polynomials

Z_n partition function

S. Corteel, L. Williams (2009)

staircase tableaux

Z_n partition function

Askey-Wilson polynomials

$$P_n(x) = P_n(x\,;\,a,b,c,d\,|\,q)$$

$$P_n(x) = a^{-n}(ab, ac, ad\,;\,q)_n \sum_{k=0}^{n} \frac{(q^{-n}, q^{n-1}abcd\,, ae^{i\theta}, ae^{-i\theta}\,;\,q)_k}{(ab, ac, ad, q\,;\,q)_k}$$

$$(a_1, a_2, .., a_s\,;\,q)_n = \prod_{r=1}^{s}\prod_{k=0}^{n-1}(1-a_r q^k)$$

${}_4\phi_3$ basic hypergeometric function

$$A_n P_{n+1}(x) + B_n P_n(x) + C_n P_{n-1}(x) = 2x P_n(x)$$

$$A_n = \frac{1-q^{n-1}abcd}{(1-q^{2n-1}abcd)(1-q^{2n}abcd)}$$

$$B_n = \frac{q^{n-1}}{(1-q^{2n-2}abcd)(1-q^{2n}abcd)}$$

$$\left[(1+q^{2n-1}abcd)(qs+abcd\,s')-q^{n-1}(1+q)abcd\,(s+qs')\right]$$

$$C_n = \frac{(1-q^n)(1-q^{n-1}ab)(1-q^{n-1}ac)(1-q^{n-1}ad)(1-q^{n-1}bc)(1-q^{n-1}bd)(1-q^{n-1}cd)}{(1-q^{2n-2}abcd)(1-q^{2n-1}abcd)}$$

$$s = a+b+c+d \qquad s' = a^{-1}+b^{-1}+c^{-1}+d^{-1}$$

$$D = \frac{1}{1-q}(1+d)$$

$$E = \frac{1}{1-q}(1+e)$$

$$de - qed = (1-q)\,\mathrm{Id}$$

$$DE = qED + E + D$$

$$d = \begin{bmatrix} d_0^\natural & d_0^\sharp & 0 & & \\ d_0^\flat & d_1^\natural & d_1^\sharp & & 0 \\ 0 & d_1^\flat & d_2^\natural & \ddots & \\ & & \ddots & \ddots & \ddots \\ & 0 & & \ddots & \ddots \end{bmatrix}$$

$$e = \begin{bmatrix} e_0^\natural & e_0^\sharp & 0 & & \\ e_0^\flat & e_1^\natural & e_1^\sharp & & 0 \\ 0 & e_1^\flat & e_2^\natural & \ddots & \\ & & \ddots & \ddots & \ddots \\ & 0 & & \ddots & \ddots \end{bmatrix}$$

$$D = \frac{1}{1-q}(1+d)$$

$$de - qed = (1-q)\,\mathrm{Id}$$

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$$DE = qED + E + D$$

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$$d = \begin{bmatrix} d_0^{\natural} & d_0^{\sharp} & 0 & & \\ d_0^{\flat} & d_1^{\natural} & d_1^{\sharp} & & 0 \\ 0 & d_1^{\flat} & d_2^{\natural} & \ddots & \\ & & \ddots & \ddots & \ddots \\ & 0 & & \ddots & \ddots \end{bmatrix}$$

$$e = \begin{bmatrix} e_0^{\natural} & e_0^{\sharp} & 0 & & \\ e_0^{\flat} & e_1^{\natural} & e_1^{\sharp} & & 0 \\ 0 & e_1^{\flat} & e_2^{\natural} & \ddots & \\ & & \ddots & \ddots & \ddots \\ & 0 & & \ddots & \ddots \end{bmatrix}$$

$$d_k^{\natural} = \frac{q^{k-1}}{(1-q^{2k-2}abcd)(1-q^{2k}abcd)} \times$$

$$\Big[bd(a+c) + (b+d)q - abcd(b+d)q^{k-1}$$

$$- \Big(bd(a+c) + (abcd)(b+d) \Big) q^{k}$$

$$- bd(a+c)q^{k+1} + ab^2cd^2(a+c)q^{2k-1} + abcd(b+d)q^{2k} \Big]$$

$$e_k^{\natural} = \frac{q^{k-1}}{(1-q^{2k-2}abcd)(1-q^{2k}abcd)} \times$$

$$\Big[ac(b+d) + (a+c)q - abcd(a+c)q^{k-1}$$

$$- \Big(ac(b+d) + abcd(a+c) \Big) q^{k}$$

$$- ac(b+d)q^{k+1} + a^2bc^2d(b+d)q^{2k-1} + abcd(a+c)q^{2k} \Big]$$

$$d_k^{\sharp} = \frac{1}{1-q^k ac}\mathcal{A}_k$$

$$d_k^{\flat} = -\frac{q^k bd}{1-q^k bd}\mathcal{A}_k$$

$$e_k^{\sharp} = -\frac{q^k ac}{1-q^k ac}\mathcal{A}_k$$

$$e_k^{\flat} = \frac{1}{1-q^k bd}\mathcal{A}_k$$

$$A_k = \left[\frac{(1-q^{k-1}abcd)(1-q^{k+1})(1-q^{k}ab)(1-q^{k}ad)}{(1-q^{2k-1}abcd)(1-q^{2k}abcd)^2} \, \frac{(1-q^{k}ac)(1-q^{k}bc)(1-q^{k}bd)(1-q^{k}cd)}{(1-q^{2k+1}abcd)} \right]^{1/2}$$

$$\langle W| = h_0^{1/2}(1,0,0,\dots)$$

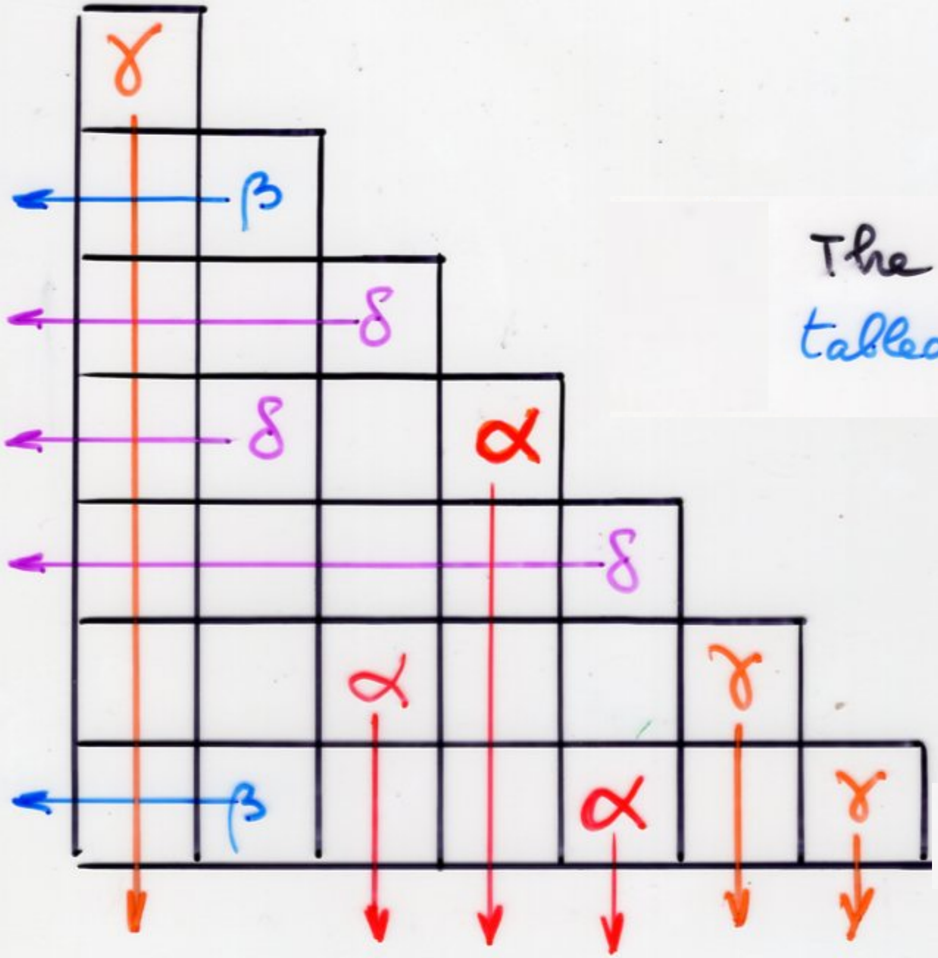
$$|V\rangle = h_0^{1/2}(1,0,0,\dots)^T$$

$$h_0 = \left[(1-q)(1-q^2)\dots\right]^{1/2}$$

$$(q;q)_\infty$$

S. Corteel, L. Williams (2009)

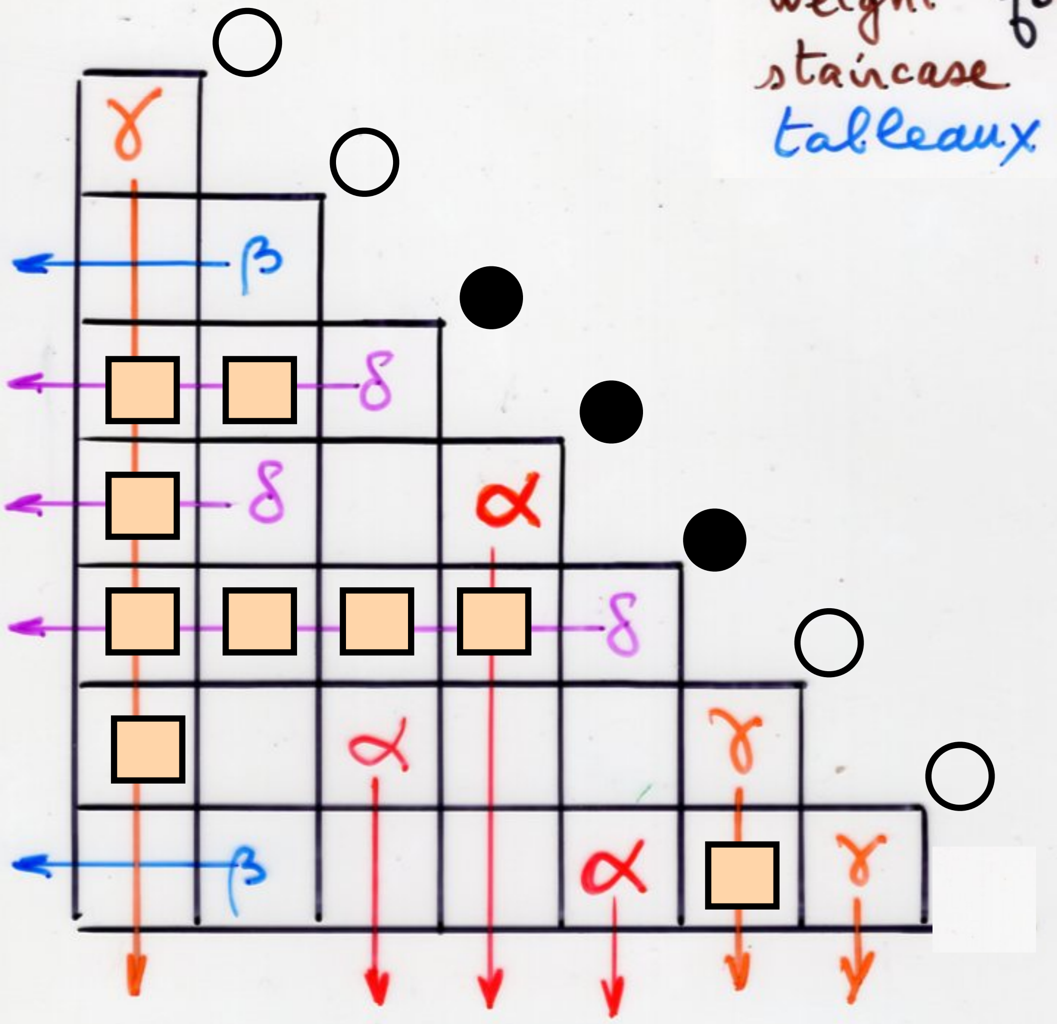
staircase tableaux



The number of staircase tableaux of size n is $4^n n!$

● α, δ ○ β, γ

weight for staircase tableaux



$\square \leftarrow \beta$

$q \leftarrow \delta$

$$\begin{tikzcd}
\beta,\gamma \arrow[d] & \\
q & \alpha,\gamma \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
\alpha,\delta \arrow[d] & \\
\square & \alpha,\gamma \arrow[l]
\end{tikzcd}$$

$\tau = (\tau_1, \ldots, \tau_n)$

$$\mathbf{Z}_{\tau} = \sum_{\substack{T \\ \text{staircase tableaux} \\ \text{size } n \\ \text{profile of } T}} v(T)$$

S. Corteel, L. Williams (2009)

$$\mathbf{Z}_n(\alpha, \beta, \gamma, \delta ; q) = \sum_{\substack{T \\ \text{staircase tableaux} \\ \text{size } n}} v(T)$$

partition function

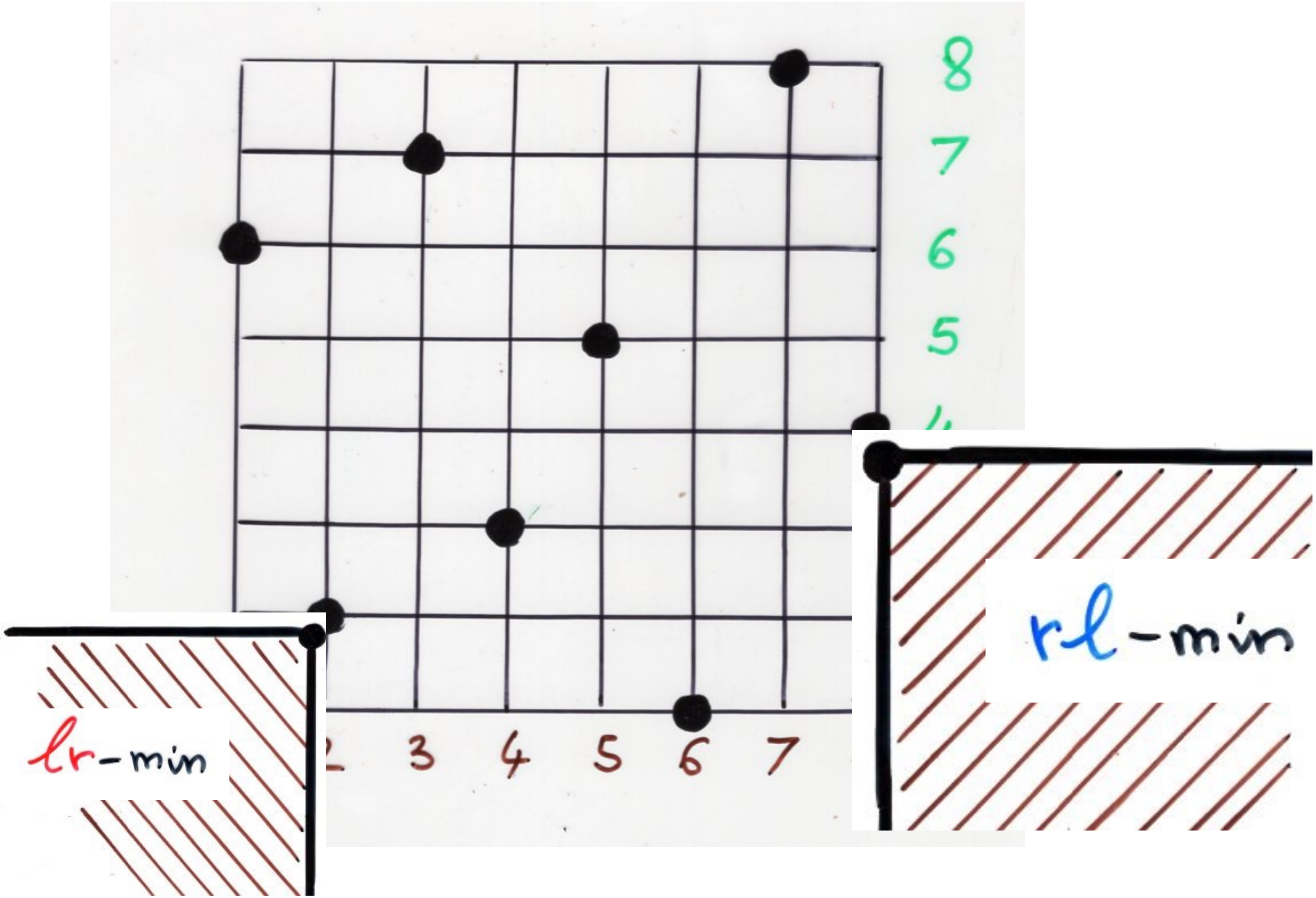
→ expression for the moments of the Askey-Wilson polynomials

S. Corteel, L. Williams
R. Stanley, D. Stanton
(2010)

Why to insist
on the 3 parameters model ?

symmetries !

my dream ...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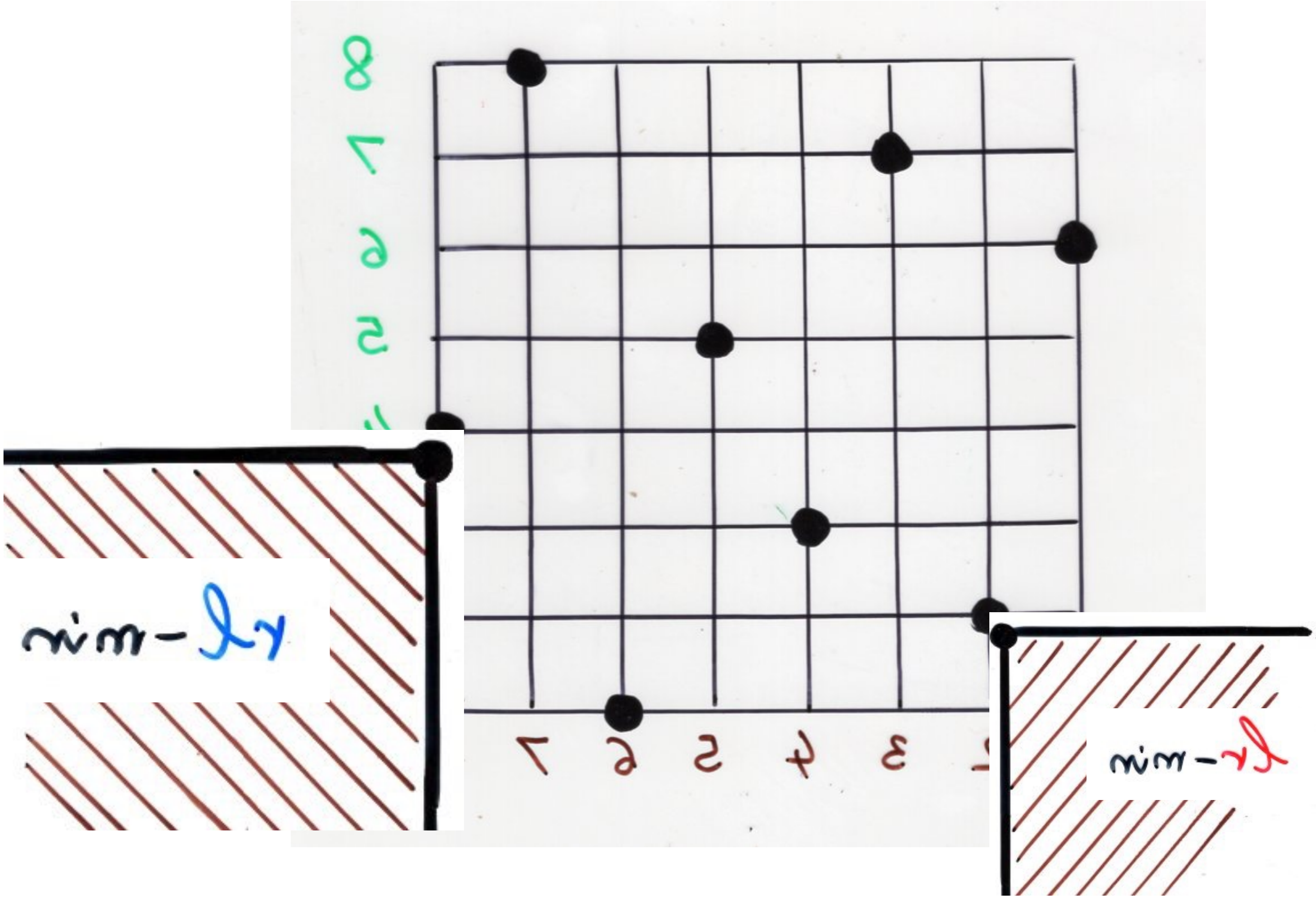

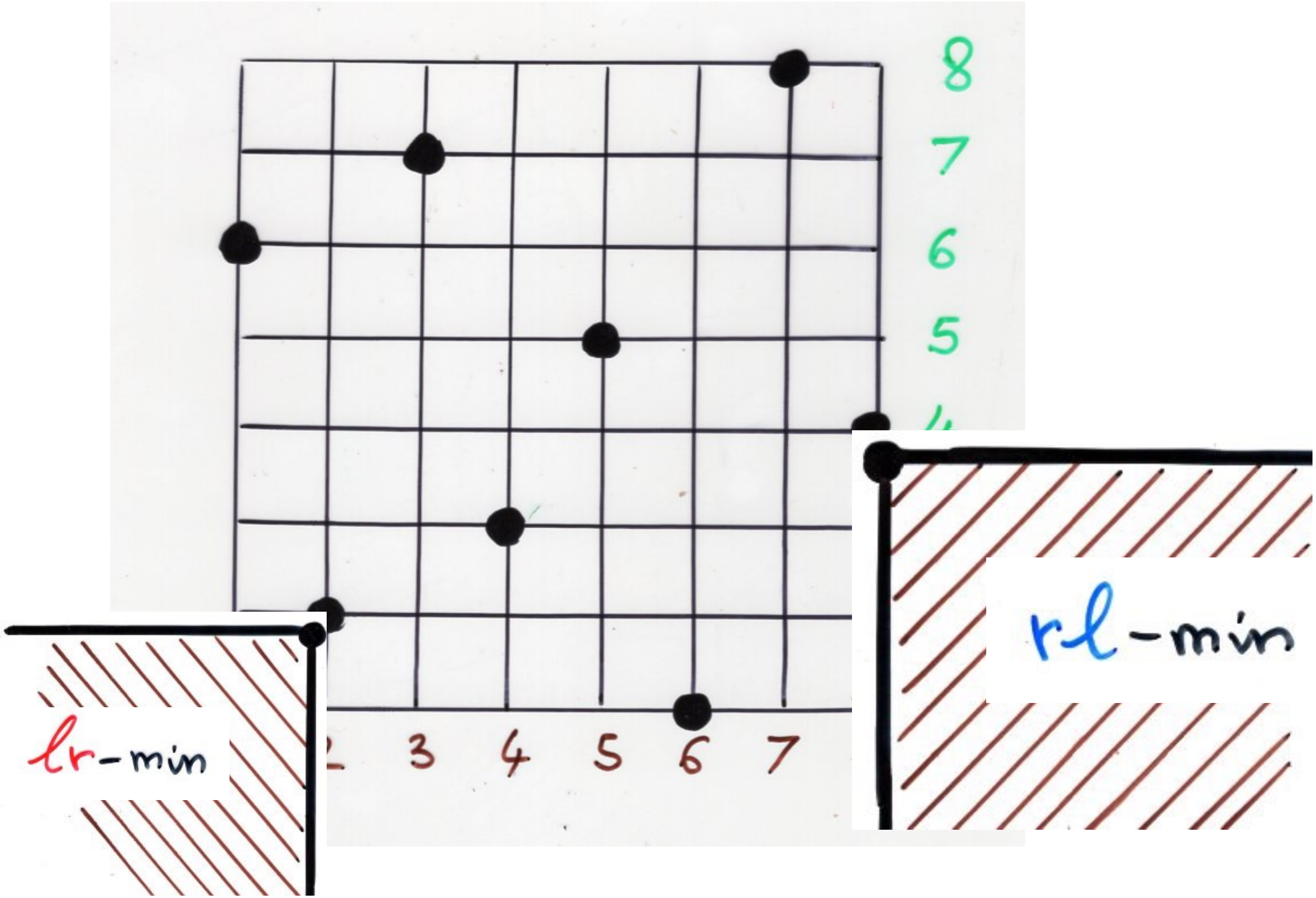
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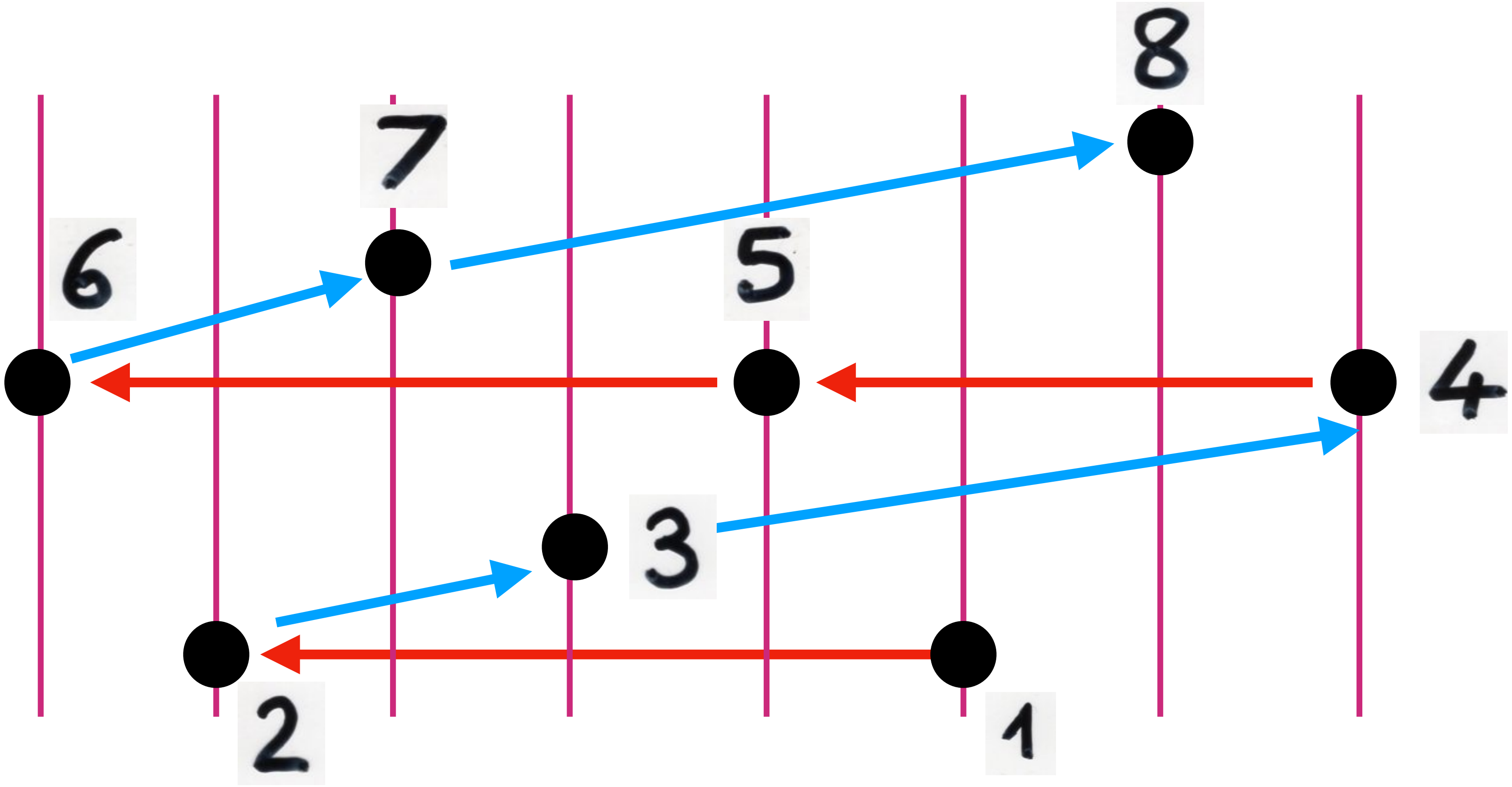
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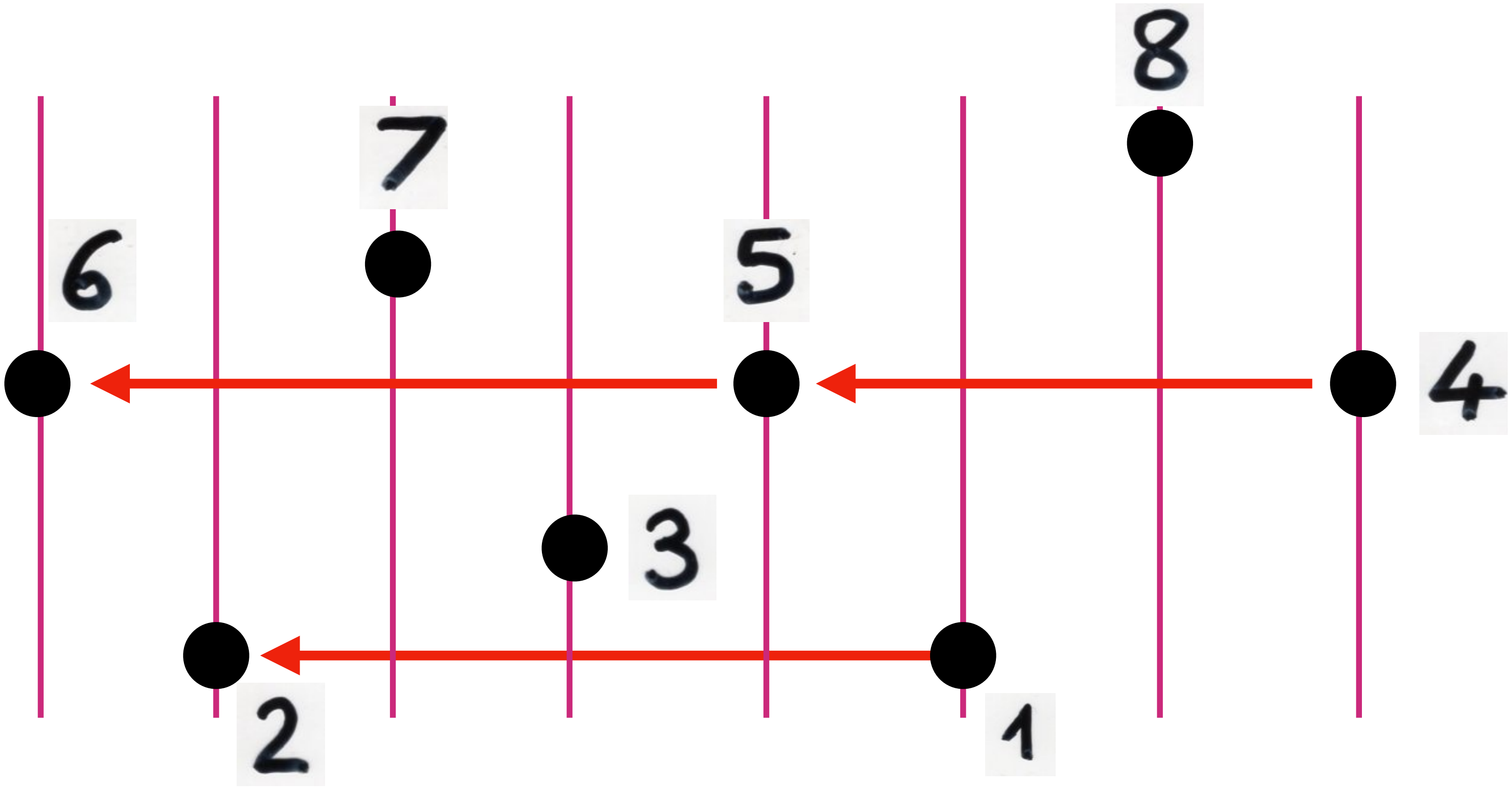
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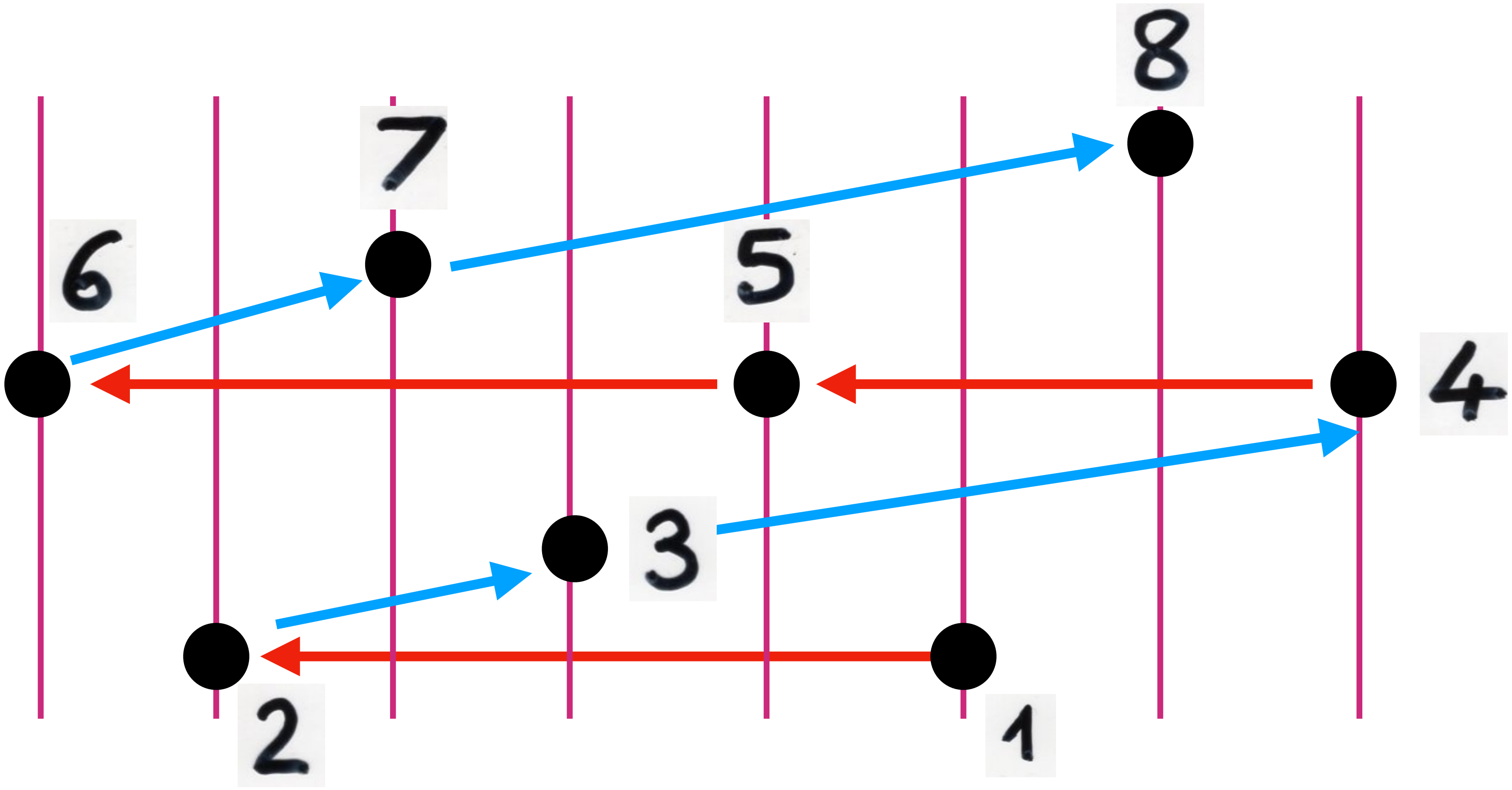
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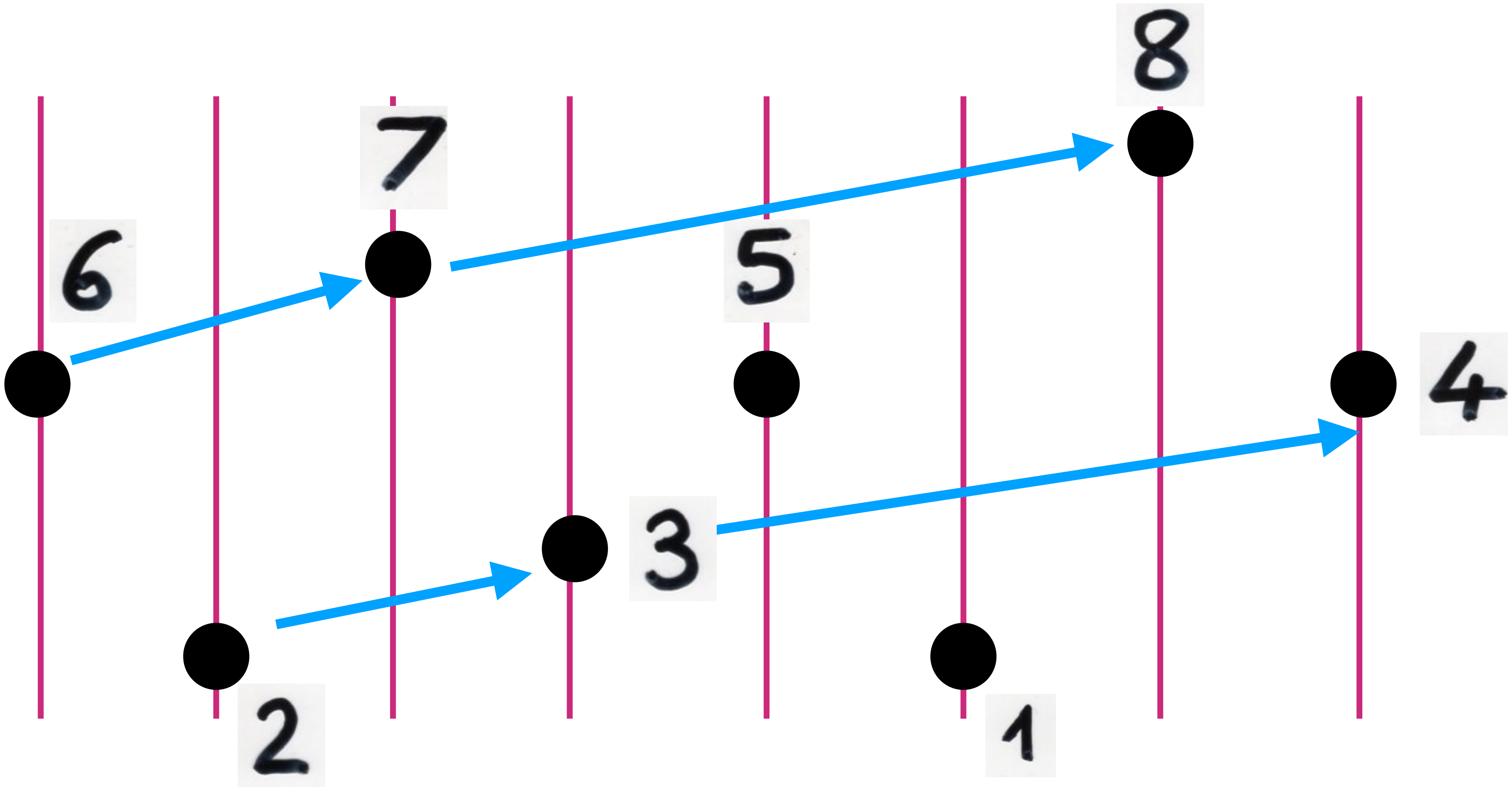
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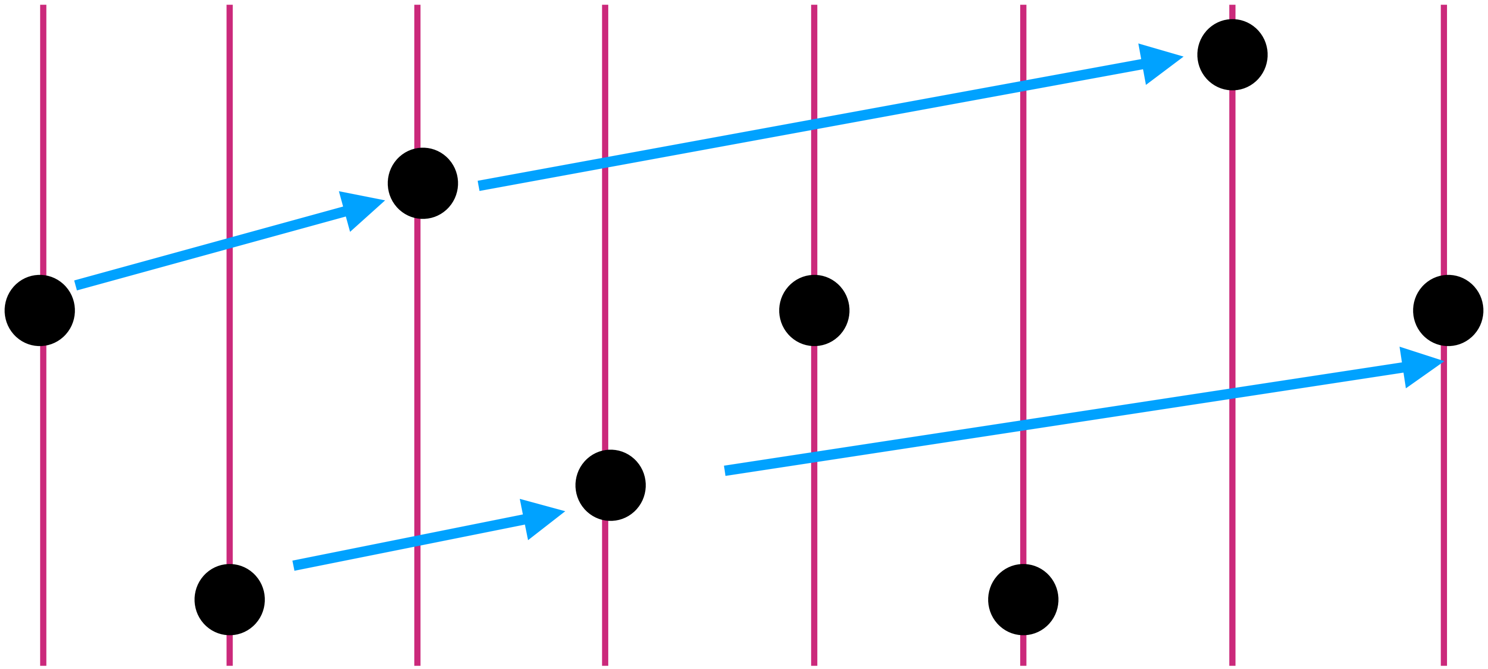
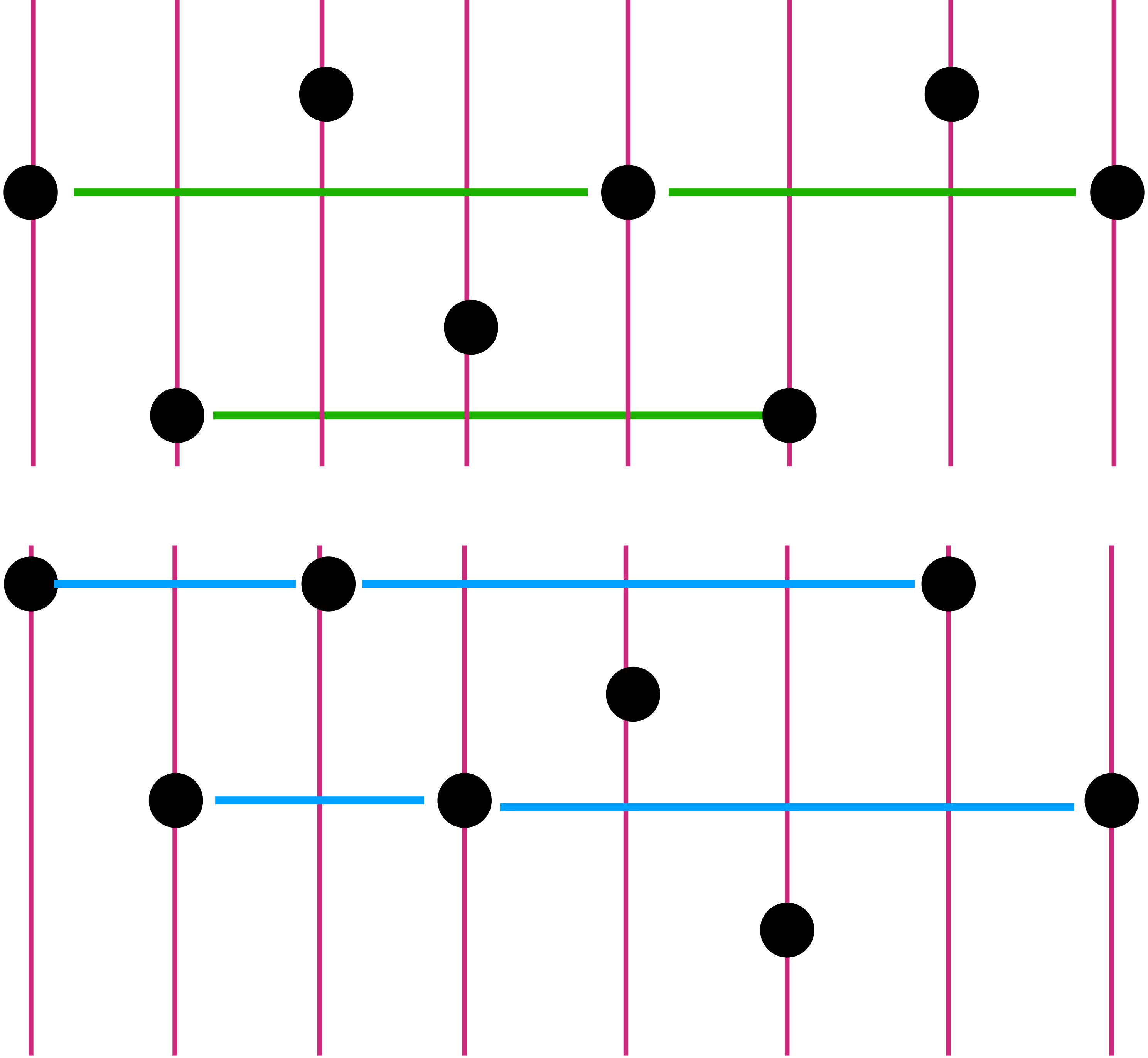




Askey-Wilson integral

The Askey-Wilson integral

$$w(\cos\theta, a,b,c,d \mid q) = \frac{(e^{2i\theta})_\infty (e^{-2i\theta})_\infty}{(ae^{i\theta})_\infty (ae^{-i\theta})_\infty (be^{i\theta})_\infty (be^{-i\theta})_\infty (ce^{i\theta})_\infty (ce^{-i\theta})_\infty (de^{i\theta})_\infty (de^{-i\theta})_\infty}$$

$$(a)_\infty = \prod_{i\geq 0} (1-aq^i)$$

$$\frac{(q)_\infty}{2\pi} \int_0^\pi w(\cos\theta, a,b,c,d \mid q)\, d\theta =$$

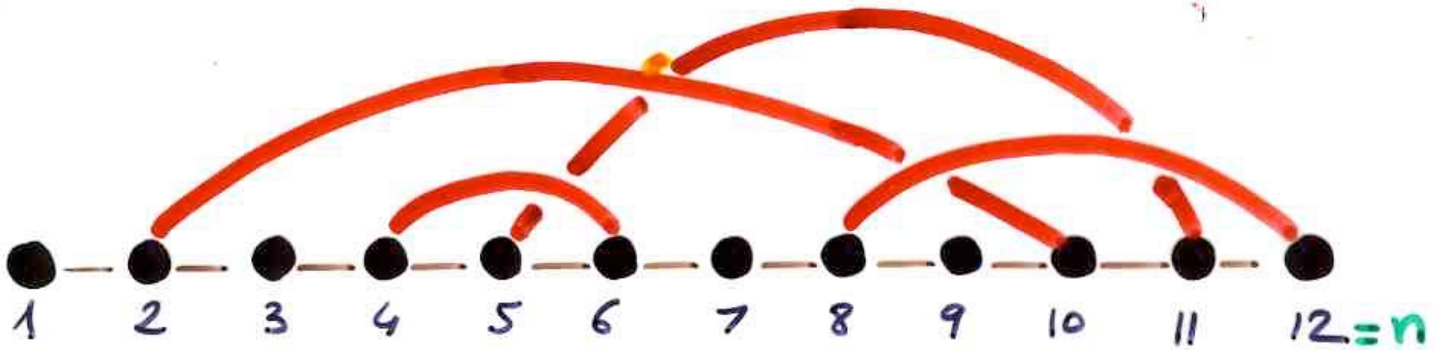
$$\frac{(abcd)_\infty}{(ab)_\infty (ac)_\infty (ad)_\infty (bc)_\infty (bd)_\infty (cd)_\infty}$$

The Askey-Wilson integral

integral of the product
of 4 q-Hermite polynomials
(type II)

Ismail, Stanton, V. (1986)

$$\frac{(q)_\infty}{2\pi}\int_0^\pi H_k(\cos\theta|q)\,H_l(\cos\theta|q)\,(e^{2i\theta})_\infty(e^{-2i\theta})_\infty=(q)_k\,\delta_{kl}$$

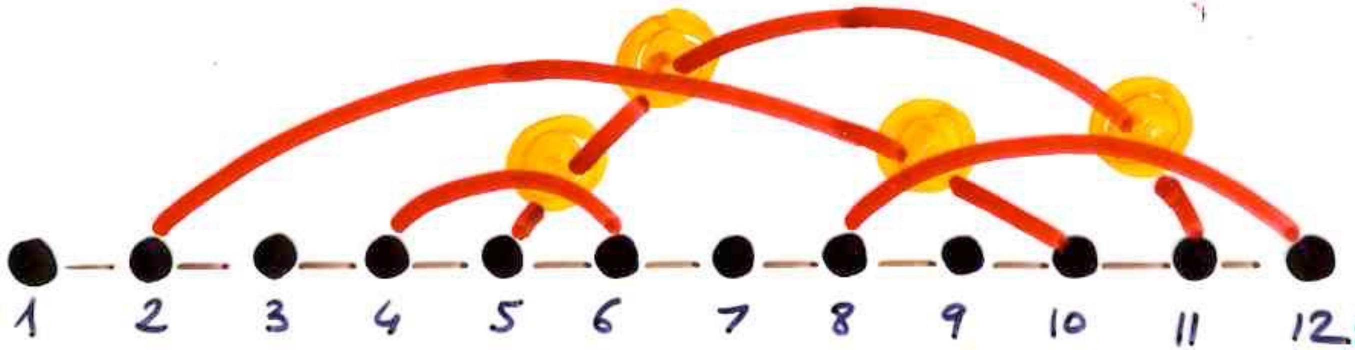


(continuous)

q-Hermite

$$H_n(x|q) = \sum_{\substack{\gamma \\ \text{matching}}} (-1)^{|\gamma|} q^{cr(\gamma)+e(\gamma)} x^{fix(\gamma)}$$

nesting



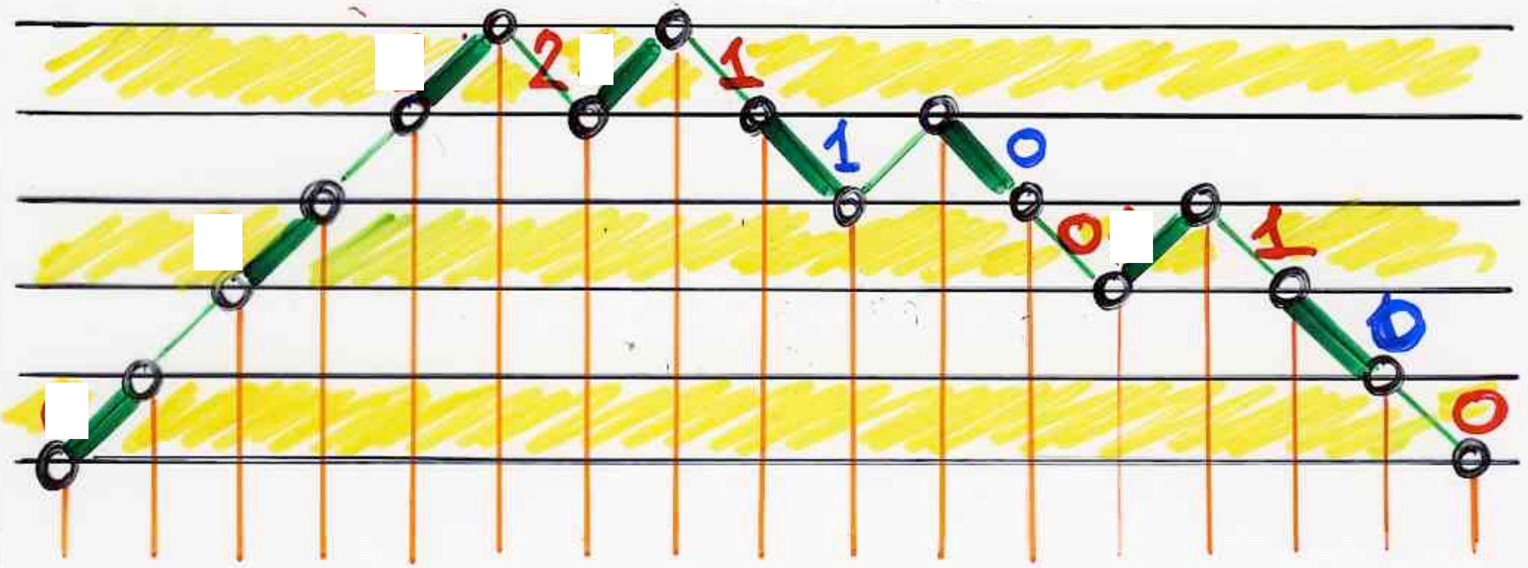
(continuous)

q-Hermite

$$H_n(x|q) = \sum_{\substack{\gamma \\ \text{matching}}} (-1)^{|\gamma|}\, q^{cr(\gamma)+e(\gamma)}\, x^{fix(\gamma)}$$

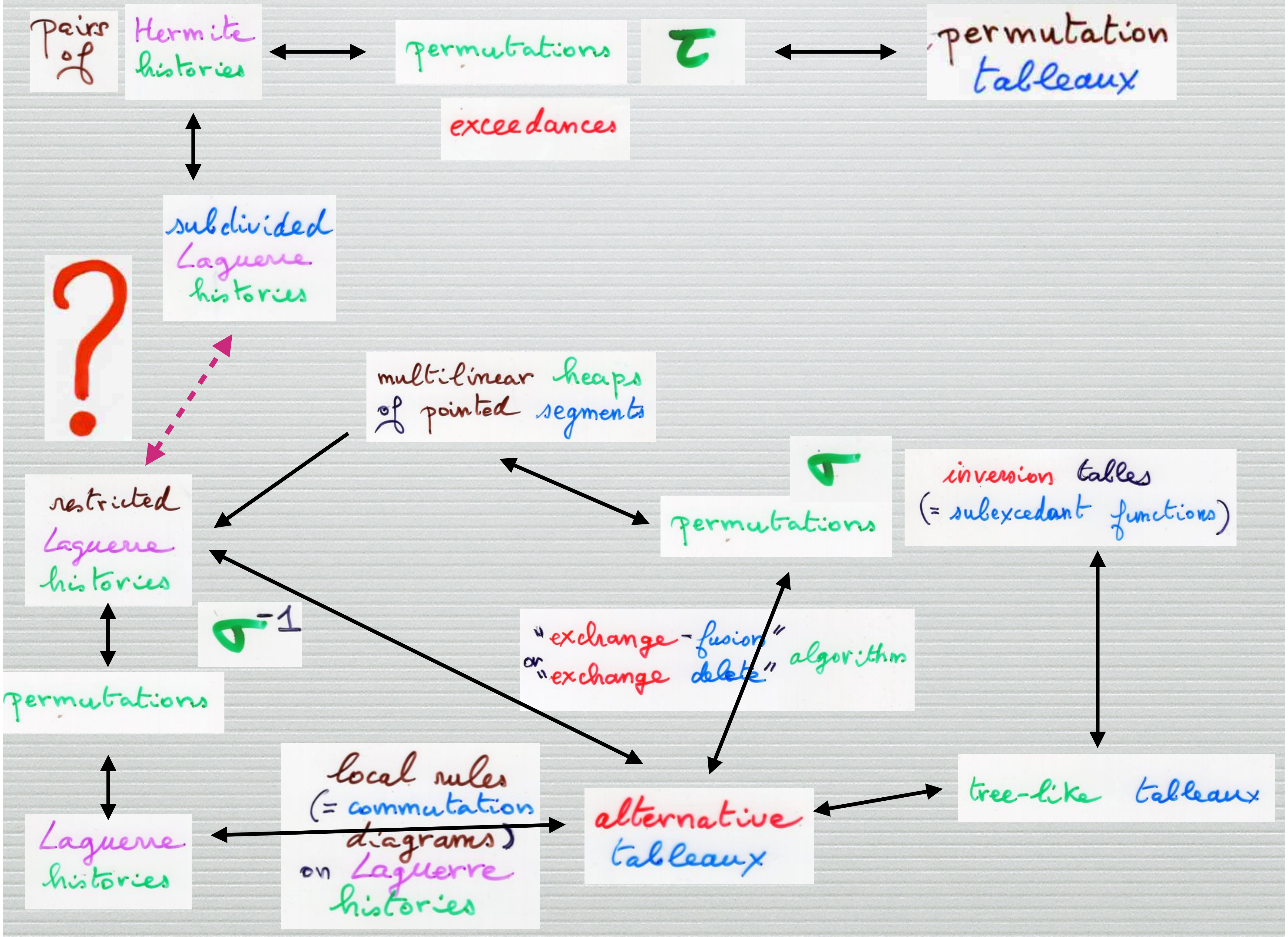
crossings

$$UD = qDU + I$$



subdivided Laguerre history

$$DE = qED + E + D$$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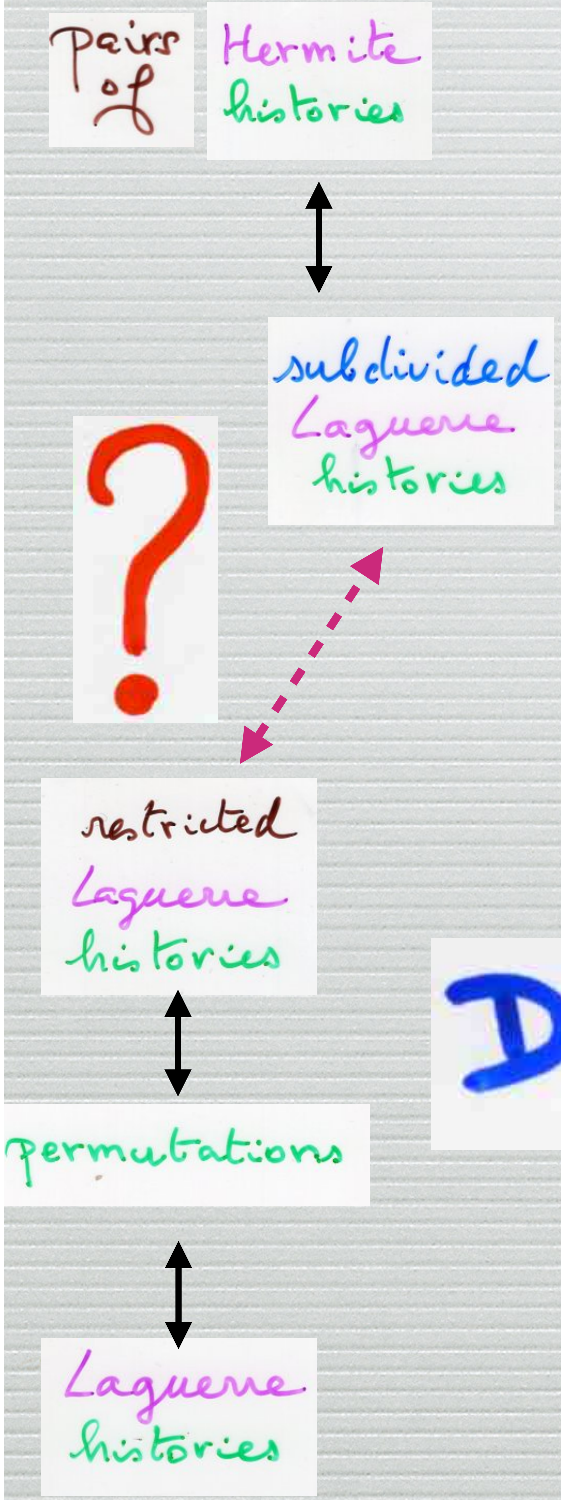


Pairs of Hermite histories

subdivided Laguerre histories

?

restricted Laguerre histories

permutations

Laguerre histories

$$UD = qDU + I$$

Hermite polynomials

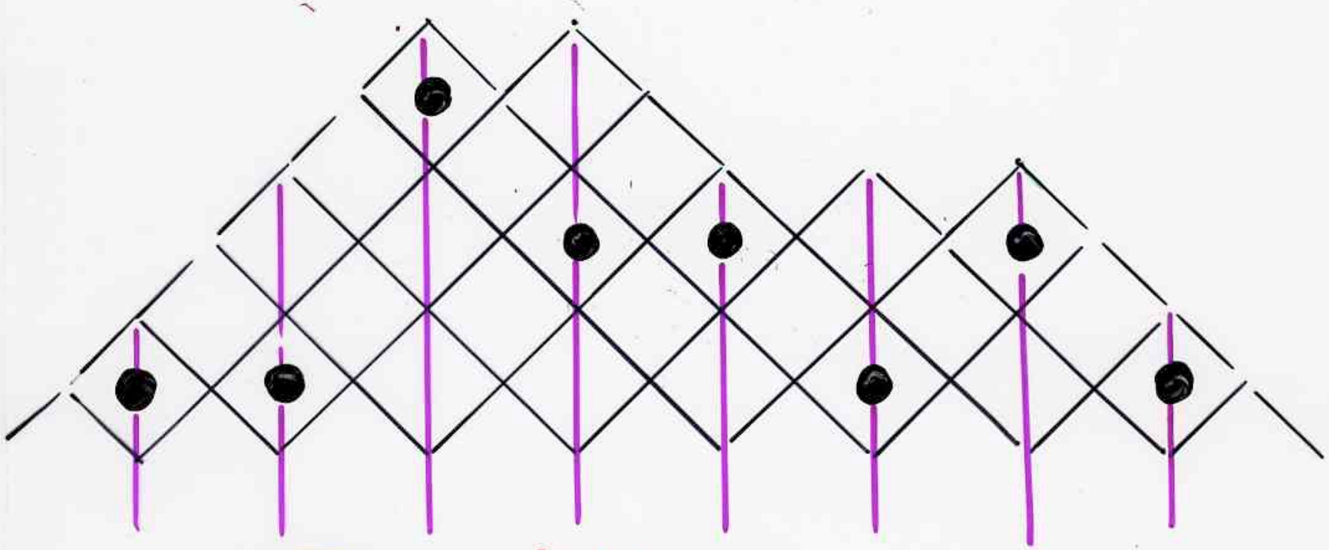
$$DE = qED + E + D$$

Laguerre polynomials

Dyck tableaux

as

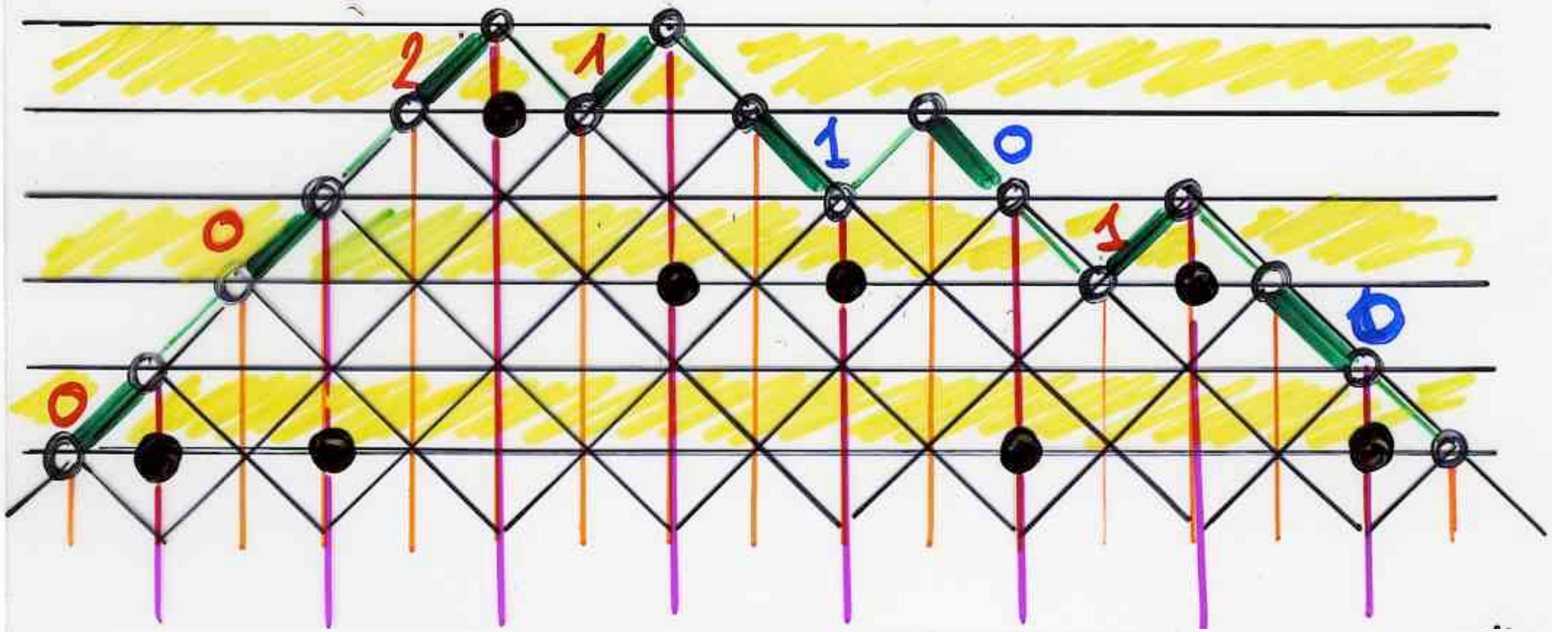
subdivided Laguerre histories



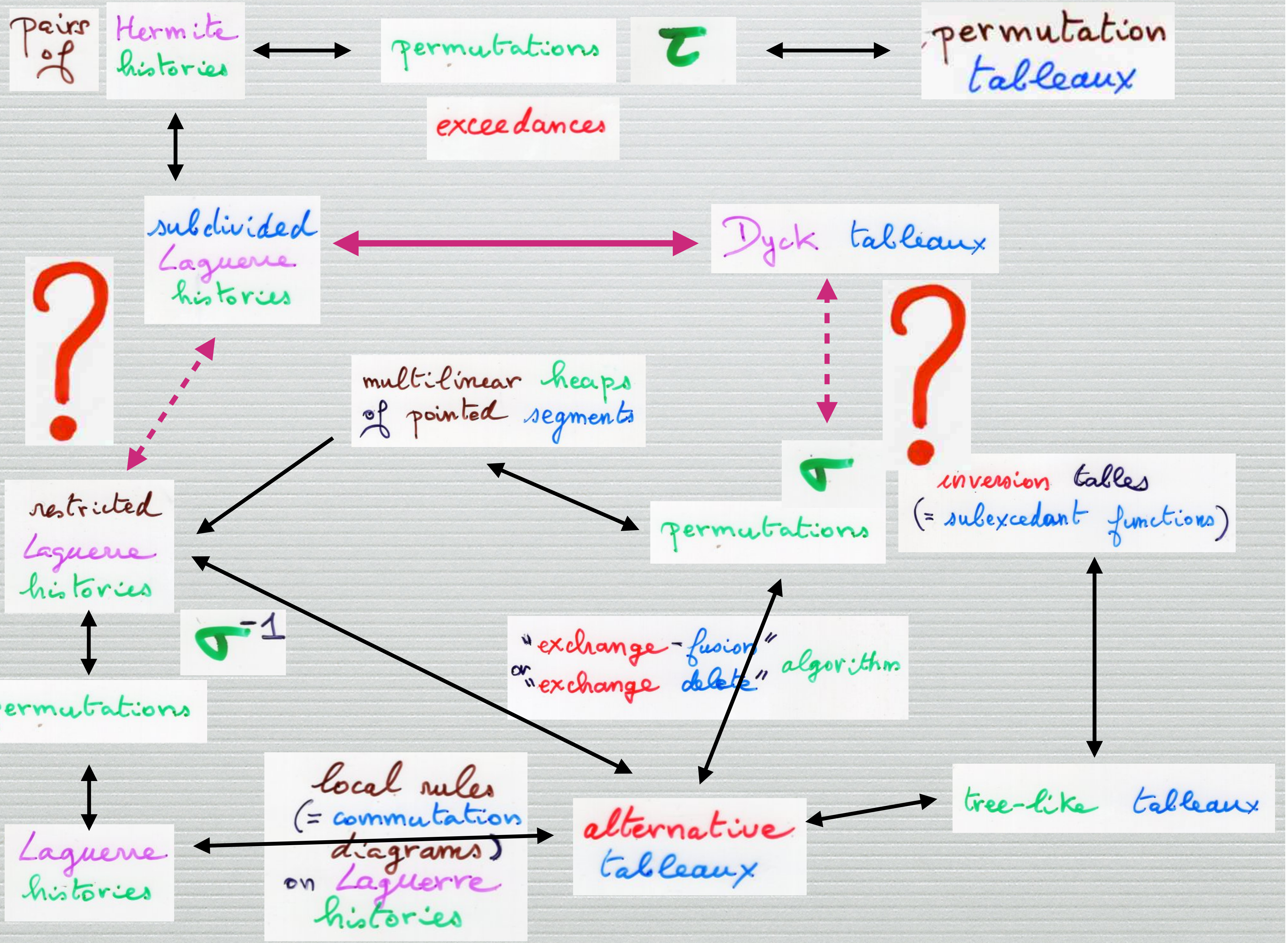
Dyck tableau

J.-C. Aval, A. Boussicault, S. Dasse-Hartaut

(2011)



Dyck tableau
as a
subdivided Laguerre history



Pairs of Hermite histories

permutations τ

permutation tableaux

exceedances

subdivided Laguerre histories

Dyck tableaux

?

multilinear heaps of pointed segments

?

σ

permutations

inversion tables (= subexcedant functions)

restricted Laguerre histories

σ^{-1}

permutations

"exchange-fusion" or "exchange delete" algorithm

Laguerre histories

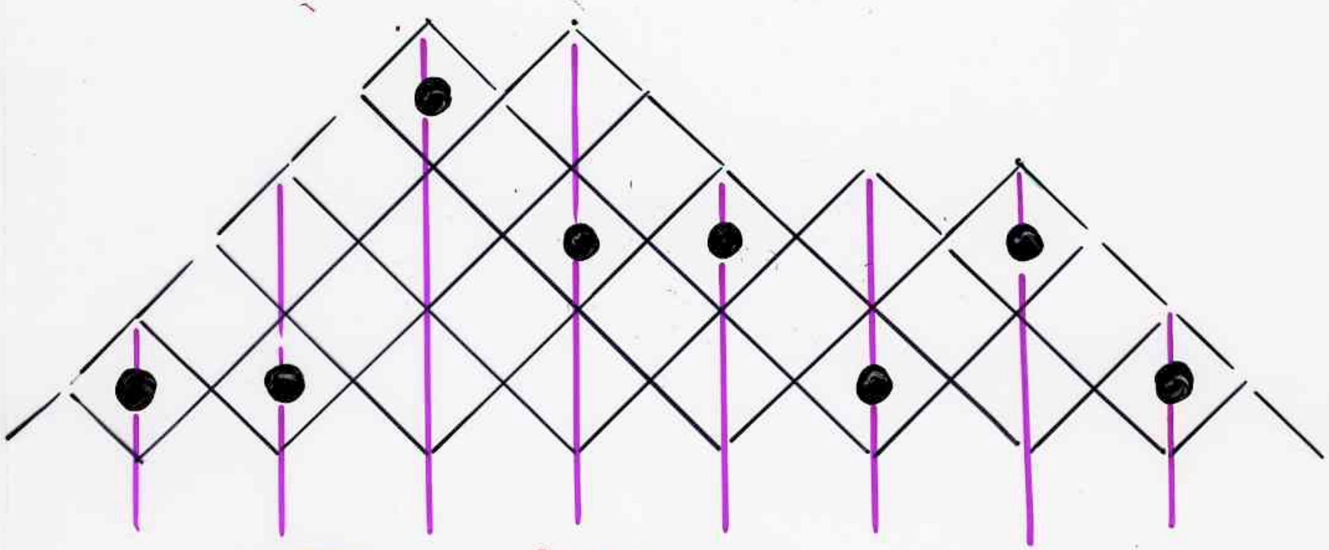
local rules (= commutation diagrams) on Laguerre histories

alternative tableaux

tree-like tableaux

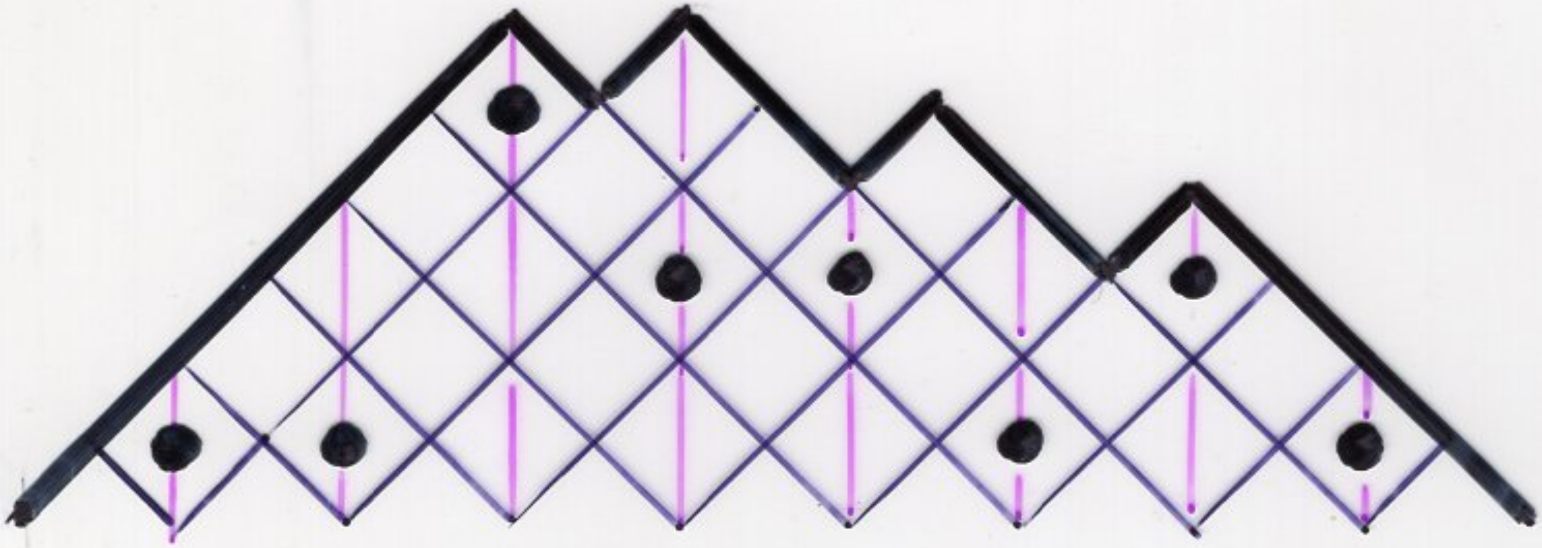
(direct) bijection

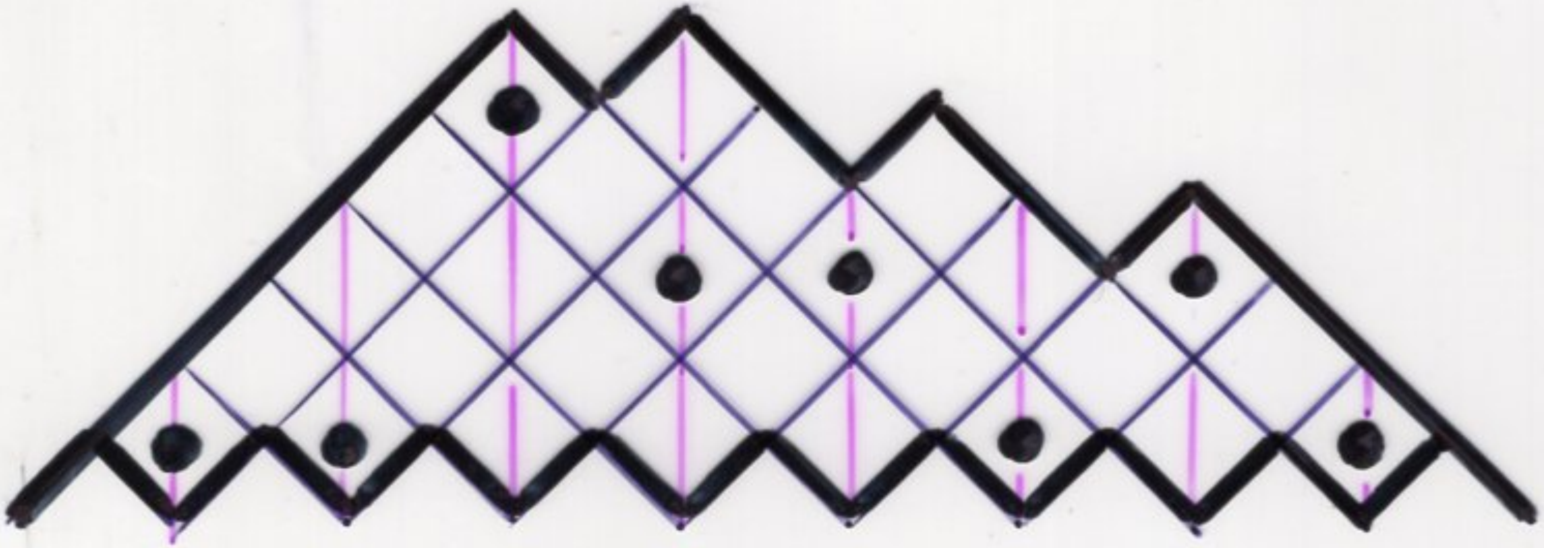
permutations → Dyck tableaux

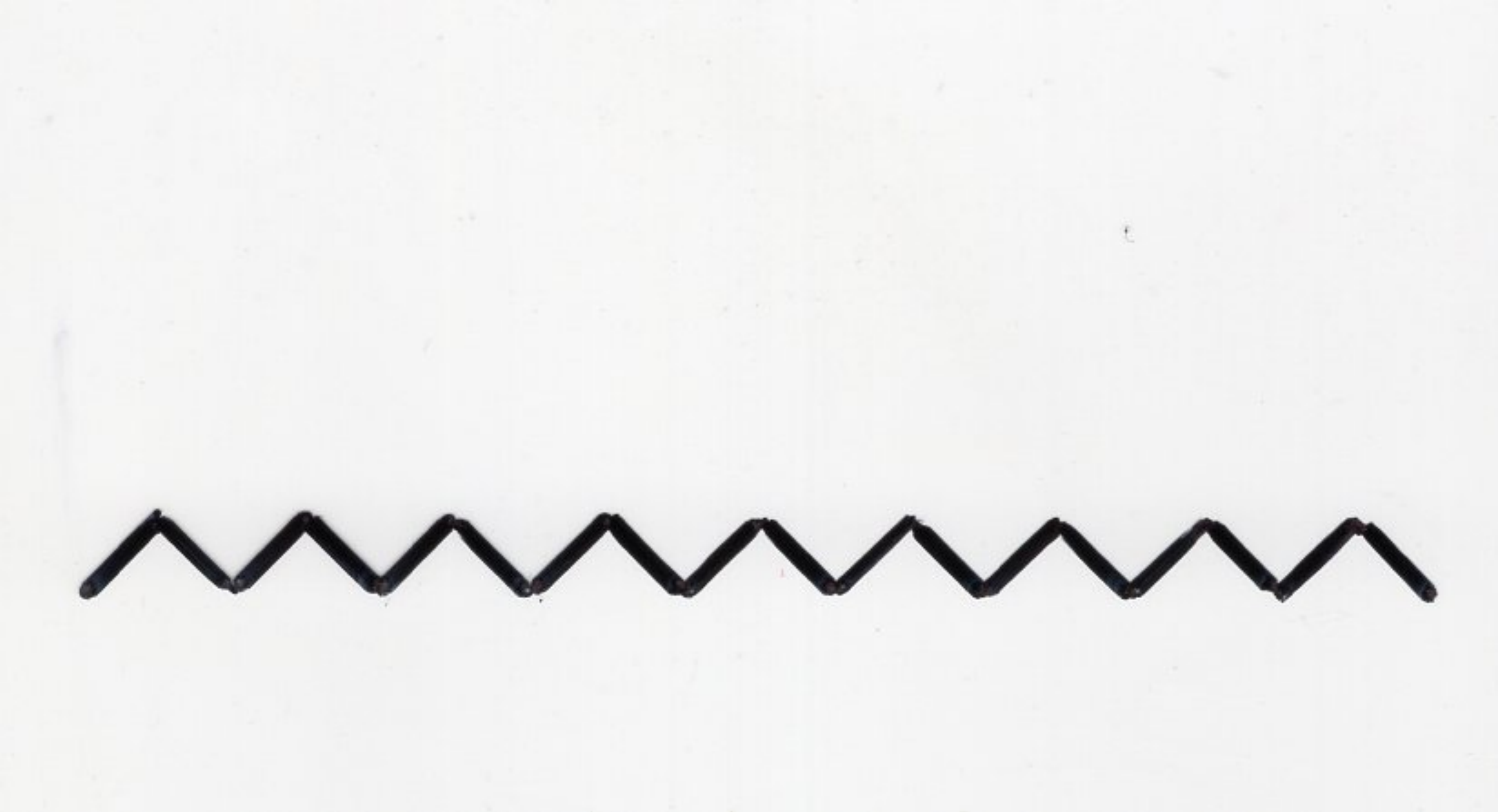


Dyck tableau

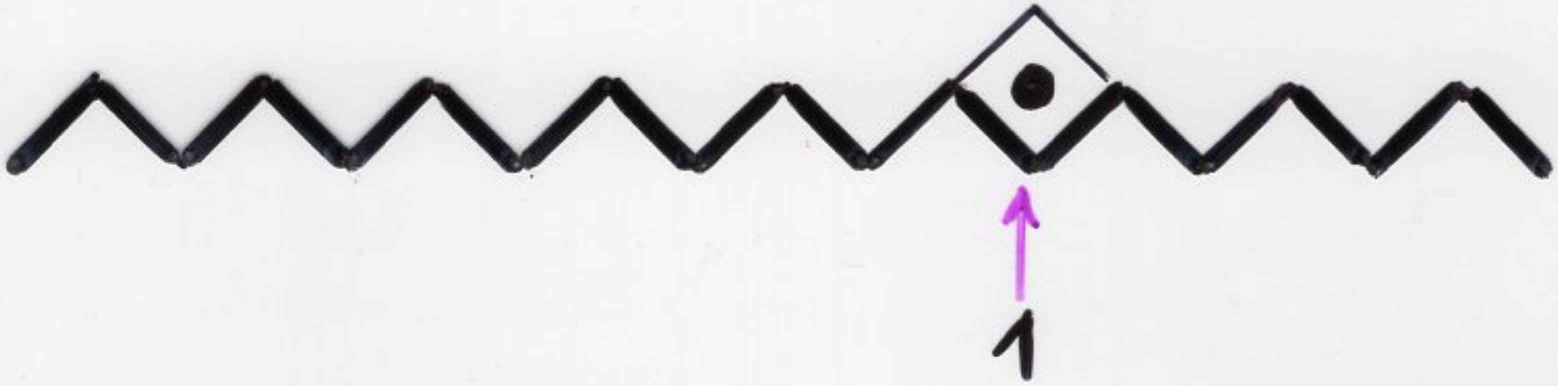
J.-C. Aval, A. Boussicault, S. Dasse-Hartaut
(2011)



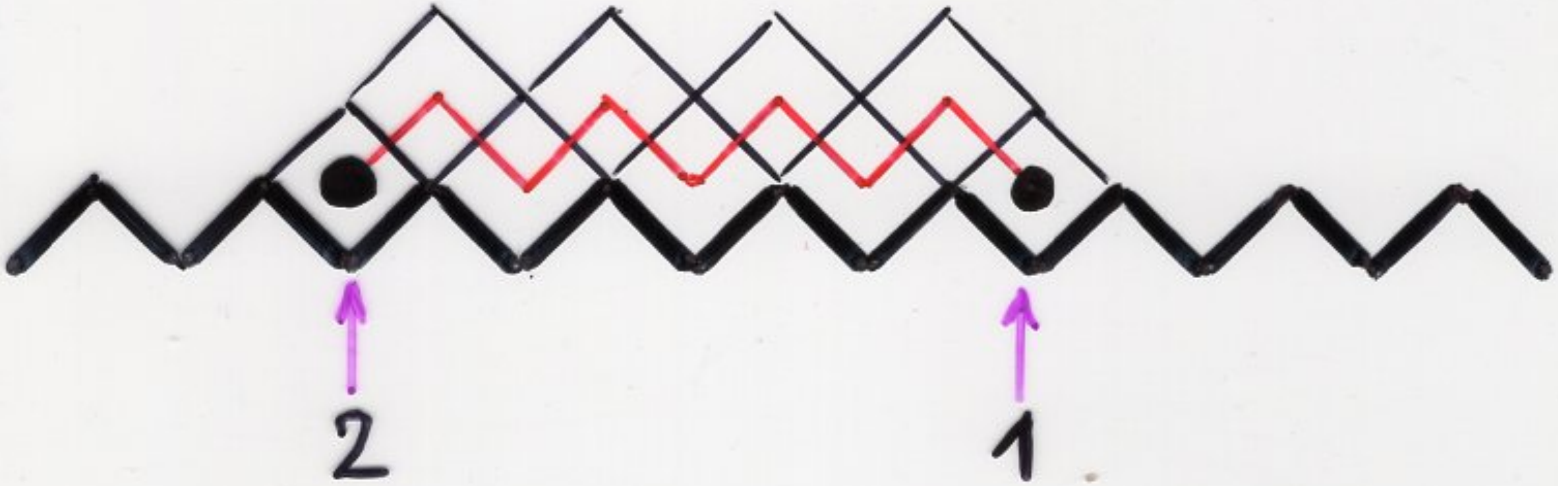




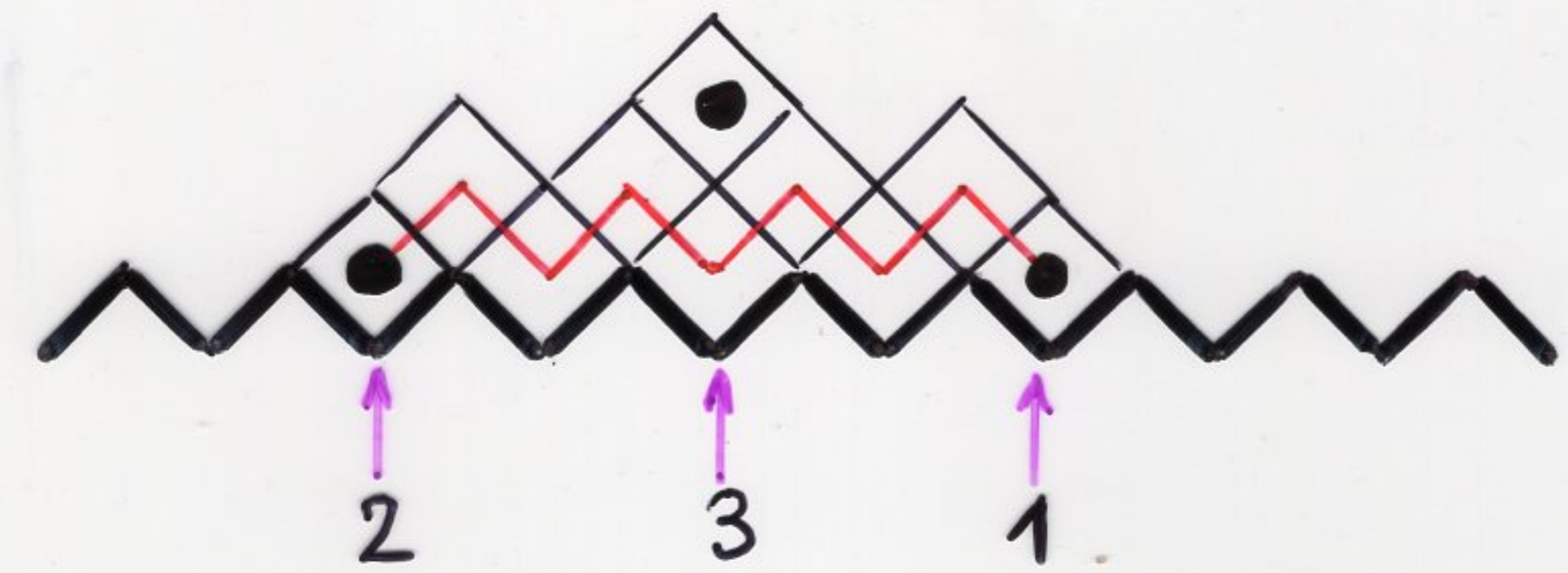
σ = 6 2 7 3 5 1 8 4



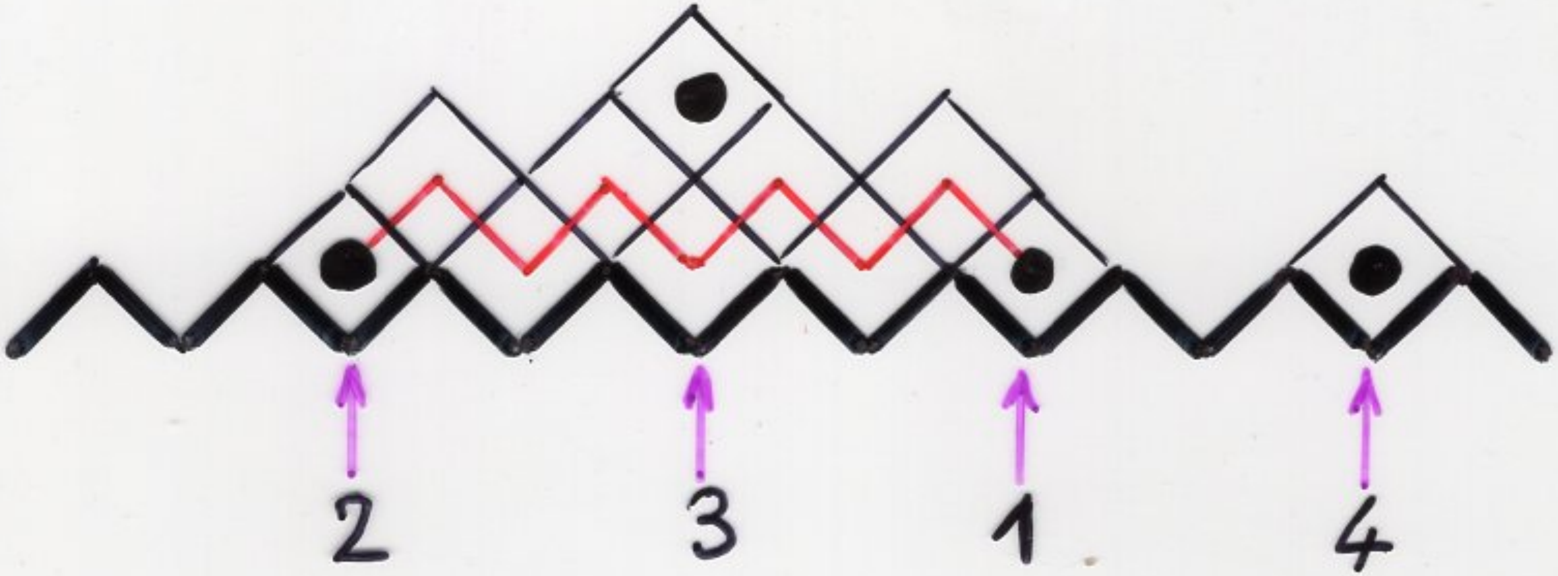
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$



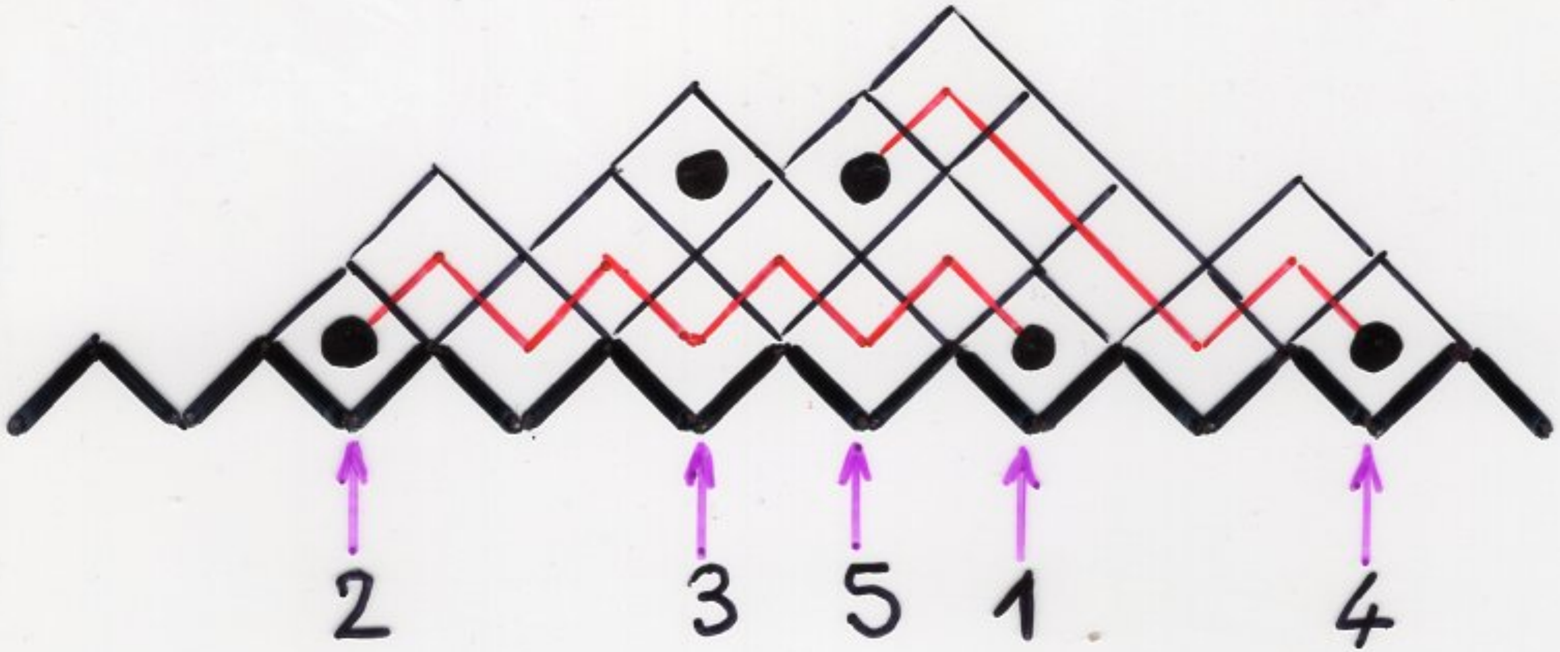
σ = 6 2 7 3 5 1 8 4



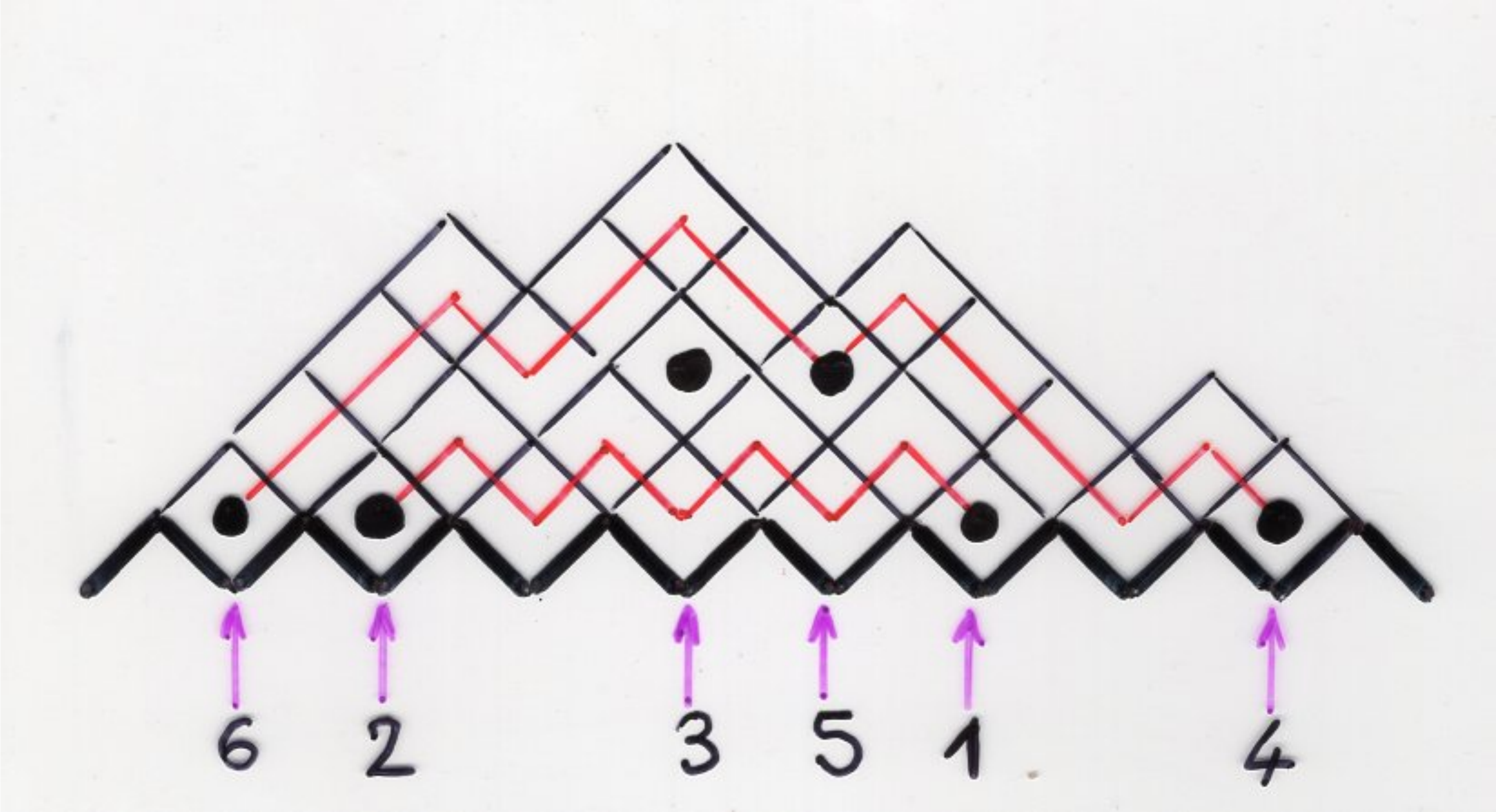
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$



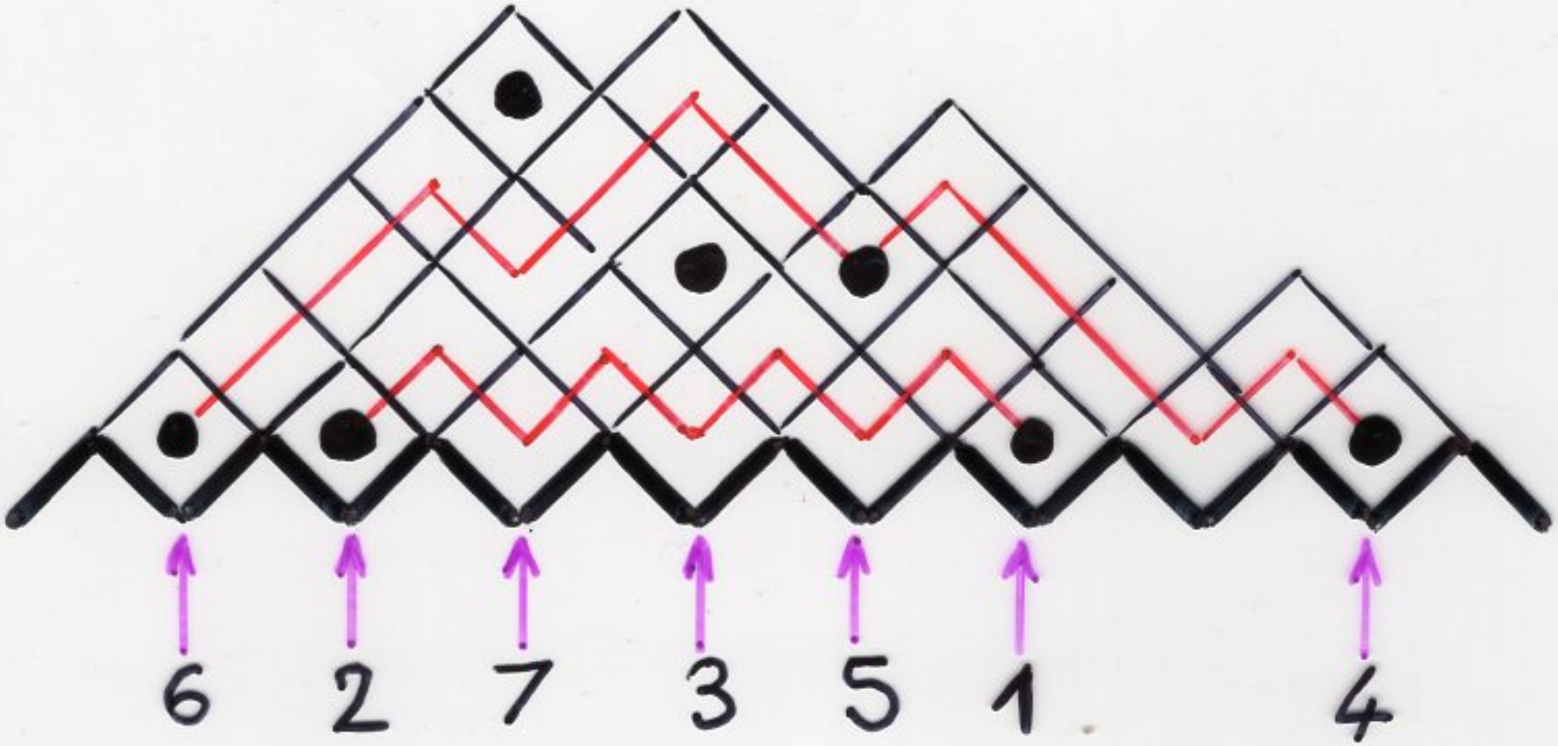
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$



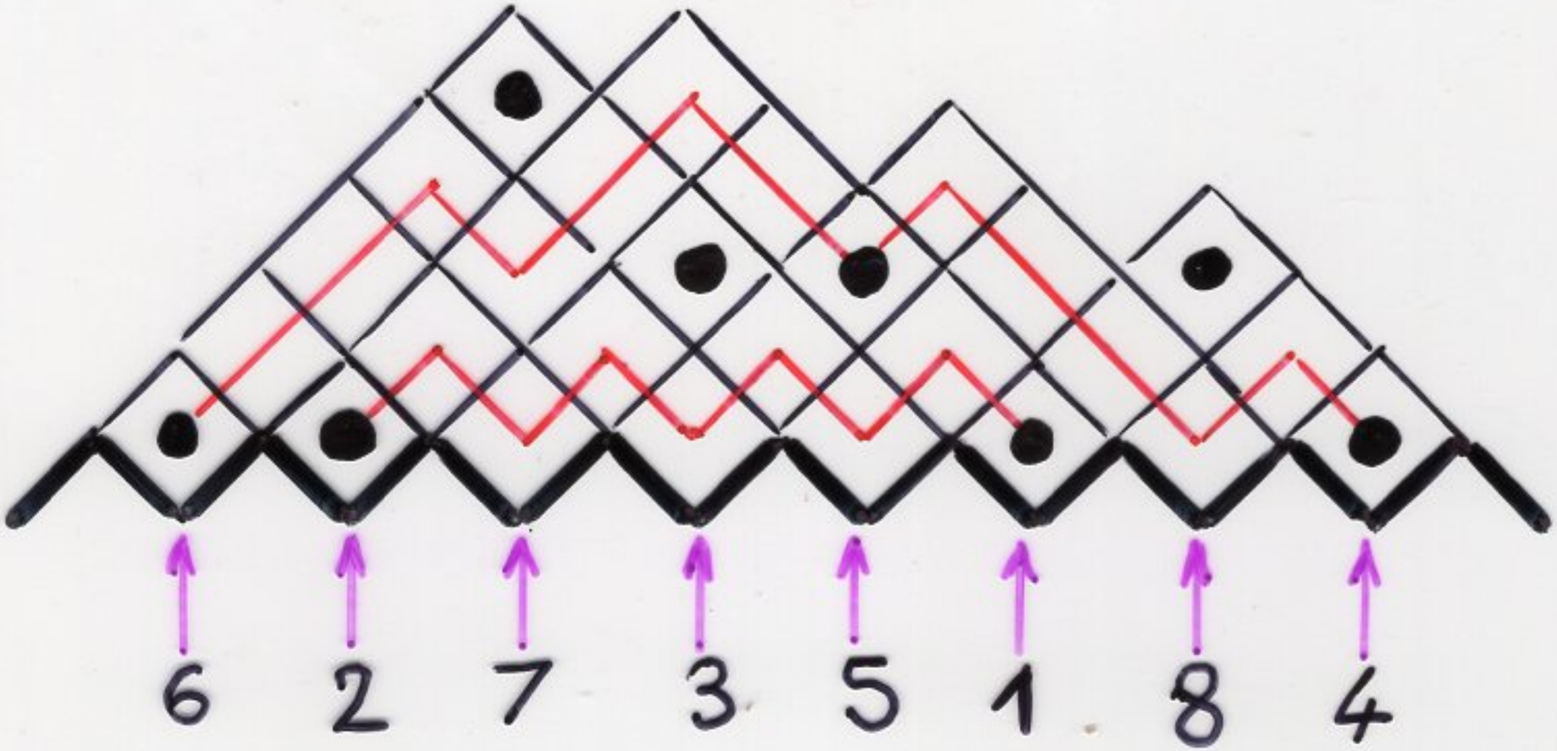
$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$



$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$



$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$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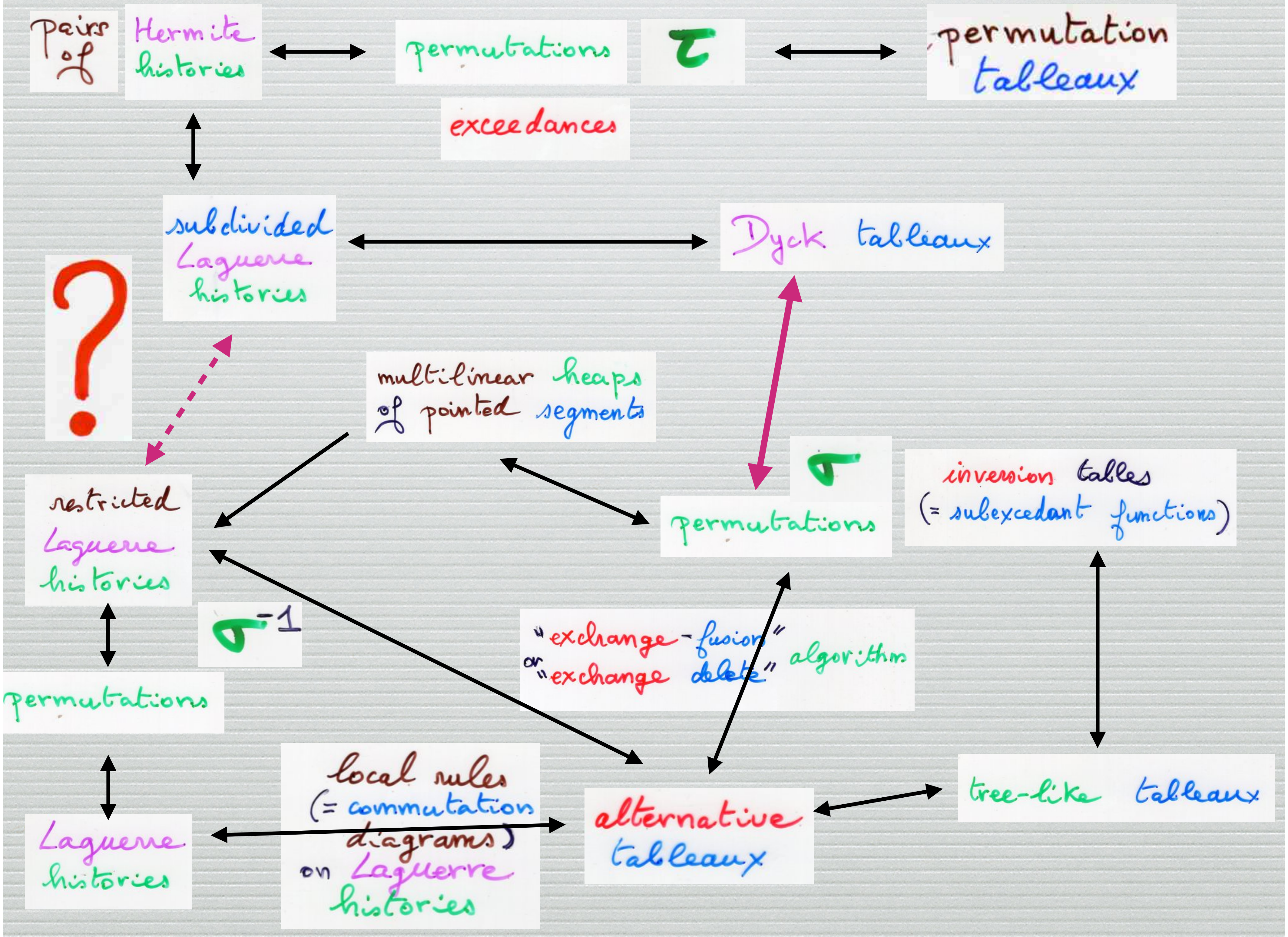

Pairs of Hermite histories
permutations
τ
permutation tableaux
exceedances
subdivided Laguerre histories
Dyck tableaux
?
multilinear heaps of pointed segments
restricted Laguerre histories
σ
permutations
inversion tables
(= subexcedant functions)
σ⁻¹
permutations
"exchange-fusion"
or "exchange delete" algorithm
Laguerre histories
local rules
(= commutation diagrams)
on Laguerre histories
alternative tableaux
tree-like tableaux

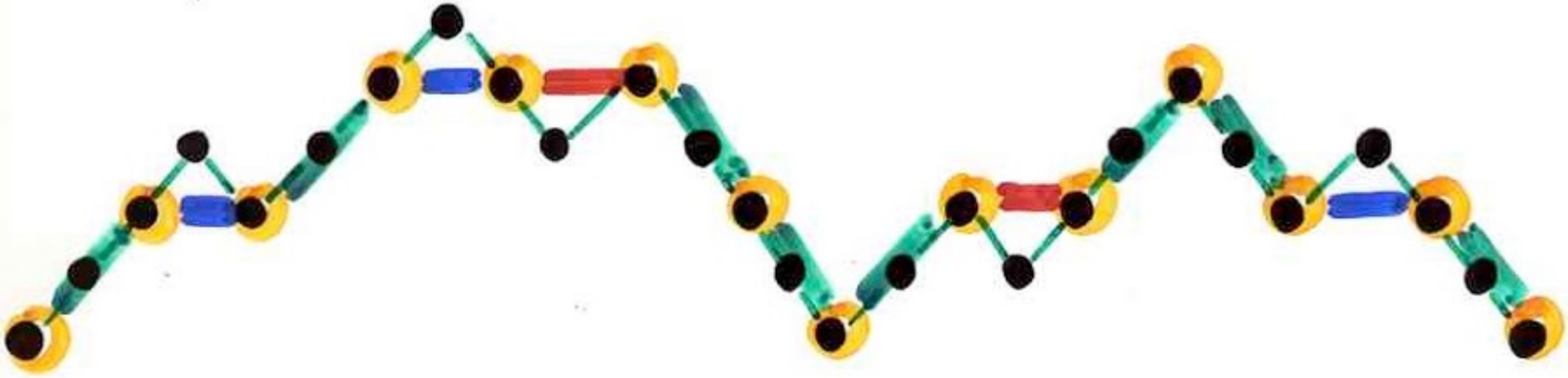
contractions

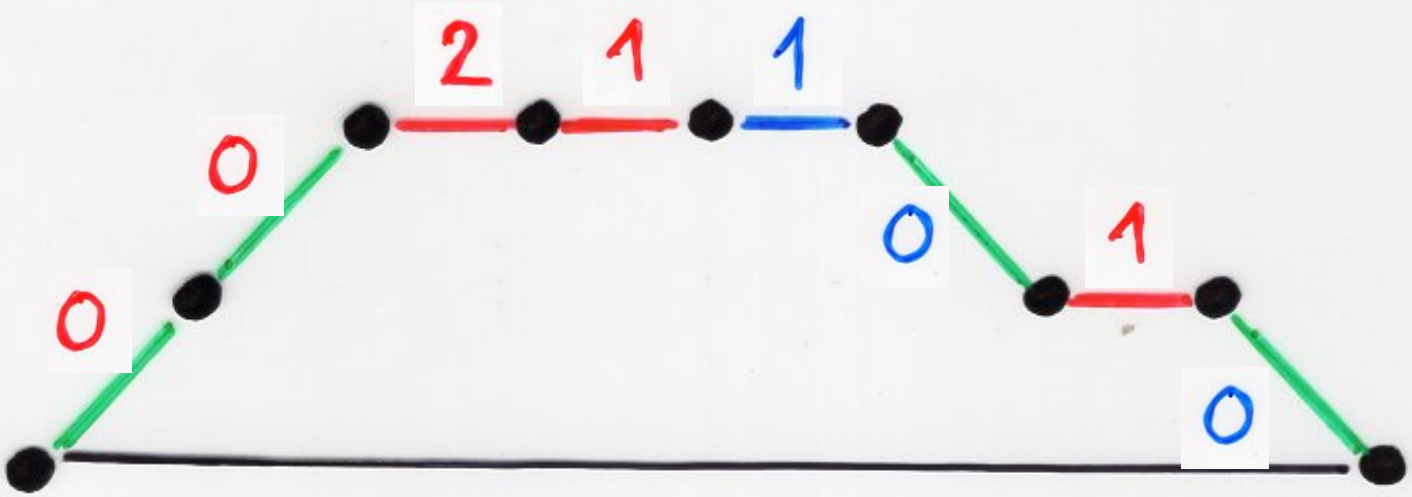
in

continued fractions

$$\sum_{n\geqslant 0} n!\, t^{n} = \cfrac{1}{1-\cfrac{1t}{1-\cfrac{1t}{1-\cfrac{2t}{1-\cfrac{2t}{1-\cfrac{3t}{\cdots}}}}}}$$

$$\sum_{n\geqslant 0} n!\, t^{n} = \cfrac{1}{1-1t-\cfrac{1^{2}t^{2}}{1-3t-\cfrac{2^{2}t^{2}}{1-5t-\cfrac{3^{2}t^{2}}{\cdots}}}}$$

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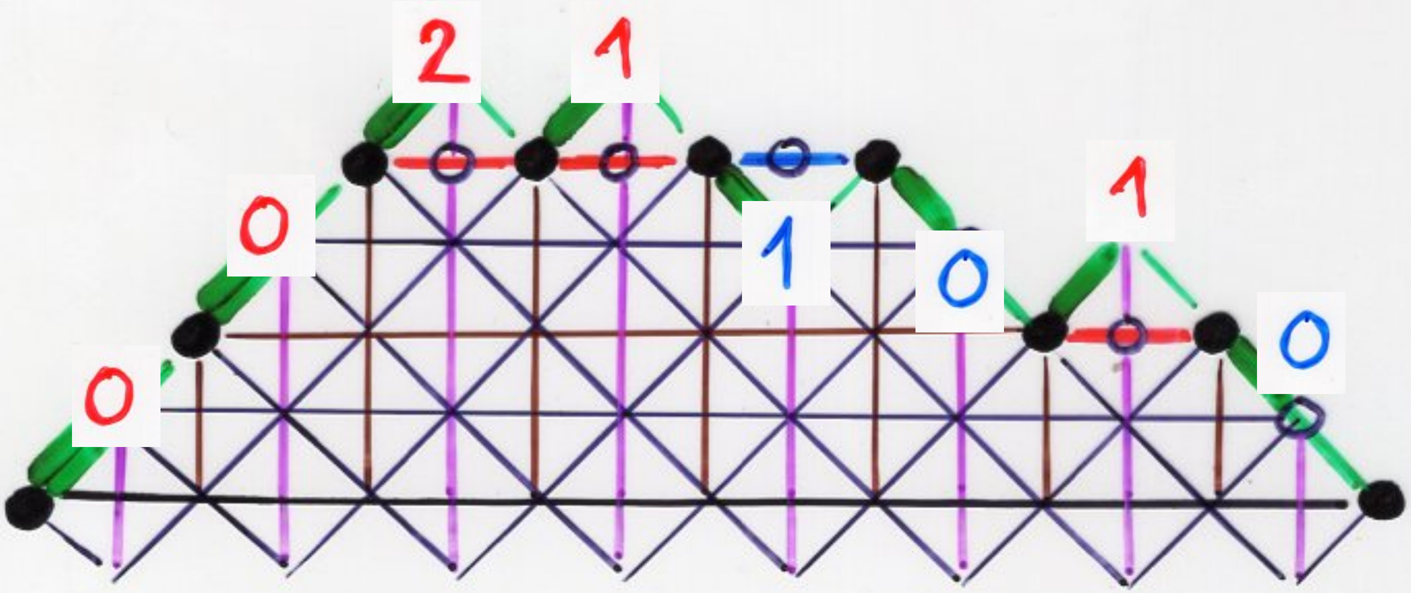
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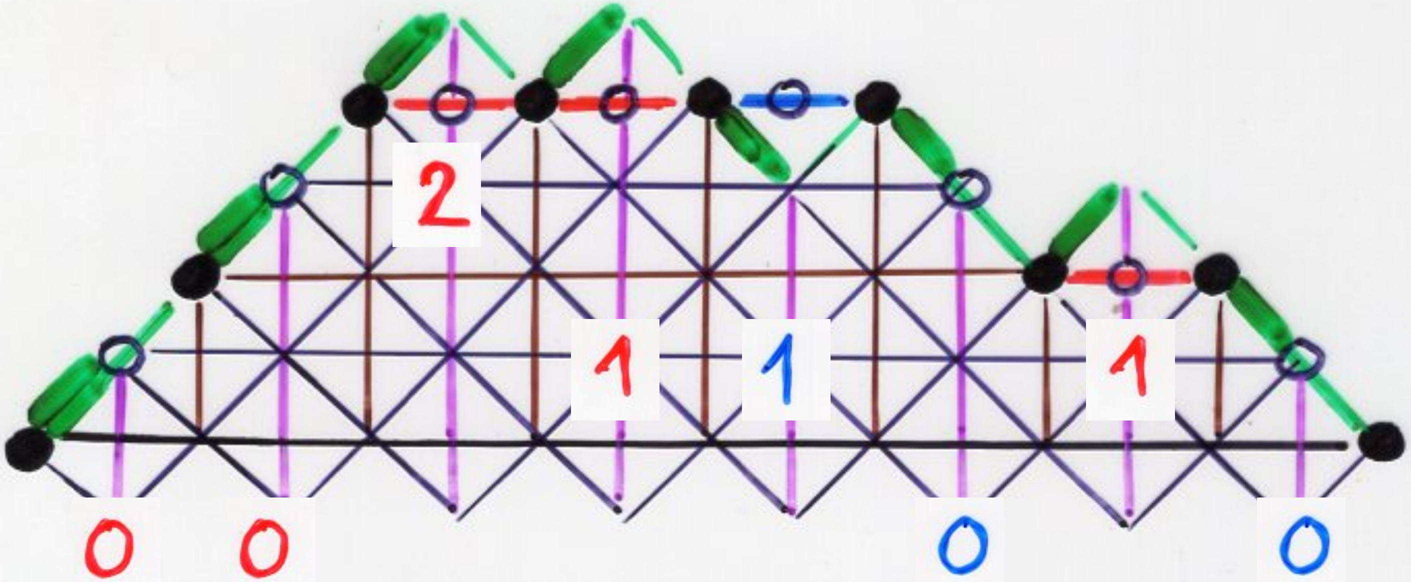
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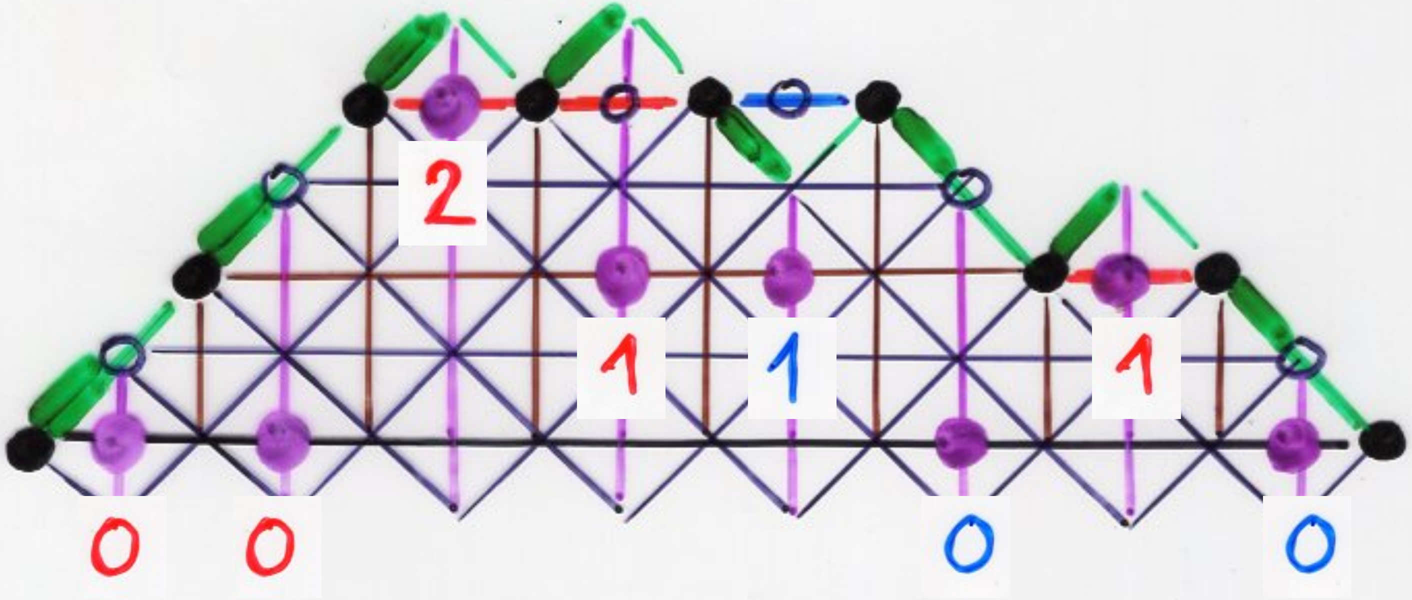
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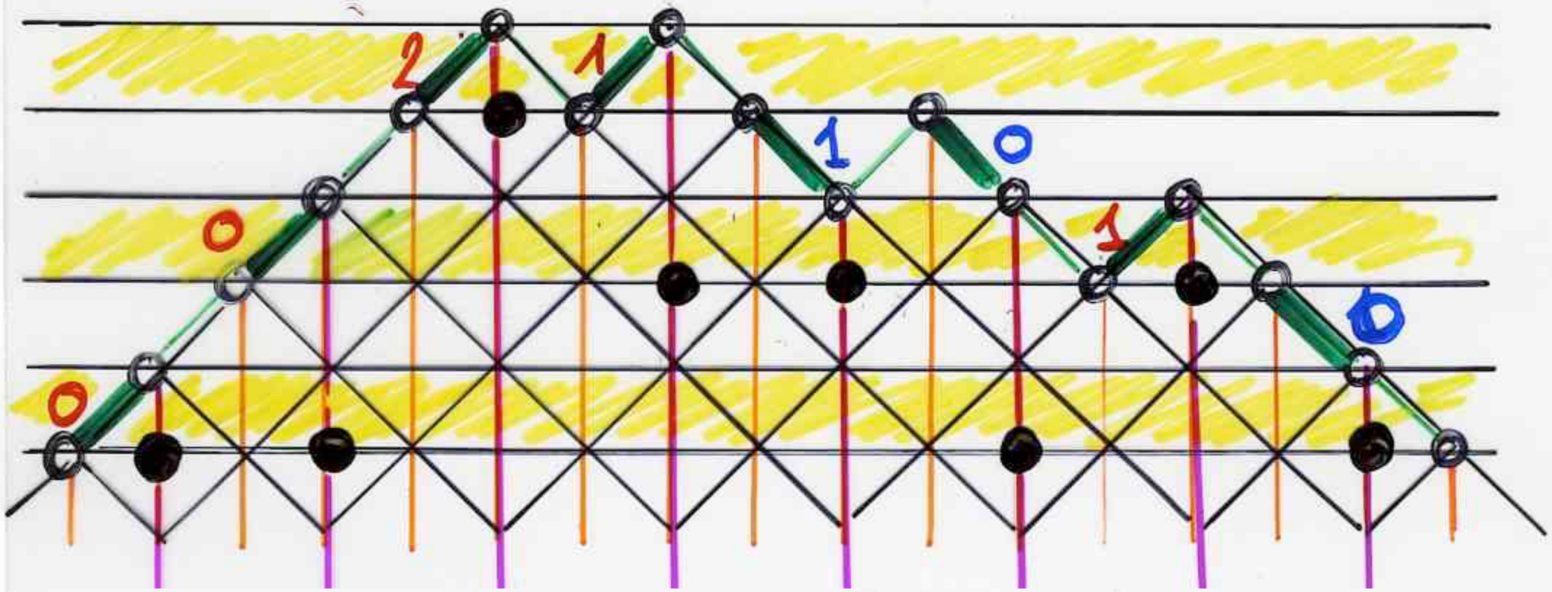
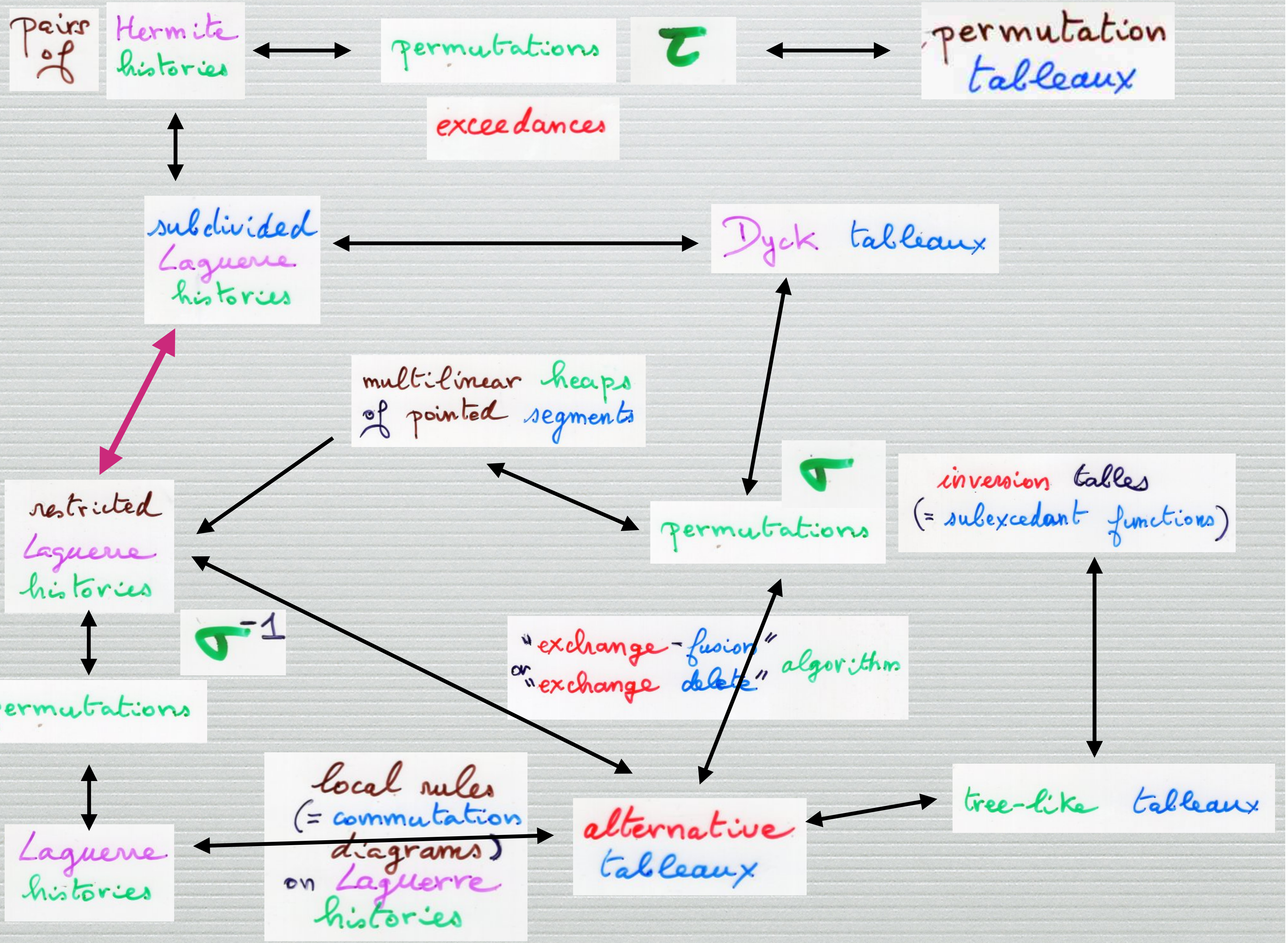


Dyck tableau as a subdivided Laguerre history



Pairs of Hermite histories

permutations τ

exceedances

permutation tableaux

subdivided Laguerre histories

Dyck tableaux

multilinear heaps of pointed segments

σ permutations

inversion tables (= subexcedant functions)

restricted Laguerre histories

σ^{-1}

permutations

"exchange-fusion" or "exchange delete" algorithms

local rules (= commutation diagrams) on Laguerre histories

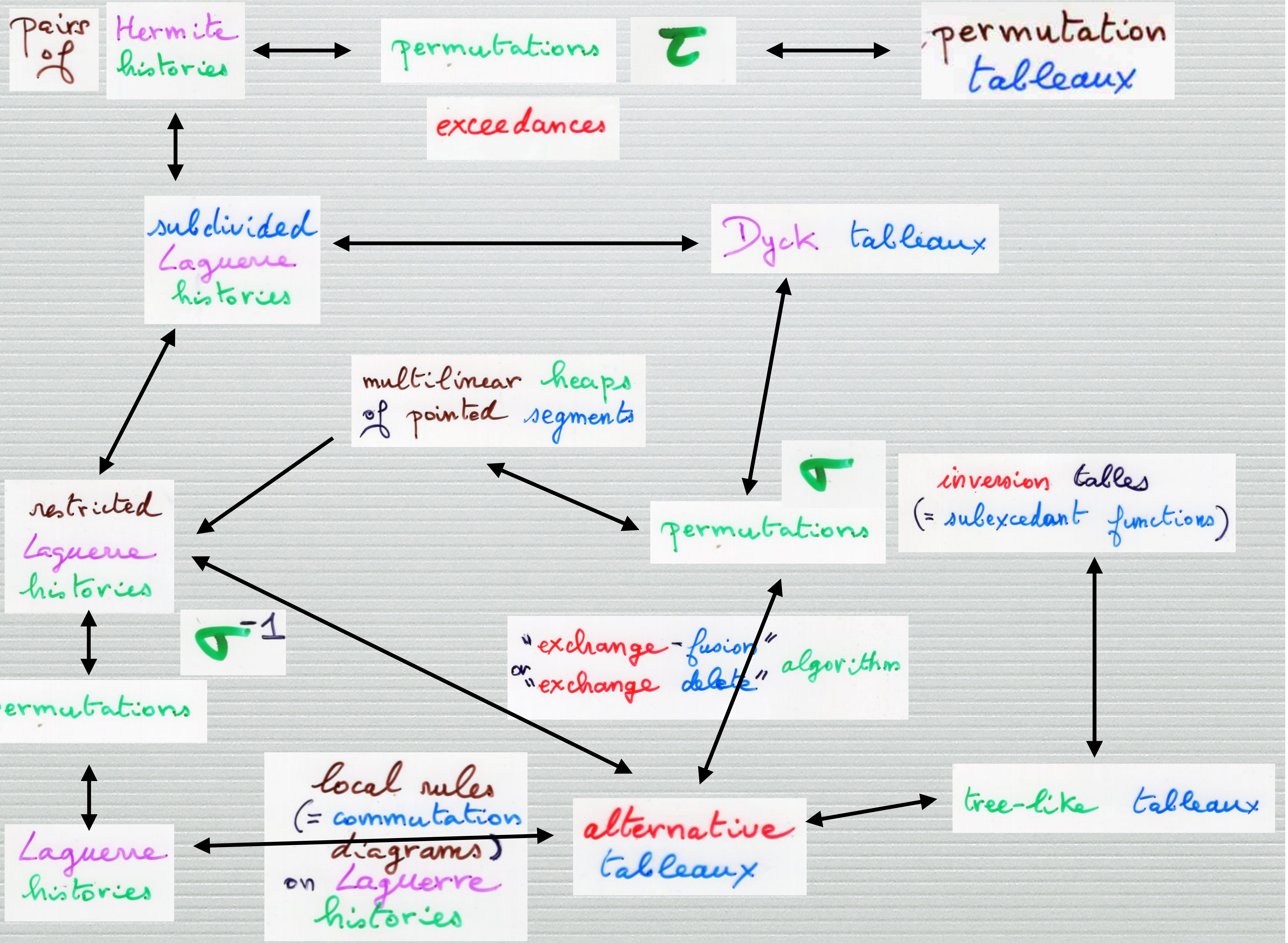
Laguerre histories

alternative tableaux

tree-like tableaux

Epilogue

The « essence » of bijections …



Pairs of Hermite histories

permutations τ

exceedances

permutation tableaux

subdivided Laguerre histories

Dyck tableaux

multilinear heaps of pointed segments

σ

permutations

inversion tables (= subexcedant functions)

restricted Laguerre histories

σ^{-1}

permutations

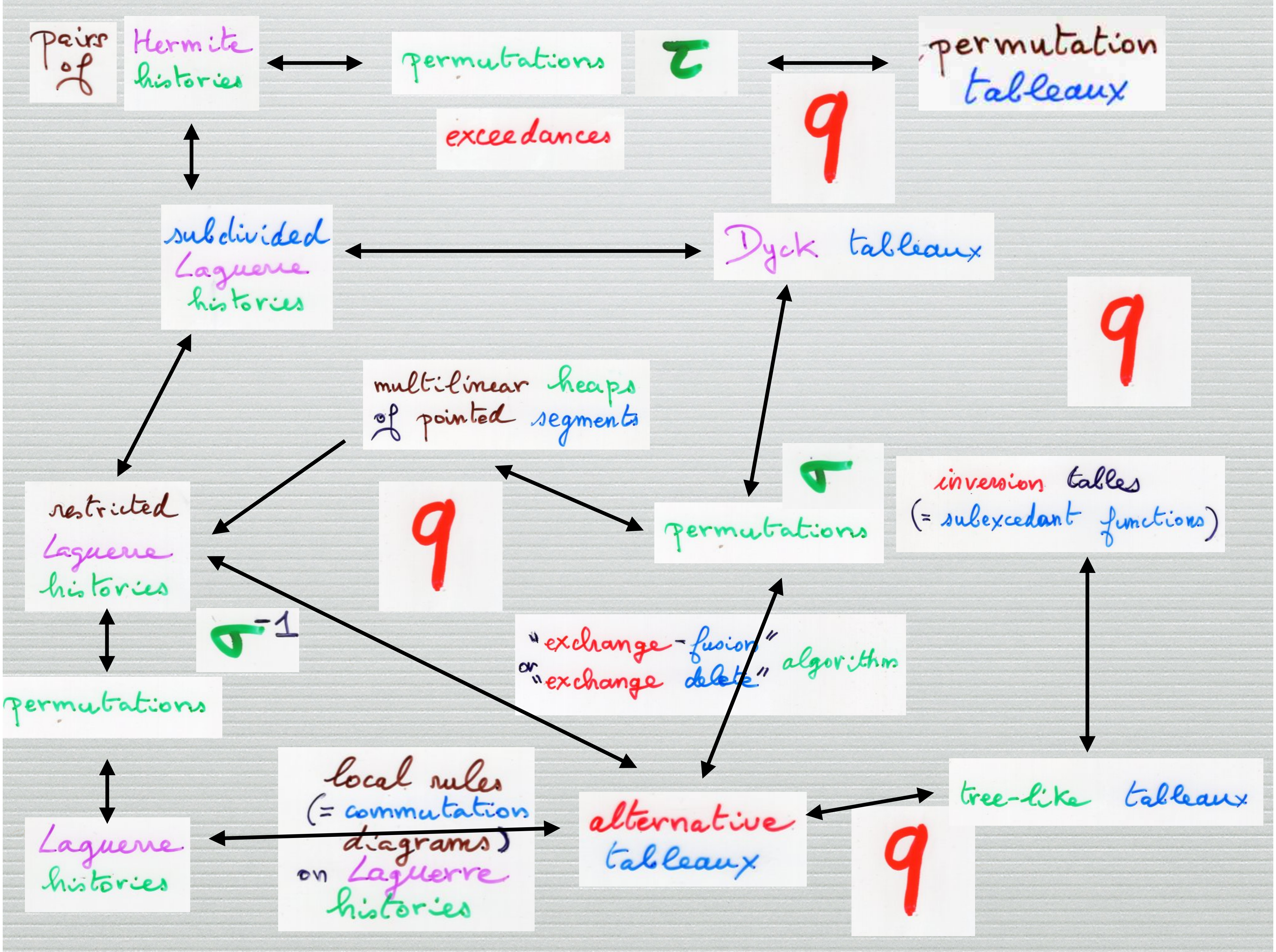
"exchange-fusion" or "exchange delete" algorithms

local rules (= commutation diagrams) on Laguerre histories

Laguerre histories

alternative tableaux

tree-like tableaux



Pairs of Hermite histories

permutations τ

permutation tableaux

exceedances

q

subdivided Laguerre histories

Dyck tableaux

q

multilinear heaps of pointed segments

restricted Laguerre histories

q

σ

permutations

inversion tables (= subexcedant functions)

σ^{-1}

permutations

"exchange-fusion" or "exchange delete" algorithms

local rules (= commutation diagrams) on Laguerre histories

Laguerre histories

alternative tableaux

tree-like tableaux

q

pairs of Hermite histories

permutations τ

permutation tableaux

exceedances

subdivided Laguerre histories

Dyck tableaux

contraction of paths

multilinear heaps of pointed segments

σ

permutations

inversion tables
(= subexcedant functions)

restricted Laguerre histories

σ^{-1}

"exchange-fusion"
or "exchange delete" algorithm

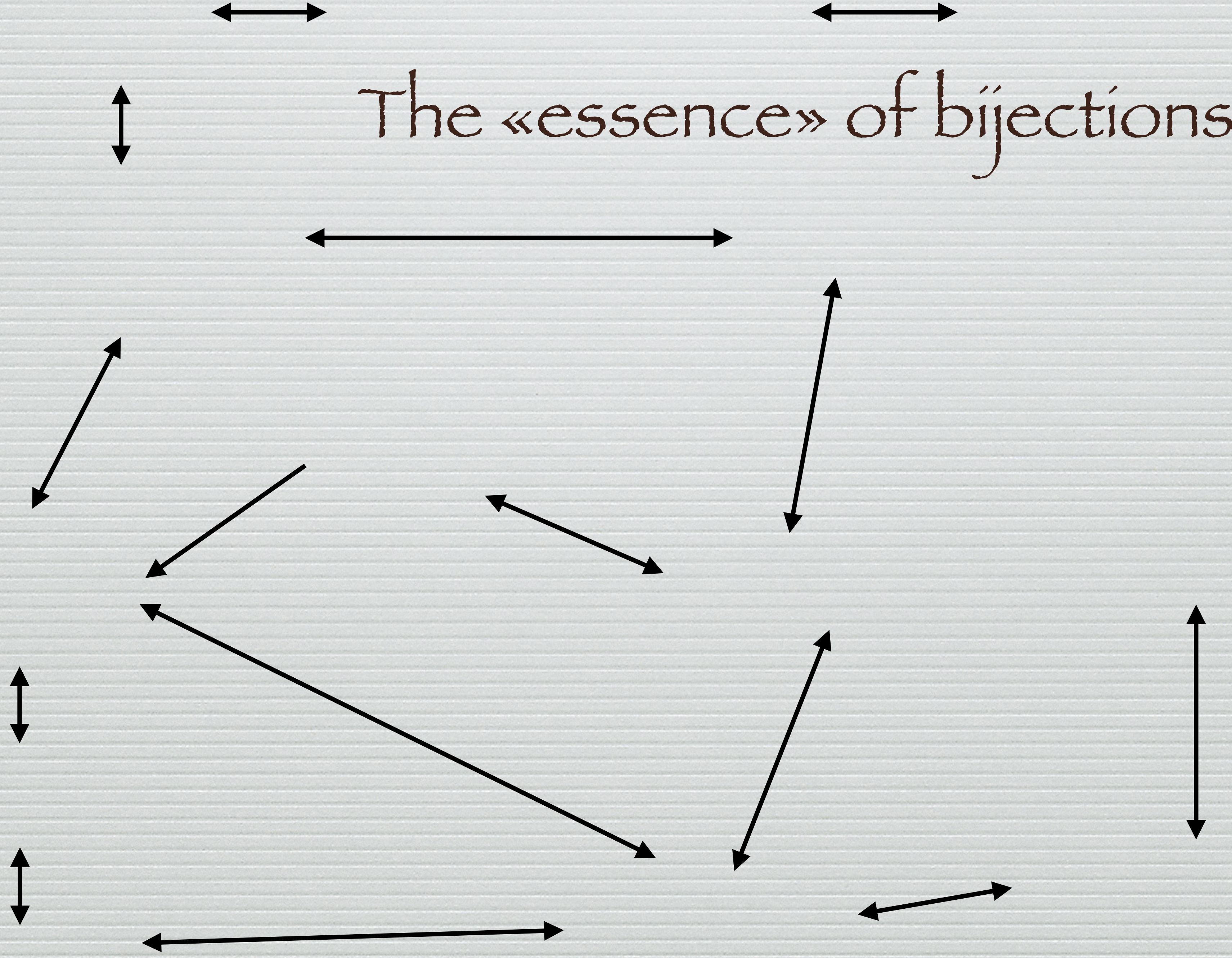
permutations

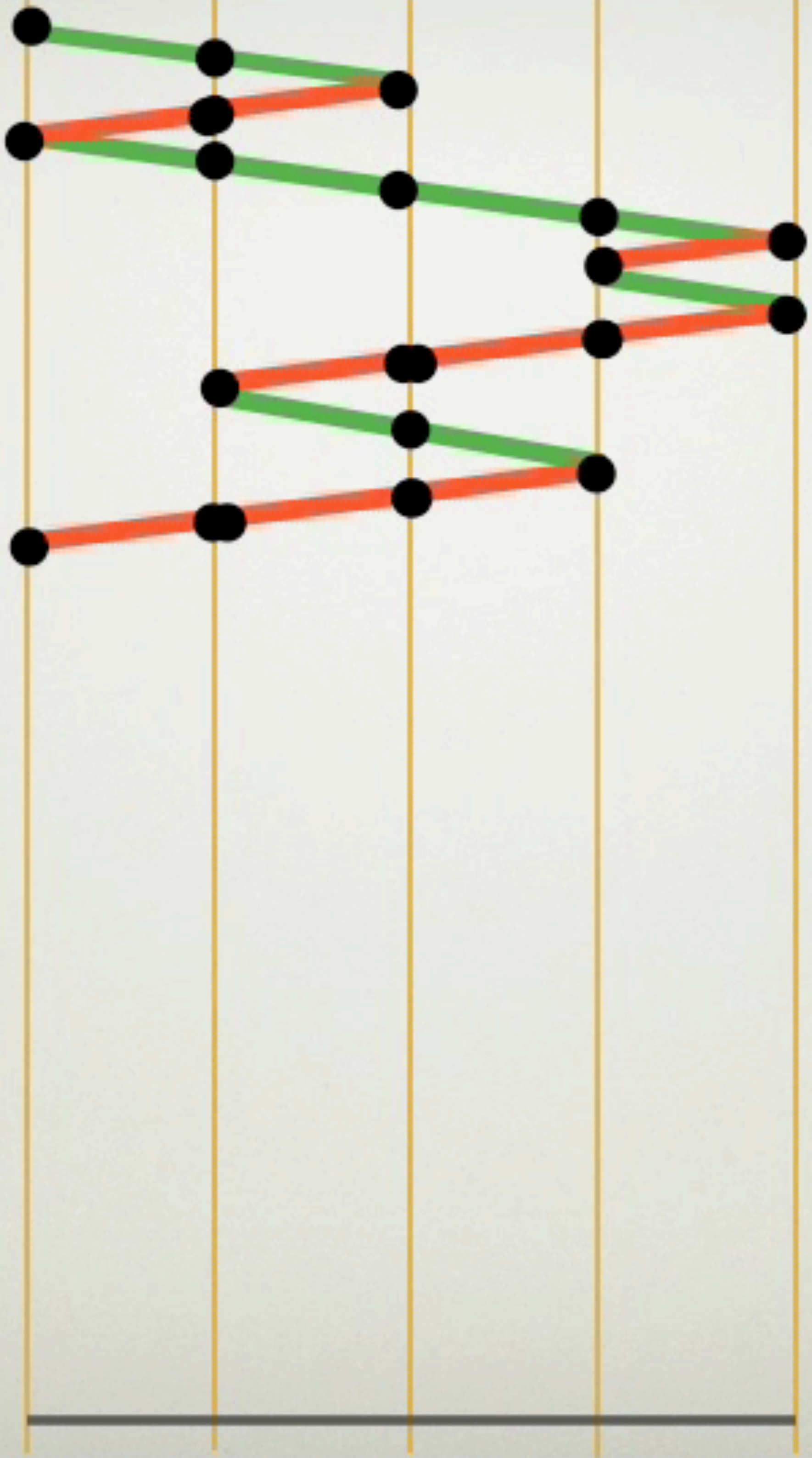
tree-like tableaux

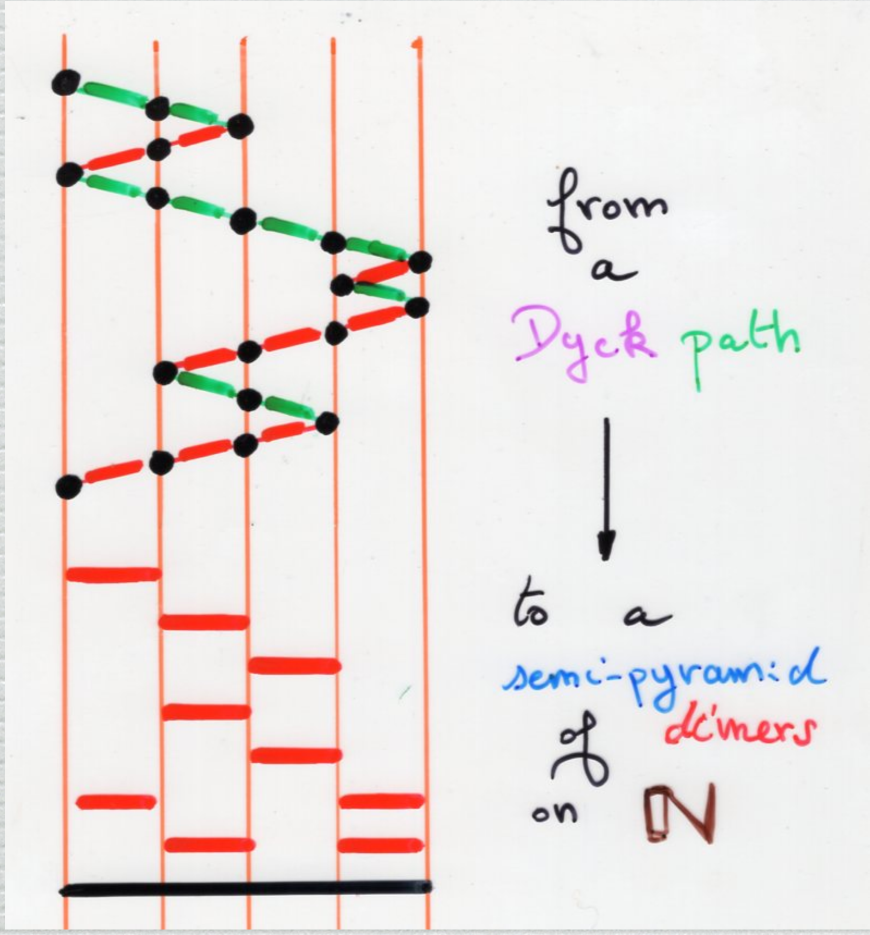
alternative tableaux

Laguerre histories

The «essence» of bijections …





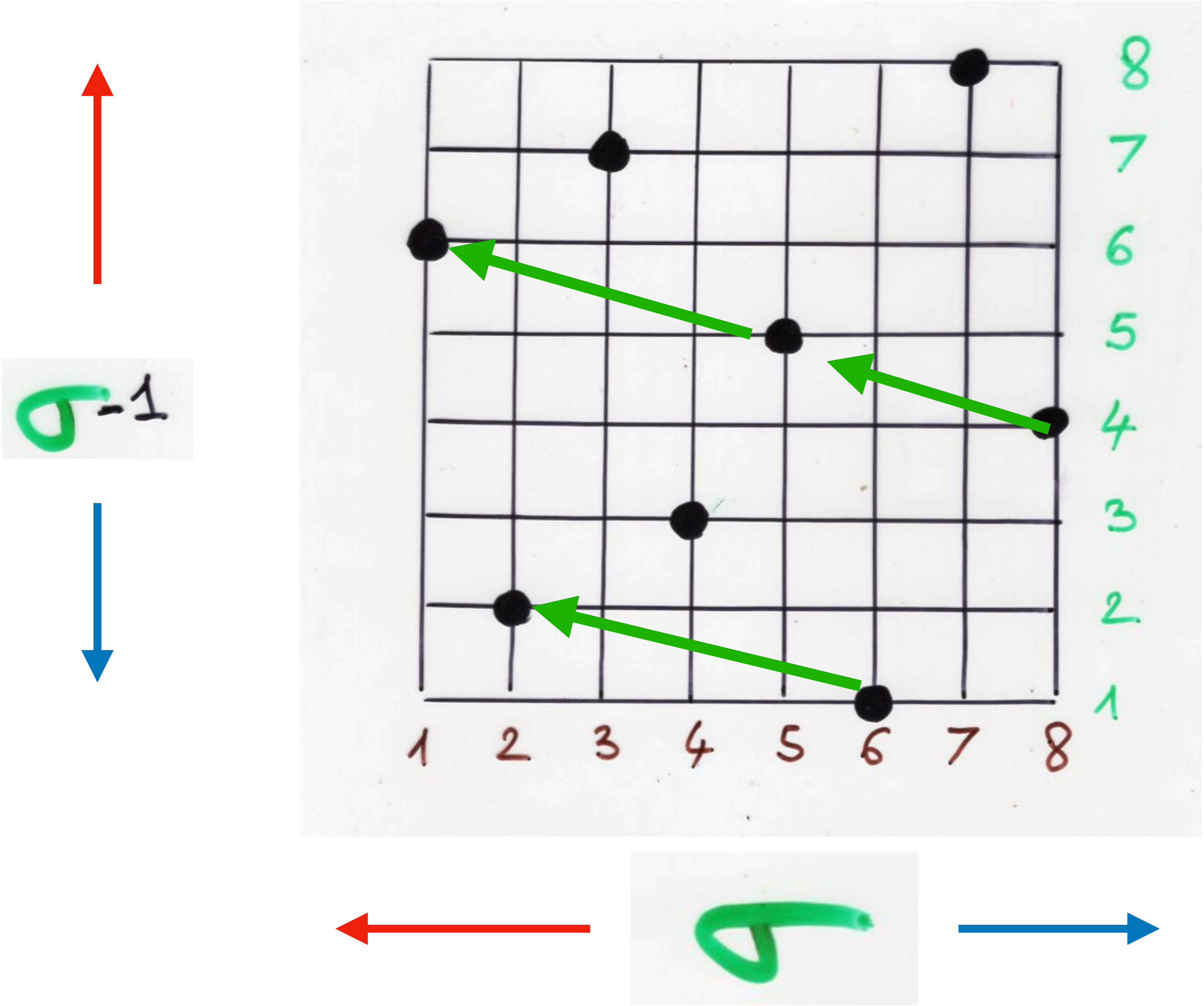


Video with violon:

violonist: Gérard Duchamp

(association Cont'Science)

Bijection paths — heaps,
see « the art of bijective combinatorics » II, Ch3b p 26-40,
and p 42,60 in the case of Dyck paths.


8
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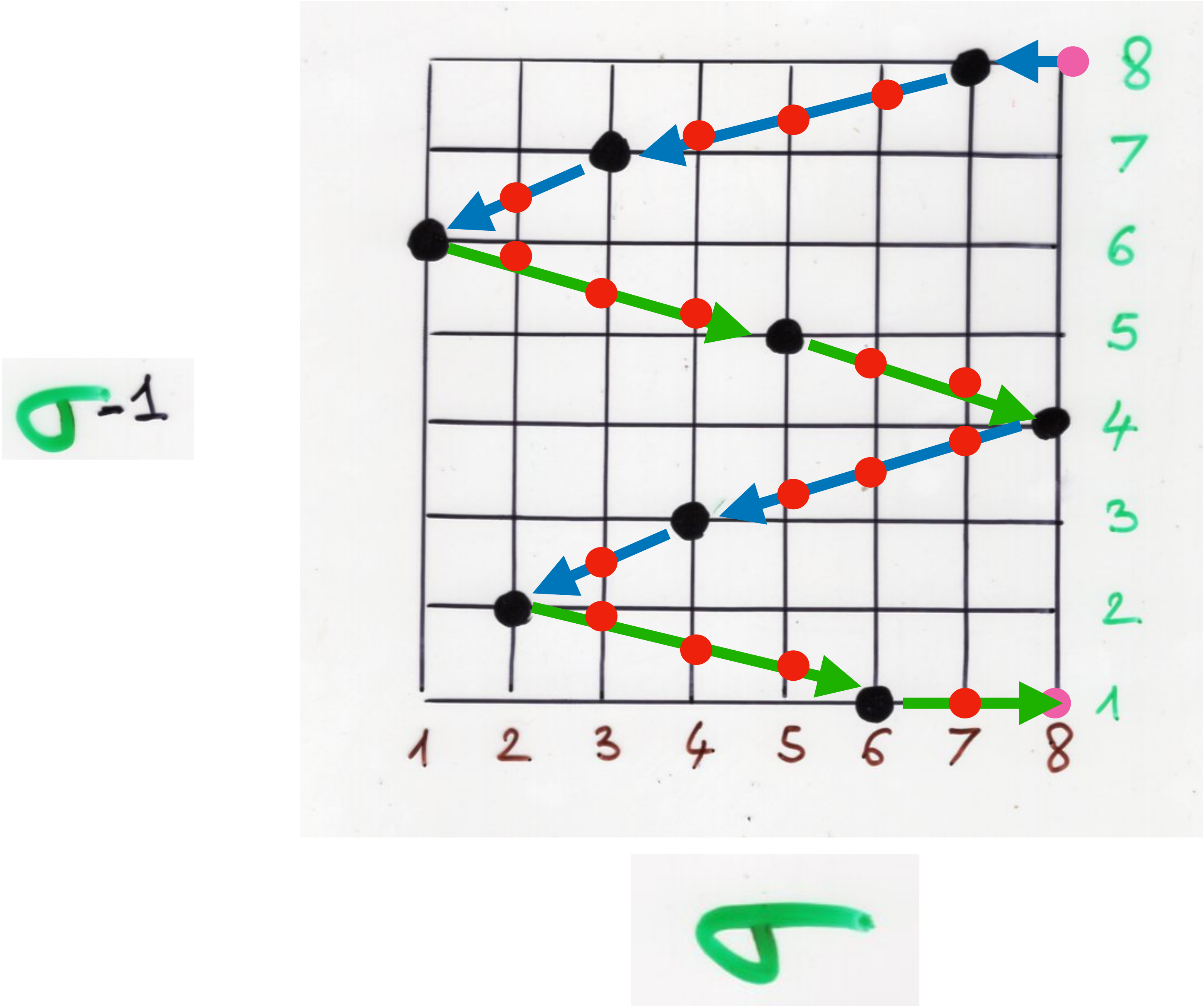
1 2 3 4 5 6 7 8
σ^{-1}
σ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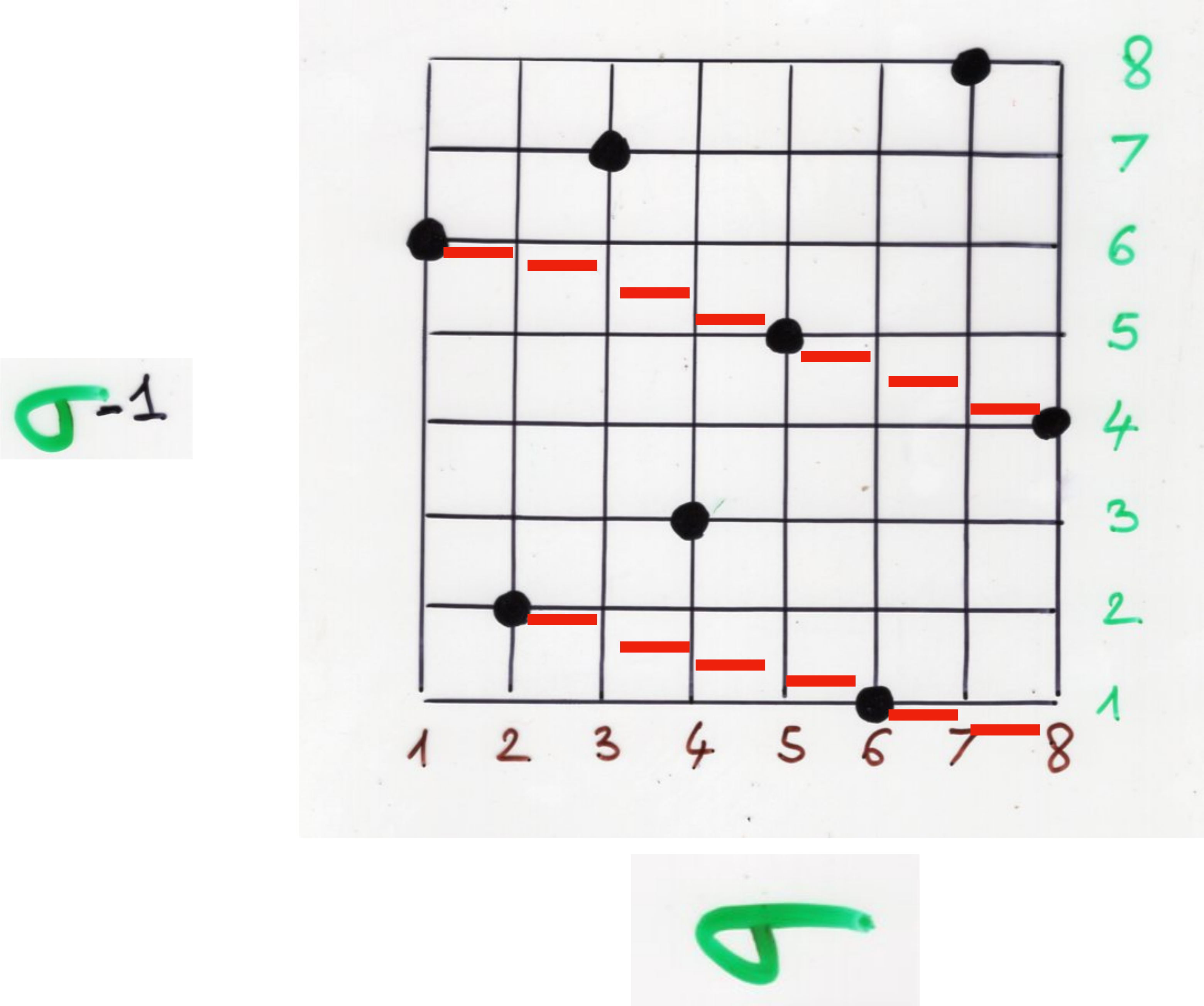
1 2 3 4 5 6 7 8
σ^{-1}
σ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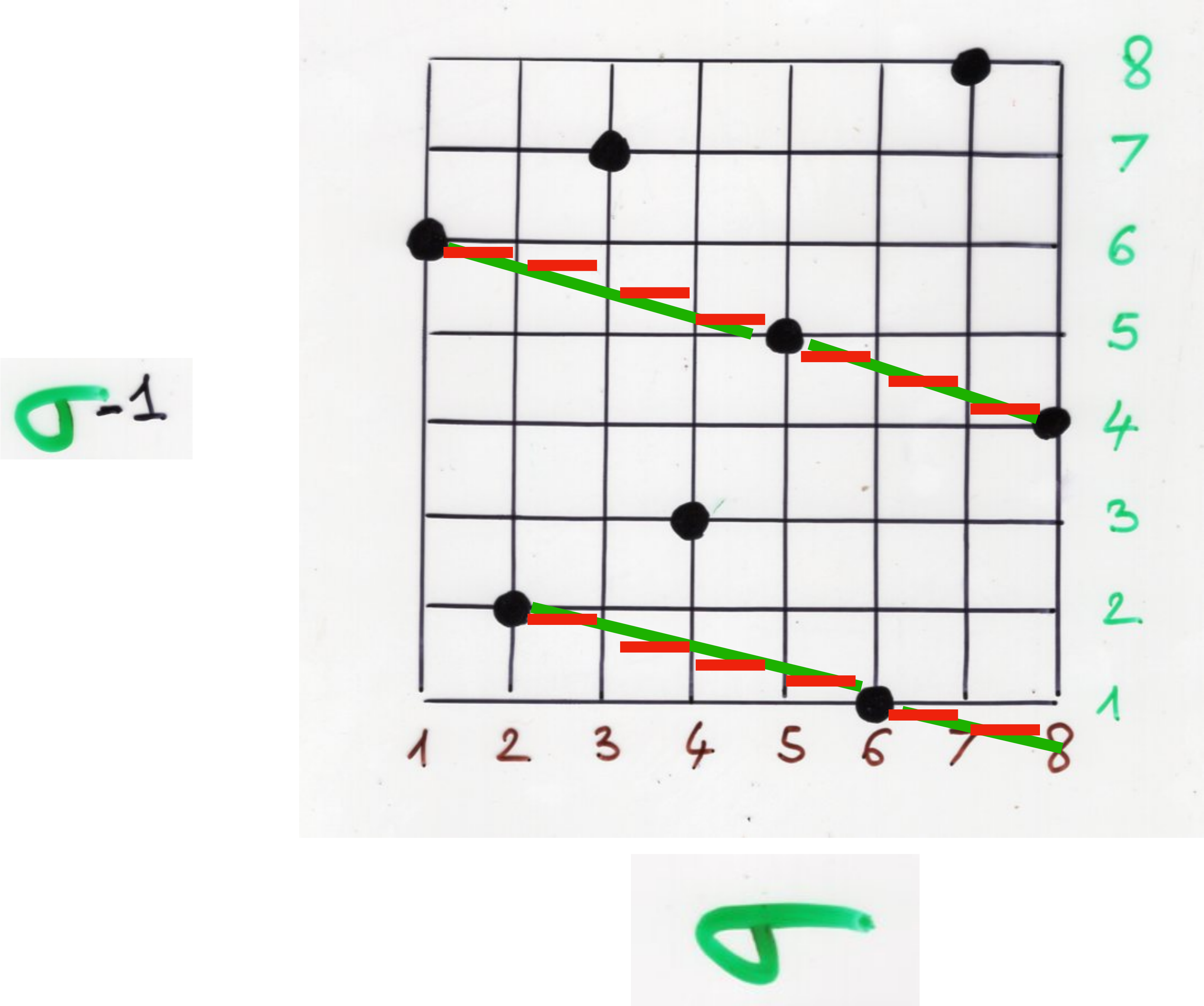
1 2 3 4 5 6 7 8
σ^{-1}
σ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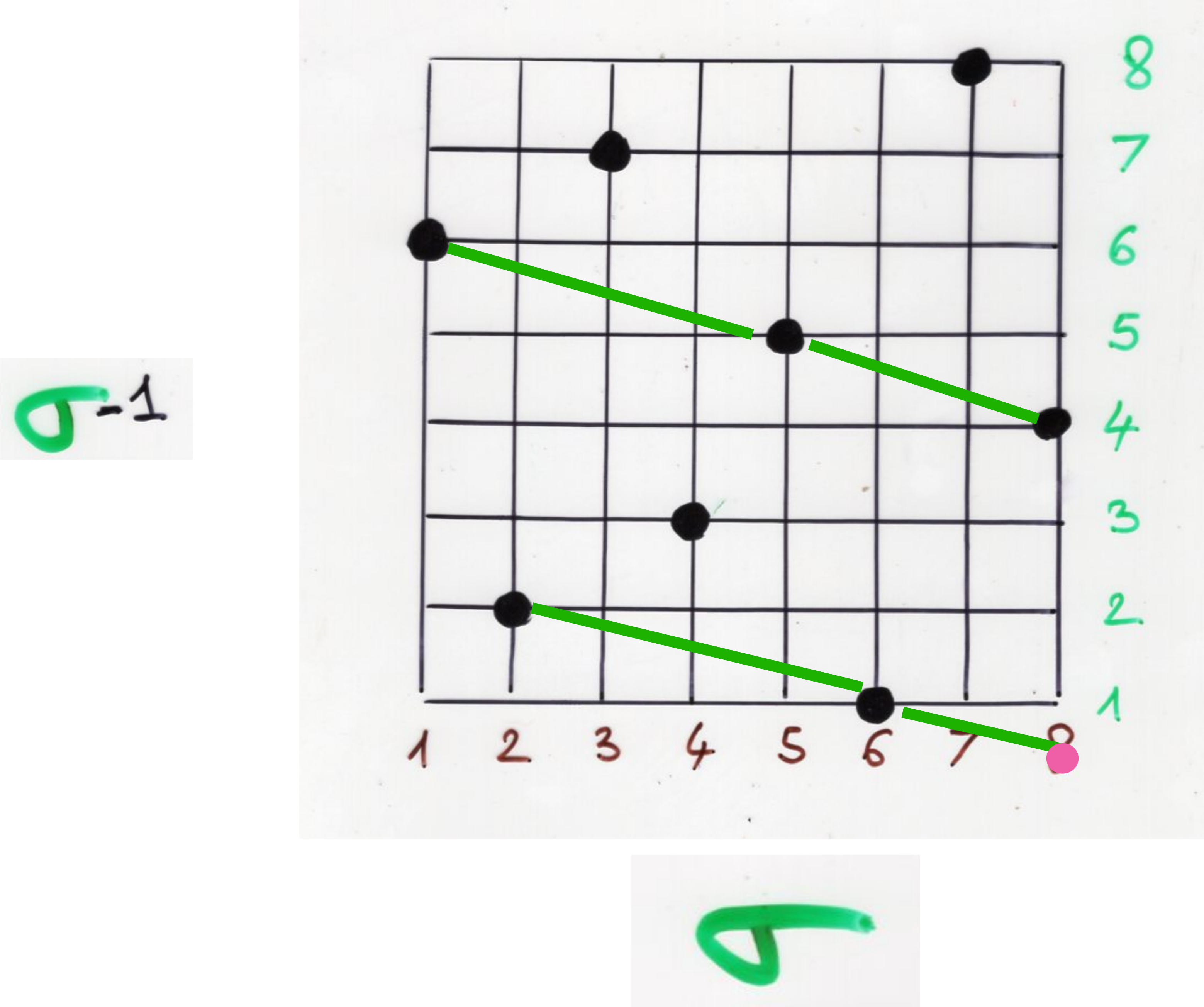
σ



8
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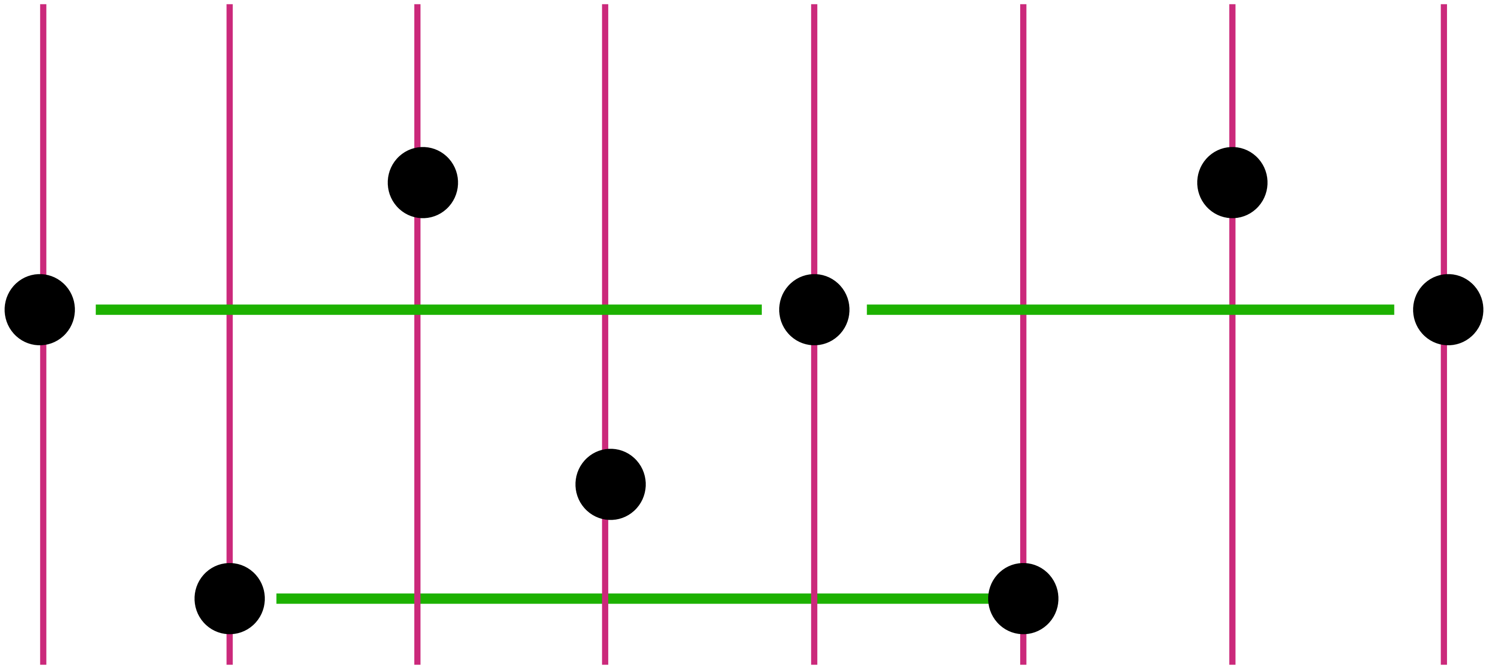
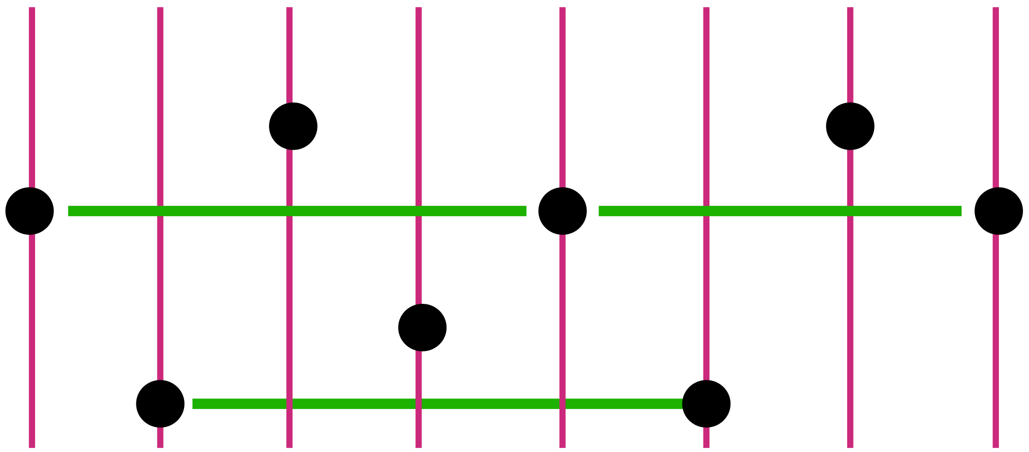
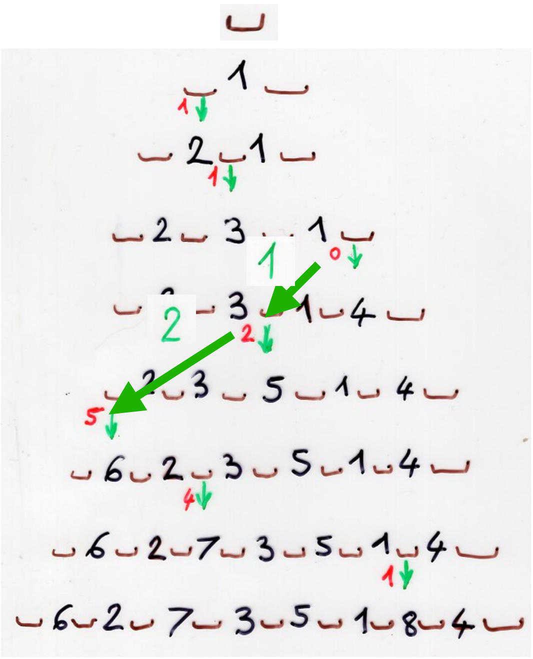
1 2 3 4 5 6 7 8
σ

$\sigma = 6 \quad 2 \quad 7 \quad 3 \quad 5 \quad 1 \quad 8 \quad 4$







␣
␣1␣
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␣2␣3␣1␣
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␣2␣3␣5␣1␣4␣
5
␣6␣2␣3␣5␣1␣4␣
4
␣6␣2␣7␣3␣5␣1␣4␣
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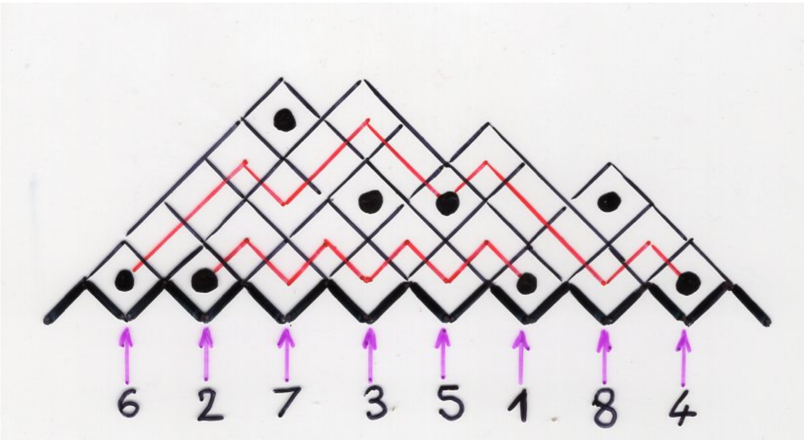
σ = 6 2 7 3 5 1 8 4



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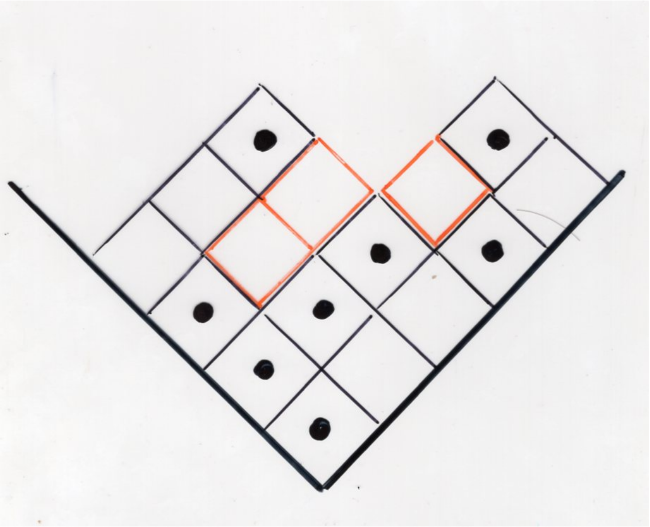
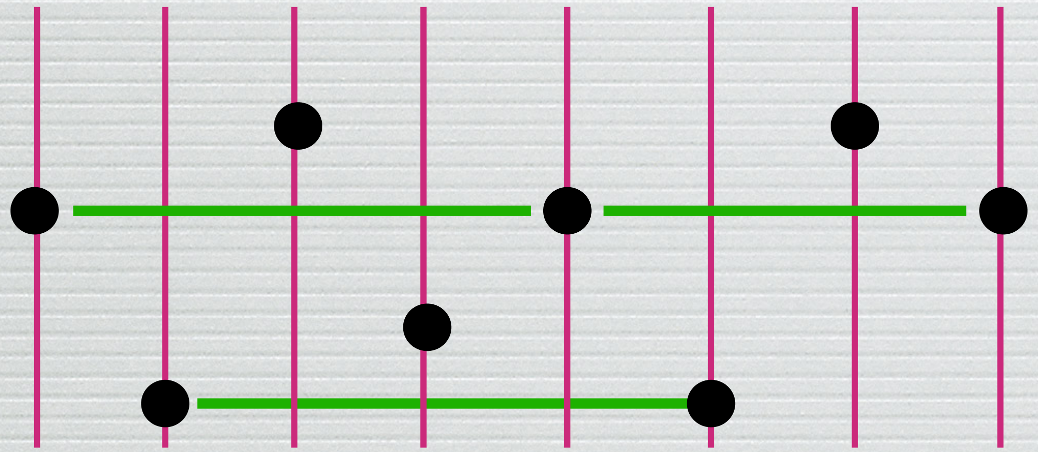


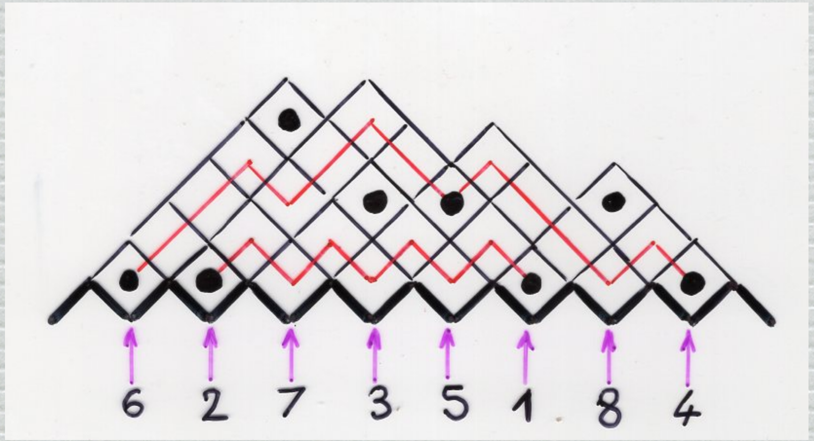


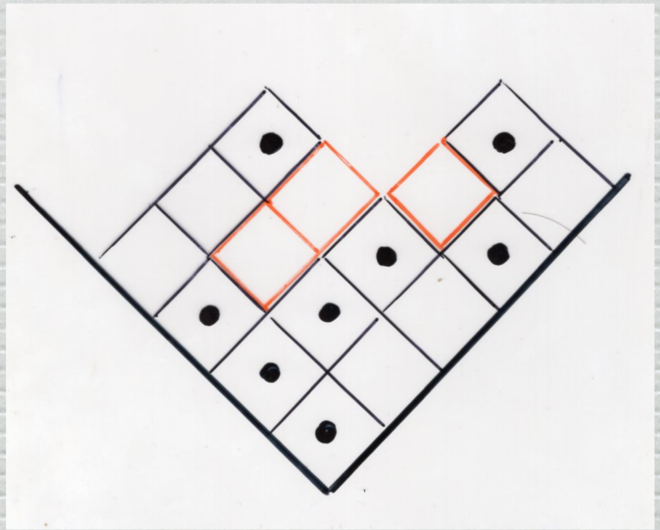
Next slides:

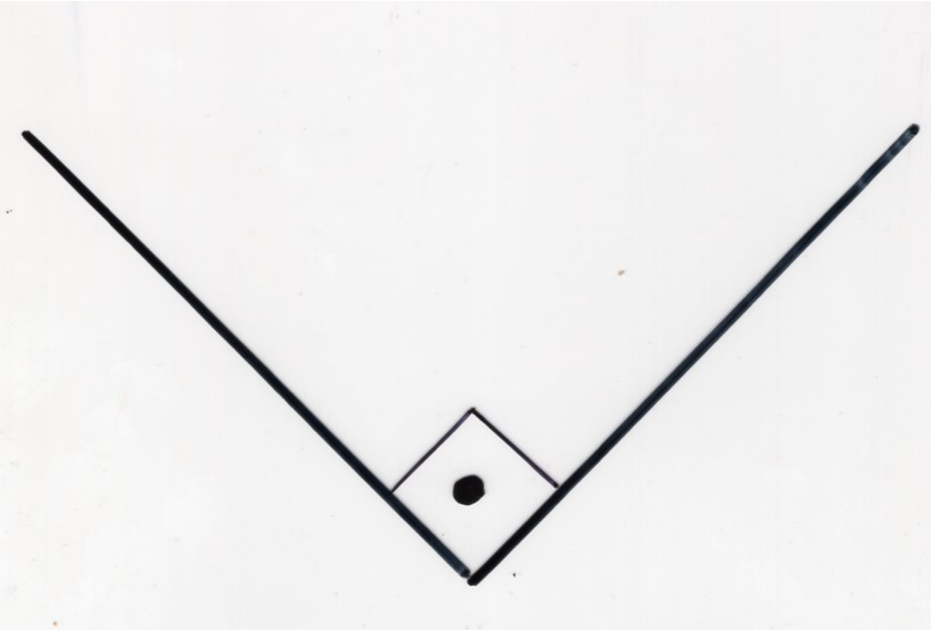
Video with violon:

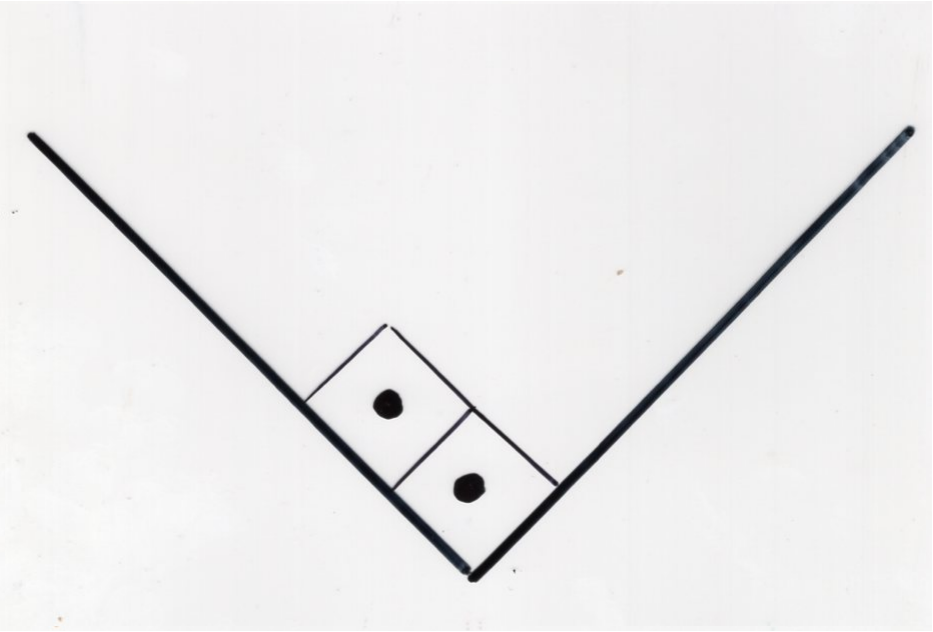
violonist: Gérard Duchamp

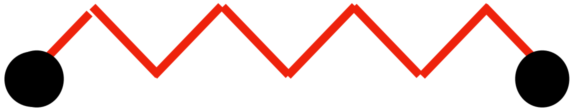
The « essence » of 3 bijections in parallel

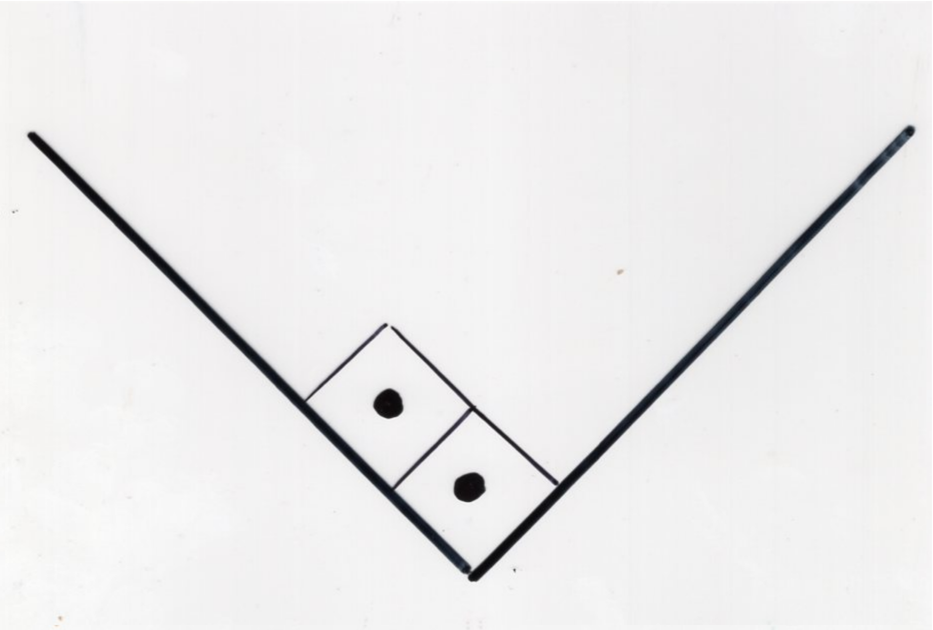


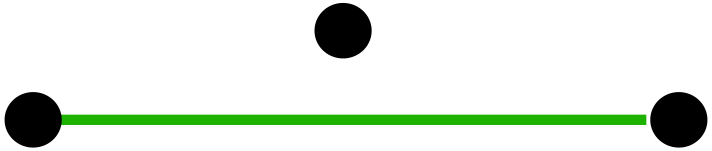


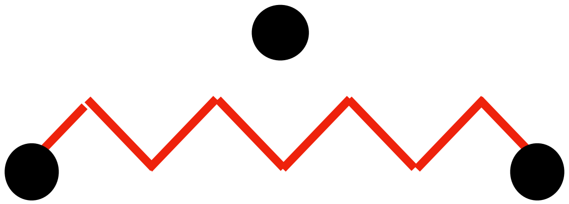


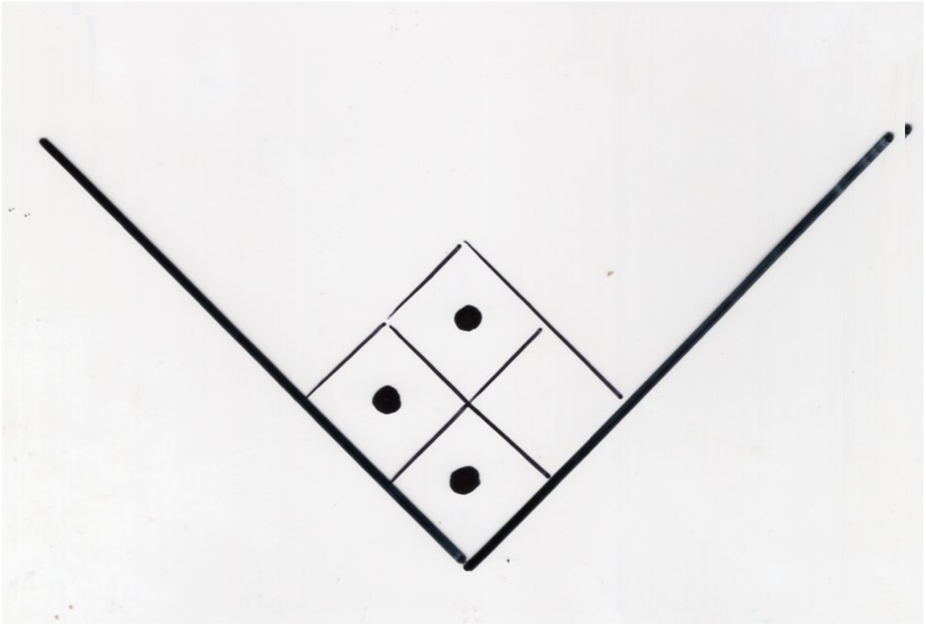


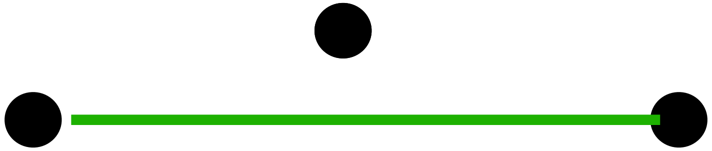


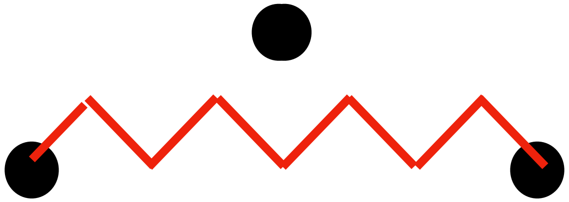


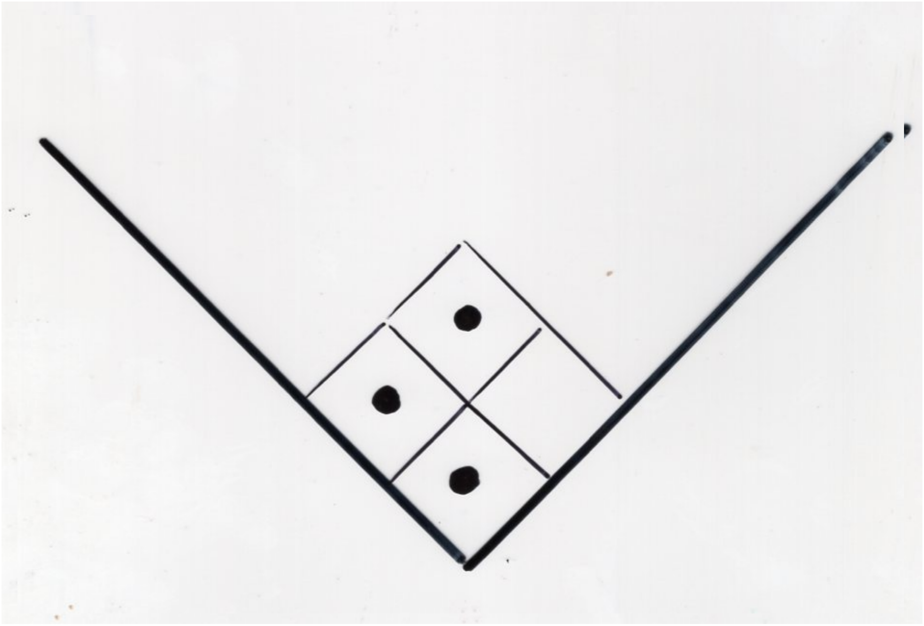


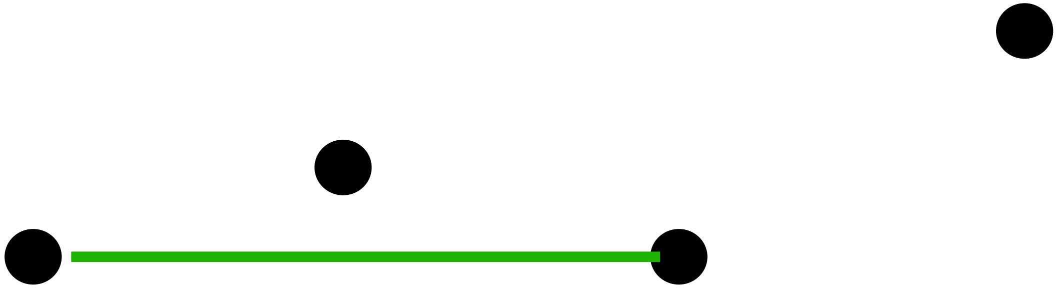


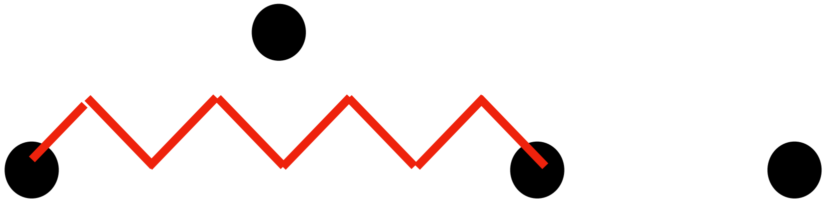


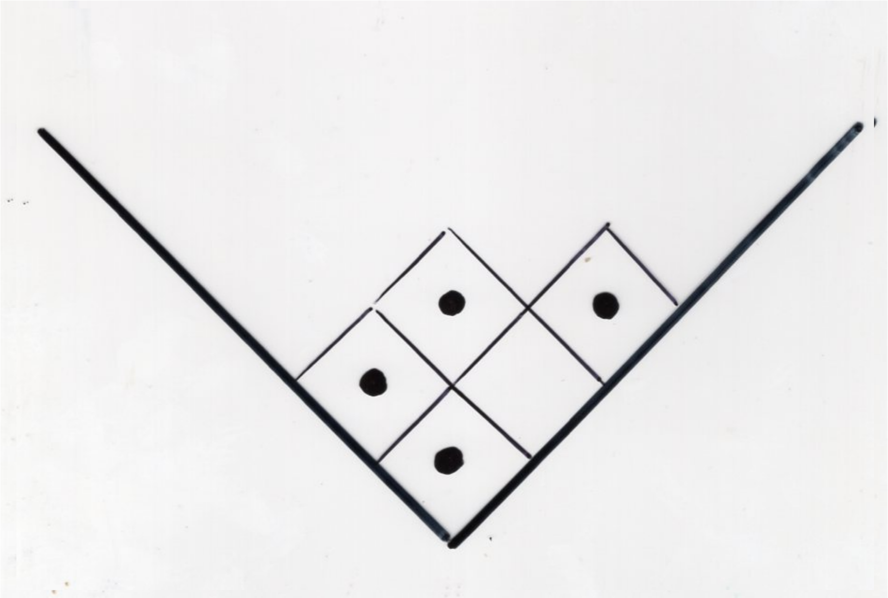


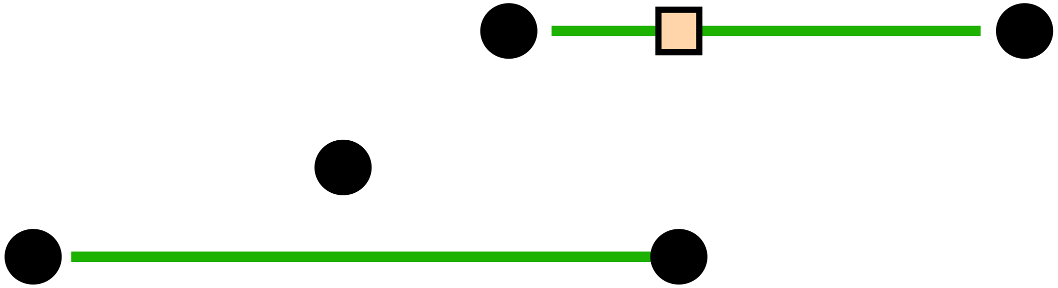


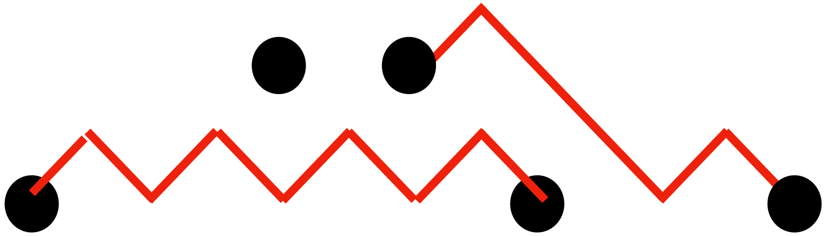


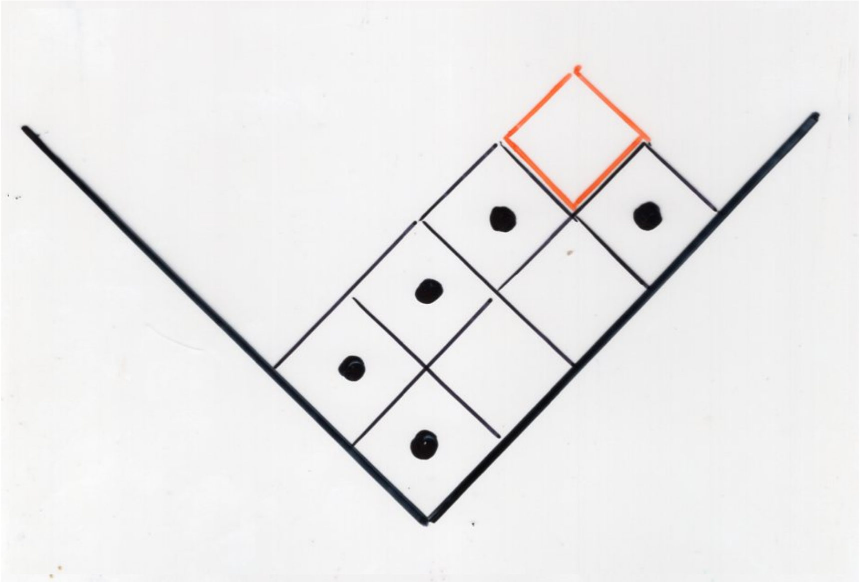


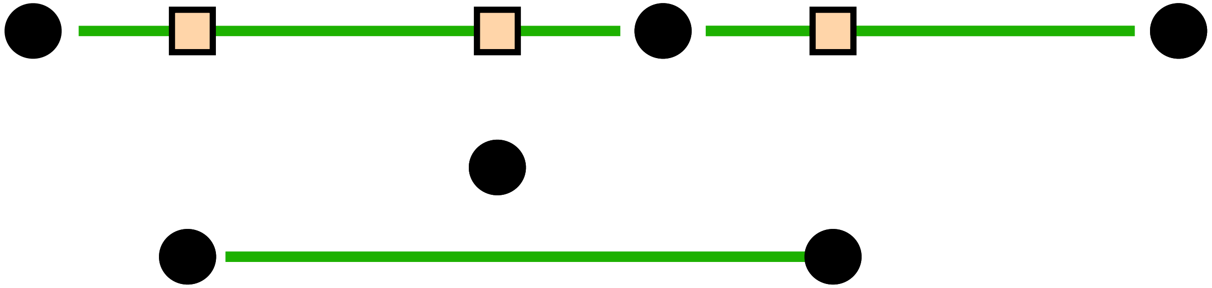


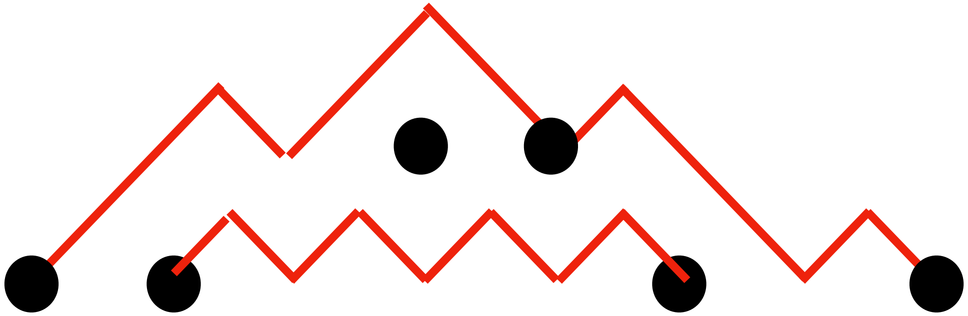


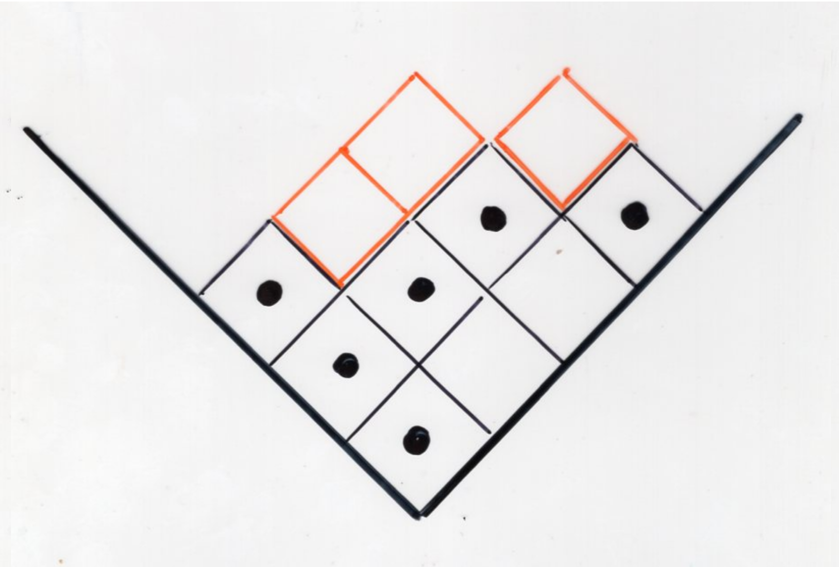


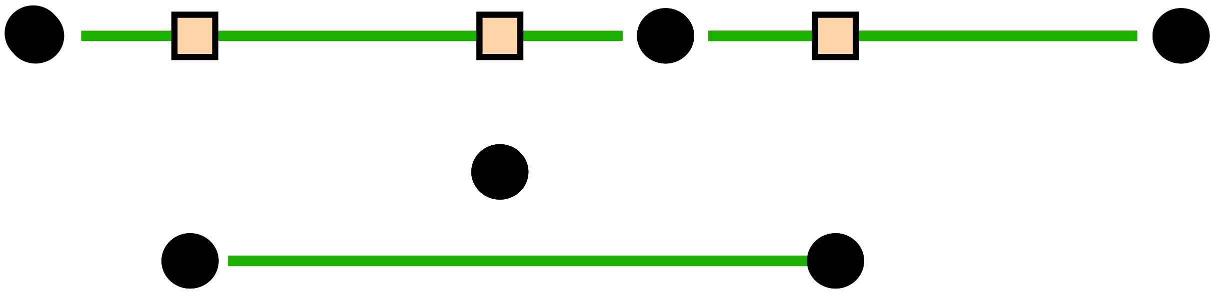


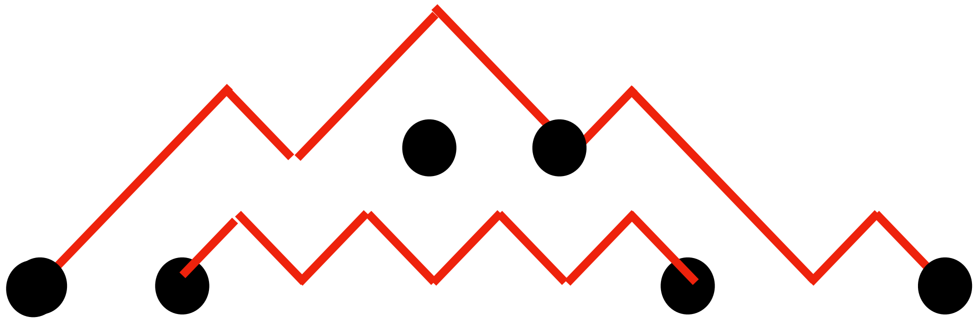


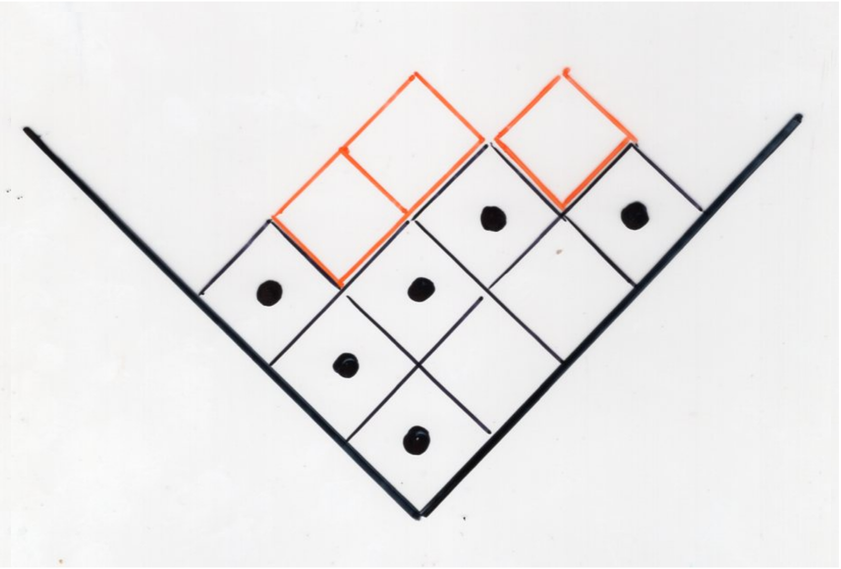


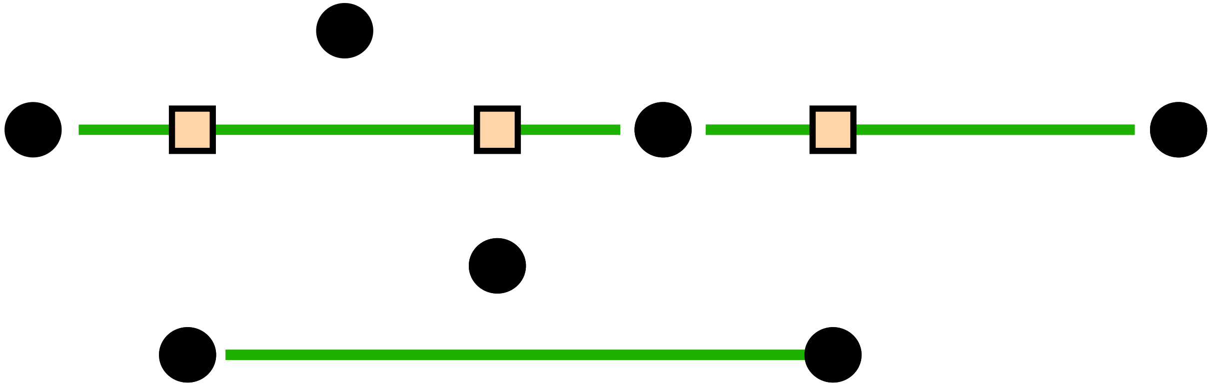


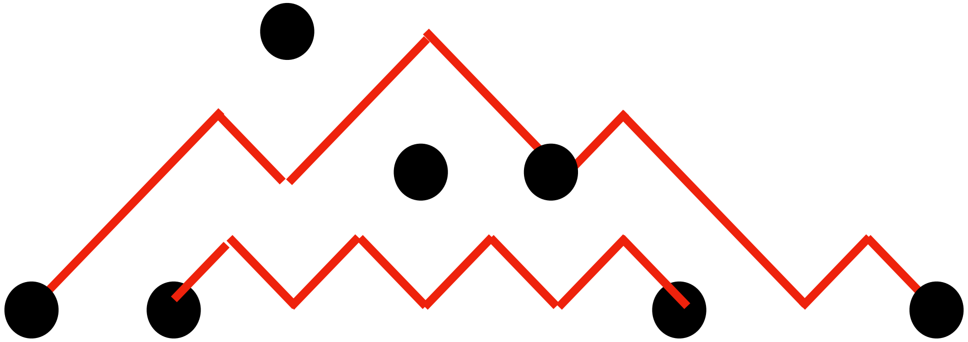


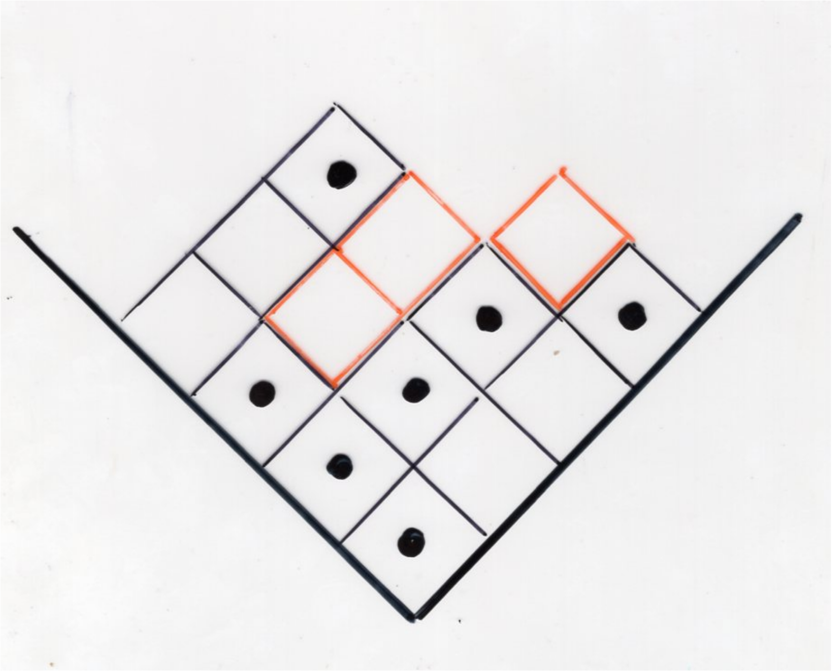


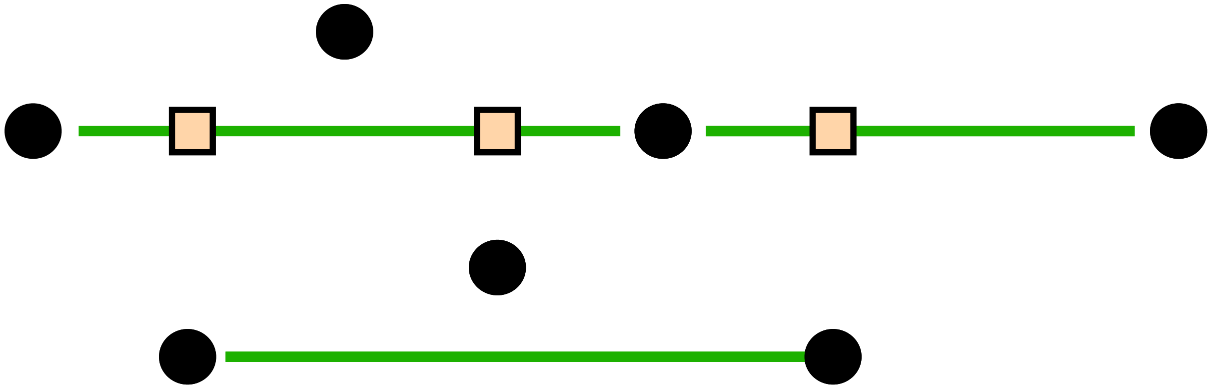


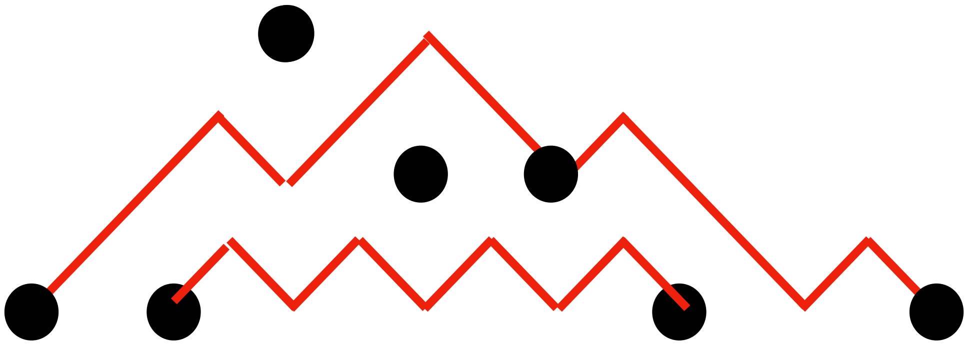


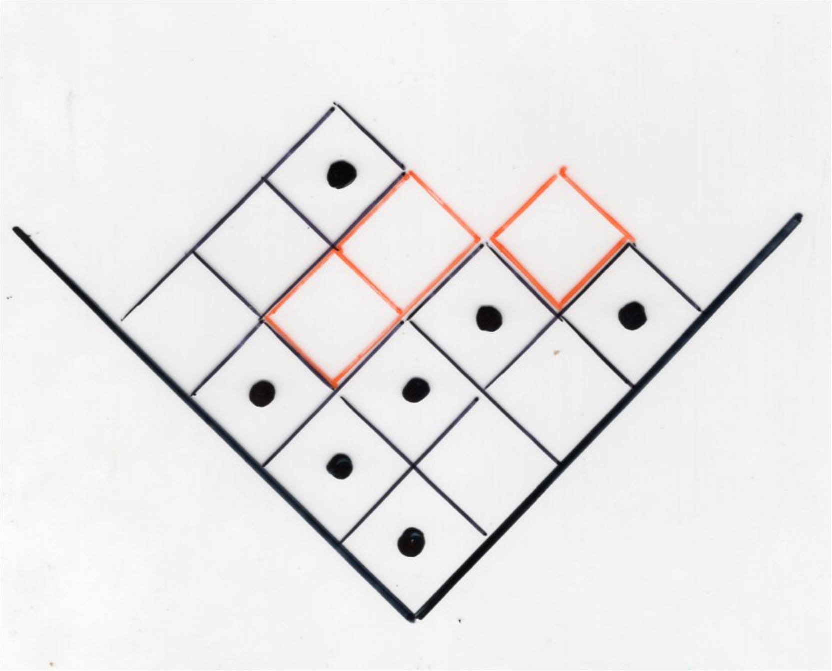


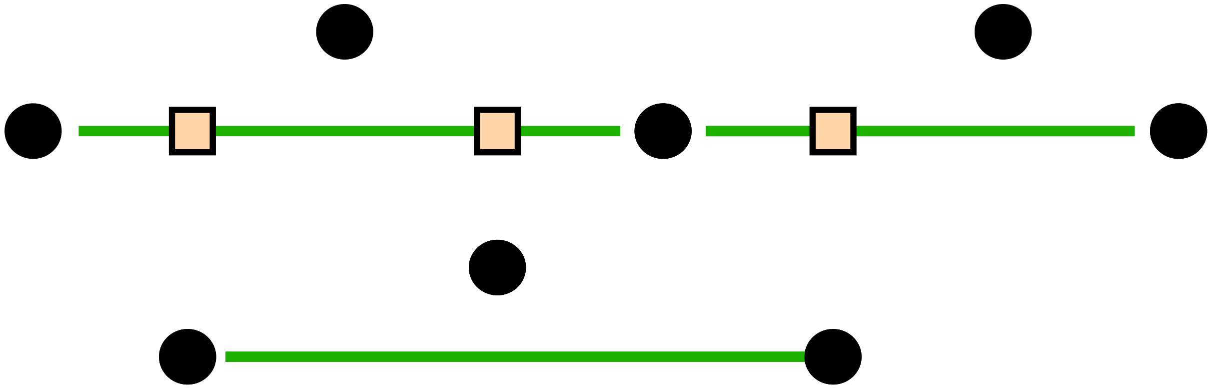


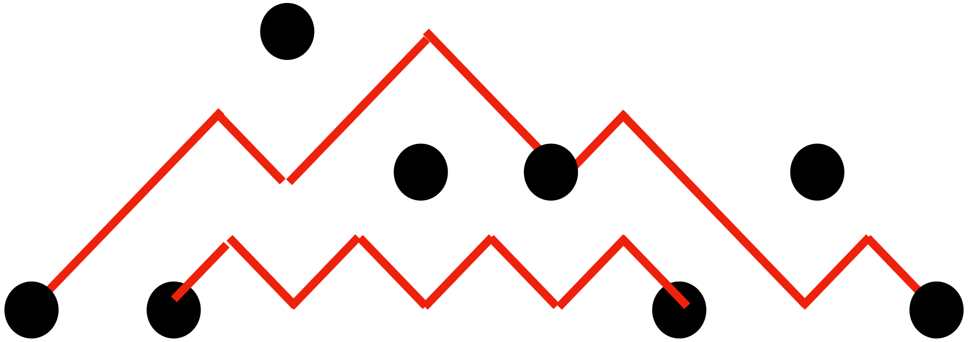


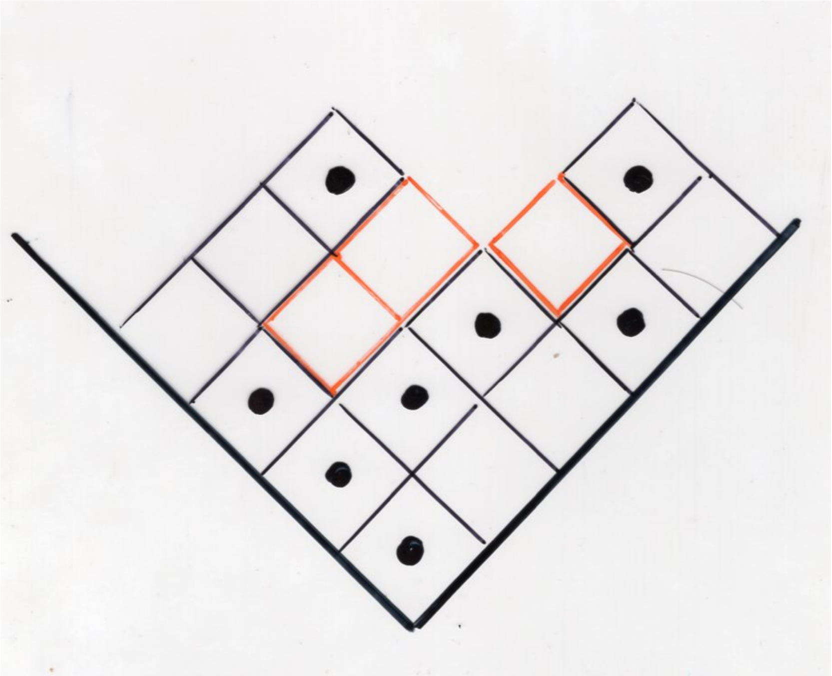


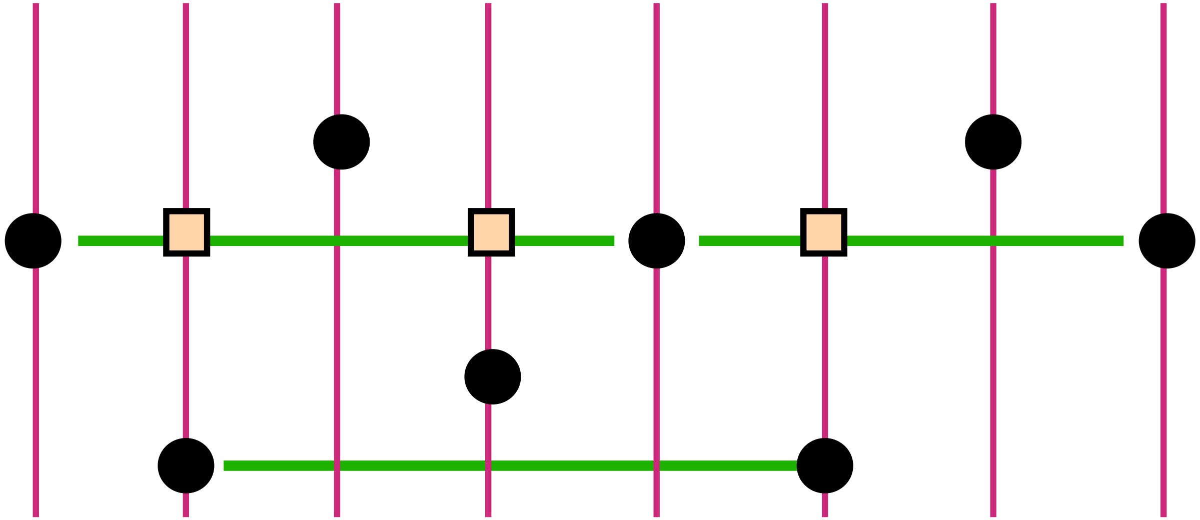


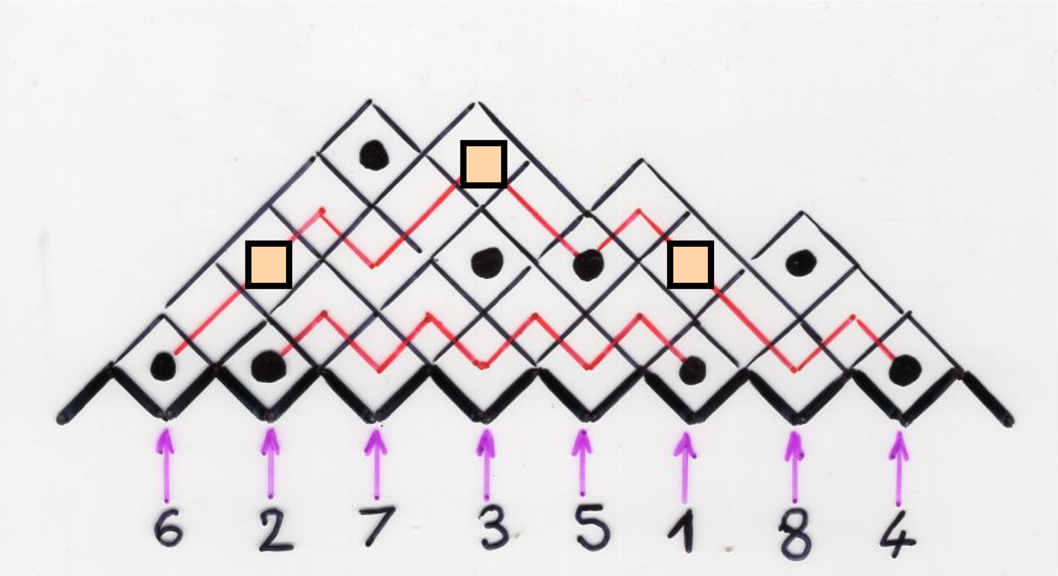








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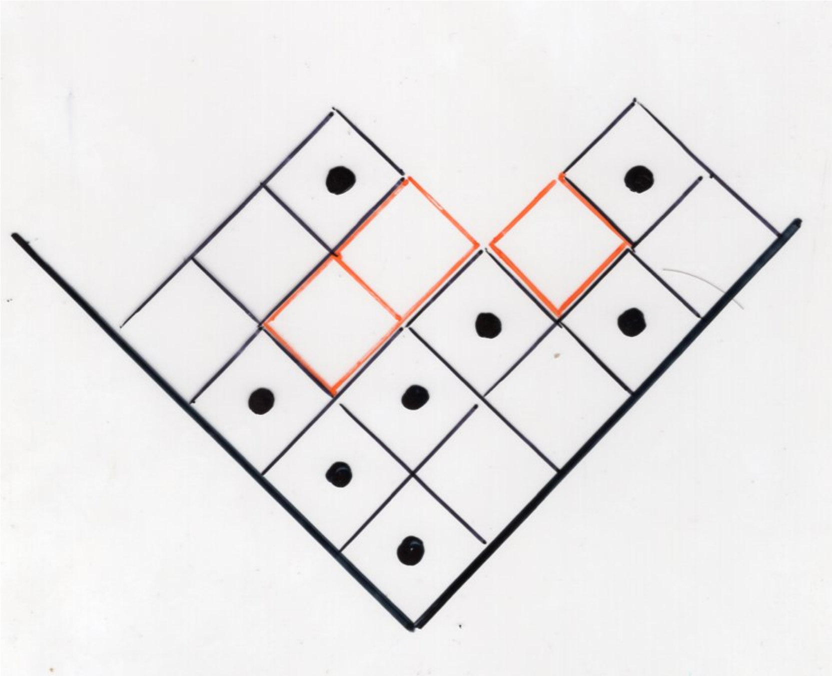

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Christian K., piano

Gérard D., violin
Marcia Pig Lagos, « contes »
Xavier, videoprojector

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des arbres dans les grains de lumières »