RIBBON TABLEAUX, HALL-LITTLEWOOD FUNCTIONS AND UNIPOTENT VARIETIES *

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Abstract

We introduce a new family of symmetric functions, which are defined in terms of ribbon tableaux and generalize Hall-Littlewood functions. We present a series of conjectures, and prove them in two special cases.

1 Introduction

Hall-Littlewood functions [**Li1**] are known to be related to a variety of topics in representation theory, geometry and combinatorics. These symmetric functions arise in the character theory of finite linear groups [**Gr**], in the geometry of unipotent varieties [**Sh1**, **HSh**], in particular as characteristics of the representations of the symmetric group in their cohomology [**HS**], and appear to be related to the Quantum Inverse Scattering Method [**KR**]. From a combinatorial point of view, their description involves the deepest aspects of the theory of Young tableaux: the multiplicative structure (plactic monoid) and the ordered structure derived from the cyclage operation [**LS2**, **La**]. There exists also a description in terms of Kashiwara's theory of crystal bases [**LLT4**].

Another kind of application of Hall-Littlewood functions is concerned with the representation theory of the complex linear group $GL(n, \mathbf{C})$. The general setting is the following. Suppose we are given a finite dimensional representation V of $G = GL(n, \mathbf{C})$. The symmetric group $\mathbf{S}_k$ acts (on the right) on the tensor space $W = V^{\otimes k}$ by

$$v_1 \otimes v_2 \otimes \cdots \otimes v_k \cdot \sigma = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(k)} \ .$$

Let $\gamma \in \mathbf{S}_k$ be a k-cycle. The eigenvalues of γ, as an endomorphism of W are ζ^r, $r = 0, \ldots, k-1$, where ζ is a primitive k-th root of unity.

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Let $W^{(i)} \subset V^{\otimes k}$ be the eigenspace of γ corresponding to the eigenvalue ζ^i. Then, $W^{(i)}$ is a sub-G-module of W, and an important problem is to compute its decomposition into irreducibles from the character of V. In terms of symmetric functions, this is a plethysm problem: if $F = \mathrm{ch}\, V$ is the formal character of V and

$$\ell_k^{(i)} = \mathrm{ch}\, \left[(\mathbf{C}^n)^{\otimes k}\right]^{(i)} = \frac{1}{k} \sum_{d|k} c(i,d) p_d^{k/d}$$

is the character of $W^{(i)}$ in the special case where $V = \mathbf{C}^n$ (the basic representation of G), the problem is to expand the plethysm $\ell_k^{(i)} \circ F$ as a linear combination of Schur functions

$$\ell_k^{(i)} \circ F = \sum_\lambda c_\lambda s_\lambda \ .$$

The coefficients c_λ are nonnegative integers, for which it is rather unlikely that a closed formula could exist, and a combinatorial interpretation of these coefficients, allowing their individual computation, should be considered as a satisfactory solution of the problem.

This problem is solved in [**LLT1**, **LLT2**], in the case where V is a tensor product of exterior or symmetric powers of the basic representation, *i.e.* $V = \Lambda^{\mu_1}\mathbf{C}^n \otimes \cdots \otimes \Lambda^{\mu_r}\mathbf{C}^n$, or $V = S^{\mu_1}\mathbf{C}^n \otimes \cdots \otimes S^{\mu_r}\mathbf{C}^n$. The answer in this case is that the character of $W^{(i)}$ can be obtained by reducing modulo $1-q^k$ a certain Hall-Littlewood function. The required combinatorial interpretation is then obtained from the one of Hall-Littlewood functions.

Another case which is completely solved is when $k = 2$ and $V = V_\lambda$ is an irreducible representation [**CL**]. This involves a new version of the Littlewood-Richardson rule, where ordinary tableaux are replaced by domino tableaux. This rule can also be formulated as a property of the reduction modulo $1-q^2$ of certain symmetric functions, also defined in terms of domino tableaux, and depending on a parameter q [**CL**, **KLLT**].

In this paper, we introduce a new family of symmetric functions $H_\lambda^{(k)}(x;q)$, defined as generating functions of certain sets of k-ribbon (or rim-hook) tableaux according to a statistic called *spin*, which contains the Hall-Littlewood functions and the domino functions as particular cases, and continue to display the same type of behaviour, at least at the experimental level.

We present a series of conjectures on these new functions, and prove them in the two extreme cases: for the shortest possible ribbons (dominoes), and for sufficiently long ones (the stable case).

The methods used for proving these two cases are quite different. In the domino case, the proofs are entierely combinatorial, and do not seem to be generalizable to other cases. This is probably due to the fact that the hyperoctahedral group is the only group in the series $\mathbf{S}_n[C_k]$, $k \geq 2$ (wreath products of a cyclic group by a symmetric group) which is also a Weyl group. In the stable case, the proofs rely upon the cell decompositions of unipotent varieties [**Sh1**, **HSh**]. We believe that the intermediate cases may also be proved by similar techniques, in terms of the geometry of other subvarieties of flag manifolds.

2 Hall-Littlewood functions and unipotent varieties

Our notations for symmetric functions will be essentially those of the book [**Mcd**], to which the reader is referred for more details.

Let $X = \{x_1, x_2, \ldots\}$ be an infinite set of indeterminates and *Sym* be the ring of symmetric functions in X, with coefficients in $\mathbf{C}(q)$, q being another indeterminate. In what follows, the scalar product $\langle \ , \ \rangle$ on *Sym* will always be the standard one, for which Schur functions form an orthonormal basis. We denote by $Q'_\mu(X;q)$ the image of the Hall-Littlewood function $Q_\mu(X;q)$ by the ring homomorphism $p_k \mapsto (1-q^k)^{-1}p_k$. That is, (Q'_μ) is the adjoint basis of the basis (P_μ) for the standard scalar product, and in λ-ring notation, $Q'_\mu(X;q) = Q(X/(1-q);q)$. In the Schur basis,

$$Q'_\mu(X;q) = \sum_\lambda K_{\lambda\mu}(q) s_\lambda(X) \tag{1}$$

where the $K_{\lambda\mu}(q)$ are the Kostka-Foulkes polynomials. The polynomial $K_{\lambda\mu}(q)$ is the generating function of a statistic c called *charge* on the set $\mathrm{Tab}\,(\lambda,\mu)$ of Young tableaux of shape λ and weight μ

$$K_{\lambda\mu}(q) = \sum_{\mathbf{t}\in \mathrm{Tab}\,(\lambda,\mu)} q^{c(\mathbf{t})} \ . \tag{2}$$

We shall also need the $\tilde{Q}'$-functions, defined by

$$\tilde{Q}'_\mu(X;q) = \sum_\lambda \tilde{K}_{\lambda\mu}(q) s_\lambda(X) = q^{n(\mu)} Q'_\mu(X;q^{-1}) \ . \tag{3}$$

The polynomial $\tilde{K}_{\lambda\mu}(q)$ is the generating function of the complementary statistic $\tilde{c}(\mathbf{t}) = n(\mu) - c(\mathbf{t})$, which is called *cocharge*. The operation of *cyclage* endows $\mathrm{Tab}\,(\lambda,\mu)$ with the structure of a rank poset, in which the rank of a tableau is equal to its cocharge (see [**La**]).

When the parameter q is interpreted as the cardinality of a finite field $\mathbf{F}_q$, it is known that $\tilde{K}_{\lambda\mu}(q)$ is equal to the value $\chi^\lambda(u)$ of the unipotent character χ^λ of $G = GL_n(\mathbf{F}_q)$ on a unipotent element u with Jordan canonical form specified by the partition μ (see [**Lu2**]).

In this specialization, the coefficients

$$\tilde{G}_{\nu\mu}(q) = \langle h_\nu \, , \, \tilde{Q}'_\mu \rangle \tag{4}$$

of the $\tilde{Q}'$-functions on the basis of monomial symmetric functions are also the values of certain characters of G on unipotent classes. Let P_ν denote a parabolic subgroup of type ν of G, for example the group of upper block triangular matrices with diagonal blocks of sizes $\nu_1, \ldots, \nu_r$, and consider the permutation representation of G over $\mathbf{C}[G/P_\nu]$. The value $\xi^\nu(g)$ of the character ξ^ν of this representation on an element $g \in G$ is equal to the number of fixed points of g on G/P_ν. Then, it can be shown that, for a unipotent u of type μ,

$$\xi^\nu(u) = \tilde{G}_{\nu\mu}(q) \ . \tag{5}$$

The factor set G/P_ν can be identified with the variety $\mathcal{F}_\nu$ of ν-flags in $V = \mathbf{F}_q^n$

$$V_{\nu_1} \subset V_{\nu_1+\nu_2} \subset \ldots \subset V_{\nu_1+\ldots\nu_r} = V$$

where $\dim V_i = i$. Thus, $\tilde{G}_{\nu\mu}(q)$ is equal to the number of $\mathbf{F}_q$-rational points of the algebraic variety $\mathcal{F}_\nu^u$ of fixed points of u in $\mathcal{F}_\nu$.

It has been shown by N. Shimomura ([**Sh1**], see also [**HSh**]) that the corresponding complex variety $\mathcal{F}_\nu^u[\mathbf{C}]$ admits a cell decomposition, involving only cells of even real dimensions. More precisely, this cell decomposition is a partition in locally closed subvarieties, each being algebraically isomorphic to an affine space. Thus, the odd-dimensional homology groups are zero, and if

$$\Pi_{\nu\mu}(t^2) = \sum_i t^{2i} \dim H_{2i}(\mathcal{F}_\nu^u, \mathbf{Z})$$

is the Poincaré polynomial of $\mathcal{F}_\nu^u$, one has $|\mathcal{F}_\nu^u| = \Pi_{\nu\mu}(q)$. But this is also equal to $\tilde{G}_{\nu\mu}(q)$, and as this is true for an infinite set of values of q, one has $\Pi_{\nu\mu}(z) = \tilde{G}_{\nu\mu}(z)$ as polynomials. That is, the coefficient of $\tilde{Q}'_\mu$ on the monomial function m_ν is the Poincaré polynomial of $\mathcal{F}_\nu^u$, for a unipotent u of type μ.

Writing

$$\tilde{Q}'_\mu = \sum_{\lambda,\nu} \tilde{K}_{\lambda\mu(q)} K_{\lambda\nu}\, m_\nu \ , \tag{6}$$

one sees that

$$\tilde{G}_{\nu\mu}(q) = \sum_{(\mathbf{t}_1,\mathbf{t}_2)\in \mathrm{Tab}\,(\lambda,\mu)\times \mathrm{Tab}\,(\nu,\mu)} q^{\tilde{c}(\mathbf{t}_1)} \ . \tag{7}$$

Knuth's extension of the Robinson-Schensted correspondence [**Kn**] is a bijection between the set

$$\coprod_\lambda \mathrm{Tab}\,(\lambda,\mu) \times \mathrm{Tab}\,(\lambda,\nu)$$

of pairs of tableaux with the same shape, and the double coset space $\mathbf{S}_\mu\backslash\mathbf{S}_n/\mathbf{S}_\nu$ of the symmetric group $\mathbf{S}_n$ modulo two parabolic subgroups. Double cosets can be encoded by two-line arrays, integer matrices with prescribed row and column sums, or by *tabloids*.

Let ν and μ be arbitrary compositions of the same integer n. A μ-tabloid of shape ν is a filling of the diagram of boxes with row lengths $\nu_1, \nu_2, \ldots, \nu_r$, the lowest row being numbered 1 (French convention for tableaux), such that the number i occurs μ_i times, and such that each row is nondecreasing. For example,

$$\begin{array}{|c|c|c|}\hline 3 \\ \cline{1-1} 1 & 1 & 1 \\ \hline 1 & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}$$

is a $(5,1,3)$-tabloid of shape $(2,3,3,1)$.

We denote by $L(\nu,\mu)$ the set of tabloids of shape ν and weight μ. A tabloid will be identified with the word obtained by reading it from left to right and top to bottom. Then,

$$\tilde{G}_{\nu\mu}(q) = \sum_{T\in L(\nu,\mu)} q^{\tilde{c}(T)} \ . \tag{8}$$

Example 2.1 To compute $\tilde{G}_{42,321}(q)$ one lists the elements of $L((4,2),(3,2,1))$, which are

$$\begin{array}{cccc} 2 & 3 & & \\ 1 & 1 & 1 & 2 \end{array} \qquad \begin{array}{cccc} 2 & 2 & & \\ 1 & 1 & 1 & 3 \end{array} \qquad \begin{array}{cccc} 1 & 3 & & \\ 1 & 1 & 2 & 2 \end{array} \qquad \begin{array}{cccc} 1 & 2 & & \\ 1 & 1 & 2 & 3 \end{array} \qquad \begin{array}{cccc} 1 & 1 & & \\ 1 & 2 & 2 & 3 \end{array}$$

Reading them as prescribed, we obtain the words

$$231112 \qquad 221113 \qquad 131122 \qquad 121123 \qquad 111223$$

whose respective charges are $2, 1, 3, 2, 4$. The cocharge polynomial is thus $\tilde{G}_{42,321}(q) = 1 + q + 2q^2 + q^3$.

In Shimomura's decomposition of the fixed point variety $\mathcal{F}_\mu^u$ of a unipotent of type ν, the cells are indexed by tabloids of shape ν and weight μ. The dimension $d(T)$ of the cell c_T indexed by $T \in L(\nu, \mu)$ is computed by an algorithm described below, and gives another combinatorial interpretation of the polynomial $\tilde{G}_{\mu\nu}(q)$, exchanging the rôles of shape and weight:

$$\tilde{G}_{\mu\nu} = \sum_{T \in L(\mu,\nu)} q^{\tilde{c}(T)} = \sum_{T \in L(\nu,\mu)} q^{d(T)} \ . \tag{9}$$

The dimensions $d(T)$ are given by the following algorithm.

1. If $T \in L(\nu, (n))$ then $d(T) = 0$;

2. If $\mu = (\mu_1, \mu_2)$ has exactly two parts, and $T \in L(\nu, \mu)$, then $d(T)$ is computed as follows. A box α of T is said to be *special* if it contains the rightmost 1 of its row. For a box β of T, put $d(\beta)$=0 if β does not contain a 2, and if β contains a 2, set $d(\beta)$ equal to the number of nonspecial 1's lying in the column of β, plus the number of special 1's lying in the same column, but in a lower position. Then
$$d(T) = \sum_\beta d(\beta) \ .$$

3. Let $\mu = (\mu_1, \ldots, \mu_k)$ and $\mu^* = (\mu_1, \ldots, \mu_{k-1})$. For $T \in L(\nu, \mu)$, let T_1 be the tabloid obtained by changing the entries k into 2 and all the other ones into 1. Let T_2 be the tabloid obtained by erasing all the entries k, *and rearranging the rows in the appropriate order.* Then,
$$d(T) = d(T_1) + d(T_2) \ . \tag{10}$$

Example 2.2 With $T = \begin{array}{ccc} 1 & 4 & \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{array} \in L(332, 4211)$, one has

$$T_1 := \begin{array}{ccc} 1 & 2 & \\ 1 & 1 & \mathbf{1} \\ 1 & 1 & \mathbf{1} \end{array} \qquad T_2 = \begin{array}{ccc} 1 & & \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{array} \qquad T_{21} = \begin{array}{ccc} 1 & & \\ 1 & \mathbf{1} & 2 \\ 1 & 1 & \mathbf{1} \end{array} \qquad T_{22} = \begin{array}{ccc} 1 & & \\ \mathbf{1} & 2 & \\ 1 & \mathbf{1} & 2 \end{array}$$

where the special entries are printed in boldtype. Thus, $d(T) = t(T_1) + d(T_2) = 2 + d(T_{21}) + d(T_{22}) = 4$.

We shall need a variant of this construction, in which the shape ν is allowed to be an arbitrary composition, and where in step 3, the rearranging of the rows is supressed. Such a variant has already been used by Terada [**Te**] in the case of complete flags.

That is, we associate to a tabloid $T \in L(\nu, \mu)$ an integer $e(T)$, defined by

1. For $T \in L(\nu, (n))$, $e(T) = d(T) = 0$;
2. For $T \in L(\nu, (\mu_1, \mu_2))$, $e(T) = d(T)$;
3. Otherwise $e(T) = e(T_1) + e(T_2)$ where T_1 is defined as above, but this time T_2 is obtained from T by erasing the entries k, without reordering.

Lemma 2.3 *Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a partition, and let $\nu = \lambda\cdot\sigma = (\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(r)})$, $\sigma \in \mathbf{S}_r$. Then, the distribution of e on $L(\nu, \mu)$ is the same as the distribution of d on $L(\lambda, \mu)$. That is,*

$$D_{\lambda\mu}(q) = \sum_{T\in L(\lambda,\mu)} q^{d(T)} = E_{\nu\mu}(q) = \sum_{T\in L(\nu,\mu)} q^{e(T)} .$$

In particular, $D_{\lambda\mu}(q) = E_{\lambda\mu}(q)$.

Proof — This could be proved by repeating word for word the geometric argument of [**Sh1**]. We give here a short combinatorial argument. As the two statistics coincide on tabloids whose shape is a partition and whose weight has at most two parts, the only thing to prove, thanks to the recurrence formula, is that e has the same distribution on $L(\beta, (\mu_1, \mu_2))$ as on $L(\alpha, (\mu_1, \mu_2))$ when β is a permutation of α. The symmetric group being generated by the elementary transpositions $\sigma_i = (i, i{+}1)$, one may assume that $\beta = \alpha\sigma_i$. We define the image $T\sigma_i$ of a tabloid $T \in L(\alpha, (\mu_1, \mu_2))$ by distinguishing among the following configurations for rows i and $i+1$:

1.
$$\begin{array}{cccccc} x_1 & \dots & x_k & 2 & 2^r \\ 1 & \dots & 1 & \mathbf{1} & 2^s \end{array} \quad \xrightarrow{\sigma_i} \quad \begin{array}{cccccc} x_1 & \dots & x_k & 2 & 2^s \\ 1 & \dots & 1 & \mathbf{1} & 2^r \end{array}$$

2.
$$\begin{array}{cccccc} 1 & \dots & 1 & \mathbf{1} & 2^r \\ x_1 & \dots & x_k & 2 & 2^s \end{array} \quad \xrightarrow{\sigma_i} \quad \begin{array}{cccccc} 1 & \dots & 1 & \mathbf{1} & 2^s \\ x_1 & \dots & x_k & 2 & 2^r \end{array}$$

3. In all other cases, the two rows are exchanged:

$$\begin{array}{cccc} x_1 & \dots & x_r & \\ y_1 & \dots & & y_s \end{array} \quad \xrightarrow{\sigma_i} \quad \begin{array}{cccc} y_1 & \dots & & y_s \\ x_1 & \dots & x_r & \end{array}$$

From this definition, it is clear that $e(T\sigma_i) = e(T)$. Moreover, it is not difficult to check that this defines an e-preserving action of the symmetric group $\mathbf{S}_m$ on the set of μ-tabloids with m rows, such that $L(\alpha, \mu)\sigma = L(\alpha\sigma, \mu)$ (the only point needing a verification is the braid relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$.

Thus, for a partition λ and a two-part weight $\mu = (\mu_1, \mu_2)$, d and e coincide on $L(\lambda, \mu)$, and for $\sigma \in \mathbf{S}_m$, $E_{\lambda\sigma,\mu}(q) = D_{\lambda\mu}(q)$. Now, by induction, for $\mu = (\mu_1, \dots, \mu_k)$,

$$D_{\lambda\mu}(q) = \sum_{T \in L(\lambda,\mu)} q^{d(T_1)} q^{d(T_2)}$$

$$= \sum_{\bar\lambda = \mathrm{shape}\,(T_1)} q^{d(T_1)} D_{\bar\lambda,\mu^*}(q) = \sum_{\bar\lambda = \mathrm{shape}\,(T_1)} e^{e(T_1)} E_{\bar\lambda,\mu^*}(q) = E_{\lambda\mu}(q)\ .$$

□

Example 2.4 Take $\lambda = (3,2,1)$, $\mu = (4,2)$ and $\nu = \lambda\sigma_1\sigma_2 = (3,1,2)$. The μ-tabloids of shape λ are

$$\begin{array}{cccccc}
 & \begin{array}{|c|cc} \hline 2 & & \\ \hline 1 & 2 & \\ \hline 1 & 1 & 1 \\ \hline \end{array} & \begin{array}{ccc} 2 & & \\ 1 & 1 & \\ 1 & 1 & 2 \end{array} & \begin{array}{ccc} 1 & & \\ 1 & 1 & \\ 1 & 2 & 2 \end{array} & \begin{array}{ccc} 1 & & \\ 2 & 2 & \\ 1 & 1 & 1 \end{array} & \begin{array}{ccc} 1 & & \\ 1 & 2 & \\ 1 & 1 & 2 \end{array} \\
T & & & & & \\
d(T) & 3 & 2 & 0 & 2 & 1
\end{array}$$

The ν-tabloids of shape λ are

$$\begin{array}{cccccc}
 & \begin{array}{ccc} 1 & 1 & 1 \\ 1 & & \\ 2 & 2 & \end{array} & \begin{array}{ccc} 1 & 1 & 1 \\ 2 & & \\ 1 & 2 & \end{array} & \begin{array}{ccc} 1 & 1 & 2 \\ 1 & & \\ 1 & 2 & \end{array} & \begin{array}{ccc} 1 & 1 & 2 \\ 2 & & \\ 1 & 1 & \end{array} & \begin{array}{ccc} 1 & 2 & 2 \\ 1 & & \\ 1 & 1 & \end{array} \\
T & & & & & \\
e(T) & 2 & 3 & 0 & 2 & 1
\end{array}$$

Thus, $D_{\lambda\mu}(q) = E_{\nu\mu}(q) = 1 + q + 2q^2 + q^3 = \tilde G_{\mu\lambda}(q)$. The tabloids contributing a term q^2 are apparied in the following way:

$$\begin{array}{ccc} 2 & & \\ 1 & 1 & \\ 1 & 1 & 2 \end{array} \longrightarrow \begin{array}{ccc} 1 & 1 & 2 \\ 2 & & \\ 1 & 1 & \end{array} \qquad \begin{array}{ccc} 1 & & \\ 2 & 2 & \\ 1 & 1 & 1 \end{array} \longrightarrow \begin{array}{ccc} 1 & 1 & 1 \\ 1 & & \\ 2 & 2 & \end{array}$$

Remark The only property that we shall need in the sequel is the equality $D_{\lambda\mu}(q) = E_{\lambda\mu}(q)$. However, it is possible to be more explicit by constructing a bijection exchanging d and e. The above action of $\mathbf{S}_m$ can be extended to tabloids with arbitrary weight, still preserving e. Suppose for example that we want to apply σ_i to a tabloid T whose restriction to rows $i, i+1$ is

$$\begin{array}{cccccccccc} 1 & 1 & 2 & 3 & 7 & 7 & 9 & & & \\ 1 & 1 & 1 & 2 & 6 & 6 & 6 & 8 & 8 & 9 \end{array}$$

One first determines the positions of the greatest entries, which are the 9's, in $T\sigma_i$. Starting with an empty diagram of the permuted shape $(10, 7)$, one constructs T_1 as above by converting all the entries 9 of T into 2 and the remaining ones into 1. Then we apply σ_i to T_1, and the positions of the 2 in $T_1\sigma_i$ give the positions of the 9 in $T\sigma_i$. Then, the entries 9 are removed from T ad the procedure is iterated until one reaches a tabloid whose rows i and $i+1$ are of equal lenghts. This tabloid is then copied (without permutation) in the remaining part of the result. On the example, this gives

$\begin{array}{llllllllll} 1&1&1&1&1&1&2 \\ 1&1&1&1&1&1&1&1&1&2 \end{array}$	$\begin{array}{llllllllll} 1&1&1&1&1&1&1&1&1&2 \\ 1&1&1&1&1&1&2 \end{array}$	$\begin{array}{cccccccccc} &&&&&&&&&9 \\ &&&&&&9 \end{array}$
$\begin{array}{llllllllll} 1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&2&2 \end{array}$	$\begin{array}{llllllllll} 1&1&1&1&1&1&1&2&2 \\ 1&1&1&1&1&1 \end{array}$	$\begin{array}{cccccccccc} &&&&&&&8&8&9 \\ &&&&&&9 \end{array}$
$\begin{array}{llllllllll} 1&1&1&1&2&2 \\ 1&1&1&1&1&1&1 \end{array}$	$\begin{array}{llllllllll} 1&1&1&1&1&1&1 \\ 1&1&1&1&2&2 \end{array}$	$\begin{array}{cccccccccc} &&&&&&&8&8&9 \\ &&&&7&7&9 \end{array}$
$\begin{array}{llllllllll} 1&1&1&1 \\ 1&1&1&1&2&2&2 \end{array}$	$\begin{array}{llllllllll} 1&1&1&1&2&2&2 \\ 1&1&1&1 \end{array}$	$\begin{array}{cccccccccc} &&&&6&6&6&8&8&9 \\ &&&&7&7&9 \end{array}$
$\begin{array}{llllllllll} 1&1&1&2 \\ 1&1&1&1 \end{array}$	$\begin{array}{llllllllll} 1&1&1&2 \\ 1&1&1&1 \end{array}$	$\begin{array}{cccccccccc} 1&1&2&3&6&6&6&8&8&9 \\ 1&1&1&2&7&7&9 \end{array}$

3 Specializations at roots of unity

As recalled in the preceding section, the Hall-Littlewood functions with parameter specialized to the cardinality q of a finite field $\mathbf{F}_q$ provide information about the characters of the linear group $GL(n, \mathbf{F}_q)$ over this field. It turns out that when the parameter is specialized to a complex root of unity, one obtains information about representations of $GL(n, \mathbf{C})$, that is, a combinatorial decomposition of certain plethysms [**LLT1**, **LLT2**]. We give now a brief review of the main results of these papers.

The first one is a factorization property of the functions $Q'_\lambda(X, q)$ when q is specialized to a primitive root of unity. This is to be seen as a generalization of the fact that when q is specialized to 1 the function $Q'_\lambda(X; q)$ reduces to $h_\lambda(X) = \prod_i h_{\lambda_i}(X)$.

Theorem 3.1 *Let $\lambda = (1^{m_1} 2^{m_2} \dots n^{m_n})$ be a partition written multiplicatively. Set $m_i = kq_i + r_i$ with $0 \le r_i < k$, and $\mu = (1^{r_1} 2^{r_2} \dots n^{r_n})$. Then, ζ being a primitive k-th root of unity,*

$$Q'_\lambda(X; \zeta) = Q'_\mu(X; \zeta) \prod_{i \ge 1} [Q'_{(i^k)}(X; \zeta)]^{q_i} \ . \tag{11}$$

The functions $Q'_{(i^k)}(X; \zeta)$ appearing in the right-hand side of (11) can be expressed as plethysms.

Theorem 3.2 *Let $p_k \circ h_n$ denote the plethysm of the complete function h_n by the power-sum p_k, which is defined by the generating series $\sum_n p_k \circ h_n(X)\, z^n = \prod_{x\in X}(1 - zx^k)^{-1}$. Then, if ζ is as above a primitive k-th root of unity, one has*

$$Q'_{(n^k)}(X;\zeta) = (-1)^{(k-1)n} p_k \circ h_n(X).$$

Example 3.3 With $k = 3$ ($\zeta = e^{2i\pi/3}$), we have

$$Q'_{444433311}(X;\zeta) = Q'_{411}(X;\zeta)\, Q'_{333}(X;\zeta)\, Q'_{444}(X;\zeta) = Q'_{411}(X;\zeta)\, p_4 \circ h_{43} \ .$$

Given two partitions λ and μ, we denote by $\lambda \vee \mu$ the partition obtained by reordering the concatenation of λ and ν, *e.g.* $(2,\,2,\,1)\vee(5,\,2,\,1,\,1) = (5,\,2^3,\,1^3)$. We write $\mu^k = \mu\vee\mu\vee\cdots\vee\mu$ (k factors). If $\mu = (\mu_1,\,\ldots,\,\mu_r)$, we set $k\mu = (k\mu_1,\,\ldots,\,k\mu_r)$.

For $k, n \in \mathbf{N}$, the *Ramanujan* or *Von Sternecksum* $c(k,n)$ (also denoted $\Phi(k,n)$) is the sum of the k-th powers of the *primitive* n-th roots of unity. Its value is given by *Hölder's formula*: if $(k,n) = d$ and $n = md$, then $c(k,n) = \mu(m)\phi(n)/\phi(m)$, where μ is the Moebius function and ϕ is the Euler totient function (see *e.g.* [**NV**]).

Let $P(q) = \sum_{k=0}^{n-1} a_k\, q^k \ \in \mathbf{Z}[q]$ be a polynomial of degree $\le n-1$. P is said to be *even modulo n* if $(i,n) = (j,n) \ \Rightarrow a_i = a_j$. The following property [**Co**] can be regarded as a generalization of the Moebius inversion formula:

Lemma 3.4 *The polynomial P is even modulo n iff for every divisor d of n, the residue of $P(q)$ modulo the cyclotomic polynomial $\Phi_d(q)$ is a constant $r_d \in Z$. In this case, one has*

$$a_k = \frac{1}{n}\sum_{d|n} c(k,d) r_d \ , \qquad r_d = \sum_{t|n} c(n/d,t) a_{n/t} \ .$$

With the aid of Ramanujan sums, we define the symmetric functions

$$\ell_n^{(k)} = \frac{1}{n}\sum_{d|n} c(k,d) p_d^{n/d} \ . \tag{12}$$

These functions were first encountered by Foulkes as Frobenius characteristics of the representations of the symmetric group induced by irreducible representations of transitive cyclic subgroup [**Fo**]. A combinatorial interpretation of the multiplicity $\langle s_\lambda\,,\, \ell_n^{(k)}\rangle$ has been given by Kraskiewicz and Weyman [**KW**]. This result is equivalent to the congruence

$$Q'_{1^n}(X;q) \equiv \sum_{0\le k\le n-1} q^k \ell_n^{(k)} \ (\bmod\ 1-q^n) \ .$$

A proof using Cohen's formula can be found in [**De**]. Taking into account Theorems 3.1 and 3.2, one obtains:

Theorem 3.5 [**LLT2**] *Let e_i be the i-th elementary symmetric function, and for $\lambda = (\lambda_1, \dots, \lambda_m)$, $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_r}$. Then, the multiplicity $\langle s_\mu \, , \, \ell_k^{(r)} \circ e_\lambda \rangle$ of the Schur function s_μ in the plethysm $\ell_k^{(r)} \circ e_\lambda$ is equal to the number of Young tableaux of shape μ' (conjugate partition) and weight λ^k whose charge is congruent to r modulo k.*

This gives as well the plethysms with product of complete functions, since

$$\langle s_{\mu'} \, , \, \ell_k^{(r)} \circ e_\lambda \rangle = \begin{cases} \langle s_\mu \, , \, \ell_k^{(r)} \circ h_\lambda \rangle & \text{if } |\lambda| \text{ is even} \\ \langle s_\mu \, , \, \tilde{\ell}_k^{(r)} \circ h_\lambda \rangle & \text{if } |\lambda| \text{ is odd} \end{cases}$$

where $\tilde{\ell}_k^{(r)} = \omega(\ell_k^{(r)}) = \ell_k^{(s)}$ with $s = k(k-1)/2 - r$.

Example 3.6 With $k = 4$, $r = 2$ and $\lambda = (2)$,

$$\ell_4^{(2)} \circ e_2 = s_{431} + s_{422} + s_{41111} + 2s_{3311} + 2s_{3221}$$

$$+2s_{32111} + s_{2222} + s_{22211} + 2s_{221111} + s_{2111111} \ .$$

To compute the coefficient $\langle s_{32111} \, , \, \ell_4^{(2)} \circ e_2 \rangle = 2$, we have to find the number of tableaux of shape $(3, 2, 1, 1, 1)' = (5, 2, 1)$, weight $(2, 2, 2, 2)$ and charge $\equiv 2 \pmod 4$. The two tableaux satisfying these constraints are:

$$\begin{array}{lllll} 3 & & & & \\ 2 & 4 & & & \\ 1 & 1 & 2 & 3 & 4 \end{array} \qquad\qquad \begin{array}{lllll} 4 & & & & \\ 2 & 3 & & & \\ 1 & 1 & 2 & 3 & 4 \end{array}$$

which both have charge equal to 6.

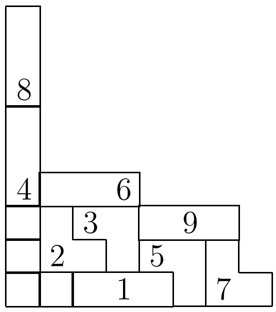
Similarly, the reader can check that $\langle s_{732} \, , \, \ell_4^{(2)} \circ e_{21} \rangle = 5$ is the number of tableaux with shape $(3, 3, 2, 1, 1, 1, 1)$, weight $(2, 2, 2, 2, 1, 1, 1, 1)$ and charge $\equiv 2 \pmod 4$.

A more combinatorial formulation of Theorems 3.1 and 3.2 can be presented by means of the notion of *ribbon tableau*, which will also provide the key for their generalization.

4 Ribbon tableaux

To a partition λ is associated a k-core $\lambda_{(k)}$ and a k-quotient $\lambda^{(k)}$ [**JK**]. The k-core is the unique partition obtained by successively removing k-ribbons (or skew hooks) from λ. The different possible ways of doing so can be distinguished from one another by labelling 1 the last ribbon removed, 2 the penultimate, and so on. Thus Figure 1 shows two different ways of reaching the 3-core $\lambda_{(3)} = (2, 1^2)$ of $\lambda = (8, 7^2, 4, 1^5)$. These pictures represent two 3-ribbon tableaux T_1, T_2 of shape $\lambda/\lambda_{(3)}$ and weight $\mu = (1^9)$.

To define k-ribbon tableaux of general weight and shape, we need some terminology. The *initial cell* of a k-ribbon R is its rightmost and bottommost cell. Let

$T_1 =$ 

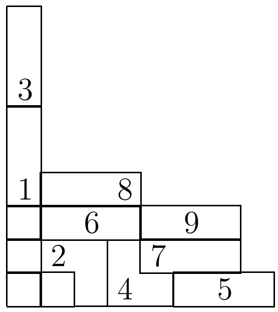
$T_2 =$ 

Figure 1:

$\theta = \beta/\alpha$ be a skew shape, and set $\alpha_+ = (\beta_1) \vee \alpha$, so that α_+/α is the horizontal strip made of the bottom cells of the columns of θ. We say that θ is a *horizontal k-ribbon strip* of weight m, if it can be tiled by m k-ribbons the initial cells of which lie in α_+/α. (One can check that if such a tiling exists, it is unique).

Now, a *k-ribbon tableau* T of shape λ/ν and weight $\mu = (\mu_1, \ldots, \mu_r)$ is defined as a chain of partitions

$$\nu = \alpha^0 \subset \alpha^1 \subset \cdots \subset \alpha^r = \lambda$$

such that α^i/α^{i-1} is a horizontal k-ribbon strip of weight μ_i. Graphically, T may be described by numbering each k-ribbon of α^i/α^{i-1} with the number i. We denote by $\mathrm{Tab}_k(\lambda/\nu, \mu)$ the set of k-ribbon tableaux of shape λ/ν and weight μ, and we set

$$K^{(k)}_{\lambda/\nu,\,\mu} = |\mathrm{Tab}_k(\lambda/\nu,\, \mu)| \ .$$

Finally we recall the definition of the k-sign $\epsilon_k(\lambda/\nu)$. Define the sign of a ribbon R as $(-1)^{h-1}$, where h is the height of R. The k-sign $\epsilon_k(\lambda/\nu)$ is the product of the signs of all the ribbons of a k-ribbon tableau of shape λ/ν (this does not depend on the particular tableau chosen, but only on the shape).

The origin of these combinatorial definitions is best understood by analyzing carefully the operation of multiplying a Schur function s_ν by a plethysm of the form $p_k \circ h_\mu$. Equivalently, thanks to the involution ω, one may rather consider a product of the type $s_\nu\, [p_k \circ e_\mu]$. To this end, since

$$p_k \circ e_\mu = (e_{\mu_1} \circ p_k) \cdots (e_{\mu_n} \circ p_k) = m_{k^{\mu_1}} \cdots m_{k^{\mu_n}}$$

one needs only to apply repeatedly the following multiplication rule due to Muir [**Mu**] (see also [**Li3**]):

$$s_\nu\, m_\alpha = \sum_\beta s_{\nu+\beta} \ ,$$

sum over all distinct permutations β of $(\alpha_1, \alpha_2, \ldots, \alpha_n, 0, \ldots)$. Here the Schur functions $s_{\nu+\beta}$ are not necessary indexed by partitions and have therefore to be standardized, this reduction yielding only a finite number of nonzero summands. For example,

$$s_{31}\, m_3 = s_{61} + s_{313} + s_{31003} = s_{61} - s_{322} + s_{314} \ .$$

Other terms such as s_{34} or s_{3103} reduce to 0. It is easy to deduce from this rule that the multiplicity

$$\langle s_\nu \, m_{k^{\mu_i}} \, , \, s_\lambda \rangle$$

is nonzero iff λ'/μ' is a horizontal k-ribbon strip of weight μ_i, in which case it is equal to $\epsilon_k(\lambda/\nu)$. Hence, applying ω we arrive at the expansion

$$s_\nu \, [p_k \circ h_\mu] = \sum_\lambda \epsilon_k(\lambda/\mu) \, K^{(k)}_{\lambda/\nu \,, \mu} \, s_\lambda$$

from which we deduce by 3.1, 3.2 that

$$K^{(k)}_{\lambda \, \mu} = (-1)^{(k-1)|\mu|} \, \epsilon_k(\lambda) \, K_{\lambda \, \mu^k}(\zeta)$$

and more generally, defining as in [**KR**] the skew Kostka-Foulkes polynomial $K_{\lambda/\nu \,, \alpha}(q)$ by

$$K_{\lambda/\nu \,, \alpha}(q) = \langle s_{\lambda/\nu} \, , \, Q'_\alpha(q) \rangle$$

we can write

$$K^{(k)}_{\lambda/\nu \,, \mu} = (-1)^{(k-1)|\mu|} \, \epsilon_k(\lambda/\nu) \, K_{\lambda/\nu \,, \mu^k}(\zeta) \ .$$

It turns out that enumerating k-ribbon tableaux is equivalent to enumerating k-uples of ordinary Young tableaux, as shown by the correspondence to be described now. This bijection was first studied by Stanton and White [**SW**] in the case of ribbon tableaux of right shape λ (without k-core) and standard weight $\mu = (1^n)$ (see also [**FS**]). We need some additional definitions.

Let R be a k-ribbon of a k-ribbon tableau. R contains a unique cell with coordinates $(x, \, y)$ such that $y - x \equiv 0 \pmod{k}$. We decide to write in this cell the number attached to R, and we define the *type* $i \in \{0, \, 1, \, \ldots, \, k-1\}$ of R as the distance between this cell and the initial cell of R. For example, the 3-ribbons of T_1 are divided up into three classes:

- 4, 6, 8, of type 0;
- 1, 2, 7, 9, of type 1;
- 3, 5, of type 2.

Define the *diagonals* of a k-ribbon tableau as the sequences of integers read along the straight lines $D_i \, : \, y - x = ki$. Thus T_1 has the sequence of diagonals

$$((8), \, (4), \, (2, \, 3, \, 6), \, (1, \, 5, \, 9), \, (7)) \ .$$

This definition applies in particular to 1-ribbon tableaux, *i.e.* ordinary Young tableaux. It is obvious that a Young tableau is uniquely determined by its sequence of diagonals. Hence, we can associate to a given k-ribbon tableau T of shape λ/ν a k-uple $(t_0, \, t_1, \, \ldots, \, t_{k-1})$ of Young tableaux defined as follows; the diagonals of t_i are obtained by erasing in the diagonals of T the labels of all the ribbons of type $\neq i$. For instance, if $T = T_1$ the first ribbon tableau of Figure 1, the sequence of diagonals of t_1 is $((2), \, (1, \, 9), \, (7))$, and

$$t_1 = \begin{array}{|c|c|}\hline 2 & 9 \\ \hline 1 & 7 \\ \hline \end{array}$$

The complete triple $(t_0,\, t_1,\, t_2)$ of Young tableaux associated to T_1 is

$$\tau^1 = \Big(\begin{array}{|c|c}\cline{1-1} 8 & \\ \hline 4 & \multicolumn{1}{c|}{6} \\ \hline \end{array} \,,\; \begin{array}{|c|c|}\hline 2 & 9 \\ \hline 1 & 7 \\ \hline \end{array} \,,\; \begin{array}{|c|c|}\hline 3 & 5 \\ \hline \end{array} \Big)$$

whereas that corresponding to T_2 is

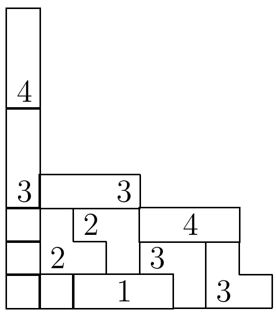
$$\tau^2 = \Big(\begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 1 & \multicolumn{1}{c|}{8} \\ \hline \end{array} \,,\; \begin{array}{|c|c|}\hline 6 & 9 \\ \hline 4 & 5 \\ \hline \end{array} \,,\; \begin{array}{|c|c|}\hline 2 & 7 \\ \hline \end{array} \Big)$$

One can show that if $\nu = \lambda_{(k)}$, the k-core of λ, the k-uple of shapes $(\lambda^0,\, \lambda^1,\, \ldots,\, \lambda^{k-1})$ of $(t_0,\, t_1,\, \ldots,\, t_{k-1})$ depends only on the shape λ of T, and is equal to the k-quotient $\lambda^{(k)}$ of λ. Moreover the correspondence $T \longrightarrow (t_0,\, t_1,\, \ldots,\, t_{k-1})$ establishes a bijection between the set of k-ribbon tableaux of shape $\lambda/\lambda_{(k)}$ and weight μ, and the set of k-uples of Young tableaux of shapes $(\lambda^0,\, \ldots,\, \lambda^{k-1})$ and weights $(\mu^0,\, \ldots,\, \mu^{k-1})$ with $\mu_i = \sum_j \mu_i^j$. (See [**SW**] or [**FS**] for a proof in the case when $\lambda_{(k)} = (0)$ and $\mu = (1^n)$).

For example, keeping $\lambda = (8,\, 7^2,\, 4,\, 1^5)$, the triple

$$\tau = \Big(\begin{array}{|c|c}\cline{1-1} 4 & \\ \hline 3 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \,,\; \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array} \,,\; \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array} \Big)$$

with weights $((0,\, 0,\, 2,\, 1),\, (1,\, 1,\, 1,\, 1),\, (0,\, 1,\, 1,\, 0))$ corresponds to the 3-ribbon tableau

$T =$ 

of weight $\mu = (1,\, 2,\, 4,\, 2)$.

As before, the significance of this combinatorial construction becomes clearer once interpreted in terms of symmetric functions. Recall the definition of ϕ_k, the adjoint of the linear operator $\psi^k :\ F \mapsto p_k \circ F$ acting on the space of symmetric functions. In other words, ϕ_k is characterized by

$$\langle \phi_k(F)\, ,\, G \rangle = \langle F\, ,\, p_k \circ G \rangle\ , \quad F,\, G \in Sym\ .$$

Littlewood has shown [**Li3**] that if λ is a partition whose k-core $\lambda_{(k)}$ is null, then

$$\phi_k(s_\lambda) = \epsilon_k(\lambda)\, s_{\lambda^0}\, s_{\lambda^1}\, \cdots\, s_{\lambda^{k-1}} \tag{13}$$

where $\lambda^{(k)} = (\lambda^0, \ldots, \lambda^{k-1})$ is the k-quotient. Therefore,

$$K^{(k)}_{\lambda\,\mu} = \epsilon_k(\lambda)\,\langle p_k \circ h_\mu\,,\, s_\lambda\rangle = \epsilon_k(\lambda)\,\langle \phi_k(s_\lambda)\,,\, h_\mu\rangle = \langle s_{\lambda^0}\, s_{\lambda^1}\,\cdots\, s_{\lambda^{k-1}}\,,\, h_\mu\rangle$$

is the multiplicity of the weight μ in the product of Schur functions $s_{\lambda^0}\cdots s_{\lambda^{k-1}}$, that is, is equal to the number of k-uples of Young tableaux of shapes $(\lambda^0, \ldots, \lambda^{k-1})$ and weights $(\mu^0, \ldots, \mu^{k-1})$ with $\mu_i = \sum_j \mu_i^j$. Thus, the bijection described above gives a combinatorial proof of (13).

More generally, if λ is replaced by a skew partition λ/ν, (13) becomes [**KSW**]

$$\phi_k(s_{\lambda/\nu}) = \epsilon_k(\lambda/\nu)\, s_{\lambda^0/\nu^0}\, s_{\lambda^1/\nu^1}\,\cdots\, s_{\lambda^{k-1}/\nu^{k-1}}$$

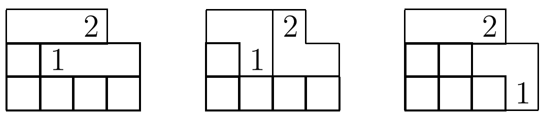
if $\lambda_{(k)} = \nu_{(k)}$, and 0 otherwise. This can also be deduced from the previous combinatorial correspondence, but we shall not go into further details.

Returning to Kostka polynomials, we may summarize this discussion by stating Theorems 3.1 and 3.2 in the following way:

Theorem 4.1 *Let λ and ν be partitions and set $\nu = \mu^k \vee \alpha$ with $m_i(\alpha) < k$. Denoting by ζ a primitive kth root of unity, one has*

$$K_{\lambda,\nu}(\zeta) = (-1)^{(k-1)|\mu|} \sum_\beta \epsilon_k(\lambda/\beta)\, K^{(k)}_{\lambda/\beta,\,\mu}\, K_{\beta,\alpha}(\zeta)\ . \tag{14}$$

Example 4.2 We take $\lambda = (4^2, 3)$, $\nu = (2^2, 1^7)$ and $k = 3$ ($\zeta = e^{2i\pi/3}$). In this case, $\nu = \mu^k \vee \alpha$ with $\mu = (1^2)$ and $\alpha = (2^2, 1)$. The summands of (14) are parametrized by the 3-ribbon tableaux of external shape λ and weight μ. Here we have three such tableaux:



so that

$$K_{443,\,221111111}(\zeta) = 2K_{41,\,221}(\zeta) - K_{32,\,221}(\zeta) = 2(\zeta^2+\zeta^3) - (\zeta+\zeta^2) = 2\zeta^2 + 3\ .$$

When $|\alpha| \le |\lambda_{(k)}|$, (14) becomes simpler. For if $|\alpha| < |\lambda_{(k)}|$ then $K_{\lambda,\nu}(\zeta) = 0$, and otherwise the sum reduces to one single term

$$K_{\lambda,\nu}(\zeta) = (-1)^{(k-1)|\mu|}\epsilon_k(\lambda/\lambda_{(k)})\, K^{(k)}_{\lambda/\lambda_{(k)},\,\mu}\, K_{\lambda_{(k)},\,\alpha}(\zeta)\ .$$

In particular, if $\nu = (1^n)$, one recovers the following theorem of Morris and Sultana [**MS**].

Theorem 4.3 *Let λ be a partition of n and ζ a primitive kth root of unity. Denote by $H(\lambda^{(k)})$ the product of the hook-lengths of the k partitions $\lambda^0, \ldots, \lambda^{k-1}$, and by $|\lambda^{(k)}|$ the sum of their weights. Set $n = kq + r$, $0 \le r < k$. If $r \ne |\lambda_{(k)}|$, then $K_{\lambda,(1^n)}(\zeta) = 0$, otherwise,*

$$K_{\lambda,(1^n)}(\zeta) = (-1)^{(k-1)q}\epsilon_k(\lambda/\lambda_{(k)})\, \frac{|\lambda^{(k)}|!}{H(\lambda^{(k)})}\, K_{\lambda_{(k)},\,1^r}(\zeta)\ .$$

Indeed, the correspondence just described between k-ribbon tableaux and k-uples of Young tableaux shows at once, in view of the classical hook-formula [**JK**], that

$$K^{(k)}_{\lambda/\lambda_{(k)},\,1^q} = \frac{|\lambda^{(k)}|!}{H(\lambda^{(k)})}\ .$$

5 H-functions

Let λ be a partition without k-core, and with k-quotient $(\lambda^0, \ldots, \lambda^{k-1})$. For a ribbon tableau T of weight μ, let $x^T = x_1^{\mu_1} x_2^{\mu_2} \cdots x_r^{\mu_r}$. Then, the Stanton-White correspondence shows that the generating function

$$\mathcal{G}^{(k)}_\lambda = \sum_{T \in \mathrm{Tab}_k(\lambda,\,\cdot\,)} x^T = \prod_{i=0}^{k-1} \sum_{\mathbf{t}_i \in \mathrm{Tab}\,(\lambda^i,\,\cdot\,)} x^{\mathbf{t}_i} = \prod_{i=0}^{k-1} s_{\lambda^i} \tag{15}$$

is a product of Schur functions. Introducing in this equation an appropriate statistic on ribbon tableaux, one can therefore obtain q-analogues of products of Schur functions. The statistic called *cospin*, described below, leads to q-analogues with interesting properties.

Let R be a k-ribbon, $h(R)$ its *heigth* and $w(R)$ its *width*.

h(R)

w(R)

The *spin* of R, denoted by $s(R)$, is defined as

$$s(R) = \frac{h(R) - 1}{2} \tag{16}$$

and the spin of a ribbon tableau T is by definition the sum of the spins of its ribbons. For example, the ribbon tableau

6
5
4 5
2 3
2 4
1 2 5
1 1 2

has a spin equal to 6.

For a partition λ without k-core, let

$$s_k^*(\lambda) = \max\{s(T) \mid T \in \mathrm{Tab}_k(\lambda, \cdot)\} \ . \tag{17}$$

The *cospin* $\tilde{s}(T)$ of a k-ribbon tableau T of shape λ is then

$$\tilde{s}(T) = s_k^*(\lambda) - s(T) \ . \tag{18}$$

Although $s(T)$ can be a half-integer, it is easily seen that $\tilde{s}(T)$ is always an integer. Also, there is one important case where $s(T)$ is an integer. This is when the shape λ of T is of the form $k\mu = (k\mu_1, k\mu_2, \ldots, k\mu_r)$. In this case, the partitions constituting the k-quotient of λ are formed by parts of μ, grouped according to the class modulo k of their indices. More precisely, $\lambda^i = \{\mu_r \mid r \equiv -i \bmod k\}$

We can now define three families of polynomials

$$\mathcal{G}_\lambda^{(k)}(X;q) = \sum_{T \in \mathrm{Tab}_k(\lambda, \cdot)} q^{\tilde{s}(T)} x^T \tag{19}$$

$$\tilde{H}_\mu^{(k)}(X;q) = \sum_{T \in \mathrm{Tab}_k(k\mu, \cdot)} q^{\tilde{s}(T)} x^T = \mathcal{G}_{k\mu}^{(k)}(X;q) \tag{20}$$

$$H_\mu^{(k)}(X;q) = \sum_{T \in \mathrm{Tab}_k(k\mu, \cdot)} q^{s(T)} x^T = q^{s_k^*(k\mu)} \tilde{H}_\mu^{(k)}(X;1/q) \ . \tag{21}$$

The parameter k will be called the *level* of the corresponding symmetric functions. There is strong experimental evidence for the following conjectures.

Conjecture 5.1 (symmetry) *The polynomials $\tilde{G}_\lambda^{(k)}$, $\tilde{H}_\mu^{(k)}$ and $H_\mu^{(k)}$ are symmetric.*

Conjecture 5.2 (positivity) *Their coefficients on the basis of Schur functions are polynomials with nonnegative integer coefficients.*

Conjecture 5.3 (monotonicity) *$H_\mu^{(k+1)} - H_\mu^{(k)}$ is positive on the Schur basis.*

Conjecture 5.4 (plethysm) *When $\mu = \nu^k$, for ζ a primitive k-th root of unity,*

$$H_{\nu^k}^{(k)}(\zeta) = (-1)^{(k-1)|\nu|} \ p_k \circ s_\nu$$

Equivalently,

$$H_{\nu^k}^{(k)}(q) \bmod 1 - q^k = \sum_{i=0}^{k-1} q^k \ell_k^{(i)} \circ s_\nu$$

The following statements will be proved in the forthcoming sections.

Theorem 5.5 *The difference $Q'_\mu - H_\mu^{(2)}$ is nonnegative on the Schur basis.*

Theorem 5.6 *For $k \geq \ell(\mu)$, $H_\mu^{(k)}$ is equal to the Hall-Littlewood function Q'_μ.*

Taking into account the results of [**LLT1**, **LLT2**] and [**CL**], this is sufficient to establish all the conjectures for $k=2$ and $k \geq \ell(\mu)$.

Example 5.7 **(i)** The 3-quotient of $\lambda=(3,3,3,2,1)$ is $((1),(1,1),(1))$ and

$$\begin{aligned}\tilde{G}_{33321}(q) &= m_{31}+(1+q)m_{22}+(2+2q+q^2)m_{211}\\ &\quad +(3+5q+3q^2+q^3)m_{1111}\\ &= s_{31}+qs_{22}+(q+q^2)s_{211}+q^3s_{1111}\end{aligned}$$

is a q-analogue of the product

$$s_1s_{11}s_1 = s_{31}+s_{22}+2s_{211}+s_{1111}\ .$$

(ii) The H-functions associated to the partition $\lambda=(3,2,1,1)$ are

$$\begin{aligned}H^{(2)}_{3211} &= s_{3211}+q\,s_{322}+q\,s_{331}+q\,s_{4111}\\ &\quad +(q+q^2)\,s_{421}+q^2\,s_{43}+q^2\,s_{511}+q^3\,s_{52}\\ H^{(3)}_{3211} &= s_{3211}+q\,s_{322}+(q+q^2)\,s_{331}+q\,s_{4111}\\ &\quad +(q+2q^2)\,s_{421}+(q^2+q^3)\,s_{43}+(q^2+q^3)\,s_{511}\\ &\quad +2q^3\,s_{52}+q^4\,s_{61}\\ H^{(4)}_{3211} &= s_{3211}+q\,s_{322}+(q+q^2)\,s_{331}+q\,s_{4111}\\ &\quad +(q+2q^2+q^3)\,s_{421}+(q^2+q^3+q^4)\,s_{43}+(q^2+q^3+q^4)\,s_{511}\\ &\quad +(2q^3+q^4+q^5)\,s_{52}+(q^4+q^5+q^6)\,s_{61}+q^7\,s_7\\ &= Q'_{3211}\end{aligned}$$

and we see that $s_{3211} < H^{(2)}_{3211} < H^{(3)}_{3211} < H^{(4)}_{3211} = Q'_{3211}$.

(iii) The plethysms of s_{21} with the cyclic characters $\ell_3^{(i)}$ are given by the reduction modulo $1-q^3$ of

$$\begin{aligned}H^{(3)}_{222111} &= q^9s_{63}+(q+1)q^7s_{621}+q^6s_{6111}+(q+1)q^7s_{54}+(q^3+2q^2+2q+1)q^5s_{531}\\ &\quad +(q^2+2q+1)q^5s_{522}+(q^3+2q^2+2q+1)q^4s_{5211}+(q+1)q^4s_{51111}\\ &\quad +(q^2+2q+1)q^5s_{441}+(q^3+2q^2+3q+2)q^4s_{432}+(2q^3+3q^2+3q+1)q^3s_{4311}\\ &\quad +(q^3+3q^2+3q+2)q^3s_{4221}+(q^3+2q^2+2q+1)q^2s_{42111}+q^3s_{411111}+(q^3+1)q^3s_{333}\\ &\quad +(2q^3+3q^2+2q+1)q^2s_{3321}+(q^2+2q+1)q^2s_{33111}+(q^2+2q+1)q^2s_{3222}\\ &\quad +(q^3+2q^2+2q+1)qs_{32211}+(q+1)qs_{321111}+(q+1)qs_{22221}+s_{222111}\end{aligned}$$

Indeed,

$$H^{(3)}_{222111} \bmod 1-q^3 \;=\; (2s_{5211}+s_{22221}+s_{321111}+3s_{4311}$$

$$
\begin{aligned}
& +2s_{32211} + s_{522} + 3s_{432} + 3s_{3321} + s_{33111} + s_{3222} + s_{51111} \\
& +3s_{4221} + 2s_{531} + 2s_{42111} + s_{54} + s_{621} + s_{441})q^2 \\
& +(2s_{5211} + s_{22221} + s_{321111} + 3s_{4311} + 2s_{32211} + s_{522} \\
& +3s_{432} + 3s_{3321} + s_{33111} + s_{3222} + s_{51111} \\
& +3s_{4221} + 2s_{531} + 2s_{42111} + s_{54} + s_{621} + s_{441})q \\
& +2s_{33111} + s_{63} + s_{6111} + 2s_{531} + 2s_{522} + 2s_{5211} + 2s_{441} \\
& +2s_{432} + 3s_{4311} + 3s_{4221} + 2s_{42111} + s_{411111} + 2s_{333} \\
& +2s_{3321} + 2s_{3222} + s_{222111} + 2s_{32211} \\
= \;& q^2 \ell_3^{(2)} \circ s_{21} + q\ell_3^{(1)} \circ s_{21} + \ell_3^{(0)} \circ s_{21} \ .
\end{aligned}
$$

6 The case of dominoes

For $k = 2$, the conjectures can be established by means of the combinatorial constructions of [**CL**] and [**KLLT**]. In this case, conjectures 5.1, 5.2 and 5.4 follow directly from the results of [**CL**], and the only point remaining to be proved is Theorem 5.5.

The important special feature of domino tableaux is that there exits a natural notion of *Yamanouchi domino tableau*. These tableaux correspond to highest weight vectors in tensor products of two irreducible GL_n-modules, in the same way as ordinary Yamanouchi tableaux are the natural labels for highest weight vectors of irreducible representations.

The *column reading* of a domino tableau T is the word obtained by reading the successive columns of T from top to bottom and left to right. Horizontal dominoes, which belong to two succesive columns i and $i+1$ are read only once, when reading column i. For example, the column reading of the domino tableau

$$
\begin{array}{cccc}
4 & & & \\
 & & & \\
3 & & & \\
 & 2 & & 2 \\
1 & & 1 &
\end{array}
$$

is $\mathrm{col}\,(T) = 431212$.

A *Yamanouchi word* is a word $w = x_1 x_2 \cdots x_n$ such that each right factor $v = x_i \cdots x_n$ of w satisfies $|v|_j \geq |v|_{j+1}$ for each j, where $|v|_j$ denotes the number of occurences of the letter j in v.

A *Yamanouchi domino tableau* is a domino tableau whose column reading is a Yamanouchi word. We denote by $\mathrm{Yam}_2(\lambda, \mu)$ the set of Yamanouchi domino tableaux of shape λ and weight μ.

It follows from the results of [**CL**], Section 7, that the Schur expansions of the H-functions of level 2 are given by

$$
H_\lambda^{(2)} = \sum_\mu \sum_{T \in \mathrm{Yam}_2(2\lambda, \mu)} q^{s(T)} s_\mu \ . \tag{22}
$$

On the other hand,

$$Q'_\lambda = \sum_\mu \sum_{\mathbf{t}\in \mathrm{Tab}\,(\mu,\lambda)} q^{c(\mathbf{t})} s_\mu \ . \tag{23}$$

To prove Theorem 5.5, it is thus sufficient to exhibit an injection

$$\eta : \quad \mathrm{Yam}_2(2\lambda, \mu) \longrightarrow \mathrm{Tab}\,(\mu, \lambda)$$

satisfying

$$c(\eta(T)) = s(T) \ .$$

To achieve this, we shall make use of a bijection described in [**BV**], and extended in [**KLLT**], which sends a domino tableau $T \in \mathrm{Tab}_2(\alpha, \mu)$ over the alphabet $X = \{1, \ldots, n\}$, to an ordinary tableau $\mathbf{t} = \phi(T) \in \mathrm{Tab}\,(\alpha, \bar\mu\mu)$ over the alphabet $\bar X \cup X = \{\bar n < \ldots < \bar 1 < 1 < \ldots < n\}$. The weight $\bar\mu\mu$ means that $\mathbf{t}$ contains μ_i occurences of i and of $\bar i$. The tableau $\phi(T)$ is invariant under Schützenberger's involution Ω, and the spin of T can be recovered from $\mathbf{t}$ by the following procedure [**KLLT2**].

Let $\alpha = 2\lambda$, $\beta = \alpha'$, $\beta_{\mathrm{odd}} = (\beta_1, \beta_3, \ldots)$ and $\beta_{\mathrm{even}} = (\beta_2, \beta_4, \ldots)$. Then, there exists a unique factorisation $\mathbf{t} = \tau_1\tau_2$ in the plactic monoid $\mathrm{Pl}\,(X \cup \bar X)$, such that τ_1 is a contretableau of shape $\alpha^1 = (\beta_{\mathrm{even}})'$ and τ_2 is a tableau of shape $\alpha^2 = (\beta_{\mathrm{odd}})'$. The spin of $T = \phi^{-1}(\mathbf{t})$ is then equal to the number $|\tau_1|_+$ of positive letters in τ_1, which is also equal to the number $|\tau_2|_-$ of negative letters in τ_2. Moreover, $\tau_2 = \Omega(\tau_1)$.

Example 6.1 With the following tableau T of shape $(4, 4, 2, 2)$, one finds

$$T = \begin{array}{cccc} & 3 & & \\ 2 & & & \\ & 1 & & 2 \\ 1 & & 1 & \end{array} \qquad \overset{\phi}{\longrightarrow} \qquad \mathbf{t} = \begin{array}{cccc} 1 & 3 & & \\ \bar 1 & 2 & & \\ \bar 2 & \bar 1 & 1 & 2 \\ \bar 3 & \bar 2 & \bar 1 & 1 \end{array}$$

By jeu-de-taquin, we find that in the plactic monoid

$$\mathbf{t} \ = \ \begin{array}{|c|c|} \hline \bar 1 & 1 \\ \hline \bar 2 & \bar 1 \\ \hline \multicolumn{1}{c|}{} & \bar 2 \\ \cline{2-2} \multicolumn{1}{c|}{} & \bar 3 \\ \cline{2-2} \end{array} \ \begin{array}{|c|c|} \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline \bar 1 & 1 \\ \hline \end{array} \ = \ \tau_1\tau_2 \ .$$

The number of positive letters of τ_1 and the number of negative letters of τ_2 are both equal to 1, which is the spin of T.

This correspondence still works in the general case (α need not be of the form 2λ) and the invariant tableau associated to a domino tableau T admits a similar factorisation $\mathbf{t} = \tau_1\tau_2$, but in general $\tau_2 \neq \Omega(\tau_1)$ and the formula for the spin is $s(T) = \frac{1}{2}(|\tau_1|_+ + |\tau_2|_-)$.

The map $\eta : \ \mathrm{Yam}_2(2\lambda, \mu) \longrightarrow \mathrm{Tab}\,(\mu, \lambda)$ is given by the following algorithm: to compute $\eta(T)$,

1. construct the invariant tableau $\mathbf{t} = \phi(T)$
2. apply the jeu-de-taquin algorithm to $\mathbf{t}$ to obtain the plactic factorization $\mathbf{t} = \tau_1\tau_2$, and keep only τ_2.
3. Apply the evacuation algorithm to the *negative* letters of τ_2, keeping track of the successive stages. After all the negative letters have been evacuated, one is left with a Yamanouchi tableau τ in positive letters.
4. Complete the tableau τ to obtain the tableau $\mathbf{t}' = \eta(T)$ using the following rule: suppose that at some stage of the evacuation, the box of τ_2 which disappeared after the elimination of $\bar{i}$ was in row j of τ_2. Then add a box numbered j to row i of τ.

Theorem 6.2 *The above algorithm defines an injection*

$$\eta : \ \mathrm{Yam}_2(2\lambda, \mu) \longrightarrow \mathrm{Tab}\,(\mu, \lambda)$$

satifying $c \circ \eta = s$.

Corollary 6.3 $H_\lambda^{(2)} \leq Q'_\lambda$

Example 6.4 Let T be the following Yamanouchi domino tableau, which is of shape $2\lambda = (6, 4, 4, 2, 2)$, of weight $\mu = (4, 3, 2)$ and has spin $s(T) = 3$

$$T = \quad \begin{array}{ccccc} 3 & & & & \\ & 3 & & & \\ 2 & & 2 & & \\ & 1 & & 2 & \\ 1 & & 1 & & 1 \end{array}$$

Then,

$$\phi(T) = \begin{array}{cccccc} 3 & 3 & & & & \\ 1 & 2 & & & & \\ \bar{1} & \bar{1} & 2 & 2 & & \\ \bar{2} & \bar{2} & \bar{1} & 1 & & \\ \bar{3} & \bar{3} & \bar{2} & \bar{1} & 1 & 1 \end{array} \quad \equiv \quad \begin{array}{ccc} \bar{1} & 1 & 3 \\ & \bar{1} & 2 \\ & \bar{2} & \bar{1} \\ & & \bar{2} \\ & & \bar{3} \end{array} \ \begin{array}{ccc} 3 & & \\ 2 & & \\ 1 & 2 & \\ \bar{2} & 1 & \\ \bar{3} & \bar{1} & 1 \end{array}$$

and the succesive stages of the evacuation process are

$$\begin{array}{ccc} 3 & & \\ 2 & & \\ 1 & 2 & \\ \bar{2} & 1 & \\ \bar{3} & \bar{1} & 1 \end{array} \ \underset{\substack{\bar{i} = \bar{3} \\ j = 5}}{\longrightarrow} \ \begin{array}{ccc} \times & & \\ 3 & & \\ 2 & 2 & \\ 1 & 1 & \\ \bar{2} & \bar{1} & 1 \end{array} \ \underset{\substack{\bar{i} = \bar{2} \\ j = 3}}{\longrightarrow} \ \begin{array}{ccc} 3 & & \\ 2 & \times & \\ 1 & 2 & \\ \bar{1} & 1 & 1 \end{array} \ \underset{\substack{\bar{i} = \bar{1} \\ j = 4}}{\longrightarrow} \ \begin{array}{ccc} \times & & \\ 3 & & \\ 2 & 2 & \\ 1 & 1 & 1 \end{array}$$

so that we find

$$\eta(T) = \begin{array}{|c|c|c|c|}\cline{1-2} 3 & 5 \\ \cline{1-3} 2 & 2 & 3 \\ \hline 1 & 1 & 1 & 4 \\ \hline\end{array}$$

a tableau of shape $\mu = (4, 3, 2)$, weight $\lambda = (3, 2, 2, 1, 1)$ and charge $c(\mathbf{t}') = 3$.

7 The stable case

As the Q'-functions are known to verify all the conjectured properties of H-functions, the stable case of the conjectures will be a consequence of theorem 5.6. This result will be proved by means of Shimomura's cell decomposition of unipotent varieties.

A tabloid $\mathbf{t}$ of shape $\nu = (\nu_1, \ldots, \nu_k)$ can be identified with a k-tuple $(w_1, \ldots, w_k)$ of words, w_i being a row tableau of lenght ν_i. The Stanton-White correspondence ψ assciates to such a k-tuple of tableaux a k-ribbon tableau $T = \psi(\mathbf{t})$. Thus, the cells of a unipotent variety $\mathcal{F}^u_\mu$ (where u is of type ν) are labelled by k-ribbon tableaux of a special kind. The following theorem, which implies the stable case of the conjectures, shows that this labelling is natural from a geometrical point of view.

Theorem 7.1 *The Stanton-White correspondence ψ sends a tabloid $\mathbf{t} \in L(\nu, \mu)$ onto a ribbon tableau $T = \psi(\mathbf{t})$ whose cospin is equal to the dimension of the cell $c_{\mathbf{t}}$ of $\mathcal{F}^u_\mu$ labelled by $\mathbf{t}$, when one uses the modified indexation for which the dimension of $c_{\mathbf{t}}$ is $e(\mathbf{t})$ (see Section 2). That is,*

$$\tilde{s}(\psi(\mathbf{t})) = e(\mathbf{t}) \ .$$

At this point, it is useful to observe, following [**Te**], that the e-statistic can be given a nonrecursive definition, as a kind of inversion number. Let $\mathbf{t} = (w_1, \ldots, w_k)$ be a tabloid, identified with a k-tuple of row tableaux. Let y be the r-th letter of w_i and x be the r-th letter of w_j, and suppose that $x < y$. Then, the pair (y, x) is said to be an e-inversion if either

(a) $i < j$

or

(b) $i > j$ and there s on the right of x in w_j a letter $u < y$

Then $e(\mathbf{t})$ is equal to the number of inversions (y, x) in $\mathbf{t}$.

Example 7.2 Let $\mathbf{t} \in L((2, 3, 2, 1), (2, 3, 1, 1, 1))$ be the following tabloid (the number under a letter y is the number of e-inversions of the form (y, x)):

$$\mathbf{t} \quad = \quad \left(\begin{array}{cc} \boxed{2}\!\boxed{3} \\ 1 \;\; 1 \end{array} \; , \; \begin{array}{c} \boxed{1}\!\boxed{1}\!\boxed{2} \\ 0 \;\; 0 \;\; 0 \end{array} \; , \; \begin{array}{c} \boxed{4}\!\boxed{5} \\ 3 \;\; 1 \end{array} \; , \; \begin{array}{c} \boxed{2} \\ 1 \end{array} \right)$$

so that $e(\mathbf{t}) = 7$. Its image under the SW-correspondence is the 4-ribbon tableau

$$
\begin{array}{ccccccccc}
 & & & 4 & & & & & \\
 & & 2 & & & & 5 & & \\
 & 2 & & & & 3 & & & \\
1 & & & & 1 & & & & 2
\end{array}
$$

whose cospin is equal to 7.

References

[**BV**] D. Barbasch and D. Vogan, *Primitive ideals and orbital integrals in complex classical groups*, Math. Ann. 259, (1982) 153-199

[**CL**] C. Carré and B. Leclerc, *Splitting the square of a Schur function into its symmetric and antisymmetric parts*, Institut Gaspard Monge, preprint, 1993 (*to appear* in J. Alg. Comb.).

[**Co**] E. Cohen, *A class of arithmetical functions*, Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 939-944.

[**De**] J. Désarménien, *Etude modulo n des statistiques mahonniennes*, Actes du 22$^{\text{e}}$ séminaire Lotharingien de Combinatoire, IRMA, Strasbourg (1990), 27-35.

[**FS**] S. Fomin and D. Stanton, *Rim hook lattices*, Mittag-Leffler institute, preprint No 23, 1991/92.

[**Fo**] H.O. Foulkes, *Characters of symmetric groups induced by characters of cyclic subgroups*, in *Combinatorics* (Proc. Conf. Comb. Math. Inst. Oxford 1972), Inst. Math. Appl., Southend-on-Sea, 1972, 141-154.

[**Ga**] D. Garfinkle, *On the classification of primitive ideals for complex classical Lie algebras, I*, Compositio Mathematica, 75 (1990) 2, 135-169

[**Gr**] J.A. Green, *The characters of the finite general linear groups*, Trans. Amer. Math. Soc. **80** (1955), 402-447.

[**HSh**] R. Hotta and N. Shimomura, *The fixed point subvarieties of unipotent transformations on generalized flag varieties and the Green functions*, Math. Ann. **241** (1979), 193-208.

[**HS**] R. Hotta and T.A. Springer, *A specialization theorem for certain Weyl groups*, Invent. Math **41** (1977), 113-127.

[**JK**] G. D. James and A. Kerber, *The representation theory of the symmetric group*, Addison-Wesley, 1981.

[**KSW**] A. Kerber, F. Sänger and B. Wagner, *Quotienten und Kerne von Young-Diagrammen, Brettspiele und Plethysmen gewöhnlicher irreduzibler Darstellungen symmetrischer Gruppen*, Mitt. Math. Sem. Giessen, **149** (1981), 131-175.

[**KKL**] A. Kerber, A. Kohnert and A. Lascoux, *SYMMETRICA, an object oriented computer-algebra system for the symmetric group*, J. Symbolic Computation 14 (1992), 195-203

[**Ki1**] A. N. Kirillov, *On the Kostka-Green-Foulkes polynomials and Clebsch-Gordan numbers*, J. of Geometry and Physics, **5**:3 (1988), 365-389.

[**KLLT**] A. N. Kirillov, A. Lascoux, B. Leclerc and J.Y. Thibon, *Séries génératrices pour les tableaux de dominos*, Preprint.

[**KLLT2**] A.N. Kirillov, A. Lascoux, B. Leclerc and J.-Y. Thibon, *Symmetric functions, configurations, and the combinatorics of domino tableaux*, in preparation.

[KR] A. N. Kirillov and N. Yu. Reshetikhin, *Bethe ansatz and the combinatorics of Young tableaux*, J. Sov. Math., **41** (1988), 925-955.

[Kn] D.E. Knuth, *Permutations, matrices and generalized Young tableaux*, Pacific. J. Math. **34** (1970), 709-727.

[KW] W. Kraskiewicz and J. Weyman, *Algebra of invariants and the action of a Coxeter element*, Preprint Math. Inst. Univ. Copernic, Toruń, Poland.

[La] A. Lascoux, *Cyclic permutations on words, tableaux and harmonic polynomials*, Proc. of the Hyderabad conference on algebraic groups, 1989, Manoj Prakashan, Madras (1991), 323-347.

[LLT1] A. Lascoux, B. Leclerc and J.Y. Thibon, *Fonctions de Hall-Littlewood et polynômes de Kostka-Foulkes aux racines de l'unité*, C.R. Acad. Sci. Paris, **316** (1993), 1-6.

[LLT2] A. Lascoux, B. Leclerc and J.Y. Thibon, *Green polynomials and Hall-Littlewood functions at roots of unity*, Europ. J. Combinatorics **15** (1994), 173-180.

[LLT4] A. Lascoux, B. Leclerc and J.Y. Thibon, *Polynômes de Kostka-Foulkes et graphes cristallins des groupes quantiques de type A_n*, Preprint (1994).

[LS1] A. Lascoux and M. P. Schützenberger, *Le monoïde plaxique*, in "Noncommutative structures in algebra and geometric combinatorics" (A. de Luca Ed.), Quaderni della Ricerca Scientifica del C. N. R., Roma, 1981.

[LS2] A. Lascoux and M. P. Schützenberger, *Sur une conjecture de H.O. Foulkes*, C.R. Acad. Sci. Paris **286A** (1978), 323-324.

[Li1] D. E. Littlewood, *On certain symmetric functions*, Proc. London Math. Soc. **43** (1961), 485-498.

[Li2] D. E. Littlewood, *The theory of group characters and matrix representations of groups*, Oxford, 1950 (second edition).

[Li3] D. E. Littlewood, *Modular representations of symmetric groups*, Proc. Roy. Soc. A. 209 (1951) 333-353

[Lu1] G. Lusztig, *Singularities, character formulas, and a q-analog of weight multiplicities*, Analyse et topologie sur les espaces singuliers (II-III), Astérisque **101-102** (1983), 208-227.

[Lu2] G. Lusztig, *Green polynomials and singularities of unipotent classes*, Adv. in Math. **42** (1981), 169-178.

[Mcd] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford, 1979.

[Mo] A.O. Morris, *On an algebra of symmetric functions*, Quart. J. Math. Oxford Ser. (2) **16** (1965), 53-64.

[MS] A. O. Morris and N. Sultana, *Hall-Littlewood polynomials at roots of* 1 *and modular representations of the symmetric group*, Math. Proc. Cambridge Phil. Soc., **110** (1991), 443-453.

[Mu] T. Muir, *A treatise on the theory of determinants*, Macmillan, London, 1882.

[NV] C.A. Nicol and H.S. Vandiver, *A Von Sterneck arithmetical function and restricted partitions with respect to a modulus*, Math. Proc. Cambridge Phil. Soc **110** (1991), 443-453.

[Ro] G. de B. Robinson, Representation theory of the symmetric group, Edimburgh 1961

[Sc] M.P. Schützenberger, *Propriétés nouvelles des tableaux de Young*, Séminaire Delange-Pisot-Poitou, 19ème année, **26**, 1977/78.

[Sh1] N. Shimomura, *A theorem of the fixed point set of a unipotent transformation of the flag manifold*, J. Math. Soc. Japan **32** (1980), 55-64.

[SW] D. Stanton and D. White, *A Schensted algorithm for rim-hook tableaux*, J. Comb. Theory A **40**, (1985), 211-247.

[Te] I. Terada, *A generalization of the length-Maj symmetry and the variety of N-stable flags*, Preprint, 1993.