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AN EXTREME POINT THEOREM FOR ORDERED POLYMATROIDS ON CHAIN ORDERS

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ABSTRACT. We consider *Ordered Polymatroids* as a generalization of polymatroids and extend the extreme point characterization of polymatroids by the greedy algorithm to the ordered case.

It is proved that a feasible point of an Ordered Polymatroid is a vertex iff it is a Greedy-Vector with respect to an appropriate primal Greedy-Procedure.

1. Introduction and Notations

In [2] Faigle and Kern considered Submodular Linear Programs of the type

$$\text{(1)} \qquad max \quad \sum_{e \in E} c_e x_e$$
$$\sum_{e \in A^+} x_e \quad \leq \quad f(A) \qquad \text{for all } A \in \mathcal{A}$$

where $P = (E, \leq)$ is a finite partially ordered set with groundset E and $|E| = n$, $c: \quad E \longrightarrow \mathbb{R}$ an objective function and f a submodular function with respect to a distributive lattice of ideals in P called $\mathcal{A}$. An ideal A of P is a subset of the groundset which satisfies the property

$$p_i \in A \text{ and } p_j < p_i \quad \Longrightarrow \quad p_j \in A.$$

Relative to the order P we associate with any element $p_i \in E$ the ideal generated by p_i

$$I(p_i) := \{p_j \in E: \ p_j \leq p_i\}.$$

Submodularity of f refers to the property

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \text{ for all } A, B \in \mathcal{A}$$

as usually. The set of maximal elements of an ideal A is denoted by A^+. This notation is used for

$$x(A^+) := \sum_{e \in A^+} x_e.$$

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Later we easily write $x^+(A)$ instead of $x(A^+)$.

We keep restricted to the special case $\mathcal{A} = 2^{(E,\leq)}$ throughout this paper, i.e. $\mathcal{A}$ consists of all ideals with respect to $(E, \leq)$. Then the polyhedron

$$\mathbb{P}(f) = \{x \in \mathbb{R}^n : \quad x^+(A) \leq f(A) \text{ for all } A \in 2^{(E,\leq)}\}$$

is called "Extended Ordered Polymatroid".

Notice that $\mathbb{P}(f)$ describes the set of all feasible vectors of (1).

The attribute "extended" will be left if all feasible x are required to be non-negative, i.e.

$$\mathbb{P}_+(f) := \mathbb{P}(f) \cap \mathbb{R}^n_+$$

defines an "Ordered Polymatroid" with rank function f. The name "Ordered Polymatroid" is reasonable because ordinary polymatroids can easily be recognized as a subclass if P is assumed to be the trivial order. The reader is referred to Chapter 10 of Grötschel, Lovász, Schrijver [4, pp. 305-329] for an introduction into Polymatroid-Theory and Submodular Functions.

Faigle and Kern developed a primal-dual greedy algorithm for the optimal solution of (1) (see [2]). Furthermore they proved total dual-integrality for pairs of Ordered Polymatroids on rooted forests, see [1]. The characterization of the extreme points of an Extended Ordered Polymatroid $\mathbb{P}(f)$ by primal Greedy-Vectors already follows from Faigle, Kern's greedy algorithm. Our proof method also works for the polytope $\mathbb{P}_+(f)$. Notice that the face lattice of the Ordered Polymatroid $\mathbb{P}_+(f)$ especially includes the faces

$$\mathbb{P}_+(f) \cap \{x_e = 0\}$$

for $e \in E$.

We mainly put our attention to chain orders as a special case of rooted forests here. The order $P = (E, \leq)$ is a chain order if it consists of totally ordered disjoint subsets. The proof of the extreme point characterization can easily be generalized to rooted forests afterwards. The order $P = (E, \leq)$ is a rooted forest if each element has at most one upper neighbour (see [2, p. 202]).

2. Feasibility of Greedy-Vectors

Let us introduce Greedy-Vectors as the main object of our interest. Therefore let $\pi = (p_1, p_2, \ldots, p_k)$ be a linear extension of the induced suborder $P' = (E', \leq)$ with respect to a subset $E' \subseteq E$. The cardinality of E' is denoted by k, i.e. $|E'| = k \leq n$. Then the result of the procedure

PRIMAL GREEDY :

$$\begin{aligned} x_{p_1} &:= f(p_1) \\ x_{p_2} &:= f(\{p_1, p_2\}) - x^+(p_1 \setminus I(p_2)) \\ &\vdots \\ x_{p_k} &:= f(\{p_1, p_2, \dots, p_k\}) - x^+(\{p_1, p_2, \dots, p_{k-1}\} \setminus I(p_k)) \\ x_{p_j} &:= 0 \quad \text{for all} \quad p_j \in E \setminus E' \end{aligned}$$

is called Greedy-Vector with respect to $\mathbb{P}_+(f)$.

We recall a basic result of Faigle, Kern [2, Theorem 4.1, Seite 202] concerning feasibility of Greedy-Vectors and adapt the proof to the special case of chain orders.

Lemma 1. *The procedure PRIMAL GREEDY yields a feasible vector x with respect to $\mathbb{P}(f)$ for the choice $E' = E$.*

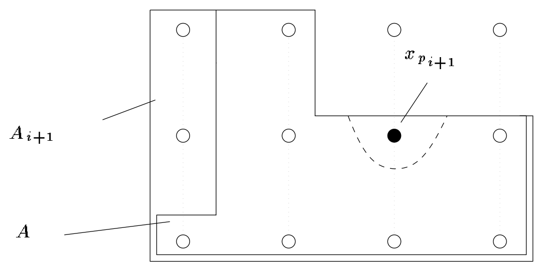

FIGURE 1. Illustration for the Proof.

Proof. We argue by induction on $|E|$ and assume that

$$\sum_{e \in A^+} x_e \leq f(A) \text{ for all } A \in 2^{(E,\leq)} \text{ and } A \subseteq A_i,$$

where

$$A_i := \{p_1, \dots, p_i\}.$$

Now, let A be an ideal with $A \subseteq A_{i+1}$ and $p_{i+1} \in A^+$. Furthermore, the set $A \setminus \{p_{i+1}\} = A \cap A_i$ is denoted by B.

We get

$$\begin{aligned} f(A) &\geq f(A \cap A_i) + f(A \cup A_i) - f(A_i) \\ &\geq f(B) + f(A_{i+1}) - f(A_i) \end{aligned}$$

by submodularity of f and

$$f(A_{i+1}) - f(A_i) = x_{p_{i+1}} - x_{p_{i+1}^-},$$

where the index $\bar{p_i}$ corresponds to the lower neighbour of p_i, if it exists and $x_{\bar{p_i}} = 0$ otherwise, by the procedure PRIMAL GREEDY.

Finally, we get

$$f(A) \geq x^+(B) + x_{p_{i+1}} - x_{p_{\bar{i+1}}} = x^+(A)$$

by $f(B) \geq x^+(B)$ which holds by assumption. $\diamondsuit$

Remark. The Greedy-Vector is feasible in $\mathbb{P}_+(f)$ for arbitrary $E' \subseteq E$ if f satisfies monotonicity, i.e.

$$A^+ \subseteq B^+ \implies f(A) \leq f(B) \text{ for } A, B \in 2^{(E,\leq)}.$$

Then

$$\begin{aligned} x_{p_i} &= f(\{p_1, p_2, \ldots, p_i\}) - x^+(\{p_1, p_2, \ldots, p_{i-1}\} \setminus I(p_i)) \\ &\geq f(\{p_1, p_2, \ldots, p_i\}) - f(\{p_1, p_2, \ldots, p_{i-1}\} \setminus I(p_i)) \\ &\geq 0 \end{aligned}$$

holds for the i-th component of the Greedy-Vector ($i = 1, \ldots, k$). $\diamondsuit$

3. Main Result

In order to state and prove the main result further notation is necessary to be introduced. From now on the symbols A_{iJ} and A_{Ij} are used to denote the i-th line and the j-th column of the matrix A. Furthermore the vector $v_{tr(i)}$ denotes the truncation of the vector v, where all components v_j of v with index $j > i$ are cut off. We have to interpret this as follows: For $1 \leq j \leq i \leq k$ we write $(a_j, \ldots, a_k)|_{tr(i)} = (a_j, \ldots, a_i)$. Now we are ready for

Theorem 1. *A feasible vector of an Ordered Polymatroid $\mathbb{P}_+(f)$ defined by a monotone rank function f is a vertex if and only if it is a Greedy-Vector for a certain linear extension π of a suitable suborder $P' = (E', \leq)$.*

Proof. Only the case $E' = E$ is outlined here. For $E' \subset E$ see the remark in the end of the proof.

It is a well-known fact from Linear-Programming-Theory that there exists a linear function $c^T x$ for each vertex x^* of a polyhedron $\mathbb{P}$ such that the maximal value of $c^T x$ subject to $x \in \mathbb{P}$ is achieved by x^*.

With any linear extension $\pi = (p_1, p_2, \ldots, p_n)$ of $P = (E, \preceq)$ we associate reduced weight coefficients $\hat{c_{p_1}}, \hat{c_{p_2}}, \ldots, \hat{c_{p_n}}$. The reduced weight coefficients are defined by the recursion

$$\begin{aligned} \hat{c_{p_n}} &:= c_{p_n} \\ \hat{c_{p_i}} &:= c_{p_i} - \sum_{i^*(i) > j > i} \hat{c_{p_j}} \quad \text{for} \quad i = n-1, \ldots, 2, 1 \end{aligned} \tag{2}$$

where

$$i^*(i) = \begin{cases} n+1 & \text{if } \; p_i \; \text{ is maximal in } \; E \\ \text{index of the upper neighbour of } \; p_i & \text{otherwise.} \end{cases}$$

Now the linear extension π is chosen such that

$$\begin{aligned} \hat{c_{p_n}} &= min\{c_{p'_n} : \; p'_n \in E^+\} \\ \hat{c_{p_i}} &= min\{c_{p'_i} - \sum_{i^*(i')>j>i'} \hat{c_{p_j}} : \; p'_i \in (E \setminus \{p_{i+1}, \dots, p_n\})^+\} \end{aligned}$$

for $i = n-1, \dots, 2, 1$. Using the reduced weight coefficients we set

$$y_A = \begin{cases} \hat{c_{p_i}} \text{ if } A = \{p_1, p_2, \dots, p_i\} \text{ for some } i \in \{1, \dots, n\} \\ 0 \quad \text{otherwise.} \end{cases}$$

and obtain a feasible solution y of the dual problem of (1) which is given by

$$\begin{aligned} \sum_{A \in 2^{(E, \leq)}} f(A) y_A &\longrightarrow min \\ \sum_{A^+ \ni p_j} y_A &\geq c_j \\ y_A &\geq 0 \,. \end{aligned} \tag{3}$$

Notice that the indices of the components of y are ideals of the poset $P = (E, \preceq)$.

For simplicity we write $\hat{f}(p_i)$ instead of $f(\{p_1, p_2, \dots, p_i\})$. We show

$$\sum_{i=1}^{n} \hat{f}(p_i) \left[c_{p_i} - \sum_{i^*(i)>j>i} \hat{c_{p_j}} \right] = \sum_{i=1}^{n} c_{p_i} \left[\hat{f}(p_i) - x^+(\{p_1 \dots p_{i-1}\} \setminus I(p_i)) \right], \tag{4}$$

which is equivalent to

$$\sum_{i=1}^{n} \hat{f}(p_i) \sum_{i^*(i)>j>i} \hat{c_{p_j}} = \sum_{i=1}^{n} c_{p_i} \cdot x^+(\{p_1, \dots, p_{i-1}\} \setminus I(p_i)). \tag{5}$$

Assume (4) holds, i.e. x and y have the same values with respect to the objective functions of (1) and it's dual. Thus x and y are optimal solutions of (1) and (3), respectively. Especially, we have $x = x^*$. Since x^* is arbitrary we know that each vertex of $\mathbb{P}(f)$ is a Greedy-Vector.

In order to prove (4) the vectors $\jmath(p_i) \in \mathbb{R}^{n-i}$ and $\wp(p_i) \in \mathbb{R}^i$ are introduced for each index $p_i \in \{p_1, p_2, \dots, p_{n-1}\}$. Both vectors $\jmath(p_i)$ and $\wp(p_i)$ only consist of 0 and 1-components. The coordinates of $\jmath(p_i)$ correspond to the elements $p_{i+1}, p_{i+2}, \ldots, p_n$ of the groundset E and the coordinates of $\wp(p_i)$ to the elements $p_1, p_2, \ldots, p_i$. The vector $\jmath(p_i)$ is defined as

$$\jmath(p_i) = (1, 1, \ldots, 1, 0, \ldots, 0),$$

such that the first 0 corresponds to the index $i^*(i)$. The vector $\wp(p_i)$ is defined by

$$\wp(p_i)_j = 1 \quad \Longleftrightarrow \quad p_j \text{ is maximal in the ideal } \{p_1, p_2, \ldots, p_{i+1}\}.$$

Similarly let $(p_j, \ldots, p_{i-1})^+$ be the 1-0-vector such that the 1-entries correspond to those elements of the set $\{p_j, \ldots, p_{i-1}\}$ which are maximal in the ideal $\{p_1, \ldots, p_{i-1}, p_i\}$.

Using this notation (4) equivalently can be written as

$$(6)\quad (\hat{f}(p_1), \hat{f}(p_2), \ldots, \hat{f}(p_n))M_1 \begin{pmatrix} c_{p_1} \\ c_{p_2} \\ \vdots \\ c_{p_n} \end{pmatrix} = (c_{p_1}, c_{p_2}, \ldots, c_{p_n})\hat{M}_n \begin{pmatrix} \hat{f}(p_1) \\ \hat{f}(p_2) \\ \vdots \\ \hat{f}(p_n) \end{pmatrix}$$

where the matrices $\hat{M}_n$ and M_1 are defined by the following recursions:

$$\begin{aligned} &\hat{M}_1 = 1; \\ &\vdots \\ &\hat{M}_{i+1} = \begin{pmatrix} & 0 \\ \hat{M}_i & \vdots \\ & 0 \\ -\wp(p_i)\cdot \hat{M}_i & 1 \end{pmatrix} \\ &i = 1, \ldots, n-1 \ . \end{aligned} \qquad \begin{aligned} &M_n = 1; \\ &\vdots \\ &M_j = \begin{pmatrix} 1 & -\jmath(p_j)\cdot M_{j+1} \\ 0 & \\ \vdots & M_{j+1} \\ 0 & \end{pmatrix} \\ &j = n-1, \ldots, 1 \end{aligned}$$

Obviously, equation (6) holds if $\hat{M}_n = {M_1}^T$, which is shown by induction now. We proceed on the subdiagonals of $\hat{M}_n$ and ${M_1}^T$.

First notice that both matrices are lower triangle matrices which consist of 1-entries in their diagonals. Considering the elements $\hat{m}_{ij} = (\hat{M}_n)_{ij}$ and $m_{ij} = ({M_1}^T)_{ij}$ we assume that the first $i-j-1$ subdiagonals of $\hat{M}_n$ and ${M_1}^T$ are equal to each other. Then we get

$$\begin{aligned} \hat{m}_{ij} &= -\wp(p_{i-1})\cdot(\hat{M}_{i-1})_{I,j} \\ &= -(p_j, \ldots, p_{i-1})^+ \cdot \begin{pmatrix} \hat{m}_{jj} \\ \hat{m}_{j+1,j} \\ \vdots \\ \hat{m}_{i-1,j} \end{pmatrix} = -(p_j, \ldots, p_{i-1})^+ \cdot \begin{pmatrix} m_{jj} \\ m_{j+1,j} \\ \vdots \\ m_{i-1,j} \end{pmatrix} \\ &\qquad \text{(use induction hypothesis)} \end{aligned}$$

$$
\begin{aligned}
&= -(p_j, p_{j+1}, \dots, p_{i-1})^+ \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\
&\quad +(p_{j+1}, p_{j+2}, \dots, p_{i-1})^+ \cdot \\
&\qquad \begin{pmatrix} 1 & & & & \\ m_{j+2,j+1} & 1 & \dots & 0 & \\ & \vdots & & & \\ m_{i-1,j+1} & m_{i-1,j+2} & & & 1 \end{pmatrix} \cdot \jmath(p_j)|_{tr(i-1)} \\
&= (p_j, p_{j+1}, p_{j+2}, \dots, p_{i-1})^+ \cdot \\
&\qquad \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ m_{j+2,j+1} & 1 & \dots & 0 & 0 \\ & \vdots & & & \\ m_{i-1,j+1} & m_{i-1,j+2} & \dots & 1 & 0 \end{pmatrix} \cdot \jmath(p_j)|_{tr(i)} \\
&\quad -(0, 0, \dots, 0, 1) \cdot \jmath(p_j)|_{tr(i)} \\
&= -(m_{i,j+1}, m_{i,j+2}, \dots, m_{ii}) \cdot \jmath(p_j)|_{tr(i)} \\
&= -(M^T_{j+1})_{iJ} \cdot \jmath(p_j) \\
&= m_{ij}
\end{aligned}
$$

by the recursive definitions of $\hat{M}_n$ and ${M_1}^T$.

For $E' \subset E$ just start the recursion (2) with $\hat{c_{p_k}} = c_{p_k}$ and set $c_{p_j} = 0$ for $p_j \in E \setminus E'$. $\diamondsuit$

Theorem 1 and its proof were motivated by the Greedy-Algorithm-Characterization of ordinary polymatroids due to Edmonds (for example see Satz 1.3.1 Girlich/Kowaljow [3, Seite 39] or Schrijver [5, pp. 27-28, Theorem 2]).

The paper is finished with an important consequence of Theorem 1.

Corollary 1. *All vertices of an Ordered Polymatroid $\mathbb{P}_+(f)$ with respect to a monotone rank function f are integral if the rank funktion f is integer-valued.*

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