THE TRIPLE, QUINTUPLE AND SEPTUPLE PRODUCT IDENTITIES REVISITED

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Dedicated to George Andrews on the occasion of his sixtieth birthday

ABSTRACT. This paper takes up again the study of the Jacobi triple and Watson quintuple identities that have been derived combinatorially in several manners in the classical literature. It also contains a proof of the recent Farkas-Kra septuple product identity that makes use only of "manipulatorics" methods.

1. Introduction

In the classical literature the Jacobi triple product appears in one of the following two forms

$$\prod_{n=1}^{\infty}(1-x^{-1}q^{n-1})(1-xq^{n})=\prod_{i=1}^{\infty}\frac{1}{(1-q^{i})}\sum_{k=-\infty}^{+\infty}(-1)^{k}\,x^{k}\,q^{k(k+1)/2}, \tag{1.1}$$

$$\prod_{n=1}^{\infty}(1-x^{-1}\,q^{2n-1})(1-x\,q^{2n-1})=\prod_{i=1}^{\infty}\frac{1}{(1-q^{2i})}\sum_{k=-\infty}^{+\infty}(-1)^{k}\,x^{k}q^{k^{2}}, \tag{1.2}$$

while the Watson quintuple product reads

$$\prod_{n=1}^{\infty}(1-x^{-1}q^{n-1})(1-xq^{n})(1-x^{-2}\,q^{2n-1})(1-x^{2}\,q^{2n-1}) \tag{1.3}$$
$$=\prod_{i=1}^{\infty}\frac{1}{(1-q^{i})}\sum_{k=-\infty}^{+\infty}q^{(3k^{2}+k)/2}(x^{3k}-x^{-3k-1}).$$

The letters x and q may be regarded as complex variables with $|q|<1$ and $x\neq 0$ or as simple indeterminates. In the latter case consider the ring $\Omega[x,x^{-1}]$ of the polynomials in the variables x and x^{-1} such that $xx^{-1}=1$ with coefficients in a ring Ω. Then the identities hold in the algebra of formal power series in the variable q with coefficients in $\Omega[x,x^{-1}]$.

As usual, let $(a;q)_n$ denote the q-ascending factorial

$$(a;q)_n=\begin{cases}1, & \text{if } n=0;\\ (1-a)(1-aq)\ldots(1-aq^{n-1}), & \text{if } n\geq 1;\end{cases}$$

$$(a;q)_\infty = \prod_{n\ge 0}(1-aq^n);$$

and let the classical q-binomial coefficient be denoted by:

$$\begin{bmatrix} n\\ k\end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k} \qquad (0\le k\le n).$$

The identities (1.1) and (1.2) have two finite versions given by

$$(x^{-1};q)_n\,(xq;q)_m = \sum_{j=-n}^{m} \begin{bmatrix} n+m\\ j+n\end{bmatrix}_q (-x)^j\, q^{j(j+1)/2}; \tag{1.4}$$

$$(x^{-2};q^2)_n\,(x^2q;q^2)_m = \sum_{j=-n}^{m} \begin{bmatrix} n+m\\ j+n\end{bmatrix}_{q^2} (-x^2)^j\, q^{j^2}. \tag{1.5}$$

Those two versions with n and m not necessarily equal are apparently due to MacMahon ([Ma15], vol. 2, § 323). He proved (1.5) by using Sylvester's [Sy82] "quasi-geometrical method of demonstration" and notes that to obtain (1.4) the variable x is to be replaced by xq and then q^2 by q. With similar substitutions (1.5) can be derived from (1.4). As those substitutions are made within finite expressions the derivations are straightforward.

Finally, as kindly mentioned to us by Garvan [Ga99], Farkas and Kra [Fa99] derived a *septuple product identity* using the algebra of k-order theta functions. If f (resp. g) is a polynomial q (resp. in x) with integral coefficients, let

$$\begin{aligned}\Theta(f,g) &:= \sum_{n\in\mathbf{Z}} q^{f(n)}\, x^{g(n)};\\ \Omega(f) &:= \sum_{n\in\mathbf{Z}} (-1)^n\, q^{f(n)};\\ \Omega(f,g) &:= \sum_{n\in\mathbf{Z}} (-1)^n\, q^{f(n)}\, x^{g(n)}.\end{aligned}$$

Then Farkas and Kra [Fa99] imagined and proved the following identity

$$\begin{aligned}\prod_{n\ge 1}&(1-q^{2n})^2(1-xq^{2n-2})(1-x^{-1}q^{2n})(1-x^2q^{4n-2})\\ &\quad\times(1-x^{-2}q^{4n-2})(1-x^2q^{4n-4})(1-x^{-2}q^{4n})\\ &= \Omega(5n^2+n)\big(\Omega(5n^2+3n,5n+3)+\Omega(5n^2-3n,5n)\big)\\ &\quad - \Omega(5n^2+3n)\big(\Omega(5n^2+n,5n+2)+\Omega(5n^2-n,5n+1)\big).\end{aligned}\tag{1.6}$$

Notice that with the substitutions $x\leftarrow x^{-2}$ and $q\leftarrow q^4$ the triple product identity (1.1) reads

$$\prod_{n\ge 1}(1-x^2q^{4n-4})(1-x^{-2}q^{4n})(1-q^{4n}) = \Omega(2n^2+2n,-2n), \tag{1.7}$$

while the quintuple product identity (1.3) with the substitutions $q \leftarrow q^2$ and $x \leftarrow x^{-1}$ takes the form:

$$\prod_{n\ge1}(1-xq^{2n-2})(1-x^{-1}q^{2n})(1-x^2q^{4n-2})(1-x^{-2}q^{4n-2})(1-q^{2n}) \\ = \Theta(3n^2+n,-3n)-\Theta(3n^2+n,3n+1). \tag{1.8}$$

At the origin our intention was to give a combinatorial proof of the quintuple product identity (1.3). A glance at the left-hand sides of identities (1.1), (1.2), (1.3) shows that (1.3) must be a consequence of (1.1) and (1.2) and the combinatorics involved, once the products on the right-hand sides of the first two identities are properly handled. This program was only partially fulfilled, because (1.3) is an easy consequence of both triple product identities and changing the "manipulatorics" needed into some combinatorial construction would have been a useless task. As will be seen in section 3, besides the two triple product identities, we only need the *Euler pentagonal number formula* (see, e.g., [An76] p. 11), another special case of those two identities, and a simple summation manipulation.

There remains to imagine the adequate bijections to prove (1.1) and (1.2). How can we dare construct such bijections, some 117 years after Sylvester [Sy82]? He already derived three different combinatorial proofs, scholarly commented by Joichi and Stanton [Jo89]. We have to admit, indeed, that any kind of new combinatorial construction for proving (1.1) and (1.2) can only be a slight variation of Sylvester's method [Sy82]. He had been the source of a long tradition of combinatorial construction makers. Even our "rectangle-moving" method that we were proud to discover did not escape his filiation. We have then decided to leave our combinatorial construction on our own home pages [Fo99] and, in the present paper, only provide with straightforward proofs for all the identities above, i.e. (1.1)—(1.6).

The first combinatorial proofs go back to Sylvester [Sy82] and have been the sources of inspiration of several subsequent ones, by Wright [Wr65], Sudler [Su66], Ewell [Ew81], Lewis [Le84], Garvan [Ga86] (see § 3.2 in his Ph.D. thesis, as it was mentioned to us by an anonymous referee). Joichi and Stanton [Jo89] discuss the various merits of those proofs. They are mostly interested in building natural involutions for proving partition identities; they also compare the approaches due to Zolnowsky [79] and Cheema [Ch64].

The other proofs are of formal nature, as in MacMahon ([Ma15], vol. 2, § 327), Bressoud [Br97] or of analytical nature, as in Andrews [An65], [An74], [An84], or in the classical treatises by Hardy and Wright [Ha38], Andrews [An76], Gupta [Gu87]. A fairly complete bibliography can be found in Gasper and Rahman [Ga90].

The quintuple product identity is originally due to Watson [Wa29]. Other proofs were given by Gordon [Go61], Carlitz and Subbarao [Ca72], Subbarao and Vidyasagar [Su70]. Hirschhorn [Hi88] proposes a generalisation of that identity and stated that there are "no fewer than twelve proofs of the quintuple product identity," in particular by Bailey [Ba51], Sears [Se52], Atkin and Swinnerton-Dyer [At54], Andrews [An74] and more recently by Alladi [Al96].

Finally, those identities are found in classical topics in Number Theory or Lie Algebra, as in Adiga, Berndt, Bhargava and Watson [Ad85], Gustafson [Gu87], Kac [Ka78], [Ka85], Lepowsky and Milne [Le78], Macdonald [Ma82], Menon [Me65], Milne [Mi85].

The paper is organized as follows. In the next section MacMahon's finite versions (1.4) and (1.5) are derived and it is shown how they imply (1.1) and (1.2). In section 3 we shall reprove (1.3) using an argument very close to the one used by Carlitz and Subbarao [Ca72]. In the final section we give our own proof of the new elected *septuple product identity* obtained by Farkas and Kra [Fa99]. We first make use of an extended Carlitz-Subbarao trick (that was sufficient for the quintuple case), then introduce two further specializations of both triple and quintuple product identities to complete the calculation. It seems that Farkas-Kra's identity is much deeper than its previous two sisters.

2. The finite and infinite versions of the triple product

As shown to us by Andrews [An98], and as it is well-known in the case $m = n$, identity (1.4) can be proved by means of the q-binomial identity in its finite form. Proceed as follows:

$$\begin{aligned}
(x^{-1};q)_n\,(xq;q)_m &= (-1)^n x^{-n} q^{n(n-1)/2} (xq^{1-n};q)_n\,(xq;q)_m \\
&= (-1)^n x^{-n} q^{n(n-1)/2} (xq^{1-n};q)_{n+m} \\
&= (-1)^n x^{-n} q^{n(n-1)/2} \sum_{j=0}^{n+m} \begin{bmatrix} n+m \\ j \end{bmatrix}_q (-xq^{1-n})^j q^{j(j-1)/2} \\
&= \sum_{j=0}^{n+m} \begin{bmatrix} n+m \\ j \end{bmatrix}_q (-x)^{j-n} q^{(j-n)(j-n+1)/2} \\
&= \sum_{j=-n}^{m} \begin{bmatrix} n+m \\ j+n \end{bmatrix}_q (-x)^j q^{j(j+1)/2}.
\end{aligned}$$

Now to deduce the "infinite" versions (1.1), (1.2) from the finite ones we only have to let n and m tend to infinity. Using (1.4) for $n = m$ the product $(x^{-1};q)_m\,(xq;q)_m\,(q;q)_\infty$ can be expressed as

$$\sum_{j=-m}^{m} (q^{m-j+1};q)_{m+j}\,(q^{m+j+1};q)_\infty\,(-x)^j\,q^{j(j+1)/2}.$$

In that sum the running term is equal to $(-x)^j\, q^{j(j+1)/2}(1-q^{m-|j|+1}a_j)$, with a_j a series in q, so that $(x^{-1};q)_m\,(xq;q)_m\,(q;q)_\infty = b_m + q^m c$, where b_m is the series $b_m = \sum_{j=-m}^{m}(-x)^j\, q^{j(j+1)/2}$ and c is a non-null series. Hence $(x^{-1};q)_\infty\,(xq;q)_\infty\,(q;q)_\infty = \lim_m b_m = \sum_{j=-\infty}^{\infty}(-x)^j\, q^{j(j+1)/2}$, which is simply (1.1).

Using the same method we can derive (1.5) that, in its turn, implies (1.2). The MacMahon finite versions (1.4) and (1.5) can be regarded as the "fundamental" triple product identities and, still, they are derived by means of the q-binomial identity in its finite form. Here we face one of the mysteries of mathematical tradition: explain why so many proofs of those identities can be found in the literature.

3. The quintuple product identity

To derive the quintuple product identity (1.3) it suffices to prove

$$\prod_{i=1}^{\infty}\frac{1}{1-q^i}\sum_{k\in\mathbb{Z}}(-1)^k x^k q^{k(k+1)/2} \times \prod_{i=1}^{\infty}\frac{1}{1-q^{2i}}\sum_{k\in\mathbb{Z}}(-1)^k x^{2k} q^{k^2} = \prod_{i=1}^{\infty}\frac{1}{1-q^i}\sum_{k\in\mathbb{Z}} q^{(3k^2+k)/2}(x^{3k}-x^{3k-1}),$$

or by using the Euler pentagonal number identity (see [An76], p. 11)

$$\prod_{i\geq 1}(1-q^i) = \sum_{k\in\mathbb{Z}}(-1)^k\, q^{(3k^2-k)/2},$$

to prove the identity

$$\sum_{k\in\mathbb{Z}}(-1)^k x^k q^{k(k+1)/2} \times \sum_{l\in\mathbb{Z}}(-1)^l x^{2l} q^{l^2} = \sum_{n\in\mathbb{Z}}(-1)^n\, q^{3n^2-n} \times \sum_{m\in\mathbb{Z}} q^{(3m^2+m)/2}(x^{3m}-x^{-3m-1}).$$

Write the product of the two series of the left-hand side as the sum of three series denoted by S_0, S_1, S_2:

$$\begin{aligned}\sum_{k,l}(-1)^{k+l}\, x^{k+2l} q^{k(k+1)/2+l^2} &= \sum_m x^{3m}\sum_{k+2l=3m}(-1)^{k+l}\, q^{k(k+1)/2+l^2}\\ &\quad+\sum_m x^{3m-1}\sum_{k+2l=3m-1}(-1)^{k+l}\, q^{k(k+1)/2+l^2}\\ &\quad+\sum_m x^{3m-2}\sum_{k+2l=3m+2}(-1)^{k+l}\, q^{k(k+1)/2+l^2}\\ &= S_0+S_1+S_2.\end{aligned}$$

For S_0 notice that $k+2l=3m$ and $l-m=n$ imply: $k+l=2m-n$ and $k(k+1)/2+l^2=(3m^2+m)/2+3n^2-n$. Hence

$$S_0=\sum_m q^{(3m^2+m)/2}\,x^{3m}\sum_n(-1)^n\,q^{3n^2-n}.$$

For S_1 the change of indices $k+2l=3m-1$ et $l-m=n$ imply: $k+l=2m-n-1$ and $k(k+1)/2+l^2=(3m^2-m)/2+3n^2+n$. Hence

$$\begin{aligned}S_1&=-\sum_m q^{(3m^2-m)/2}\,x^{3m-1}\sum_n(-1)^n\,q^{3n^2+n}\\&=-\sum_m q^{(3m^2+m)/2}\,x^{-3m-1}\sum_n(-1)^n\,q^{3n^2-n}.\end{aligned}$$

Finally, for S_2 make the change of indices $k+2l=3m-2$ and $l-m=n$, so that $k+l=2m-n-2$ et $k(k+1)/2+l^2=(3m^2-3m+2)/2+3n^2+3n$. Hence

$$S_2=\sum_m x^{3m-2}\,q^{(3m^2-3m+2)/2}\sum_n(-1)^n\,q^{3n^2+3n}.$$

But $\sum\limits_{n\in\mathbb{Z}}(-1)^n\,(q^3)^{n(n+1)}=0$, and $S_2=0$. The sum S_0+S_1 is exactly the right-hand side of the quintuple product identity (1.3). $\square$

4. The septuple product identity

Let E be the left-hand side of identity (1.6), i.e.,

$$\begin{aligned}E:=\prod_{n\ge1}&(1-q^{2n})^2(1-xq^{2n-2})(1-x^{-1}q^{2n})(1-x^2q^{4n-2})\\&\times(1-x^{-2}q^{4n-2})(1-x^2q^{4n-4})(1-x^{-2}q^{4n})\end{aligned}$$

Taking both identities (1.7) and (1.8) into account and using the identity

$$\prod_{m\ge1}(1-q^{4m})=\prod_{m\ge1}(1-q^{2m})(1+q^{2m}),$$

we may write:

$$\prod_{m\ge1}(1+q^{2m})\,E=\Omega(2n^2+2n,-2n)\big(\Theta(3n^2+n,-3n)-\Theta(3n^2+n,3n+1)\big).$$

We now use the method of the previous section. However this time each product of the right-hand side of the previous formula is transformed into a sum of *five* products of two Ω-series. We may write:

$$\Omega(2n^2+2n,-2n)\,\Theta(3n^2+n,-3n)=S_0+S_1+S_2+S_3+S_4,\tag{4.1}$$

where for each $k=0,1,2,3,4$ we let

$$S_k:=\sum_a z^{5a+k}\sum_{-2i-3j=5a+k}(-1)^i q^{2i^2+2i+3j^2+j}.$$

In the second summation of each expression S_k we make a change of variables indicated in Table 1 below. For instance, as shown in the first row of the table, when $-2i-3j=5a$ (first column), we let $a+j=-2n$ (second column) noting that $a+j$ is necessarily even. Hence $i\equiv a+n \pmod 2$ (third column). Finally, the exponent of q is transformed into $30n^2+4n+5a^2-3a$ (fourth column).

$-2i-3j$	$a+j$	$i\equiv \pmod 2$	$2i^2+2i+3j^2+j$
$5a$	$-2n$	$a+n$	$30n^2+4n+5a^2-3a$
$5a+1$	$-2n-1$	$a+n+1$	$30n^2+28n+6+5a^2-a$
$5a+2$	$2n$	$a+n+1$	$30n^2+8n+5a^2+a$
$5a+3$	$-2n-1$	$a+n$	$30n^2+16n+2+5a^2+3a$
$5a+4$	$2n$	$a+n$	$30n^2+20n+4+5a^2+5a$

Table 1

With those changes of variables we get

$$\begin{aligned}(4.2)\quad \Omega(2n^2+2n,-2n)\,\Theta(3n^2+n,-3n)&\\ =\Omega(5n^2-3n,5n)\,\Omega(30n^2+4n)&\\ -\Omega(5n^2-n,5n+1)\,\Omega(30n^2+28n+6)&\\ -\Omega(5n^2+n,5n+2)\,\Omega(30n^2+8n)&\\ +\Omega(5n^2+3n,5n+3)\,\Omega(30n^2+16n+2)&\\ +\Omega(5n^2+5n,5n+4)\,\Omega(30n^2+20n+4).&\end{aligned}$$

In the same manner, let

$$(4.3)\quad \Omega(2n^2+2n,-2n)\,\Theta(3n^2+n,3n+1)=T_0+T_1+T_2+T_3+T_4,$$

where for each $k=0,1,2,3,4$ we let

$$T_k:=\sum_a z^{5a+k}\sum_{-2i+3j+1=5a+k}(-1)^i q^{2i^2+2i+3j^2+j}.$$

Again, the changes of variables made in each T_k are indicated in Table 2.

$-2i-3j+1$	$a-j$	$i\equiv \pmod 2$	$2i^2+2i+3j^2+j$
$5a$	$2n+1$	$a+n+1$	$30n^2+16n+2+5a^2-3a$
$5a+1$	$-2n$	$a+n$	$30n^2+8n+5a^2-a$
$5a+2$	$2n+1$	$a+n$	$30n^2+28n+6+5a^2+a$
$5a+3$	$2n$	$a+n+1$	$30n^2+4n+5a^2+3a$
$5a+4$	$-2n-1$	$a+n$	$30n^2+20n+4+5a^2+5a$

Table 2

Those changes of variables yield:

$$
\begin{aligned}
(4.4)\quad \Omega(2n^2+2n,-2n)\,\Theta(3n^2+n,3n+1)
&= -\Omega(5n^2-3n,5n)\,\Omega(30n^2+16n+2)\\
&\quad + \Omega(5n^2-n,5n+1)\,\Omega(30n^2+8n)\\
&\quad + \Omega(5n^2+n,5n+2)\,\Omega(30n^2+28n+6)\\
&\quad - \Omega(5n^2+3n,5n+3)\,\Omega(30n^2+4n)\\
&\quad + \Omega(5n^2+5n,5n+4)\,\Omega(30n^2+20n+4).
\end{aligned}
$$

In particular, we notice that $S_4 = T_4$. When taking the difference (4.2) − (4.4) we simply get:

$$
\begin{aligned}
(4.5)\quad \prod_{m\ge1}(1+q^{2m})\,E &= \big(\Omega(5n^2+3n,5n+3)+\Omega(5n^2-3n,5n)\big)\\
&\qquad\times\big(\Omega(30n^2+4n)+\Omega(30n^2+16n+2)\big)\\
&\quad-\big(\Omega(5n^2+n,5n+2)+\Omega(5n^2-n,5n+1)\big)\\
&\qquad\times\big(\Omega(30n^2+28n+6)+\Omega(30n^2+8n)\big).
\end{aligned}
$$

Now if we compare the last identity with the septuple identity (1.6) we see that the latter is the consequence of the next Lemma.

Lemma. *We have:*

$$(4.6)\quad \prod_{m\ge1}(1+q^{2m})\,\Omega(5n^2+n)=\Omega(30n^2+16n+2)+\Omega(30n^2+4n);$$

$$(4.7)\quad \prod_{m\ge1}(1+q^{2m})\,\Omega(5n^2+3n)=\Omega(30n^2+28n+6)+\Omega(30n^2+8n).$$

The proof of the Lemma has very much the flavor of the proofs derived by our friend Mike Hirschhorn [Hi88] when he masterly plays with the triple product identity for various specializations. Here the quintuple identity also gets into the picture. In the sequel, the range of the index m (resp. n) is $\mathbb{N}\setminus\{0\}$ (resp. $\mathbb{Z}$).

Proof of the Lemma. The triple product (1.1) with $q := q^{10}$ and $x := q^{-4}$ reads

$$(4.8)\quad \prod_m(1-q^{10m})(1-q^{10m-4})(1-q^{10m-6})=\Omega(5n^2+n),$$

while with $q := q^{10}$ and $x := q^{-2}$ yields

$$(4.9)\quad \prod_m(1-q^{10m})(1-q^{10m-2})(1-q^{10m-8})=\Omega(5n^2+3n).$$

Next the quintuple product (1.3) with $q := q^{20}$ and $x := -q^{-2}$ reads

$$\prod_m (1-q^{20m})(1+q^{20m-18})(1+q^{20m-2})(1-q^{40m-16})(1-q^{40m-24}) \\ = \Omega(30n^2+16n+2)+\Omega(30n^2+4n), \tag{4.10}$$

that is, the right-hand side of (4.6). In the same manner the quintuple product with $q := q^{20}$ and $x := -q^{-6}$ takes the form

$$\prod_m (1-q^{20m})(1+q^{20m-14})(1+q^{20m-6})(1-q^{40m-8})(1-q^{40m-32}) \\ = \Omega(30n^2+28n+6)+\Omega(30n^2+8n), \tag{4.11}$$

which is the right-hand side of (4.7).

The two identities (4.6) and (4.7) may then rewritten as:

$$\prod_m (1+q^{2m}) \times (\text{l.-h. s. of (4.8)}) = \text{l.-h. s. of (4.10)}, \tag{4.6$'$}$$

$$\prod_m (1+q^{2m}) \times (\text{l.-h. s. of (4.9)}) = \text{l.-h. s. of (4.11)}. \tag{4.7$'$}$$

As

$$\prod_m (1+q^{2m}) = \prod_m (1+q^{10m})(1+q^{10m-2})(1+q^{10m-4})(1+q^{10m-6})(1+q^{10m-8}),$$

the left-hand side of (4.6$'$) is equal to

$$\prod_m (1-q^{20m})(1-q^{20m-8})(1-q^{20m-12})(1+q^{10m-2})(1+q^{10m-8}).$$

Now as

$$\prod_m (1+q^{10m-2})(1+q^{10m-8}) \\ = \prod_m (1+q^{20m-2})(1+q^{20m-12})(1+q^{20m-8})(1+q^{20m-18}),$$

we see that the left-hand side of (4.6$'$) is equal to

$$\begin{aligned}\prod_m &(1-q^{20m})(1-q^{20m-8})(1-q^{20m-12}) \\ &\times (1+q^{20m-2})(1+q^{20m-12})(1+q^{20m-8})(1+q^{20m-18}) \\ = \prod_m &(1-q^{20m})(1+q^{20m-2})(1+q^{20m-18}) \\ &\times (1-q^{20m-8})(1-q^{20m-12})(1+q^{20m-12})(1+q^{20m-8}) \\ = \prod_m &(1-q^{20m})(1+q^{20m-2})(1+q^{20m-18})(1-q^{40m-16})(1-q^{40m-24}),\end{aligned}$$

which is the left-hand side of (4.10). Hence (4.6) is proved.

In the same manner, the left-hand side of (4.7′) is equal to

$$\prod_m (1-q^{20m})(1-q^{20m-4})(1-q^{20m-16})(1+q^{10m-4})(1+q^{10m-6}).$$

As

$$\begin{aligned}\prod_m &(1+q^{10m-4})(1+q^{10m-6})\\ &= \prod_m (1+q^{20m-4})(1+q^{20m-6})(1+q^{20m-14})(1+q^{20m-16}),\end{aligned}$$

we also see that the left-hand side of (4.7′) is equal to

$$\begin{aligned}\prod_m &(1-q^{20m})(1-q^{20m-4})(1-q^{20m-16})\\ &\qquad\times (1+q^{20m-4})(1+q^{20m-6})(1+q^{20m-14})(1+q^{20m-16})\\ &= \prod_m (1-q^{20m})(1+q^{20m-6})(1+q^{20m-14})\\ &\qquad\times (1-q^{20m-4})(1-q^{20m-16})(1+q^{20m-4})(1+q^{20m-16})\\ &= \prod_m (1-q^{20m})(1+q^{20m-6})(1+q^{20m-14})(1-q^{40m-8})(1-q^{40m-32}),\end{aligned}$$

which is the left-hand side of (4.11). This achieves the proof of (4.7) and then of the Lemma. ∎

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