THREE CLASSICAL RESULTS ON REPRESENTATIONS OF A NUMBER

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Introduction

Three classical results concern the number of representations of the positive integer n in the form $x^2 + 3y^2$ with $x, y \in \mathbb{Z}$, the form $(x^2 + x)/2 + 3(y^2 + y)/2$ with $x, y \in \mathbb{Z}^+$ and the form $x^2 + xy + y^2$ with $x, y \in \mathbb{Z}$. Indeed, if $s(n)$, $t(n)$ and $u(n)$ respectively denote the three numbers, then

$$s(n) = 2\Big(d_{1,3}(n) - d_{2,3}(n)\Big) + 4\Big(d_{4,12}(n) - d_{8,12}(n)\Big), \tag{1}$$

$$t(n) = d_{1,3}(2n+1) - d_{2,3}(2n+1) \tag{2}$$

and

$$u(n) = 6\Big(d_{1,3}(n) - d_{2,3}(n)\Big). \tag{3}$$

where $d_{r,m}(n)$ is the number of divisors d of n with $d \equiv r \pmod{m}$.
(1) is equivalent to the q-series identity

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}} q^{m^2+3n^2} = 1 + 2\sum_{n\ge0}\left(\frac{q^{3n+1}}{1-q^{3n+1}} - \frac{q^{3n+2}}{1-q^{3n+2}}\right)\\ + 4\sum_{n\ge0}\left(\frac{q^{12n+4}}{1-q^{12n+4}} - \frac{q^{12n+8}}{1-q^{12n+8}}\right)\end{aligned} \tag{4}$$

or to

$$\sum_{m,n\in\mathbb{Z}} q^{m^2+3n^2} = 1 + 2\sum_{n\ge0}\left(\frac{q^{3n+1}}{1-(-1)^n q^{3n+1}} - \frac{q^{3n+2}}{1+(-1)^n q^{3n+2}}\right), \tag{5}$$

(2) is equivalent to the q-series identity

$$\sum_{m,n\in\mathbb{Z}^+} q^{(m^2+m)/2+3(n^2+n)/2} = \sum_{n\geq 0}\left(\frac{q^{3n}}{1-q^{6n+1}} - \frac{q^{3n+2}}{1-q^{6n+5}}\right) \tag{6}$$

and (3) is equivalent to the q-series identity

$$\sum_{m,n\in\mathbb{Z}} q^{m^2+mn+n^2} = 1 + 6\sum_{n\geq 0}\left(\frac{q^{3n+1}}{1-q^{3n+1}} - \frac{q^{3n+2}}{1-q^{3n+2}}\right). \tag{7}$$

Both (5) and (6) appear in Ramanujan's second Notebook [12, p.239], and Berndt [2, pp.223-224 and p.116] shows how they follow from Ramanujan's ${}_1\psi_1$ summation [12, p.196], [2, p.32]. (7) appears in Berndt and Rankin [5, p.196] and a proof is given by Berndt [3]. The reader is referred also to [6], [7], [9] and [4] for related developments and generalisations, and to [1] and [10] for applications in statistical mechanics.

It seems that Dirichlet (1840) may have known (1), since he gives [8, p.463] the corresponding results for the forms $x^2 + y^2$ and $x^2 + 2y^2$, and continues "And so on in similar fashion." ("Et ainsi de suite.")

However, Lorenz (1871) [11, p.420] states both (4) and (1), and in reference to (1) says (my translation) "From this equation one can deduce a theorem which must be considered new in the theory of numbers because it cannot immediately be deduced from known theorems:

If a number N contains prime factors $p_1, p_2, \cdots$ of the form $3m+1$ with exponents $a_1, a_2, \cdots$ and if the prime factors of the form $3m+2$ appear to nothing but even powers, the number of solutions of the equation $m^2 + 3n^2 = N$ is given by

$$\rho_N = 2(a_1+1)(a_2+1)\ldots$$

if N is odd and by

$$\rho_N = 6(a_1+1)(a_2+1)\ldots$$

if N is even. If, on the contrary, N contains a prime factor of the form $3m+2$ to an odd power, one has $\rho_N = 0$."

Lorenz also [11, p.424] states (7), and a proof is provided by his reviewer/translator Valentiner [11, p.430].

So perhaps credit rests with Lorenz.

We shall give proofs of (4), (6) and (7) which demonstrate that all three results are intimately related.

2. Proof of the result involving $s(n)$

Let $a(q)$ denote the left hand side of (7). Then

$$a(q) + 2a(q^4) = 3 \sum_{k,l \in \mathbb{Z}} q^{k^2+3l^2}. \tag{8}$$

For,

$$a(q) = \sum_{\substack{m \text{ odd} \\ n \text{ even}}} q^{m^2+mn+n^2} + \sum_{\substack{m \text{ odd} \\ n \text{ odd}}} q^{m^2+mn+n^2} + \sum_{\substack{m \text{ even} \\ n \text{ odd}}} q^{m^2+mn+n^2} + \sum_{\substack{m \text{ even} \\ n \text{ even}}} q^{m^2+mn+n^2}.$$

In the first sum, let $k = m + \frac{n}{2},\ l = \frac{n}{2}$ (and conversely, $m = k - l,\ n = 2l$), in the second sum, $k = \frac{m-n}{2},\ l = \frac{m+n}{2}$ (conversely $m = k+l,\ n = l-k$), in the third sum, $k = \frac{m}{2} + n,\ l = \frac{m}{2}$ (conversely $m = 2l,\ n = k - l$) and in the fourth sum, $k = \frac{m-n}{2},\ l = \frac{m+n}{2}$ (conversely $m = k + l,\ n = l - k$),
and we find

$$a(q) = 3 \sum_{k \not\equiv l \pmod 2} q^{k^2+3l^2} + \sum_{k \equiv l \pmod 2} q^{k^2+3l^2}. \tag{9}$$

Also,

$$a(q^4) = \sum_{m,n \in \mathbb{Z}} q^{4m^2+4mn+4n^2} = \sum_{m,n \text{ even}} q^{m^2+mn+n^2} = \sum_{k \equiv l \pmod 2} q^{k^2+3l^2}, \tag{10}$$

as with the fourth sum above. (8) follows from (9) and (10). (4) follows from (7) and (8).

3. Proof of the result involving $t(n)$

We can write (4)

$$\sum_{k,l \in \mathbb{Z}} q^{k^2+3l^2} = 1 + 2 \sum_{n \geq 0} \left(\frac{q^{6n+1}}{1 - q^{12n+2}} - \frac{q^{6n+5}}{1 - q^{12n+10}} \right) + 6 \sum_{n \geq 0} \left(\frac{q^{12n+4}}{1 - q^{12n+4}} - \frac{q^{12n+8}}{1 - q^{12n+8}} \right).$$

If we extract the even powers of q we obtain

(11) $$\sum_{k\equiv l \pmod 2} q^{k^2+3l^2} = 1+6\sum_{n\geq 0}\left(\frac{q^{12n+4}}{1-q^{12n+4}}-\frac{q^{12n+8}}{1-q^{12n+8}}\right).$$

(Note, incidentally, that (7) follows from (10) and (11), and that (11) follows from (7) and (10)!)

From (11) we deduce

(12)
$$\sum_{k,l\in\mathbb{Z}} q^{k^2+3l^2} + 4q\sum_{k,l\in\mathbb{Z}^+} q^{k^2+k+3l^2+3l} = 1+6\sum_{n\geq 0}\left(\frac{q^{3n+1}}{1-q^{3n+1}}-\frac{q^{3n+2}}{1-q^{3n+2}}\right).$$

If we subtract (4) from (12) we find

(13)
$$\begin{aligned} 4q\sum_{k,l\in\mathbb{Z}^+} q^{k^2+k+3l^2+3l} &= 4\sum_{n\geq 0}\left(\frac{q^{3n+1}}{1-q^{3n+1}}-\frac{q^{3n+2}}{1-q^{3n+2}}\right) \\ &\qquad - 4\sum_{n\geq 0}\left(\frac{q^{12n+4}}{1-q^{12n+4}}-\frac{q^{12n+8}}{1-q^{12n+8}}\right) \\ &= 4\sum_{n\geq 0}\left(\frac{q^{6n+1}}{1-q^{12n+2}}-\frac{q^{6n+5}}{1-q^{12n+10}}\right). \end{aligned}$$

∎

Finally, if we divide (13) by $4q$ and replace q^2 by q we obtain (6).

4. Proof of the result involving $u(n)$

We begin by showing that

(14) $$\sum_{m,n\in\mathbb{Z}} \omega^{m-n} q^{m^2+mn+n^2} = \frac{(q)_\infty^3}{(q^3)_\infty}$$

where $\omega^3 = 1,\ \omega \neq 1$.

Let $\mathbf{CT}_a\left\{\sum_{-\infty}^{\infty} a^n f_n(q)\right\}$ denote $f_0(q)$, the "**C**onstant **T**erm" of the Laurent series in a. Then

$$\begin{aligned} \sum_{m,n\in\mathbb{Z}} \omega^{m-n} q^{m^2+mn+n^2} &= \sum_{m+n+p=0} \omega^{m-n} q^{(m^2+n^2+p^2)/2} \\ &= \mathbf{CT}_a\left\{\sum_{-\infty}^{\infty} a^m\omega^m q^{m^2/2}\sum_{-\infty}^{\infty} a^n\omega^{-n}q^{n^2/2}\sum_{-\infty}^{\infty} a^p q^{p^2/2}\right\} \end{aligned}$$

$$
\begin{aligned}
&= \mathbf{CT}_a \Bigg\{ \prod_{n\geq 1}(1+a\omega q^{n-\frac12})(1+a^{-1}\omega^{-1}q^{n-\frac12})(1-q^n). \\
&\qquad \cdot \prod_{n\geq 1}(1+a\omega^{-1}q^{n-\frac12})(1+a^{-1}\omega q^{n-\frac12})(1-q^n). \\
&\qquad \cdot \prod_{n\geq 1}(1+aq^{n-\frac12})(1+a^{-1}q^{n-\frac12})(1-q^n) \Bigg\} \\
&= \mathbf{CT}_a \left\{ \prod_{n\geq 1}(1+a^3q^{3n-\frac32})(1+a^{-3}q^{3n-\frac32})(1-q^n)^3 \right\} \\
&= \frac{(q)_\infty^3}{(q^3)_\infty}.\mathbf{CT}_a \left\{ \prod_{n\geq 1}(1+a^3q^{3n-\frac32})(1+a^{-3}q^{3n-\frac32})(1-q^{3n}) \right\} \\
&= \frac{(q)_\infty^3}{(q^3)_\infty}.\mathbf{CT}_a \left\{ \sum_{-\infty}^{\infty} a^{3n}q^{3n^2/2} \right\} \\
&= \frac{(q)_\infty^3}{(q^3)_\infty},
\end{aligned}
$$

as claimed.

Now the left hand side of (14) can be written

$$
\sum_{m-n\equiv 0 \pmod 3} q^{m^2+mn+n^2} + \omega \sum_{m-n\equiv 1 \pmod 3} q^{m^2+mn+n^2} + \omega^{-1} \sum_{m-n\equiv -1 \pmod 3} q^{m^2+mn+n^2}.
$$

In the first sum, let $k = \dfrac{m-n}{3},\ l = \dfrac{m+2n}{3}$ (conversely $m = 2k+l,\ n = l-k$),
in the second sum, $k = \dfrac{m-n-1}{3},\ l = \dfrac{m+2n-1}{3}$ (conversely $m = 2k+l+1,\ n = l-k$) and
in the third sum, $k = \dfrac{n-m-1}{3},\ l = \dfrac{n+2m-1}{3}$ $(m = l-k,\ n = 2k+l+1)$, and the left hand side of (14) is seen to be

$$
\sum_{k,l\in\mathbb{Z}} q^{3k^2+3kl+3l^2} + \omega \sum_{k,l\in\mathbb{Z}} q^{3k^2+3kl+3l^2+3k+3l+1} + \omega^{-1} \sum_{k,l\in\mathbb{Z}} q^{3k^2+3kl+3l^3+3k+3l+1}
$$

$$= a(q^3) - qc(q^3),$$

where

$$c(q) = \sum_{m,n\in\mathbb{Z}} q^{m^2+mn+n^2+m+n}.$$

Thus (14) becomes

$$a(q^3) - qc(q^3) = \frac{(q)_\infty^3}{(q^3)_\infty}. \tag{15}$$

Now, it is a celebrated identity of Jacobi that

$$(q)_\infty^3 = \sum_{n\geq 0}(-1)^n(2n+1)q^{(n^2+n)/2}. \tag{16}$$

We split this sum according to the residue modulo 3 of n. For $n \equiv 0 \pmod 3$, we write $3n$ $(n \geq 0)$, for $n \equiv 1 \pmod 3$, we write $3n+1$ $(n \geq 0)$, and for $n \equiv -1 \pmod 3$ we write $-3n-1$ $(n \leq -1)$, and the right hand side of (16) becomes

$$\begin{aligned}\sum_{n\geq 0}(-1)^n(6n+1)q^{(9n^2+3n)/2} &- \sum_{n\geq 0}(-1)^n(6n+3)q^{(9n^2+9n+2)/2} \\ &- \sum_{n\leq -1}(-1)^n(-6n-1)q^{(9n^2+3n)/2} \\ &= \sum_{-\infty}^{\infty}(-1)^n(6n+1)q^{(9n^2+3n)/2} - 3q(q^9)_\infty^3.\end{aligned}$$

So (15) becomes

$$a(q^3) - qc(q^3) = \frac{1}{(q^3)_\infty}\left\{\sum_{-\infty}^{\infty}(-1)^n(6n+1)q^{3(3n^2+n)/2} - 3q(q^9)_\infty^3\right\}. \tag{17}$$

It follows that

$$a(q) = \frac{1}{(q)_\infty}\sum_{-\infty}^{\infty}(-1)^n(6n+1)q^{(3n^2+n)/2} \tag{18}$$

and

$$c(q) = 3\frac{(q^3)_\infty^3}{(q)_\infty}.$$

Now (18) becomes

$$\begin{aligned}
a(q) &= \frac{1}{(q)_\infty}\left[\frac{d}{da}\sum_{-\infty}^{\infty}(-1)^n a^{6n+1}q^{(3n^2+n)/2}\right]_{a=1} \\
&= \frac{1}{(q)_\infty}\left[\frac{d}{da}\left\{a\prod_{n\geq 1}(1-a^6q^{3n-1})(1-a^{-6}q^{3n-2})(1-q^{3n})\right\}\right]_{a=1} \\
&= \frac{1}{(q)_\infty}\left[\prod_{n\geq 1}(1-a^6q^{3n-1})(1-a^{-6}q^{3n-2})(1-q^{3n})\times\right. \\
&\qquad\left.\times\left\{1+6\sum_{n\geq 1}\left(\frac{a^{-6}q^{3n-2}}{1-a^{-6}q^{3n-2}}-\frac{a^6q^{3n-1}}{1-a^6q^{3n-1}}\right)\right\}\right]_{a=1} \\
&= \frac{1}{(q)_\infty}(q)_\infty\left\{1+6\sum_{n\geq 1}\left(\frac{q^{3n-2}}{1-q^{3n-2}}-\frac{q^{3n-1}}{1-q^{3n-1}}\right)\right\} \\
&= 1+6\sum_{n\geq 0}\left(\frac{q^{3n+1}}{1-q^{3n+1}}-\frac{q^{3n+2}}{1-q^{3n+2}}\right)
\end{aligned}$$

which is (7).

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