**A new characteristic property of the palindrome prefixes of a standard Sturmian word**

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We use notions and terminology of theoretical computer science (see [1, 2, 3, 4, 5]).

The words $u$ and $v$ are conjugate and we write $u \sim v$ if there exist words $s$ and $t$ such that $u = st$ and $v = ts$.

Let $\varphi : \{a,b\}^* \to \{a,b\}^*$ be the morphism given by $\varphi(a) = ab$, $\varphi(b) = a$. Let $f_0 = b$ and, for $n \geq 0$,

$$f_{n+1} = \varphi(f_n).$$

For $n \geq 2$, let $g_n = f_{n-2}f_{n-1}$ and let $h_n$ be the longest common prefix of $f_n$ and $g_n$.

For example, for $n \leq 5$, we have:

$f_1 = a$,
$f_2 = ab$,
$f_3 = aba$,
$f_4 = abaab$,
$f_5 = abaababa$,
$g_2 = ba$
$g_3 = aab$,
$g_4 = ababa$,
$g_5 = abaabaab$
$h_2 = \epsilon$,
$h_3 = a$,
$h_4 = aba$,
$h_5 = abaaba$.

Notice that, for $n \geq 0$, $|f_n|$ is the $n^{th}$ element of the sequence of Fibonacci numbers $F_n$. We have $F_0 = 1$, $F_1 = 1$ and, for each $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$; for $0 \leq n \leq 13$ the Fibonacci numbers $F_n$ are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377. For each $n \geq 2$, $f_n = f_{n-1}f_{n-2}$ implying that for each $n \geq 1$, $f_n$ is a prefix of $f_{n+1}$.

Hence there exists a unique infinite word, namely the Fibonacci word (see [1, 2, 3, 4, 5]) denoted by $f$ such that, for each $n \geq 1$, $f_n$ is a prefix of $f$ and we have

$$f = abaababaabaababaababaabaababaabaababaababa\ldots.$$

Some very well known facts concerning $f$ are collected in Lemma 1.

**Lemma 1.** *For each $n \geq 2$, i) (Near–commutative property) $f_{n+2} = f_{n+1}f_n = f_ng_{n+1} = h_{n+2}xy$ and $g_{n+2} = f_nf_{n+1} = f_{n+1}g_n = h_{n+2}yx$, where $x, y \in \{a,b\}$, $x \neq y$ and*

$$xy = \begin{cases} ab & \text{if } n \text{ is even} \\ ba & \text{if } n \text{ is odd.} \end{cases}$$

*ii) the words $h_n ab$ and $h_n ba$ are conjugate, i.e. $f_n \sim g_n$ (for instance, $aba \sim aab$, $abaab \sim ababa$, $abaababa \sim abaababa$ and so on).*

**Definition.** *An infinite word $s = s_1 s_2 s_3 \cdots, s_i \in \{a, b\}$ is Sturmian if there exist reals $\alpha, \rho \in [0, 1]$, such that either for all $i$*

$$s(i) = a \text{ if } \lfloor \rho + (n+1)\alpha \rfloor = \lfloor \rho + n\alpha \rfloor, \;\; s(i) = b \text{ otherwise}$$

*or for all $i$*

$$s(i) = a \text{ if } \lceil \rho + (n+1)\alpha \rceil = \lceil \rho + n\alpha \rceil, \;\; s(i) = b \text{ otherwise.}$$

*The infinite word is proper Sturmian if $\alpha$ is irrational.*
*The infinite word is periodic Sturmian if $\alpha$ is rational.*
*The infinite word is standard Sturmian if $\rho = 0$.*

The Fibonacci word is a very particular case of Sturmian word (see [1, 2, 3, 4, 5]). We are convinced that looking **carefully** at the property of the Fibonacci word one can discover properties of Sturmian words. The previous mentioned fact suggested us the following result.

**Proposition.** *A word $w$ is a palindrome prefix of some standard Sturmian word if and only if $wab$ and $wba$ are conjugate.*

**Proof. "only if" part.** It is possible to say that this part is well known, in any case for an accurate proof one must use some notions defined in [2] ($PAL$, $PER$, *Stand*, ...).

**"if" part.** This part seems to be unknown and, in our knowledge, it is never mentioned in the literature.

Now let $w$ be written on an arbitrary alphabet and be such that $wab \sim wba$.

If $|w| = 0$ or $|w| = 1$ the statement is easily verified. So suppose that $|w| > 1$

We know that, for some words $u, v$ we have $wab = uv$ and $wba = vu$.

If $|u| = 1$ then $u = a$ and $w = a^{|w|}$ which is a palindrome prefix of a standard Sturmian word.

So suppose $|u| \geq 2$.

If $|v| = 1$ then $v = b$ and $w = b^{|w|}$ which is a palindrome prefix of a standard Sturmian word.

So we can suppose $|u| \geq 2$ and $|v| \geq 2$. If, without loss of generality, for instance $|u| \leq |v|$ we easily see that $u$ is a prefix of $v$ and so $w$ is a fractional power of both $u$ and $v$.

Pose $p = |u|$ and $q = |v|$ and $d = MCD(p, q)$. If $d$ were greater than 1 then $|w| = p + q - 2 \geq p + q - d$ and, by a result of Fine and Wilf [5], $w$ should have period $d$ and consequently $u = h^{p/d}$ and $v = h^{q/d}$ for some non empty word $h$ which is clearly impossible as $u$ has suffix $ba$ and $v$ has suffix $ab$. So $d = 1$.

Now let us show that $alph(w) = \{a, b\}$. If it is not the case, suppose that $|alph(w)| \geq 3$ and $w$ is one the words of minimal length such that $wab \sim wba$. Suppose $|u| < |v|$ and put $q = kp + r$, $0 \leq r < p$. We can write $v = u^k v_1$ with $v_1$ a prefix of $u$ of length $r$. Put also $v_1 = v''ab$. By the conjugation relation of $wab$ and $wba$ we get $u'bav'' = v''abu' = w'$ for some word $w'$. We have $w'ab \sim w'ba$. As $alph(w'ab) = alph(wab)$ and $|w'| < |w|$ we have a contradiction. Consequently $alph(wab) = \{a, b\}$.

As $d = 1$ we can apply Theorem 2 of [2] and so w is a palindrome prefix of a standard Sturmian word.

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