# Schur's Determinants and Partition Theorems

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**Abstract**

Garrett, Ismail, and Stanton gave a general formula that contains the Rogers–Ramanujan identities as special cases. The theory of associated orthogonal polynomials is then used to explain determinants that Schur introduced in 1917 and show that the Rogers–Ramanujan identities imply the Garrett, Ismail, and Stanton seemingly more general formula. Using a result of Slater a continued fraction is explicitly evaluated.

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**1. Introduction**. In a recent paper [6] Garrett, Ismail, and Stanton prove, amongst many other things, the following generalization of the celebrated Rogers–Ramanujan identities:

$$\begin{aligned}(1.1)\qquad 1+\sum_{n=1}^{\infty}\frac{q^{n^2+mn}}{(1-q)(1-q^2)\dots(1-q^n)} &= (-1)^m q^{-\binom{m}{2}}E_{m-2}\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})}\\ &\quad -(-1)^m q^{-\binom{m}{2}}D_{m-2}\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})},\end{aligned}$$

with the *Schur* polynomials, defined by

$$(1.2)\qquad D_m = D_{m-1}+q^m D_{m-2},\quad D_0=1,\ D_1=1+q,$$

$$(1.3)\qquad E_m = E_{m-1}+q^m E_{m-2},\quad E_0=1,\ E_1=1.$$

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Another proof, based on generalized Engel expansions, can be found in [3].

Schur [8] has computed the limits

$$(1.4)\qquad D_\infty = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}, \qquad E_\infty = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.$$

The aim of this note is to explain the result (1.1) within the context of associated orthogonal polynomials and their disguised appearance in Schur's work in the form of determinants. One basic ingredient is a sequence of orthogonal polynomials studied by Al-Salam and Ismail in [1]. In §2 we show how the polynomials studied by Schur in [8] and the Rogers–Ramanujan identities can be used to give a proof of (1.1). In §3 the more general polynomials of [1] are used to formulate a more general identity than (1.1). An application to some of the Slater identities [9] is included in §4.

The Al-Salam and Ismail polynomials $\{U_n(x; a, b)\}$ are defined by

$$(1.5)\qquad U_0(x; a, b) = 1, \quad U_1(x; a, b) = x(1 + a),$$

$$(1.6)\qquad U_{n+1}(x; a, b) = x(1 + aq^n)U_n(x; a, b) - bq^{n-1}U_{n-1}(x; a, b), \quad n \geq 1.$$

To indicate the dependence of $U_n(x; a, b)$ on $q$, when necessary we will use the notation $U_n(x; a, b|q)$. In accordance with the theory of orthogonal polynomials [4], the numerator polynomials $\{U^*_n(x; a, b)\}$ satisfy the recursion in (1.6) and the initial conditions

$$(1.7)\qquad U^*_0(x; a, b) = 0, \quad U^*_1(x; a, b) = 1 + a.$$

Therefore

$$(1.8)\qquad U^*_n(x; a, b) = (1 + a)U_{n-1}(x; qa, qb).$$

Schur [8] actually considered the polynomials $U_n(1; 0, -q)$ and $U^*_n(1; 0, -q)$. In the notation of (1.2) and (1.3) we have

$$(1.9)\qquad D_n = U_{n+1}(1; 0, -q), \qquad E_n = U^*_{n+1}(1; 0, -q) = U_n(1; 0, -q^2).$$

Let

$$(1.10)\qquad F(z; a, q) = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{(q; q)_n(-a; q)_n} q^{n(n-1)}$$

where we have used the standard notation for shifted factorials $(a;q)_n$ found in [2], [5]. Al-Salam and Ismail [1] proved that the limiting relation

$$(1.11)\qquad \lim_{n\to\infty} z^{-n} U_n(z;a,b) = (-a;q)_\infty\, F(b/z^2;a,q),$$

holds uniformly on compact subsets of the complex $z$-plane which do not contain $z=0$. They also gave the explicit representation

$$(1.12)\qquad U_n(x;a,b) = \sum_{k=0}^{\lfloor n/2\rfloor} \frac{(-a;q)_{n-k}(q;q)_{n-k}x^{n-2k}}{(-a;q)_k(q;q)_k(q;q)_{n-2k}}(-b)^k q^{k(k-1)}.$$

In §4 we recast (1.12) in the form of a generating function with generating function variables $a$ and $b$. In order to do so, we need the following form of the $q$-binomial theorem [2], [5]

$$(1.13)\qquad (-u;q)_n = \sum_{k=0}^{n} \frac{(q;q)_n q^{k(k-1)/2}}{(q;q)_k(q;q)_{n-k}}\, u^k.$$

**2. Schur's determinants**. We first show how Schur would have proved (1.1) in 1917. Consider the following determinant of Schur:

$$\mathsf{Schur}(b) := \begin{vmatrix} 1 & bq^{1+m} & & & & \dots \\ -1 & 1 & bq^{2+m} & & & \dots \\ & -1 & 1 & bq^{3+m} & & \dots \\ & & -1 & 1 & bq^{4+m} & \dots \\ & & & \ddots & \ddots & \ddots \end{vmatrix}.$$

Expanding the determinant with respect to the first column ("top–recursion") we get

$$\mathsf{Schur}(b) = \mathsf{Schur}(bq) + bq^{1+m}\,\mathsf{Schur}(bq^2).$$

Setting

$$\mathsf{Schur}(b) = \sum_{n=0}^{\infty} a_n b^n,$$

we get, upon comparing coefficients,

$$a_n = q^n a_n + q^{1+m} q^{2n-2} a_{n-1},$$

or

$$a_n = \frac{q^{2n-1+m}}{1-q^n} a_{n-1}.$$

Since $a_0 = 1$, iteration leads to

$$a_n = \frac{q^{n^2+mn}}{(1-q)(1-q^2)\dots(1-q^n)},$$

and thus the left hand side of (1.1) can be expressed by $\mathsf{Schur}(1)$.

On the other hand, $\mathsf{Schur}(1)$ is the limit of the *finite* determinants

$$\mathsf{Schur}_n := \begin{vmatrix} 1 & q^{1+m} & & & & \dots \\ -1 & 1 & q^{2+m} & & & \dots \\ & -1 & 1 & q^{3+m} & & \dots \\ & & -1 & 1 & q^{4+m} & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ & & & -1 & 1 & q^{n+m} \\ & & & & -1 & 1 \end{vmatrix}.$$

Expanding this determinant with respect to the last row ("bottom–recursion") we get

$$\mathsf{Schur}_n = \mathsf{Schur}_{n-1} + q^{n+m}\,\mathsf{Schur}_{n-2}\,. \tag{2.1}$$

We see that the sequences $\langle D_{n+m}\rangle_n$ and $\langle E_{n+m}\rangle_n$ satisfy the recursion (2.1) , and thus any linear combination will satisfy the same recurrence relation.

Set

$$\mathsf{Schur}_n = \lambda D_{n+m} + \mu E_{n+m}. \tag{2.2}$$

We can determine the parameters $\lambda$ and $\mu$ using the initial conditions $\mathsf{Schur}_0 = 1$, $\mathsf{Schur}_1 = 1 + q^{1+m}$, which leads to the evaluations

$$\lambda = \frac{E_m - E_{m-1}}{D_{m-1}E_m - D_m E_{m-1}} = \frac{q^m E_{m-2}}{D_{m-1}E_m - D_m E_{m-1}}, \tag{2.3}$$

$$\mu = \frac{D_m - D_{m-1}}{D_m E_{m-1} - D_{m-1}E_m} = \frac{q^m D_{m-2}}{D_m E_{m-1} - D_{m-1}E_m}. \tag{2.4}$$

The denominators in (2.3) and (2.4) are Casorati determinants, the discrete version of a Jacobian, and can be computed explicitly [7]. Indeed

$$D_{m-1}E_m - D_m E_{m-1} = (-1)^m q^{\binom{m+1}{2}}. \tag{2.5}$$

The proof of (2.5) is by induction on $m$. The beginning $m = 0$ is trivial; the induction step goes like this:

$$\begin{aligned} D_m E_{m+1} - D_{m+1} E_m &= D_m(E_m + q^{m+1} E_{m-1}) - (D_m + q^{m+1} D_{m-1}) E_m \\ &= q^{m+1}(D_m E_{m-1} - D_{m-1} E_m) \\ &= -q^{m+1}(-1)^m q^{\binom{m+1}{2}} = (-1)^{m+1} q^{\binom{m+2}{2}}. \end{aligned}$$

This replaces (2.3)–(2.4) by the nicer forms

(2.6) $$\lambda = (-1)^m q^{-\binom{m}{2}} E_{m-2}, \qquad \mu = (-1)^m q^{-\binom{m}{2}} D_{m-2}.$$

Thus the above analysis has led to

(2.7) $$\mathsf{Schur}_n = (-1)^m q^{-\binom{m}{2}} E_{m-2} D_{n+m} - (-1)^m q^{-\binom{m}{2}} D_{m-2} E_{n+m}.$$

Performing the limit $n \to \infty$ this turns into

(2.8) $$\mathsf{Schur}(1) = (-1)^m q^{-\binom{m}{2}} E_{m-2} D_\infty - (-1)^m q^{-\binom{m}{2}} D_{m-2} E_\infty,$$

which is (1.1).

## 3. Associated orthogonal polynomials.

The proof outlined in §2 can be considered in the context of orthogonal polynomials which satisfy three term recurrences such as (1.2)–(1.3). In this section we give in Lemma 3.3 a result for general orthogonal polynomials which specializes to the proof in §2. Some applications of Lemma 3.3 to the Al-Salam-Ismail polynomials are also given.

Any sequence of orthogonal polynomials $\{p_n(x)\}$ satisfies a three term recurrence relation

(3.1) $$p_{n+1}(x) = (A_n x + B_n) p_n(x) + C_n p_{n-1}(x), \quad n \geq 1,$$

and we assume the initial conditions

(3.2) $$p_0(x) = 1, \quad p_1(x) = A_0 x + B_0.$$

The analogue of Schur's finite determinant is the well-known tridiagonal determinant

(3.3) $\quad p_n(x) =$

$$= \begin{vmatrix} A_0x+B_0 & C_1 & & & & \cdots \\ -1 & A_1x+B_1 & C_2 & & & \cdots \\ & -1 & A_2x+B_2 & C_3 & & \cdots \\ & & -1 & A_3x+B_3 & C_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ & & & -1 & A_{n-2}x+B_{n-2} & C_n \\ & & & & -1 & A_{n-1}x+B_{n-1} \end{vmatrix}.$$

If $A_n = 1$, $B_n = 0$, $C_n = q^{n+m}$, then $p_{n+1}(1) = \mathsf{Schur}_n$. To see that the polynomials defined by (3.3) satisfy (3.1) expand the determinant representing $p_{n+1}(x)$ about the last row. We then verify that $p_1(x)$ and $p_2(x)$ of (3.3) agree with $p_1(x)$ from (3.2) and $p_2(x)$ which arises from (1.1) using the initial conditions (1.2).

Recall that the numerator polynomials $\{p_n^*(x)\}$ [4], [7] associated with $\{p_n(x)\}$ are defined to be solutions of

(3.4) $$z_{n+1}(x) = (A_nx + B_n)\, z_n(x) + C_n\, z_{n-1}(x), \quad n \geq 1,$$

with the initial conditions

(3.5) $$p_0^*(x) = 0, \quad p_1^*(x) = A_0.$$

The two sets of polynomials $\{p_n(x)\}$ and $\{p_n^*(x)\}$ form a basis for solutions of the three-term recurrence (3.4). One could also consider $\{p_{n+1}^*(x)\}$ as a solution to the three-term recurrence relation which has the indices shifted up by one. More generally, the $m$th associated polynomials are defined to be the solution to

(3.6) $$p_{n+1}^{(m)}(x) = (A_{n+m}x + B_{n+m})p_n^{(m)}(x) + C_{n+m}p_{n-1}^{(m)}(x), \quad n \geq 1,$$

with

(3.7) $$p_0^{(m)}(x) = 1, \quad p_1^{(m)}(x) = A_mx + B_m.$$

Thus we see that if $A_n = 1$, $B_n = 0$, $C_n = q^n$, then

(3.8) $$p_{n+1}^{(m)}(1) = \mathsf{Schur}_n, \quad D_n = p_{n+1}(1), \quad E_n = p_n^{(1)}(1).$$

**Theorem 3.1** *The polynomials $U_n(x;a,-b|q)$ satisfy the polynomial identity*

$$(3.9)\qquad (-b)^{m-1}q^{\binom{m-1}{2}}U_n(x;aq^m,-bq^m|q) = U_{m-1}(x;a,-b|q)U_{n+m-1}(x;aq,-bq|q) - U_{m-2}(x;aq,-bq|q)U_{n+m}(x;a,-b|q),$$

*for $m \geq 1, n \geq 0$, with $U_{-1}(x;a,b|q) := 0$.*

The relationship (3.9) is an extension of (2.7). After applying (1.11), the $n \to \infty$ limit of (3.9) becomes the following corollary.

**Corollary 3.2** *We have the following generalization of (1.1)*

$$(3.10)\qquad \begin{aligned}&(-b/x)^{m-1}q^{\binom{m-1}{2}}F(-bq^m/x^2;aq^m,q)\\ &= \left[(-aq;q)_{m-1}U_{m-1}(x;a,b)F(-bq/x^2;aq,q)\right.\\ &\quad \left. - (-a;q)_m xU_{m-2}(x,aq,-bq)F(-b/x^2;a,q)\right].\end{aligned}$$

The proof of Theorem 3.1 depends on a Lemma well-known to those who are familiar with the analytic theory of continued fractions and orthogonal polynomials. We include its proof only to make this work as self-contained as possible.

**Lemma 3.3** *The associated polynomials $\{p_n^{(m)}(x)\}$ satisfy*

$$(3.11)\qquad p_n^{(m)}(x) = \frac{p^*_{m-1}(x)p_{n+m}(x) - p_{m-1}(x)p^*_{n+m}(x)}{(-1)^m C_1C_2\dots C_{m-1}A_0}$$

**Proof**.

Fix $m \geq 2$. As a function of $n$, $\{p_{n-m}^{(m)}(x)\}_{n=m}^{\infty}$ also satisfies (3.4), so it is a linear combination of $\{p_n(x)\}$ and $\{p_n^*(x)\}$ with coefficients that are independent of $n$, but may depend upon $m$ and $x$. Thus we use initial conditions (3.7) to find coefficients $A_m(x)$ and $B_m(x)$ in

$$p_n^{(m)}(x) = A_m(x)p_{n+m}(x) + B_m(x)p^*_{n+m}(x).$$

The result is

$$(3.12)\qquad A_m(x) = p^*_{m-1}(x)/\Delta_m(x), \quad B_m(x) = p_{m-1}(x)/\Delta_m(x),$$

where

$$(3.13)\qquad \Delta_m(x) = p^*_{m-1}(x)p_m(x) - p_{m-1}(x)p^*_m(x).$$

It is clear that $\Delta_m(x)$ is a discrete Wronskian (Casorati Determinant) of $p_m(x)$ and $p_m^*(x)$ and can be evaluated from

$$\Delta_{m+1}(x) = -C_m \Delta_m(x),$$

which follows from (3.4). Thus

$$\Delta_m(x) = (-1)^m C_1 C_2 \dots C_{m-1} A_0,$$

since $\Delta_0(x) = -A_0$.

Let

$$A_n = 1 + aq^n, \quad B_n = 0, \quad C_n = bq^{n-1}, \tag{3.14}$$

so that the $p_n(x) = U_n(x; a, -b)$, see (1.5) and (1.6). Thus the $m$th associated polynomials are

$$p_n^{(m)}(x) = U_n(x; aq^m, -bq^m), \quad \text{for} \quad m, n \geq 0. \tag{3.15}$$

In view of (1.9) and (3.8), it follows that (3.9) and (3.10) generalize (2.7) and (2.8), respectively. In other words Schur's proof is the case $a = 0$, $b = q$.

**Remark**: The analogue of (1.1) for Rogers–Ramanujan identities of any moduli have been found by Garrett. She also combinatorially proved (1.1) by an involution, and generalized (1.1) to partitions whose parts differ by at least $d$.

**4. Further results**. The relationship (1.1) is the case $a = 0$, $x = 1$ and $b = q$ of (3.10), if we assume the Rogers–Ramanujan identities, that is assume (1.4). Another interesting result is found by choosing $a = -q^{1/2}$, $x = 1$, $b = q$, and then replacing $q$ by $q^2$.

In our notation (38) and (39) in [9] are

$$F(-q^2; -q, q^2) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q)_{2n}} = \prod_{n=0}^{\infty} \frac{(1+q^{3+8n})(1+q^{5+8n})(1-q^{8n+8})}{(1-q^{2n+2})}, \tag{4.1}$$

$$\frac{F(-q^4; q^3, q^2)}{(1-q)} = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q)_{2n+1}} = \prod_{n=0}^{\infty} \frac{(1+q^{1+8n})(1+q^{7+8n})(1-q^{8n+8})}{(1-q^{2n+2})}. \tag{4.2}$$

These are Rogers–Ramanujan identities of order 8 as indicated in [9]. It is interesting to note that with the choices $a = -q^{1/2}$, $x = 1$, $b = q$, formula (3.10) is

$$
\begin{aligned}
(4.3)\qquad & \sum_{n=0}^{\infty} \frac{q^{2mn+n^2}}{(q;q)_n(q^{2n+1};q)_m} \\
&= (-1)^m q^{\binom{m}{2}} \left[U_{m-2}(1;-q^3,-q^4|q^2)F(-q^2;-q,q^2)\right. \\
&\qquad \left. - U_{m-1}(1;-q,-q^2|q^2)F(-q^4;-q^3,q^2)/(1-q)\right].
\end{aligned}
$$

Al-Salam and Ismail [1] established the continued fraction representation

$$
(4.4)\qquad \frac{F(-qbz^{-2};qa,q)}{zF(-bz^{-2};a,q)} = \frac{1+a}{(1+aq)z+}\ \frac{b}{(1+aq^2)z+}\ \frac{bq}{(1+aq^3)z+}\ \cdots
$$

The special case $z=1, b=q, a=-\sqrt{q}$ gives, via (4.1)–(4.2),

$$
(4.5)\qquad \frac{1}{(1-q^3)+}\ \frac{q^2}{(1-q^5)+}\ \frac{q^4}{(1-q^7)+}\cdots = \prod_{n=0}^{\infty} \frac{(1+q^{1+16n})(1+q^{14+16n})}{(1+q^{6+16n})(1+q^{10+16n})}.
$$

Recall that a Gaussian (or $q$-)binomial and trinomial coefficients are

$$
(4.6)\qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, \qquad \begin{bmatrix} n \\ j,k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_j(q;q)_k(q;q)_{n-j-k}}.
$$

The polynomials $\{U_n(x;a,b|q)\}$ contain a redundant parameter. In fact it is clear from (1.12) that $x^{-n}U_n(x;a,bx^2|q)$ is independent of $x$. As orthogonal polynomials the $x$ variable is important and we can scale away the $b$ parameter. Set

$$
(4.7)\qquad V_n(a,b|q) = x^{-n}U_n(x;a,-bx^2|q) = \sum_{k=0}^{\lfloor n/2 \rfloor} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \frac{(-a;q)_{n-k}}{(-a;q)_k} b^k q^{k(k-1)}.
$$

Since $(-a;q)_{n-k}/(-a;q)_k = (-aq^k;q)_{n-2k}$ we can expand the quotient using the $q$-binomial theorem (1.13) and obtain

$$
(4.8)\qquad V_n(a,b|q) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{n-2k} \begin{bmatrix} n-k \\ j,k \end{bmatrix}_q a^j b^k q^{k(k+j-1)+j(j-1)/2}.
$$

For example the coefficient of $a^j b^k q^m$ in $V_n(a,b|q)$ has a combinatorial interpretation in terms of counting pairs of partitions. This combinatorial study is still in progress.

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