Séminaire Lotharingien de Combinatoire **47** (2001), Article B47b

$\mathbf{2kn - \binom{2k+1}{2}}$
A Note on Extremal Combinatorics of Cyclic Split Systems

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[*]This work was carried out whilst the author was a Distinguished Visiting Professor at the Dept. of Chem. Eng., City College, CUNY, New York.

[†]The author thanks C.M.A.F., Lisbon, for hosting him during part of this work.

[‡]The author thanks the Swedish National Research Council (NFR) for its support (grant# M12342-300), and both C.M.A.F., Lisbon, and The Institute of Fundamental Sciences, Massey University, Palmerston North for hosting him during part of this work.

Abstract

It is shown that every cyclic split system $\mathbf{S}$ defined on an n-set with $\#\mathbf{S} > 2kn - \binom{2k+1}{2}$ for some $k \leq \frac{n-1}{2}$ always contains a subset of $k+1$ pairwise incompatible splits provided one has $\min(k, n-(2k+1)) \leq 3$. In addition, some related old and new conjectures are also discussed.

1 Introduction

In this note, we show that every split system $\mathbf{S}$ defined on an n-set X contains a subset of $k+1$ pairwise incompatible splits provided one has $n \geq 2k+1$, $\#\mathbf{S} > 2kn - \binom{2k+1}{2}$, and $\min(k, n-(2k+1)) \leq 3$.

Recall that a *split* $S = \{A, B\}$ of a set X is a bipartition of X into two sets A, B; in particular, we have $B = \overline{A} := X - A$. We denote by $\mathbf{S}(X)$ the set of all splits of X; any subset $\mathbf{S}$ of $\mathbf{S}(X)$ is called a *split system* (defined on X). Two splits S, S' of a set X are called *compatible* if there exist subsets $A \in S$ and $A' \in S'$ with $A \cap A' = \emptyset$, otherwise S and S' are called *incompatible*[1]. We call a split system $\mathbf{S} \subseteq \mathbf{S}(X)$ *k-compatible* if it does not contain a subset of $k+1$ pairwise incompatible splits.

At the end of the seventies, due to newly discovered results on multicommodity flow problems such as those appearing in [6, 7], it became of increasing interest to determine upper bounds for the cardinality of a k-compatible split system $\mathbf{S} \subseteq \mathbf{S}(X)$, defined on an n-set X. It is well known that every maximal 1-compatible split system contains exactly $2n-3$ distinct splits, and it was observed by M. Lomonosov that

$$\#\mathbf{S} \leq n + \frac{kn}{2} + \frac{kn}{3} + \cdots + \frac{kn}{\lfloor n/2 \rfloor} < n(1 + k\log_2(n))$$

always holds for a k-compatible split system defined on an n-set (see Section 2). In [6], A. Karzanov conjectured that there is some universal constant c so that $\#\mathbf{S} \leq cn$ holds for all 2-compatible split systems defined on an n-set X, a conjecture that was established in [9] by P. Pevzner, who showed[2] that

$$karz_k(n) := \max(\#\mathbf{S} \mid \mathbf{S} \subseteq \mathbf{S}([n]), \mathbf{S}\ k\text{-compatible}\ )$$

[1] Note that incompatible splits have also been called *qualitatively independent partitions* in [10, p.16] and *crossing sets* in [5, 6, 7, 8, 9]

[2] In [5], a possible flaw in Pevzner's proof – probably due to poor translation – is pointed out.

is bounded by $6n$ in case $k := 2$. Recently, T. Fleiner improved on this bound [5], showing that $karz_2(n) \leq 5n$ must hold.

Here, we will restrict our attention to *cyclic* split systems, that is, split systems $\mathbf{S} \subseteq \mathbf{S}(X)$ for which there exists a bijection

$$\phi : [n] := \{1, \dots, n\} \to X \quad (n := \#X)$$

such that $\mathbf{S}$ is contained in the split system $\mathbf{S}(\phi)$ consisting of all splits S of the form

$$\{\{\phi(i), \phi(i+1), \dots, \phi(j-1)\}, \overline{\{\phi(i), \phi(i+1), \dots, \phi(j-1)\}}\},$$

with $1 \leq i < j \leq n$ (cf. [1, 8], for example). For $k, n \in \mathbf{N}$, let $cycl_k(n)$ denote the maximal cardinality of a k-compatible cyclic split system $\mathbf{S}$, defined on $[n]$. As every cyclic split system defined on $[n]$ is $\lfloor n/2 \rfloor$-compatible, we clearly have $cycl_k(n) = \binom{n}{2}$ whenever $2k + 1 \geq n$ holds. We will show later on that, in the remaining cases $2k+1 \leq n$, there are k-compatible cyclic split systems $\mathbf{S}$ of cardinality $2kn - \binom{2k+1}{2}$ that are maximal among all such split systems, and we conjecture

Conjecture 1 $cycl_k(n) = 2kn - \binom{2k+1}{2}$ for all $k, n \in \mathbf{N}$ with $n \geq 2k + 1$.

It is easy to see that there exists a non-cyclic 3-compatible split system $\mathbf{S}_0$ defined on $\{1, 2, \dots, 7\} = [7]$ with $\#\mathbf{S}_0 = cycl_3(7) + 3 = 24$: Just take the union of the cyclic split system $\mathbf{S}(Id_{[7]})$ with the three splits $S_{\{1,3\}}, S_{\{4,6\}}$, and $S_{\{5,7\}}$. This example can be generalized; see [4] for more details.

However, it is not unlikely that the upper bound conjectured for cyclic k-compatible split systems also holds for k-compatible split systems that, though not necessarily cyclic, are at least *weakly compatible*, i.e. for k-compatible split systems $\mathbf{S}$ with $A_1 \cap A_2 \cap A_3 = \emptyset$, or $\overline{A_1} \cap \overline{A_2} \cap A_3 = \emptyset$, or $\overline{A_1} \cap A_2 \cap \overline{A_3} = \emptyset$, or $A_1 \cap \overline{A_2} \cap \overline{A_3} = \emptyset$ for any three subsets A_1, A_2, A_3 with $\{A_i, \overline{A_i}\} \in \mathbf{S}$, $1 \leq i \leq 3$ (cf. [1]). Clearly, every cyclic split system is weakly compatible, and so we have necessarily $weak_k(n) \geq cycl_k(n)$ where - of course - $weak_k(n)$ now denotes the maximal cardinality of a weakly and k-compatible split system, defined on an n-set. Moreover, as any 2-compatible split system is weakly compatible, we have $karz_2(n) = weak_2(n)$; so, if $weak_k(n) = cycl_k(n) = 2kn - \binom{2k+1}{2}$ would hold for $k = 2$ and all n, this would imply $karz_2(n) = 4n - 10$ for $n \geq 5$ (while $karz_2(n) = \binom{n}{2}$ clearly holds for $n \leq 5$), thus implying sharp upper bounds for all n in case $k = 2$ for Karzanov's conjecture.

In addition, the result that every weakly compatible split system defined on $[n]$ that is of maximal cardinality among all such split systems actually is a cyclic split system (cf. [1, Theorem 5]) suggests that this might also hold for weakly *and* k-compatible split systems, too. In other words, we conjecture

Conjecture 2 Every weakly and k-compatible split system $\mathbf{S}$ with

$$\#\mathbf{S} \geq 2kn - \binom{2k+1}{2}$$

is cyclic (and, hence, of cardinality $2kn - \binom{2k+1}{2}$ if $n \geq 2k+1$ and Conjecture 1 holds).

In this note, we will establish the following two results:

Theorem 1 *Assume $k, n \in \mathbf{N}$ and $n \geq 2k+1$.*
(i) There exist cyclic k-compatible split systems $\mathbf{S}$ defined on an n-set that are maximal among all weakly and k-compatible split systems and have cardinality

$$\#\mathbf{S} = 2kn - \binom{2k+1}{2}.$$

(ii) If $\mathbf{S}$ is a maximal cyclic and k-compatible split system contained in $\mathbf{S}(Id_{[n]})$ such that $\{\{i, i+1, \ldots, i+k\}, \overline{\{i, i+1, \ldots, i+k\}}\} \in \mathbf{S}$ holds for $i = 1, 2, \ldots, k-1$, then the split system $\mathbf{S}'$ induced on $Y := [n] - \{k\}$ by eliminating the split $\{\{k\}, X - \{k\}\}$ and restricting all other splits in $\mathbf{S}$ to Y, is a maximal cyclic and k-compatible split system defined on Y, and one has $\#\mathbf{S} = \#\mathbf{S}' + 2k$.

Theorem 2 *Assume $k, n \in \mathbf{N}$ and $n \geq 2k+1$. Then one has $cycl_k(n) = 2kn - \binom{2k+1}{2}$ provided $\min(k, n-(2k+1)) \leq 3$ holds.*

The rest of this paper is organized as follows: In Section 2, we present the well-known proof for Lomonosov's bound. In Section 3, we establish Theorem 1. In Sections 4 and 5, we prove that Conjecture 1 holds for 2-compatible and 3-compatible cyclic split systems, respectively (see Theorems 3 and 4), and in the final section we show that Conjecture 1 also holds for k-compatible cyclic split systems on an n-set with $n - (2k+1) \leq 3$ (see Theorem 5). Theorem 2 follows immediately from these facts.

Remark 1 *Since submission of this paper, Conjecture 1 has been established in [3] while it has been established in [2] Conjecture 2 holds in case $k = 2$.*

Acknowledgment: The authors would like to thank an anonymous referee for his (unusually constructive, helpful and thorough) criticism.

2 Lomonosov's bound

In this section, we present – for the reader's convenience – the well-known simple proof for Lomonosov's bound.

We begin by recalling some further terminology from split theory. Given a finite set X and a split system $\mathbf{S} \subseteq \mathbf{S}(X)$, define the *size* of a split $S = \{A, B\} \in \mathbf{S}$ to be the smaller of the two numbers $\#A$ and $\#B$, that is,

$$\mathrm{size}(\{A, B\}) := \min(\#A, \#B).$$

A split of size m will also be called an *m-split*. If $S = \{A, B\}$ is an m-split and $m = \#A < \#B$ holds, A will also be called the *m-part* of S. For every proper subset A of X, we denote by S_A the split $S_A := \{A, \overline{A}\}$, induced by A. Whenever a subset A of X consists of one element $x \in X$ only, we may also write 'x' instead of '$\{x\}$' as long as no confusion can arise. In particular, we will write $\overline{x}$ instead of $\overline{\{x\}}$ and $S_x = \{x, \overline{x}\}$ instead of $S_{\{x\}} = \{\{x\}, \overline{\{x\}}\}$, for every $x \in X$. For every element $x \in X$ and split $S \in \mathbf{S}(X)$, we denote by $S(x)$ that subset, A or B, in S that contains x, and by $\overline{S}(x)$, we denote its complement $\overline{S}(x) := \overline{S(x)} = X - S(x)$. A split $S \in \mathbf{S}$ *seperates* distinct elements $x, y \in X$ if $S(x) \neq S(y)$. Finally, we denote the set $\{A \subseteq X \mid S_A \in \mathbf{S}\} = \{S(x) \mid S \in \mathbf{S}, x \in X\}$ by $\cup\mathbf{S}$.

We now give a proof for Lomonosov's bound in the language of split theory.

Lemma 1 *For any two positive integers k, n, we have*

$$karz_k(n) \leq n + \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} \lfloor \frac{kn}{i} \rfloor.$$

Furthermore, if n is even, say $n = 2m$, then

$$karz_k(n) \leq n + k + \sum_{i=2}^{m-1} \lfloor \frac{kn}{i} \rfloor.$$

Proof: Let $\mathbf{S}$ be a k-compatible split system on X with $\#X = n$, and assume $2 \leq t \leq n/2$, $t \in \mathbf{N}$. Define

$$(\cup\mathbf{S})_t := \{A \in \cup\mathbf{S} \mid \#A = t\}.$$

For $A, B \in (\cup\mathbf{S})_t$, the splits S_A and S_B are compatible if and only if $A = B$ or $A \cap B = \emptyset$. This means that, for all $x \in X$, there are at most k *distinct* sets $A \in (\cup\mathbf{S})_t$ with $x \in A$. By counting the pairs (A, x) with $x \in A, A \in (\cup\mathbf{S})_t$ we find

$$\#(\cup\mathbf{S})_t \leq \frac{kn}{t}.$$

The lemma follows now easily by noting that the number of 1-splits is bounded by n and that, if $n = 2m$, then the number of m-splits in $\mathbf{S}$ is half the size of $(\cup\mathbf{S})_m$. ∎

3 Cyclic split systems

From now on, we assume $X = [n]$ and, when we talk about a cyclic split system $\mathbf{S}$, we tacitly assume $\mathbf{S} \subseteq \mathbf{S}^{(n)} := \mathbf{S}(Id_{[n]})$ (i.e. the cyclic split system obtained by putting $\phi = Id_{[n]}$ in the definition for $\mathbf{S}(\phi)$ given in the introduction). For every $m \in \mathbf{Z}$, we denote by $\widehat{m}$ the unique element in $[n]$ that is congruent to m modulo n. We also introduce the notation $[p, q] := \{\widehat{p}, \widehat{p+1}, \ldots, \widehat{q-1}\}$, and $S_i^{(j)} := S_{[i,i+j-1]} = \{[i, i+j-1], [n] - [i, i+j-1]\}$ for $i, j \in [n]$. *Rotating* a split system $\mathbf{S} \subseteq \mathbf{S}(X)$ by "i", $1 \leq i \leq n$, we obtain a new split system $\vec{\mathbf{S}}^i$ defined by $\vec{\mathbf{S}}^i := \{A - i, B - i \,:\, \{A, B\} \in \mathbf{S}\}$, where

$$C - i := \{\widehat{c - i} \,:\, c \in C\}$$

for any $C \subseteq [n]$. Note that if $\mathbf{S}$ is cyclic, k-compatible, or incompatible, then so is $\vec{\mathbf{S}}^i$.

In what follows we will present rather abstract arguments when dealing with cyclic split systems. However, the reader is advised to visualize these arguments by drawing similar diagrams to that presented in Figure 1, a figure that can be used in visualizing the proof of the following result on cyclic split systems.

Lemma 2 *Suppose that $\mathbf{S} \subseteq \mathbf{S}(X)$ is cyclic, and that $n \geq 2k + 1$. If $S \in \mathbf{S}$ has size less than $k+1$, then S cannot be contained in any incompatible subset $\mathbf{S}'$ of $\mathbf{S}$ of cardinality $k + 1$.*

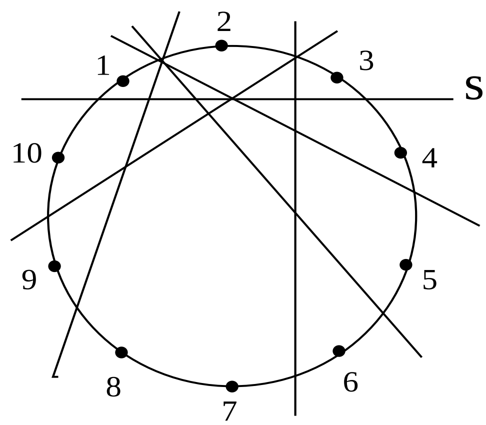

Figure 1: *Visualizing cyclic split systems: In this figure, cyclic splits of the set* $\{1, 2, \ldots, 10\}$ *are represented by diagonals connecting mid points of the edges* $\{i, i+1\}$, $i = 1, \ldots, 10 \ mod\, 10$. *Two splits are incompatible if and only if the corresponding diagonals have a point of intersection in the interior of the circle. Clearly, there is no system consisting of four pairwise incompatible cyclic splits of* $\{1, 2, \ldots, 10\}$ *containing the split* $S = S_1^{(3)}$ *(just as claimed in Lemma 2 below in the case* $k := 3$*).*

Proof: Without loss of generality, suppose that $S := S_1^{(j)}$, $1 \leq j \leq k$. If S were contained in a set $\mathbf{S}'$ of $k+1$ pairwise incompatible splits, then k splits from $\mathbf{S}'$ would have to be incompatible with S. So, each of these splits would have to separate some pair $\{i, i+1\}$ for some i with $1 \leq i \leq j-1 < k$ and so, by the pigeonhole principle, at least one pair of splits from $\mathbf{S}' - \{S\}$ would have to be compatible, a contradiction. ∎

For each $n \geq 2k + 1$, $k \geq 1$, define $\mathbf{S}_{n,k} \subseteq \mathbf{S}^{(n)}$ to be the cyclic split system on $[n]$ consisting of all splits in $\mathbf{S}^{(n)}$ of size less than $k + 1$, so that $\#\mathbf{S}_{n,k} = kn$. We define a split system $\mathbf{S}$ to be a *k-reduced* split system if all splits in $\mathbf{S}$ have size at least $k + 1$. So, a cyclic split system $\mathbf{S} \subseteq \mathbf{S}^{(n)}$ is k-reduced if and only if $\mathbf{S} \cap \mathbf{S}_{n,k} = \emptyset$ holds.

Corollary 1 *The split system* $\mathbf{S}_{n,k}$ *is k-compatible.*

Corollary 2 *A cyclic split system* $\mathbf{S} \subseteq \mathbf{S}^{(n)}$ *is k-compatible if and only if the union* $\mathbf{S} \cup \mathbf{S}_{n,k}$ *is k-compatible.*

Remark 2 *In view of this last corollary, our conjecture that* $cycl_k(n) = 2kn - \binom{2k+1}{2}$ *can be reformulated as* $\#\mathbf{S} = kn - \binom{2k+1}{2}$ *for all* k*-reduced and* k*-compatible split systems* $\mathbf{S} \subseteq \mathbf{S}^{(n)}$ *(*$n \geq 2k+1$*).*

We now give a recursive construction of maximal k-compatible cyclic split systems $\mathbf{S} \subseteq \mathbf{S}^{(n)}$ of cardinality $2kn - \binom{2k+1}{2}$, for all $n \geq 2k+1$, $k \geq 1$:

Construction A Fix $k \geq 1$. If $n = 2k+1$, then $\mathbf{S}^{(n)} = \mathbf{S}_{n,k}$ and $\#\mathbf{S}_{n,k} = k(2k+1) = 2k(k+1) - \binom{2k+1}{k}$, so that $\mathbf{S}_{n,k}$ provides us with the required split system.

Now suppose we have a k-compatible split system $\mathbf{S} \subseteq \mathbf{S}^{(n)}$ with $\mathbf{S}_{n,k} \subseteq \mathbf{S}$ of any cardinality $\#\mathbf{S}$, and that $n \geq 2k+1$ holds. We will construct a k-compatible cyclic system on $[n+1]$ of cardinality $\#\mathbf{S} + 2k$. Note that since $\mathbf{S}_{n,k} \subseteq \mathbf{S}$, there is a split of size at least k separating 1 from n. Let S_0 be any such split and put $A_0 := S_0(1)$ and $B_0 := S_0(n)$. We now extend the split system $\mathbf{S}$ to a new split system on $[n+1]$ as follows:
(a) Put $S_1 := \{A_0 \cup \{n+1\}, B_0\}$ and $S_2 := \{A_0, B_0 \cup \{n+1\}\}$.
(b) If $S \in \mathbf{S}$ does not separate 1 and n, then define $S^* := \{S(1) \cup \{n+1\}, \overline{S}(1)\}$.
(c) If $S \in \mathbf{S} - \{S_0\}$ does separate 1 and n, we have either $A_0 \subset S(1)$ or $S(1) \subset A_0$. In the first case, define $S^* := \{S(1) \cup \{n+1\}, S(n)\}$, in the second case $S^* := \{S(1), S(n) \cup \{n+1\}\}$. Put

$$\mathbf{S}^* := \{S^* \,:\, S \in \mathbf{S} - \{S_0\}\} \cup \{S_1\} \cup \{S_2\}.$$

Then $\mathbf{S}^* \subseteq \mathbf{S}^{(n+1)}$ is clearly k-compatible, and $\#\mathbf{S}^* = \#\mathbf{S} + 1$ holds. Note that $\#S(n+1) > k$ holds for all $S \in \mathbf{S}^*$ with $1 \notin S(n+1)$ or $n \notin S(n+1)$. Hence, by including those $2k-1$ splits in $\mathbf{S}_{n+1,k}$ that separate $n+1$ from n or from 1 or from both, we obtain a k-compatible split system $\mathbf{S}' \subseteq \mathbf{S}^{(n+1)}$ in view of Corollary 2, that contains $\mathbf{S}_{n+1,k}$ and has $2k$ more splits than $\mathbf{S}$, thus producing a k-compatible cyclic split system on $[n+1]$ of size $2k + \#\mathbf{S}$, as required.

It is straight forward to see that $\mathbf{S}'$ is a maximal weakly and k-compatible split system whenever $\mathbf{S}$ is: This is trivial in case $k = 1$ and, in case $k \geq 2$, it follows from the fact that every split $S_* \in \mathbf{S}([n+1]) - \mathbf{S}'$ for which $\mathbf{S}' \cup \{S_*\}$ is weakly and k-compatible, must reduce to a split

$$S := \{S_*(n+1) - \{n+1\}, \overline{S}_*(n+1)\} \in \mathbf{S}$$

because S_* cannot coincide with $S_{\{n+1\}}$ and because we have assumed $\mathbf{S}$ to be a maximal weakly and k-compatible split system in $\mathbf{S}([n])$. Hence, S must be distinct from S_0 because both extensions of S_0 are in $\mathbf{S}'$, and S must separate 1 and n because S_* - being distinct from S^* - would otherwise necessarily be of the form $S_* = \{S(1), \overline{S}(1) \cup \{n+1\}\} = \{S(n), \overline{S}(n) \cup \{n+1\}\}$ in contradiction to the fact that then $S_*, S_{\{n,n+1\}}$ and $S_{\{n+1,1\}}$ would form a system of three not weakly compatible splits in view of

$$\begin{aligned}
n+1 \in S_*(n+1) \cap \{n, n+1\} \cap \{n+1, 1\} &\neq \emptyset, \\
n \in \overline{S}_*(n+1) \cap \{n, n+1\} \cap ([n+1] - \{n+1, 1\}) &\neq \emptyset, \\
1 \in \overline{S}_*(n+1) \cap ([n+1] - \{n, n+1\}) \cap \{n+1, 1\} &\neq \emptyset, \\
S_*(n+1) \cap ([n+1] - \{n, n+1\}) \cap ([n+1] - \{n+1, 1\}) = \overline{S}(1) &\neq \emptyset.
\end{aligned}$$

So, we must have $S(1) \neq S(n)$ and either $A_0 \subseteq S(1)$ and $S_* = \{S(1), S(n) \cup \{n+1\}\}$ or $S(1) \subseteq A_0$ and $S_* = \{S(1) \cup \{n+1\}, S(n)\}$, and it is easy to see that the $k-1$ splits in $\mathbf{S}'$ of size $k+1$ that contain $\{n, n+1, 1\}$ together with S_* and with S_1 in the first case or S_2 in the second case, would then form a system of $k+1$ pairwise incompatible splits. Hence we established:

Proposition 1 *Given any cyclic and k-compatible split system $\mathbf{S} \subseteq \mathbf{S}([n])$ that is maximal among all weakly and k-compatible split systems contained in $\mathbf{S}([n])$ for some k and n with $n \geq 2k+1$, there exists a k-compatible and cyclic split system $\mathbf{S}' \subseteq \mathbf{S}([n+1])$ of cardinality $2k + \#\mathbf{S}$ that is maximal among all weakly and k-compatible split systems contained in $\mathbf{S}([n+1])$.*

Starting with $\mathbf{S} := \mathbf{S}^{(n)}$ in case $n = 2k+1$, this leads to

Corollary 3 *There exist maximal weakly and k-compatible split systems $\mathbf{S} \subseteq \mathbf{S}([n])$ of cardinality $2kn - \binom{2k+1}{2}$, for any $k, n \in \mathbf{N}$ with $2k+1 \leq n$.*

Clearly, this establishes the first assertion in Theorem 1. To show that also the second one must hold, assume that $\mathbf{S}$ is a maximal cyclic and k-compatible split system contained in $\mathbf{S}^{(n)}$ and that $\mathbf{S}$ contains the $k-1$ pairwise incompatible splits S_i^{k+1} $(i = 1, 2, \ldots, k-1)$.

As every split $S \in \mathbf{S}^{(n)} - \mathbf{S}_{n,k}$ for which either $S(k-1) \neq S(k)$ or $S(k+1) \neq S(k)$ holds, is necessarily incompatible with every split S_i^{k+1}, with $i = 1, 2, \ldots, k-1$, any two splits $S, S' \in \mathbf{S}_{>k} := \mathbf{S} - \mathbf{S}_{n,k}$ with $S(k-1) \neq S(k)$

and $S'(k+1) \neq S'(k)$ must be compatible. Hence, $A_+ \cap A_- = \emptyset$ must hold for any two sets

$$A_- \in \mathcal{A}_- := \{S(k-1) \mid S \in \mathbf{S}_{>k},\, S(k-1) \neq S(k)\}$$

and

$$A_+ \in \mathcal{A}_+ := \{S(k+1) \mid S \in \mathbf{S}_{>k},\, S(k+1) \neq S(k)\}.$$

Consequently, there exists at most one pair of distinct splits S, S' in $\mathbf{S}_{>k-1}$ with $S|_{[n]-\{k\}} = S'|_{[n]-\{k\}}$: Indeed, any such pair in $\mathbf{S}^{(n)}_{>k-1}$ is necessarily of the form $S^{(i)}_k$, $S^{(i-1)}_{k+1}$ for some integer i with $k \leq i \leq n-k$; hence, if there were two such pairs $S^{(i)}_k$, $S^{(i-1)}_{k+1}$ and $S^{(j)}_k$, $S^{(j-1)}_{k+1}$ with, say, $i < j$, we would have

$$S^{(i)}_k(k-1) = [k+i, k-1] \in \mathcal{A}_-$$

and

$$S^{(j-1)}_{k+1}(k+1) = [k+1, k+j-1] \in \mathcal{A}_+$$

in contradiction to

$$k+i \in [k+i, k-1] \cap [k+1, k+j-1].$$

Thus, we must have

$$\#\mathbf{S} - 2k - 1 \leq \#\mathbf{S}|_{[n]-\{k\}} \leq \#\mathbf{S} - 2k$$

in view of the fact that there is exactly one split in $\mathbf{S}$ that does not give rise to any split in the induced split system $\mathbf{S}|_{[n]-\{k\}}$, viz. S^1_k, and there are exactly $2(k-1)$ pairs of distinct splits S, S' in $\mathbf{S} \cap \mathbf{S}_{n,k}$ with $S|_{[n]-\{k\}} = S'|_{[n]-\{k\}}$, viz. the pairs $S^{(k-i)}_i, S^{(k-i+1)}_i, i = 1, \ldots, k-1$ and $S^{(i)}_k, S^{(i-1)}_{k+1}, i = 2, \ldots, k$ and we have

$$\#\mathbf{S}|_{[n]-\{k\}} = \#\mathbf{S} - 2k$$

if and only if there exists a pair of distinct splits S, S' in $\mathbf{S}_{>k-1}$ with $S|_{[n]-\{k\}} = S'|_{[n]-\{k\}}$.

Yet, if no such pair would exist, we could use our construction above relative to any split $S \in \mathbf{S}|_{[n]-\{k\}}$ with $S(k-1) \neq S(k+1)$ of size at least k to construct a k-compatible split system $\mathbf{S}' \subseteq \mathbf{S}^{(n)}$ with $\#\mathbf{S}' = 1 + \#\mathbf{S}$ in contradiction to our assumption that $\mathbf{S}$ is a maximal k-compatible split system in $\mathbf{S}^{(n)}$.

Thus, we must have $\#\mathbf{S}|_{[n]-\{k\}} = \#\mathbf{S} - 2k$, $\mathbf{S}|_{[n]-\{k\}}$ must be a maximal cyclic k-compatible split system defined on the set $[n] - \{k\}$, there exists a (necessarily unique) pair of distinct splits S, S' in $\mathbf{S}_{>k-1}$ with $S|_{[n]-\{k\}} = S'|_{[n]-\{k\}} := S_0$, and $\mathbf{S}$ can be constructed from $\mathbf{S}|_{[n]-\{k\}}$ as above, using S_0 as the 'cut-off split'.

This establishes the second assertion of Theorem 1.

4 2-compatible cyclic split systems

In this section, we prove the following result:

Theorem 3 *Suppose that $n \geq 5$. If $\mathbf{S} \subseteq \mathbf{S}^{(n)}$ is a 2-compatible split system, then $\#\mathbf{S} \leq 4n - 10$.*

This result follows easily from the following lemma:

Lemma 3 *Let $\mathbf{S} \subseteq \mathbf{S}^{(n)}$ be a maximal k-compatible and k-reduced split system. Then all minimal elements, with respect to inclusion, in $\cup\mathbf{S}$ have size $k+1$.*

Proof: Assume that $A \in \cup\mathbf{S}$ is minimal in $\cup\mathbf{S}$ and, without loss of generality, assume $A = \{1, 2, \ldots, i\}$ for some $i > k$. If $i = k+1$, we are done. Otherwise, the maximality of $\mathbf{S}$ together with $\{1, 2, \ldots, i-1\} \notin \cup\mathbf{S}$ implies that there must exist k splits $S_1, S_2, \ldots, S_k$ in $\mathbf{S}$ that are pairwise incompatible and incompatible with $S_{\{1,\ldots,i-1\}}$ while one of those splits must be compatible with $S_A = S_{\{1,\ldots,i\}}$. So, this split must be of the form $S_{\{j,j+1,\ldots,i\}}$ for some j with $1 < j \leq i - k$, contradicting the minimality of A in $\cup\mathbf{S}$. ∎

Proof of Theorem 3: We proceed by induction with respect to $n \geq 5$. If $n = 5$, then clearly $\#\mathbf{S} \leq 4n - 10 = 10 = \binom{5}{2}$. Suppose that $\#\mathbf{S} \leq 4n - 10$ for all 2-compatible split systems $\mathbf{S} \subseteq \mathbf{S}^{(n)}$. Let $\mathbf{S} \subseteq \mathbf{S}^{(n+1)}$ be a maximal 2-compatible split system. Then, by Lemma 3, $\mathbf{S}$ contains a 3-split. Hence, by Theorem 1(ii),

$$\#\mathbf{S} \leq cycl_2(n) + 4 = 4(n+1) - 10$$

which completes the proof. ∎

Remark 3 *It immediately follows from Lemma 3 and Theorem 1(ii) that* $\mathbf{S} \subseteq \mathbf{S}^{(n)}$ *is a 2-compatible split system of cardinality* $4n-10$ *if and only if* $\mathbf{S}$ *can be constructed using rotations and Construction A.*

In a separate publication, we will deal more explicitly with the number $K_2(n)$ of maximal 2-compatible split systems in $\mathbf{S}^{(n)}$, and with the complexity of algorithms that allow one to construct all of them.

5 3-compatible cyclic split systems

In this section, we give a proof for the following theorem.

Theorem 4 *Suppose that* $n \geq 7$. *If* $\mathbf{S} \subseteq \mathbf{S}^{(n)}$ *is a* 3*-reduced,* 3*-compatible split system, then* $\#\mathbf{S} \leq 3n-21$.

Before we begin this proof, we introduce an important concept for pairs of splits: Given some $x \in X$, a pair of splits S, T of X is called an *x-pair* if $S(x) \cap T(x) = \{x\}$ and $S(x) \cup T(x) = X$ hold. Note that every x-pair consists of two compatible splits, and that two distinct splits $\{A, B\}$ and $\{A', B'\}$ form an x-pair if and only if their restrictions $\{A-\{x\}, B-\{x\}\}$ and $\{A'-\{x\}, B'-\{x\}\}$ to $X-\{x\}$ coincide. For cyclic split systems x-pairs are easy to visualize; for example, in Figure 2 we picture two x-pairs, S_1, S_2 and S_3, S_4 in $\mathbf{S}^{(n)}$ with $x=3$.

Proof of Theorem 4: Note that the theorem is clearly true for $n=7$. Now, suppose that there is a 3-reduced and 3-compatible split system $\mathbf{S} \subseteq \mathbf{S}^{(n)}$ with $\#\mathbf{S} > 3n-20$ and that Theorem 4 holds for all 3-reduced and 3-compatible cyclic split systems $\mathbf{S}'$ defined on a set X of smaller cardinality. We have to show then that $\#\mathbf{S} = 3n-21$ must hold.

In view of Theorem 1, (ii), we may assume that $\mathbf{S}$ does not contain two consecutive 4-splits and that, consequently, no $x \in [n]$ can be contained in the 4-part of three or more 4-splits in $\mathbf{S}$.

By comparing – for each $x \in [n]$ – the given split system $\mathbf{S}$ with the split system

$$\mathbf{S}_x^* := \{\{A, B\} \in \mathbf{S}([n]-\{x\}) :$$

$$\text{size}(\{A, B\}) \geq 4 \text{ and } \{A \cup \{x\}, B\} \in \mathbf{S} \text{ or } \{A, B \cup \{x\}\} \in \mathbf{S}\}$$

we see that it suffices to find one $x \in [n]$ so that either at most one x-pair exists or so that at most two x-pairs exist, yet x is contained in at most one subset A from $\cup\mathbf{S}$ with $\#A = 4$.

So, assume from now on that two x-pairs exist for every $x \in [n]$, and define a pair of incompatible splits $S := \{A, B\}, T := \{C, D\}$ in $\mathbf{S}$ to be *minimal* if $\#(A \cup C)$ is minimal under the condition $\#(A \cap C) = \#A - 1$. Note first that $\#(A \cup C) \geq 6$ must then hold because, by assumption, the two splits S and T cannot be two consecutive 4-splits.

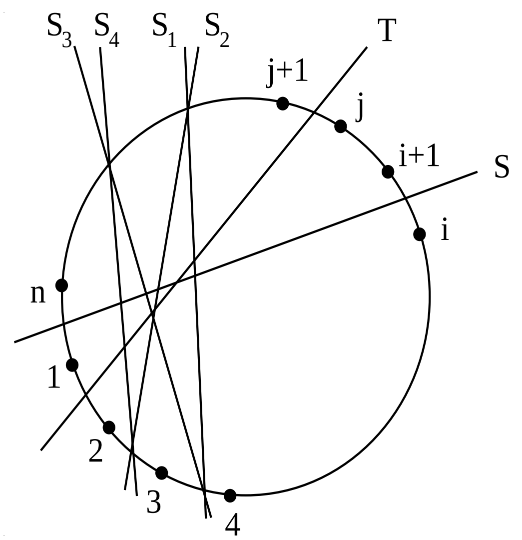

Figure 2: *An impossible configuration.*

Now, without loss of generality, we may assume that $S = S_1^{(i)}$, $T = S_2^{(j)}$, where $j \geq i \geq 4$ and $j \geq 5$. Consider the element $3 \in [n]$. As shown above, $\mathbf{S}$ must contain at least two 3-pairs S_1, S_2 and S_3, S_4 with, say, S_1 and S_3 separating 3 and 4. Note that $S_1, \ldots, S_4$ cannot all be compatible with T because this would be in contradiction to the the minimality of S, T. Moreover, we can't have $S_1, \ldots, S_4$ all incompatible with T because then (cf. Figure 2) S, T, S_2, and S_3 would form a quartet of pairwise incompatible splits. Hence, there must be *precisely* two 3-pairs, and – without loss of generality – we can assume that S_1, S_2 are both compatible with T and that S_3, S_4 are both incompatible with T.

All we need to observe now to complete the proof is that 3 cannot be contained in the 4-part of two or more 4-splits in $\mathbf{S}$. However (cf. Figure 3),

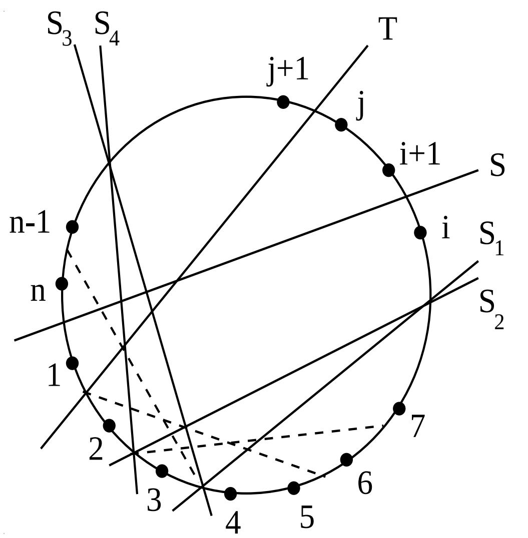

Figure 3: *Three impossible 4-splits: Neither* $S_2^{(4)} = S_{[2,5]}$ *nor* $S_3^{(4)} = S_{[3,6]}$ *can be contained in* $\mathbf{S}$ *because otherwise we see that either the pair* $S_2^{(4)}$ *and* $S_2 = \{\{3,4,\ldots,h\},\{h+1,\ldots,n,1,2\}\}$ *(*$h \leq j$*) or the pair* $S_3^{(4)}$ *and* $S_1 = \{\{4,5,\ldots,h\},\{h+1,\ldots,n,1,2,3\}\}$ *would contradict the minimality of* S *and* T*. And the split* $S_n^{(4)}$ *cannot be contained in* $\mathbf{S}$ *because, together with* S, T *and* S_4*, it forms a quartet of pairwise incompatible splits.*

the minimality of S, T implies that indeed neither $S_2^{(4)} = S_{[2,5]}$ nor $S_3^{(4)} = S_{[3,6]}$ can be contained in $\mathbf{S}$ whereas $S_n^{(4)}$ cannot be contained in $\mathbf{S}$ because, together with S, T and S_4, it forms a set of four pairwise incompatible splits. This establishes Theorem 4. ∎

Remark 4 *In contrast to Remark 3, not every* 3*-compatible cyclic split system with size* $6n - 21$ $(n \geq 7)$ *can be constructed using Construction A. For example, let* $X := \{1, \ldots, 12\}$*, and*

$$\mathbf{S} := \{S_1^{(4)}, S_3^{(4)}, \ldots, S_9^{(4)}, S_{11}^{(4)}, S_1^{(5)}, S_3^{(5)}, \ldots, S_9^{(5)}, S_{11}^{(5)}, S_1^{(6)}, S_5^{(6)}, S_9^{(6)}\} \cup \mathbf{S}_{12,3}.$$

Then it can be shown that $\mathbf{S} \subseteq \mathbf{S}^{(12)}$ *is a* 3*-compatible split system with size* $51 = 72 - 21$*, such that the union of no two* 4*-parts of any two* 4*-splits in* $\mathbf{S}$ *has cardinality five. From this, and Lemma 4 it follows that* $\mathbf{S}$ *cannot be constructed using Construction A.*

6 k-compatible cyclic split systems with $n-2k$ small

In this section we prove a theorem concerning the behavior of k-compatible cyclic split systems when $n-2k$ is small. Our approach relies on the identity

$$2kn - \binom{2k+1}{2} = \binom{n}{2} - \binom{n-2k}{2},$$

and the fact that $\#\mathbf{S}^{(n)} = \binom{n}{2}$ holds.

Theorem 5 *Suppose that $n \geq 2k+1$. If $\mathbf{S} \subseteq \mathbf{S}^{(n)}$ is a k-compatible split system on $[n]$, and $n-(2k+1) \leq 3$, then*

$$\#\mathbf{S} \leq 2kn - \binom{2k+1}{2},$$

that is, $\mathbf{S}$ conforms with Conjecture 1.

Proof: We show that

$$\#\mathbf{S}^{(n)} - \#\mathbf{S} \geq \binom{n-2k}{2}$$

when $0 \leq n-(2k+1) \leq 3$, from which the theorem immediately follows in view of the identity stated above.

First, note that this equation clearly holds for $n = 2k+1$. Now, put $b := n-2k$. We have to show that

$$\#\mathbf{S}^{(n)} - \#\mathbf{S} \geq \binom{b}{2} \tag{1}$$

holds for $b = 2, 3, 4$.

b = 2: Since $\mathbf{S}^{(n)}$ contains exactly $k+1$ splits of size $k+1$ in this case, we must remove at least one of these splits from $\mathbf{S}^{(n)}$ to obtain a k-compatible cyclic split system $\mathbf{S}$. So $\#\mathbf{S}^{(n)} - \#\mathbf{S} \geq 1$ must hold.

b = 3: In this case, there are $2k+3$ splits of size $k+1$, viz. $S_i^{(k+1)}$ for $i = 1, \dots, n$. We may assume, without loss of generality, that $S_1^{(k+1)}$ is not contained in $\mathbf{S}$. Now, since $\mathbf{S}$ is k-compatible, no $k+1$ consecutive $(k+1)$-splits can be in $\mathbf{S}$. Let $i > 1$ be minimal such that $S_i^{(k+1)}$ fails to lie in $\mathbf{S}$.

Then, since $i \leq k+2$, we have $i+k+1 \leq 2k+3$, and so at least one of the splits $S_{i+1}^{(k+1)}, \ldots, S_{i+k+1}^{(k+1)}$ must also fail to lie in $\mathbf{S}$. Thus, we see that $\#\mathbf{S}^{(n)} - \#\mathbf{S} \geq 3$ must hold, as required.

b = 4: Since any two $(k+2)$-splits are necessarily incompatible, we may assume that at least two $(k+2)$-splits lie in the complement of $\mathbf{S}$ in $\mathbf{S}^{(n)}$. Without loss of generality, let two of these splits be $S_1^{(k+2)} = S_{k+3}^{(k+2)}$ and $S_i^{(k+2)} = S_{i+k+2}^{(k+2)}$ where $1 < i \leq k+2$. Similarly, applying the argument given in the case $b = 3$ to the set of $(k+1)$-splits in $\mathbf{S}^{(n)}$, we see that at least three $(k+1)$-splits must also be contained in $\mathbf{S}^{(n)} - \mathbf{S}$. Now, consider the following four collections of splits:

$$\begin{array}{l}
\{S_{k+3+i}^{(k+1)}, \ldots, S_n^{(k+1)}, S_1^{(k+1)}, S_2^{(k+2)}, \ldots, S_{i-1}^{(k+2)}\} \\
\{S_2^{(k+1)}, \ldots, S_i^{(k+1)}, S_{i+1}^{(k+2)}, \ldots, S_{k+2}^{(k+2)}\} \\
\{S_{i+1}^{(k+1)}, \ldots, S_{k+3}^{(k+1)}, S_{k+4}^{(k+2)}, \ldots, S_{k+1+i}^{(k+2)}\} \\
\{S_{k+4}^{(k+1)}, \ldots, S_{k+2+i}^{(k+1)}, S_{k+3+i}^{(k+2)}, \ldots, S_{2k+4}^{(k+2)}\}.
\end{array}$$

Note that each of these collections consists of exactly $k+1$ pairwise incompatible splits, and that every $(k+1)$-split from $\mathbf{S}^{(n)}$ occurs exactly once, while none of the two $(k+2)$-splits $S_1^{(k+2)}$ and $S_i^{(k+2)}$ occurs in either of these four families. Hence, there exits either a third $(k+2)$-split in $\mathbf{S}^{(n)} - \mathbf{S}$, or there exists a fourth $(k+1)$-split in $\mathbf{S}^{(n)}$ that is not contained in $\mathbf{S}$. In any case, there must be a least six splits in $\mathbf{S}^{(n)} - \mathbf{S}$, exactly as required. ∎

Note that Theorems 3, 4 and 5 together establish Theorem 2.

Remark 5 *The above proof implies in particular that – up to isomorphism – there are as many k-compatible split systems in $\mathbf{S}^{(2k+3)}$ of size $\binom{2k+3}{2} - 3$ as there are partitions of $2k$ into three numbers $a, b, c \in \mathbb{N}_0$ with $a \leq b \leq c \leq k$, a number that is straight forward to compute and coincides with*

$$\sum_{c=\lceil \frac{2k}{3} \rceil}^{k} (1 + \lfloor k - \frac{c}{2} \rfloor - 2k + 2c)$$

and, hence, with $1 + 3\kappa + 3\kappa^2$ in case $k = 6\kappa$ for some $\kappa \in \mathbb{N}$, and with $i(1+\kappa) + 3\kappa + 3\kappa^2$ in case $k = 6\kappa + i$ for some $\kappa \in \mathbb{N}$ and some $i \in \{1, \ldots, 5\}$. Of course, only $1 + \lfloor \frac{k}{2} \rfloor$ of these can be constructed from smaller systems (of $b = 2$ type) using Construction A.

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