Séminaire Lotharingien de Combinatoire **47** (2002), Article B47f

COMBINATORIAL FANS, LATTICE-ORDERED GROUPS, AND THEIR NEIGHBOURS: A SHORT EXCURSION

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ABSTRACT. Just like the celebrated dictionary between integral convex geometry and toric varieties bridged until then remote mathematical worlds, the less well-known duality between (certain) lattice-ordered groups and rational polyhedral sets brought combinatorial techniques to bear on the investigation of these abstract algebraic structures. We offer a brief excursion into these relatively new territories, with special emphasis on the novel and admittedly unclear rôle played by order-theoretical considerations.

1. INTRODUCTION

In an inaugural address ([25]) delivered in Florence, Italy, in 1993, G.-C. Rota maintained that

> [while] one of the leading trends that is visible in present-day mathematics [is] 'the return to concreteness', [...] the relationship between combinatorics and algebra today [...] is not a forgetting of the past. [...] some outstanding work in combinatorics that is going on today is greatly benefiting the algebra of yesterday.

As a prominent example, Rota mentioned

> the discovery of toric varieties, [...] a dictionary whereby results of algebraic geometry can be translated into results of convex integral geometry in the style of Minkowski and Hadwiger. [...] The theorems of algebraic geometry now find new and unexpected life in the newly added class of toric varieties, coming from an altogether different source, a combinatorial source to be sure.

The main theme of this paper is a similar interaction between another realm of abstract algebra, the theory of lattice-ordered groups, and a version of "convex integral geometry in the style of Minkowski and Hadwiger" ([25]), the eminently combinatorial theory of rational polyhedral sets. *Prima facie*, ordered groups are no less far removed from polyhedra than algebraic geometry — and yet, much as in the case of toric varieties, the connection is so intimate as to be a genuine duality.

Key words and phrases. Fans, polyhedral cones, convex geometry, lattice-ordered groups, toric varieties.

We begin with a survey of basic results on complexes of polyhedral cones and their subdivisions. We then introduce lattice-ordered abelian groups, and sketch enough of their theory to establish the relationship with polyhedral complexes. Along the way, as a sort of counterpoint to our *leitmotiv*, we indicate the algebro-geometric significance of some of the material introduced. It is only in the last section that we bring this second theme into the foreground for a necessarily short excursion into the world of toric varieties.

2. Combinatorial Convexity: Fans

We introduce a linear version of polyhedral complexes known as fans. The terminology is due to M. Demazure, who called these objects *eventails*. It is quite a descriptive name, provided it is interpreted as 'possibly torn fans' ([23]). A comprehensive treatment may be found in [15].

Throughout this paper, we fix a free abelian group N of rank n. Upon tensoring N with $\mathbb{R}$ we obtain a real n-dimensional vector space $N_{\mathbb{R}} = \mathbb{R} \otimes N$. We let $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ denote the $\mathbb{Z}$-module dual to N, and $M_{\mathbb{R}} = \mathbb{R} \otimes M$ the vector space dual to $N_{\mathbb{R}}$.

In down-to-earth terms, $N \cong \mathbb{Z}^n$, $N_{\mathbb{R}} \cong \mathbb{R}^n$ and N is the group of integral n-tuples in $\mathbb{R}^n$. Similarly, M can be identified with $\mathbb{Z}^n$ and $M_{\mathbb{R}}$ with $\mathbb{R}^n$. These identifications, however, only make sense if we choose a basis of N (and therefore of M, $N_{\mathbb{R}}$, and $M_{\mathbb{R}}$).[1] Whenever convenient, we shall tacitly assume we have fixed one such basis.

Let $\{\vec{v}_1, \ldots, \vec{v}_m\} \subseteq N$ be a finite set of integral vectors. The *rational polyhedral cone* σ, or simply the *cone* σ generated by $\{\vec{v}_1, \ldots, \vec{v}_m\}$ is their positive linear hull over $\mathbb{R}$, i.e.,

$$\sigma = \langle \vec{v}_1, \ldots, \vec{v}_m \rangle = \{p_1\vec{v}_1 + \cdots + p_m\vec{v}_m \mid 0 \leq p_1, \ldots, p_m \in \mathbb{R}\}.$$

A cone σ is *k-dimensional* iff the $\mathbb{R}$-linear space it spans is k-dimensional. A *face* of σ is any nonempty convex subset $\tau \subseteq \sigma$ such that every line segment in σ which has an interior point in τ lies entirely in τ. Lest we burden this paper with an inordinate amount of detail, we shall assume a number of intuitive and easily proved statements about cones — e.g., the faces of a cone are finite in number, the set of faces is a (finite) lattice under inclusion (whence every face is contained in some maximal proper face, a *facet*), and so on.

Upon intersecting a finite number of integral half-spaces (i.e., sets of the form $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid z_1x_1 + \cdots + z_nx_n \geq 0\}$, $z_i \in \mathbb{Z}$) one clearly obtains a cone in $N_{\mathbb{R}}$. That this is no accident is a basic result in convex geometry.

The Fundamental Theorem of Polyhedra. *A subset $\sigma \subseteq N_{\mathbb{R}}$ is a cone iff it is the intersection of finitely many integral half spaces of $N_{\mathbb{R}}$.*

[1] It turns out that this is equivalent to a choice of coordinates on an algebraic variety — see Section 4.

A cone $\sigma = \langle \vec{v}_1, \ldots, \vec{v}_m \rangle$ is *simplicial* iff $\{\vec{v}_1, \ldots, \vec{v}_m\}$ is linearly independent over $\mathbb{R}$. A nonzero vector $\vec{v} \in N$ is *primitive* iff its coordinates (with respect to the canonical basis) are relatively prime. The *vertices* of a simplicial cone σ are the (uniquely determined) primitive vectors $\vec{v}_i \in N$ such that $\sigma = \langle \vec{v}_1, \ldots, \vec{v}_m \rangle$.

A *rational polyhedral fan* (in $N_{\mathbb{R}}$), in short, a *fan*, is a finite set Σ of polyhedral cones such that

(1) Every face of every cone of Σ belongs to Σ.
(2) Any two cones of Σ intersect in a common face.

The *support* of Σ, denoted $|\Sigma|$, is the union of all its cones. A fan Σ is *simplicial* iff all its cones are simplicial.

A polyhedral cone σ carries a natural fan structure Σ_σ which is given by the collection of all its faces (including $\emptyset$ and σ itself). With this notation, $|\Sigma_\sigma| = \sigma$. We shall not be pedantic about the distinction between Σ_σ and σ.

Subdivisions. Subdivisions of rational polyhedral cones, besides being of interest in themselves, encode the birational geometry of toric varieties. Thus, we discuss various notions of refinement for fans (see Figure 1). Given two fans Σ and Δ in $N_{\mathbb{R}}$, Δ is a *subdivision* of Σ, or it *refines* Σ, iff $|\Sigma| = |\Delta|$ and every cone of Δ is contained in some cone of Σ. We write $\Delta \leq \Sigma$ to denote the fact that Δ refines Σ. In this case, every cone of Σ is a union of cones of Δ. More generally, following Danilov ([11]) we say that Σ is *inscribed* in Δ iff every cone of Σ is a union of cones of Δ, whence we have the (possibly strict) inclusion $|\Sigma| \subseteq |\Delta|$.

Just like a polyhedron can be triangulated without adding new vertices (say, by generalising the elementary plane construction of 'adding diagonals'), a fan can always be subdivided into a simplicial fan without adding 1-dimensional cones. The *m-skeleton* $\Sigma^{(m)}$ of a fan Σ is the set of m-dimensional cones of Σ. (It is not itself a fan, for it does not contain, together with a cone, all of its faces). Then there exists a simplicial fan Δ_1 with the same 1-skeleton of Σ such that $\Delta_1 \leq \Sigma$.

In many contexts, simplicial subdivisions are not fine enough — it turns out that a certain proper subclass of simplicial cones plays a key rôle in the theory of toric varieties. A cone $\sigma = \langle \vec{v}_1, \ldots, \vec{v}_m \rangle \subseteq N_{\mathbb{R}}$ is *unimodular* (alternative terminology, *regular*) iff $\{\vec{v}_1, \ldots, \vec{v}_m\}$ can be completed to a basis of N. Hence, in particular, σ is simplicial and the set of its vertices is $\{\vec{v}_1, \ldots, \vec{v}_m\}$. A fan Σ is unimodular iff all its cones are unimodular.

Clearly, not every simplicial cone is unimodular. However, it is always possible to subdivide a given cone into unimodular subcones. Further, this can always be done through *stellar subdivisions*. Let Σ be a fan in $N_{\mathbb{R}}$, and let $\sigma \in \Sigma$ be an m-dimensional cone. The *star* in Σ of a face $F \subseteq \sigma$, denoted $\operatorname{star} F$, is the set of cones in Σ having F as a face; the *closed star* of F is the fan $\operatorname{cstar} F = \{\tau \mid \tau \text{ is a face of a cone } \theta \in \operatorname{star} F\}$. The *relative interior* of σ, denoted $\operatorname{relint} \sigma$, is the topological interior of $\sigma \cap V$, where V is the linear subspace of $N_{\mathbb{R}}$ spanned by σ. Let $\vec{w} \in \operatorname{relint} \sigma \cap N$. Then there is a unique minimal face F of σ such that $\vec{w} \in \operatorname{relint} F$, namely, the intersection of all faces containing $\vec{w}$. The *stellar subdivision*[2] of Σ along σ through $\vec{w}$ is the fan obtained from Σ removing $\operatorname{star} F$ and joining $\vec{w}$ to the boundary $(\operatorname{cstar} F - \operatorname{star} F)$ of the star of F. If Δ is obtained from Σ via a finite number (may be zero) of stellar subdivisions, we write $\Sigma \leq^* \Delta$. Clearly, $\Sigma \leq^* \Delta$ implies $\Sigma \leq \Delta$, but not conversely.

Unimodular Refinements. *Let Σ be a fan. There exists a simplicial fan Δ_1 with the same 1-skeleton of Σ such that $\Delta_1 \leq \Sigma$. Further, there exists a unimodular fan Δ_2 such that $\Delta_2 \leq^* \Delta_1 \leq \Sigma$, that is, Δ_2 is obtained from Δ_1 by a finite number of stellar subdivision.*

Remark. Unimodularity is the combinatorial counterpart of the central algebro-geometric notion of *smoothness* ([23], [16], [15]).

For self-contained proofs of the existence of simplicial and unimodular refinements, see e.g. [15].

A special sort of stellar subdivision is particularly important for the contents of this paper. Let $\sigma = \langle \vec{v}_1, \ldots, \vec{v}_m \rangle$ be an m-dimensional cone of a fan Σ in $N_{\mathbb{R}}$. Suppose $\tau = \langle \vec{v}_{i_1}, \ldots \vec{v}_{i_d} \rangle$ is a d-dimensional face of σ. We call $\vec{w} = \sum \vec{v}_{i_j}$ the *barycenter* of τ. The fan obtained from Σ by stellar subdivision along τ through $\vec{w}$ is a *barycentric stellar subdivision of* Σ. We write $\Delta \preceq \Sigma$ to denote the fact that Δ is obtained from Σ by a finite number (maybe zero) of stellar barycentric subdivisions; if all barycenters belong to 2-dimensional cones, we write $\Delta \preceq_2 \Sigma$ and speak of *binary starring.* It is easy to show but important to realize that if $\Delta \preceq \Sigma$ and Σ is unimodular, then Δ is unimodular — *barycentric subdivisions preserve unimodularity.* This fails for general stellar subdivisions.[3]

If Σ is simplicial but not unimodular, a fan Δ such $\Delta \preceq \Sigma$ cannot be unimodular. Thus barycentric subdivisions cannot be used to refine a given fan into a unimodular one. On the other hand, if Σ happens to be unimodular, binary starrings produce most general subdivisions, in the following sense.

The De Concini-Procesi Lemma. *Let Σ, Δ be two unimodular fans in $N_{\mathbb{R}}$, and assume $|\Sigma| = |\Delta|$. There exists a unimodular fan Σ^* in $\mathbb{R}^n$ such that*

(1) $\Sigma^* \prec_2 \Sigma$;
(2) $\Sigma^* \leq \Delta$.

The De Concini-Procesi Lemma was first established in [14], and may be interpreted as a statement on the birational geometry of toric varieties. The interested reader can find a self-contained elementary proof in Panti's paper [24].

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[2]Stellar subdivisions were introduced by James W. Alexander in [3] for abstract simplicial complexes, and in [2] for geometric complexes. Their importance in algebraic topology is well-known — see e.g. [4].

[3]The barycenter of a rational simplex is known to geometers of numbers as the *(multidimensional) Farey mediant* of its vertices, a classical notion intimately connected with (multidimensional) continued fraction expansions. See [10], [22] and references therein for further information.

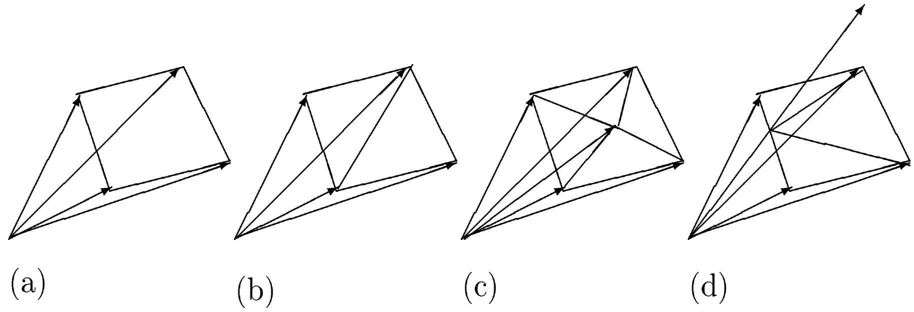

FIGURE 1. Subdivisions of a cone. (a) A cone C. (b) A subdivision (triangulation) of C. (c) A stellar subdivision of C. (d) A barycentric stellar subdivision of C.

MORPHISMS OF FANS. Let Σ be a fan in $N_{\mathbb{R}}$. Let N' be a free abelian group of rank n', $N'_{\mathbb{R}} = \mathbb{R} \otimes N'$, Σ' a fan in $N'_{\mathbb{R}}$. A *map of fans* is an element of $\mathrm{Hom}_{\mathbb{Z}}(N, N')$ — a group homomorphism $\phi : N \to N'$ — such that its (unique) scalar extension $\phi_{\mathbb{R}} : N_{\mathbb{R}} \to N'_{\mathbb{R}}$ maps cones into cones; explicitly, for each $\sigma \in \Sigma$ there is $\sigma' \in \Sigma'$ such that $\phi_{\mathbb{R}}(\sigma) \subseteq \sigma'$. In the fan-toric dictionary included in Section 4, maps of fans correspond to morphisms of toric varieties. The latter provided the original motivation for the definition of a map of fans.

Let us restrict attention to an endomorphism ϕ of fans with the same support — that is, assume $N' = N$ and $|\Sigma'| = |\Sigma|$. A moment's reflection shows that ϕ induces a subdivision of Σ into Σ'. Conversely, any fan Σ' in $N_{\mathbb{R}}$ such that $\Sigma' \leq \Sigma$ canonically induces an endomorphism ϕ of fans which fixes the support $|\Sigma|$. Thus, all forms of subdivisions we have introduced above may be regarded as *morphisms* (of fans or of toric varieties). The combinatorially fundamental notion of stellar subdivision gives rise to morphisms of great interest in algebraic geometry, namely *blow-ups* — the basic tool used to resolve singularities of algebraic varieties. As we have seen above, every fan admits a unimodular refinement. This translates to the statement that *every toric variety admits a resolution of singularities.* While we refer to [23], [16] and [15] for further information on the algebro-geometric aspects of desingularizations, we include a fan-theoretic treatment of resolutions of two-dimensional toric singularities, both by way of illustration and with the intent of showing interesting connections with the geometry of numbers.

A cone $\sigma \subseteq N_{\mathbb{R}}$ is *strongly convex* (alternative terminology, has *apex* at the origin) iff it does not contain a linear subspace of $N_{\mathbb{R}}$, except $\{0\}$.[4] For the remaining part

[4]This is an interesting notion for many reasons, amomgst which is the following characterization: for $m \geq 2$, an m-dimensional cone is strongly convex iff it is the positive linear hull of a polytope of dimension $m-1$. A polytope (= bounded polyhedron) is the convex hull of a finite set of points in $\mathbb{R}^n$.

of this section, set $n = 2$, and let Σ be a fan in $N_{\mathbb{R}} \cong \mathbb{R}^2$. To avoid tedious degenerate cases, we assume all cones of Σ are strongly convex. Let $\mathcal{S}_\Sigma$ be the set of all unimodular fans with the same support as Σ. Clearly, refinement is a partial order on $\mathcal{S}_\Sigma$. We now claim that the poset $\langle \mathcal{S}_\Sigma, \leq \rangle$ is in fact a lattice.[5] Furthermore, we claim we can *compute* finite lattice joins and meets in $\langle \mathcal{S}_\Sigma, \leq \rangle$. It suffices to show this much: Given unimodular fans Σ_1,$\Sigma_2 \in \mathcal{S}_\Sigma$, one can compute $\Delta \in \mathcal{S}_\Sigma$ with the property that if $\Sigma' \leq \Sigma_1$, Σ_2 for some $\Sigma' \in \mathcal{S}_\Sigma$, then $\Delta \leq \Sigma'$. The argument for the existence and computability of joins is similar.

Let Σ_1,$\Sigma_2 \in \mathcal{S}_\Sigma$, and consider the set $\Theta = \{\sigma_1 \cap \sigma_2 \mid \sigma_i \in \Sigma_i, i = 1, 2\}$. It is promptly seen that Θ is a fan such that $|\Theta| = |\Sigma|$. Further, Θ jointly refines Σ_1 and Σ_2, and it is clear that it is the coarsest such refinement. Unfortunately, Θ need not be unimodular, whence in general $\Theta \notin \mathcal{S}_\Sigma$. Our claims are settled if we can show that *it is possible to compute a coarsest (hence, canonical) unimodular refinement of* Θ.

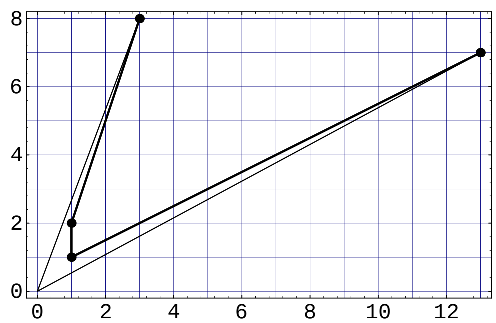

FIGURE 2. Canonical unimodular refinement of a 2-dimensional cone.

Clearly, it suffices to proceed locally and restrict attention to a single strongly convex 2-dimensional cone σ in $N_{\mathbb{R}}$, whence, in particular, σ is simplicial. Consider the convex hull P in $N_{\mathbb{R}}$ of all integral points $(\sigma \cap N) - \{0\}$. (See Figure 2.) This is an unbounded polygon with a finite number of compact edges. Let us display the (necessarily integral) endpoints of such edges as the finite set of vectors

$$\{\vec{v}_0, \ldots, \vec{v}_m\}$$

such that $\vec{v}_i$ and $\vec{v}_{i+1}$ are adjacent[6], for $0 \leq i < m$. To fix ideas, suppose that the order in which we have listed the $\vec{v}_i$'s is counterclockwise. For each i, $\sigma_i = \langle \vec{v}_i, \vec{v}_{i+1} \rangle$ is a 2-dimensional simplicial cone whose vertices are precisely the vectors $\vec{v}_i$, $\vec{v}_{i+1}$ — if some $\vec{v}_i$ were not primitive, the primitive generator of $\langle \vec{v}_i \rangle$ would not belong to P, a contradiction. Furthermore, each σ_i is unimodular, as is easy to check. For $0 \leq i < m$, the collection of all cones σ_i, together with their 1-dimensional faces $\vec{v}_i$ and the origin of $N_{\mathbb{R}}$, is a unimodular fan Σ^σ refining σ. A moment's reflection

[5] With maximum, but with no minimum, if Σ has at least one 2-dimensional cone.

[6] That is, their convex hull in $N_{\mathbb{R}}$ does not contain any other $\vec{v}_k$, $k \neq i$, $k \neq i + 1$.

shows that *any unimodular subdivision of* σ *must refine* Σ^σ. We have thus shown the existence of the coarsest desingularization of σ. Finally, we exhibit an explicit algorithm to compute Σ^σ, given a presentation $\sigma = \langle \vec{u}_1, \ldots, \vec{u}_t\rangle$ of the cone σ. Notice that from any such presentation one can compute the vertices of σ — first compute rational vectors $\vec{r}_1$, $\vec{r}_2$ such that $\sigma = \langle \vec{r}_1, \vec{r}_2\rangle$, and then use the Euclidean algorithm to compute the vertices. Without loss of generality, then, we may assume that σ is presented as $\sigma = \langle \vec{w}_1, \vec{w}_2\rangle$, where $\{\vec{w}_1, \vec{w}_2\}$ is the set of its vertices, again listed in counterclockwise order. Notice that $\vec{w}_1 = \vec{v}_0$ and $\vec{w}_2 = \vec{v}_m$. Since these vertices are primitive linearly independent vectors, another application of the Euclidean algorithm yields a primitive vector $\vec{n}_1$ and integers p, q such that

- $\{\vec{w}_1, \vec{n}_1\}$ is a basis of the $\mathbb{Z}$-module N — in other words, $\langle \vec{w}_1, \vec{n}_1\rangle$ is unimodular;
- $0 \leq p < q$ with p and q relatively prime;
- $\vec{w}_2 = p\vec{w}_1 + q\vec{n}_1$.

In case $q = 1$, we are done, for $p = 0$ and σ is unimodular. Otherwise, we consider the *Hirzebruch-Jung continued fraction expansion*[7]

$$\frac{q}{q-p} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \cdots}}$$

with $b_j \geq 2$, which must terminate since $\frac{q}{q-p}$ is rational. Then it can be proved that, upon inductively defining

$$\vec{n}_{i+1} = (b_i - 2)\vec{v}_{i-i} + (b_i - 1)\vec{n}_i \ ,$$

the following equalities hold:

$$\vec{v}_i = \vec{v}_{i-1} + \vec{n}_i \ ,$$

whence the vectors $\vec{v}_i$ can be computed, as claimed. Thus, the Hirzebruch-Jung continued fraction expansion is precisely what is needed for the effective desingularization of 2-dimensional toric varieties.

Remark. In higher dimensions, things are far more complicated and not yet well-understood. Already in dimension 3, in general there is no coarsest unimodular refinement of a given strongly convex cone $\sigma \subseteq \mathbb{R}^3$. However, the set of maximally coarse refinements is finite, and its classification was carried out by M. Reid and V. I. Danilov (see [23] for details and references). However, the Danilov-Reid treatment is non-effective. Algorithmic desingularization of 3-dimensional toric variety was attained in the paper [1], to which we refer the interested reader for further information.

[7]This is different from the usual continued fraction expansion, sometimes called *regular*, where minus signs are replaced by plus signs. See [16].

3. Abelian lattice-ordered groups as functions on fans

This section is devoted to show that bringing together ordered algebraic structures and fans is in the nature of things — indeed, there is an intimate connection between (finitely presented) lattice-ordered groups and integral piecewise linear geometry. We begin with an admittedly cursory introduction to ordered groups. Some standard references are [17] and [13].

A *partially ordered abelian group* is an abelian group G together with a submonoid G^+ of G such that the induced binary relation

$$a \geq b \quad \text{iff} \quad a - b \in G^+$$

is a partial order which is translation invariant with respect to addition, i.e.,

$$a \geq b \quad \text{implies} \quad a + t \geq b + t \quad \text{for all} \quad a, b, t \in G$$

The submonoid G^+ is the *positive cone* of G. Notice that $0 \in G^+$. An element $p \in G$ is *positive* iff $p \in G^+$, and it is *strictly positive* iff $p \in G^+ - \{0\} = G_+$. Accordingly, $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}_+ = \mathbb{N}$.

A partially ordered abelian group G is *directed* iff any two elements of G have a lower and an upper bound, and it is *unperforated* iff $ng \in G^+$, $n \in \mathbb{N}$, implies $g \in G^+$. Further, G has the *Riesz Interpolation Property* iff for any $x_1, x_2, y_1, y_2 \in G$, $x_i \leq y_j$ implies that there exists $z \in G$ such that $x_i \leq z \leq y_j \quad (i, j = 1, 2)$. Homomorphisms of partially ordered abelian groups are order-preserving group homomorphisms. A subgroup $J \subseteq G$ is an *o-ideal* iff it is *convex* ($a \leq b \leq c$ and $a, c \in J$ imply $b \in J$) and directed. Kernels of homomorphisms, then, are precisely o-ideals.

A *lattice-ordered abelian group* (ℓ-group, for short) is a partially ordered abelian group which is a lattice with respect to the order relation. Equivalently, an ℓ-group is a structure $(G, +, 0, \wedge, \vee)$ such that $(G, +, 0)$ is an abelian group, $(G, \wedge, \vee)$ is a lattice, and $+$ distributes over $\vee$ and $\wedge$. By an *ℓ-subgroup* of an ℓ-group we mean a subgroup which is also a sublattice. Given a subset $S \subseteq G$ of an ℓ-group, we denote by $\langle S \rangle$, $[S]$ and $[[S]]$ the ℓ-subgroup, the subgroup and the submonoid generated by S.

Homomorphisms of ℓ-groups (*ℓ-homomorphisms*) are maps preserving both the group and the lattice structure. We call their kernels *ℓ-ideals*. It is easy to check that ℓ-ideals are the same thing as convex ℓ-subgroups. One easily deduces appropriate versions of the usual isomorphisms theorems. Let G be an ℓ-group, J an ℓ-ideal of G, ϕ_J the canonical ℓ-homomorphism associated to J. We denote by G/J the ℓ-group quotient of G by J, or, equivalently, the image of G under ϕ_J. We use the symbol '$\cong$' to denote isomorphism of ℓ-groups.

Free ℓ-groups. Since ℓ-groups are equationally definable, they are a variety and thus admit free (more generally, finitely presented) objects. We denote by $\mathcal{F}_\kappa$ the free ℓ-group over κ generators, for any cardinal κ. A very strong representation theorem for $\mathcal{F}_\kappa$ holds. Its interesting but slightly involved history is more thoroughly traced in [19], where precise references to the original results can be found — here we summarize the mathematical facts. Let $n \in \mathbb{N}$. By an *ℓ-function* (of n variables) we mean a continuous piecewise linear homogeneous map $f : \mathbb{R}^n \to \mathbb{R}$ with integral coefficients.[8] The set $\mathcal{C}_n$ of all continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$ is an ℓ-group under pointwise addition and order. Let $\mathcal{L}_n \subseteq \mathcal{C}_n$ denote the ℓ-subgroup of all ℓ-functions of n variables. Further, let $\mathcal{D}_n \subseteq \mathcal{C}_n$ denote the ℓ-subgroup generated by the projection functions $\pi_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, n$. Clearly, $\mathcal{D}_n \subseteq \mathcal{L}_n \subset \mathcal{C}_n$. Besides, a trivial universal-algebraic argument shows that $\mathcal{D}_n$ is a homomorphic image of $\mathcal{F}_n$. Non-trivial geometric and algebraic arguments yield the following strengthening of the previous statements.[9]

Weinberg-Beynon/Chang-McNaughton Representation. $\mathcal{F}_n \cong \mathcal{D}_n = \mathcal{L}_n$.

Remark. One half of the theorem ($\mathcal{F}_n \cong \mathcal{D}_n$, or $\mathbb{Z}$ generates the variety of abelian ℓ-groups), is due to Weinberg ([26]). The other half ($\mathcal{D}_n = \mathcal{L}_n$, or the ℓ-group of ℓ-functions is generated by the projection functions) is due to Beynon ([7]). As proved by the second author ([21]; also see [10] for self-contained proofs), abelian ℓ-groups with strong order unit[10] are categorically equivalent to *M(any)V(alued)-algebras*, a (natural) generalization of boolean algebras introduced by C. C. Chang in his algebraic analysis of Łukasiewcz infinite-valued propositional logic. MV-algebraic versions of Weinberg's and Beynon's aforementioned results were proved by Chang ([9]) and McNaughton ([20]), respectively. For the interplay between ordered groups and MV-algebras see [19] and [10].

On the basis of the above representation theorem, we shall tacitly identify $\mathcal{F}_n$ and $\mathcal{L}_n$ (equivalently, $\mathcal{D}_n$) whenever convenient.

FINITELY PRESENTED ℓ-GROUPS. Let G be an ℓ-group. An ℓ-ideal J of G is *generated* by a set $S \subseteq G$, written $J = \langle\langle S \rangle\rangle$, iff the intersection of all ℓ-ideals of G containing S is precisely J. We say that $J = \langle\langle S \rangle\rangle$ is *finitely* generated iff S can be chosen finite. Further, J is *principal* iff it is generated by a singleton $\{g\} \subseteq G$. In this case, we write $J = \langle\langle g \rangle\rangle$. It is an exercise to check that every finitely generated ℓ-ideal is principal. We call an ℓ-group G *finitely presented* iff $G \cong \mathcal{F}_n/\langle\langle g \rangle\rangle$ for some $n \in \mathbb{N}$. A useful description of the principal ideal $\langle\langle g \rangle\rangle$ is the following: $f \in \langle\langle g \rangle\rangle$ iff there is some $n \in \mathbb{N}$ such that $ng \geq f$ (in $\mathcal{F}_n$). Given $f \in \mathcal{F}_n$, the *zeroset* of f is the set $\mathbb{V}(f) = \{\vec{x} \in \mathbb{R}^n \mid f(\vec{x}) = 0\}$. More generally, for any subset $S \subseteq \mathcal{F}_n$, let us define $\mathbb{V}(S) = \bigcap_{f \in S} \mathbb{V}(f)$. From the description of $\langle\langle g \rangle\rangle$ given a moment ago, it follows that $\mathbb{V}(\langle\langle g \rangle\rangle) = \mathbb{V}(g)$. We now claim that the zeroset of an ℓ-function g

[8] In detail, a function $f : \mathbb{R}^n \to \mathbb{R}$ is an ℓ-function iff it is continuous and there exist finitely many homogeneous linear functions $l_i : \mathbb{R}^n \to \mathbb{R}$ ($i \in \{1, \ldots, m\}$) with integer coefficients, the *linear constituents* of f, such that for every $\vec{x} \in \mathbb{R}^n$ there is some $i \in \{1, \ldots, m\}$ such that $f(\vec{x}) = l_i(\vec{x})$.

[9] Generalization to an arbitrary number of generators is straightforward.

[10] An element $u \in G$ is a *strong order unit* of the ℓ-group G iff for every $p \in G_+$ there exists $n \in \mathbb{N}$ such that $nu \geq p$.

is the support of a fan. Let $g \in \mathcal{F}_n$ be an ℓ-function, and let $\{l_i\}$ ($i \in \{1,\ldots,m\}$) be the set of its linear constituents (cfr. footnote 8). Let $\mathcal{S}_m$ denote the group of permutations of m letters $\{1,\ldots,m\}$ (the symmetric group). For each $\sigma \in \mathcal{S}_m$, we define the *basic cone*

$$C_\sigma = \{\vec{x} \in \mathbb{R}^n \mid l_{\sigma(1)}(\vec{x}) \geq \ldots \geq l_{\sigma(m)}(\vec{x})\}\,.$$

It is clear that C_σ is a closed set; also notice that it may well be the singleton $\{0\}$. In fact, C_σ is the intersection of finitely many integral half-spaces, as is read off directly from its definition. Hence, by the Fundamental Theorem of Polyhedra, it is a (rational polyhedral) cone. Furthermore, the set of all such C_σ and their finite intersections $C_{\sigma_1} \cap \cdots \cap C_{\sigma_s}$, as $\sigma_1,\ldots,\sigma_s$ vary in $\mathcal{S}_m$, is a fan Σ_g in $N_\mathbb{R} \cong \mathbb{R}^n$. Since g is linear over each C_σ, its zeroset $\mathbb{V}(g)$ can be described as the support of a fan Θ_g inscribed in Σ_g, as we claimed. In analogy with the time-honored tradition of classical algebraic geometry, it makes sense to investigate the relationship between the ℓ-group of all ℓ-functions in $\mathcal{F}_n$ restricted to $|\Theta_g|$ — a geometric entity — and the ℓ-group $\mathcal{F}_n/\langle\langle g\rangle\rangle$ — an algebraic object. In order to do this at the natural level of generality, we introduce some further notions from the theory of ordered groups.

Let G be an ℓ-group and $J \subseteq G$ an ℓ-ideal. We say that J is *maximal* iff for every ℓ-ideal $I \subseteq G$, $J \subseteq I$ implies $J = I$. Further, J is *proper* iff $J \neq G$. The intersection of any collection of ℓ-ideals is an ℓ-ideal. The *radical* of G is the intersection of all its maximal ℓ-ideals. Finally, G is *archimedean* iff for every $p, q \in G_+$ such that $p \leq q$, there is $n \in \mathbb{N}$ such that $np \not\leq q$. Assume that G is finitely generated. Then G is archimedean iff its radical is $\{0\}$, as is easy to see.

Quite generally, if $J \subseteq \mathcal{F}_n$ is an ℓ-ideal of the free ℓ-group over n variables, then $\mathbb{V}(J)$ is a closed conical[11] subset of $N_\mathbb{R}$. Conversely, given a closed conical subset $C \subseteq N_\mathbb{R}$, let $\mathbb{I}(C)$ denote the set of all ℓ-functions of n variables vanishing over C. Then $\mathbb{I}(C)$ is an ℓ-ideal. Clearly, $\mathbb{V}(\mathbb{I}(C)) = C$ for any closed conical set C. However, $J \subseteq \mathbb{I}(\mathbb{V}(J))$ may well be a strict inclusion. The following result, characterizing those ℓ-ideals J such that $J = \mathbb{I}(\mathbb{V}(J))$, is the analogue of Hilbert's Nullstellensatz.

Archimedean representation.[12] Let G be a finitely generated ℓ-group, and assume $G \cong \mathcal{F}_n/J$. Let $\mathcal{F}_n \restriction \mathbb{V}(J)$ denote the ℓ-group of ℓ-functions of n variables restricted to $\mathbb{V}(J)$. Then $G \cong \mathcal{F}_n \restriction \mathbb{V}(J)$ iff G is archimedean iff $J = \mathbb{I}(\mathbb{V}(J))$.

Thus, returning to the finitely presented ℓ-group $G \cong \mathcal{F}_n/\langle\langle g\rangle\rangle$, we need to ascertain whether G can violate the archimedean condition. It cannot.

Wójcicki-Baker's Theorem.[13] A finitely presented ℓ-group is archimedean.

[11] A subset $C \subseteq \mathbb{R}^n$ is *conical* iff it contains, together with every point $\vec{x} \in C$, the ray $\{p\vec{x} \mid p \in \mathbb{R}^+\}$ generated by that point.

[12] Fair attribution of this result would take up an amount of space we are not ready to devote to the matter here. See [10] and [19].

[13] This is due to Baker (ℓ-group version, [6]) and Wójcicki (MV-algebraic version, [27]).

As an immediate consequence of the above results, a finitely presented ℓ-group can be identified with the ℓ-group of ℓ-functions of n-variables restricted to the support of a fan. Conversely, given a fan Σ in $N_{\mathbb{R}}$, it is possible to describe its support as the zeroset of an ℓ-function g, thus obtaining a finitely presented ℓ-group $\mathcal{F}_n/\langle\langle g\rangle\rangle$ associated with Σ. It turns out that this duality between fans and ℓ-groups is the germ of a full-scale categorical equivalence.

MORPHISMS OF FINITELY PRESENTED ℓ-GROUPS. The maps of fans we have hitherto considered are motivated by algebro-geometric considerations — they are the direct combinatorial translation of the notion of toric morphism. We now introduce another class of morphisms of fans which is of equal importance for the contents of this paper. Let N' be a free abelian group of rank n', $N'_{\mathbb{R}} = \mathbb{R} \otimes N'$, Σ' a fan in $N'_{\mathbb{R}}$. Let Σ be a fan in $N_{\mathbb{R}}$ and Σ' a fan in $N'_{\mathbb{R}}$. A function $\phi : |\Sigma| \to |\Sigma'|$ is an *ℓ-map* iff $\phi(\vec{x}) = (f_1(\vec{x}), \ldots, f_{n'}(\vec{x}))$ for every $\vec{x} \in |\Sigma|$, where f_i is an ℓ-function of n variables. The fan Σ is *ℓ-embedded* in Σ' iff ϕ is injective; Σ' is an *ℓ-homomorphic image* of Σ iff ϕ is surjective; Σ and Σ' are *ℓ-homeomorphic* iff there is an ℓ-map $\psi : |\Sigma'| \to |\Sigma|$ such that $\psi \circ \phi$ is the identity function on Σ.

Baker-Beynon Duality.[14]

(1) An abelian ℓ-group P is finitely presented iff it is isomorphic to the ℓ-group of ℓ-functions of $n \in \mathbb{N}$ variables restricted to the support of some fan Σ in $N_{\mathbb{R}}$. We call Σ a fan *associated* with P.

(2) Two finitely presented ℓ-groups P_1, P_2 are isomorphic iff for every two fans Σ_1, Σ_2 associated with P_1, P_2, respectively, one has that Σ_1 and Σ_2 are ℓ-homeomorphic.

We add a few remarks on onto and into morphisms to round off the above statement.[15] Let $P \twoheadrightarrow Q$ be a surjective homomorphism of finitely presented ℓ-groups P and Q,[16] and let Σ be a fan associated with P. Then there is a fan Σ' associated with Q such that $|\Sigma'| \subseteq |\Sigma|$, the latter inclusion being an ℓ-embedding of fans. (However, it can also be shown that there exists a surjective ℓ-map of fans $\rho : |\Sigma| \twoheadrightarrow |\Sigma'|$ which *retracts* Σ onto Σ', i.e., ρ restricted to Σ' is the identity.) Conversely, consider an ℓ-embedding of fans $\phi : |\Sigma| \rightarrowtail |\Sigma'|$, and let P and Q be the finitely presented ℓ-groups associated with Σ and Σ'. We identify $|\Sigma|$ with a subset of $|\Sigma'|$. Restricting ℓ-functions on $|\Sigma'|$ (= elements of Q) to the subset $|\Sigma|$, we obtain that P is a homomorphic image of Q. (However, it can indeed be shown that P is also a subalgebra of Q, because there is an ℓ-map $\rho : |\Sigma'| \twoheadrightarrow |\Sigma|$ which retracts $|\Sigma'|$ onto $|\Sigma|$ — that is, $\rho \circ \phi$ is the identity on $|\Sigma|$.) Hence we see that ℓ-embeddings in the category of fans

[14] The original references are [6],[7],[8].

[15] Some of the assertions we are about to make are not trivial to prove. Some of the material may be novel, whence the absence of detailed references. What is not new can be found in [6],[7],[8].

[16] Notice that it is not possible to do without the assumption that Q be finitely presented — P may have non-archimedean (let alone non-finitely presentable) homomorphic images.

correspond to surjections of finitely presented ℓ-groups. The analogous results for onto ℓ-maps of fans can also be established. Finally, in the light of (2) above, these statements hold for *any* two fans associated with P and Q. The foregoing could be easily recast into a categorical equivalence between fans and their ℓ-maps, on the one hand, and finitely presented ℓ-groups and their homomorphisms, on the other.

Δ-LINEAR SUPPORT FUNCTIONS. Perusal of any textbook on toric varieties will reveal no reference to lattice-ordered groups — in its long and distinguished history, geometry has seldom taken advantage of order-theoretical considerations.[17] We shall now argue that finitely presented ℓ-groups do arise in a natural manner from the theory of toric varieties.

Let Δ be a fan in $N_{\mathbb{R}}$. Let $|\Delta|_{\mathbb{Z}} = N \cap |\Delta|$ be the set of integral points in $|\Delta|$. A function $f : |\Delta| \to \mathbb{R}$ is a *Δ-linear support function* iff it is linear on each $\sigma \in \Delta$, and $f \restriction |\Delta|_{\mathbb{Z}}$ is integer-valued. We shall denote by $\mathrm{SF}(\Delta)$ the set of Δ-linear support functions. It is an abelian group under addition. (We remark in passing that each element of the dual lattice M yields an integral linear — *a fortiori* Δ-linear — support function, but this correspondence is injective iff $|\Delta|$ spans the whole space $N_{\mathbb{R}}$.) ¿From what we expounded above, up to isomorphism there is a unique finitely presented ℓ-group P_Δ associated with Δ, namely $P_\Delta \cong \mathcal{F}_n/\mathbb{I}(|\Delta|)$. It is easy to check that a Δ-linear support function is the same thing as an ℓ-function on $|\Delta|$ which is linear over each $\sigma \in \Sigma$. In other words, $\mathrm{SF}(\Delta) \subseteq P_\Delta$. We thus see that $\mathrm{SF}(\Delta)$ inherits a partial order from P_Δ turning $\mathrm{SF}(\Delta)$ into a partially ordered abelian group. Notice that $\mathrm{SF}(\Delta)$ is not an ℓ-subgroup of P_Δ — it is not closed under lattice operations. Given $s \in \mathrm{SF}(\Delta)$, to each ray (= 1-dimensional cone) $\rho_i = \langle \vec{r_i} \rangle \in \Delta^{(1)}$ generated by its (unique) vertex $\vec{r_i}$, we can associate the integral value $s(\vec{r_i})$; if the 1-skeleton $\Delta^{(1)}$ has cardinality t, we thus obtain for each s an element $(s(\vec{r}_1), \ldots, s(\vec{r}_t))$ of the free abelian group $\mathbb{Z}^t$. Further, the sum of two Δ-linear support functions corresponds to the coordinatewise sum of the associated elements of $\mathbb{Z}^t$. In other words, we have an embedding of abelian groups $\phi : \mathrm{SF}(\Delta) \rightarrowtail \mathbb{Z}^t$. An ℓ-group is *simplicial* iff it is isomorphic (as an ℓ-group) to $\langle \mathbb{Z}^k, \leq \rangle$, where $(v_1, \ldots, v_k) \leq (w_1, \ldots, w_k)$ iff $v_i \leq w_i$ for all $i \in \{1, \ldots, k\}$ (*coordinatewise* or *simplicial order*). Through the group embedding ϕ, the simplicial order on $\mathbb{Z}^t$ induces a partial order on $\mathrm{SF}(\Delta)$. It is an exercise to check that the latter coincides with the order induced on $\mathrm{SF}(\Delta)$ by the inclusion $\mathrm{SF}(\Delta) \subseteq P_\Delta$. Thus ϕ is naturally an embedding of partially ordered abelian groups. It is not hard to show that *ϕ is onto iff Δ is unimodular*. To put it differently, when Δ is unimodular, for every assignment $\alpha : \{\vec{r}_1, \ldots, \vec{r}_k\} \to \mathbb{Z}$ of integral values to the vertices $\vec{r_i}$, there is a unique Δ-linear support function $s_\alpha \in \mathrm{SF}(\Delta)$ such that $s_\alpha(\vec{r_i}) = \alpha(\vec{r_i})$ for all $i \in \{1, \ldots, k\}$; whereas when Δ is not unimodular, there exists an assignment α which cannot be thus realized by any $s \in \mathrm{SF}(\Delta)$.

[17] One notable exception being real semi-algebraic geometry.

Now assume Δ is unimodular and *complete*, that is, $|\Delta| = N_{\mathbb{R}}$, whence $P_\Delta \cong \mathcal{F}_n$ is the free abelian ℓ-group over n generators. We remark in passing that any (unimodular) fan Σ can be extended to a complete (unimodular) fan — intuitive as this may sound, the proof is not trivial, see [15].[18] Then $\phi : \mathrm{SF}(\Delta) \rightarrowtail \mathbb{Z}^t$ is an isomorphism of partially ordered groups. In any simplicial group S, there exists a unique set of strictly positive linearly independent generators. We call such a set the *simplicial basis* of S. Let $\{\vec{h}_1, \ldots, \vec{h}_t\} \subseteq \mathbb{Z}^t$ be the simplicial basis of $\mathbb{Z}^t$. Via the injection $\mathrm{SF}(\Delta) \subseteq P_\Delta$ and the isomorphism ϕ, we identify $\{\vec{h}_i\}$ with a subset of P_Δ, of which we are now going to provide an explicit combinatorial description. Let $\rho = \langle \vec{r} \rangle \in \Delta^{(1)}$, where $\vec{r}$ is the vertex of ρ. The *Schauder hat* at ρ ([22], [10]) is the unique continuous piecewise-linear homogeneous function $h_\rho\colon N_{\mathbb{R}} \to \mathbb{R}$ such that

(1) $h_\rho(\vec{r}) = 1$,
(2) $h_\rho(\vec{v}) = 0$ for every vertex $\vec{v} \neq \vec{r}$ of any cone in $\Delta^{(1)}$,
(3) h_ρ is homogeneous linear on each cone of Σ.

As a consequence of the unimodularity of Δ, a Schauder hat is guaranteed to have linear pieces with integral coefficients. Thus, h_ρ is an element of $\mathcal{F}_n$. We denote by $\mathbf{H}_\Delta$ the set of all Schauder hats at the rays of the complete unimodular fan Δ, and call it a *Schauder basis* of $\mathcal{F}_n$.[19] The subgroup $[\mathbf{H}_\Delta]$ generated by $\mathbf{H}_\Delta$ in $\mathcal{F}_n$ is easily seen to be free (as a group). In fact, $\mathrm{SF}(\Delta) = [\mathbf{H}_\Delta] \cong \mathbb{Z}^t$ as groups. But $[\mathbf{H}_\Delta]$ inherits a partial order from $\mathcal{F}_n$, and $\mathrm{SF}(\Delta) = [\mathbf{H}_\Delta] \cong \mathbb{Z}^t$ as simplicial groups. The positive cone in $[\mathbf{H}_\Delta]$ is precisely the monoid $[[\mathbf{H}_\Delta]]$. Notice that from $\mathbf{H}_\Delta$ it is possible to recover Δ completely, and conversely. Further, this correspondence is algorithmic — see [10] and references therein. Thus Schauder bases are the exact ℓ-group-theoretical analogue of complete unimodular fans.

Simplicial approximations. We remark that Δ-linear support functions are the combinatorial counterpart of linear systems and divisors — formal integral linear combinations of subvarieties of codimension 1 — on toric varieties (see [23] or [15]). Thus, we are considering a partial order on (what essentially is) the group of divisors $\mathrm{SF}(\Delta)$ of a toric variety. This is not entirely foreign to algebraic geometers, who call an element of $\mathrm{SF}(\Delta)$ *effective* iff, in our terminology, it is positive in the simplicial order of $\mathrm{SF}(\Delta)$. However, we are also *order*-embedding[20] $\mathrm{SF}(\Delta)$ into a larger lattice-ordered group P_Δ associated with Δ, something which, to the best of our knowledge, has no established counterpart in the algebro-geometric language. We shall now show fan-theoretically that P_Δ — a finitely presented structure amenable to computation — completely encodes the birational geometry of the associated toric variety. We continue to use the notation introduced in the previous

[18] Completeness is the exact counterpart of algebro-geometric compactness, and completion of a fan corresponds to compactification of the associated toric variety.

[19] Generalizing this construction to finitely presented ℓ-groups is straightforward.

[20] But not lattice-embedding.

subsection, and to assume that Δ is complete and unimodular. Let $p \in \mathcal{F}_n^+$ be a positive ℓ-function such that $p \notin \mathrm{SF}(\Delta)$, and suppose Σ is a complete fan in $N_{\mathbb{R}}$ such that $p \in \mathrm{SF}(\Sigma)$. As we have seen, there exists a complete unimodular refinement $\Sigma' \leq \Sigma$. Clearly, $p \in \mathrm{SF}(\Sigma')$. By the De Concini-Procesi Lemma, there exists a complete unimodular fan $\Delta_1 \preceq_2 \Delta$ such that $\Delta_1 \leq \Sigma' \leq \Sigma$, whence $p \in \mathrm{SF}(\Delta_1)$.[21] Equivalently, $p \in [[\mathbf{H}_{\Delta_1}]]$. In any ℓ-group G, every $g \in G$ decomposes uniquely as $g = g^+ - g^-$ for elements $g^+, g^- \in G^+$ such that $g^+ \wedge g^- = 0$; thus, the fact that we have argued for $p \in \mathcal{F}_n^+$ is not restrictive. Iterating this construction we obtain:

Let Δ be a complete unimodular fan in $N_{\mathbb{R}}$. There exists a sequence of complete unimodular fans $\{\Delta_i\}_{i\in\mathbb{Z}^+}$ such that

(1) $\Delta_0 = \Delta$;
(2) $\Delta_{i+1} \preceq_2 \Delta_i$ for all $i \in \mathbb{Z}^+$;
(3) For all $p \in \mathcal{F}_n^+$, there exists $i \in \mathbb{Z}^+$ such that $p \in [[\mathbf{H}_{\Delta_i}]]$, hence for all $f \in \mathcal{F}_n$ there is $i \in \mathbb{Z}^+$ such that $f \in [\mathbf{H}_{\Delta_i}]$.

We call a sequence $\{\Delta_i\}_{i\in\mathbb{Z}^+}$ satisfying the properties of the above statement a *De Concini-Procesi sequence* for Δ.

A moment's reflection shows that $[[\mathbf{H}_{\Delta_0}]] \subseteq [[\mathbf{H}_{\Delta_1}]]$, whence $[\mathbf{H}_{\Delta_0}] \subseteq [\mathbf{H}_{\Delta_1}]$. Let t_0 and t_1 be the cardinalities of $\Delta_0^{(1)}$ and $\Delta_1^{(1)}$, respectively. Since $[\mathbf{H}_{\Delta_0}] \cong \mathbb{Z}^{t_0}$ and $[\mathbf{H}_{\Delta_1}] \cong \mathbb{Z}^{t_1}$, we obtain an embedding of free abelian groups $\phi_1 : \mathbb{Z}^{t_0} \rightarrowtail \mathbb{Z}^{t_1}$; since $[[\mathbf{H}_{\Delta_0}]] \subseteq [[\mathbf{H}_{\Delta_1}]]$, ϕ_1 is order-preserving (with respect to the simplicial order), that is, it sends positive elements to positive elements. In other words, with respect to simplicial bases in $\mathbb{Z}^{t_0}$ and $\mathbb{Z}^{t_1}$, ϕ_1 is represented by a unique full-rank integral $t_1 \times t_0$ matrix with nonnegative entries. Given a De Concini-Procesi sequence $\{\Delta_i\}$ as in the above, we thus obtain a sequence of simplicial groups $\{\mathbb{Z}^{t_i}\}$ and injective order-preserving group homomorphisms $\{\phi_{i+1} : \mathbb{Z}^{t_i} \rightarrowtail \mathbb{Z}^{t_{i+1}}\}$. Quite generally, given a sequence of partially ordered abelian groups $\{G_i\}$ and order-preserving (not necessarily injective) group homomorphisms $\{\phi_{i+1} : G_i \to G_{i+1}\}$, the couple

$$\langle G_i, \phi_{i+1} \rangle$$

is called a *direct system* of partially ordered abelian groups. Every such direct system admits a unique *limit group* $G = \lim G_i$, which is again a partially ordered abelian group. When all homomorphisms are injective, the limit is simply the group $G = \bigcup_{i\in\mathbb{Z}^+} G_i$, with positive cone $G^+ = \bigcup_{i\in\mathbb{Z}^+} G_i^+$. In this case, G is called *ultrasimplicial*. Thus:

Simplicial approximations of free ℓ-groups. Let Δ be a complete unimodular fan in $N_{\mathbb{R}}$. The ℓ-group $P_\Delta \cong \mathcal{F}_n$ is ultrasimplicial. More precisely, given any De Concini-Procesi sequence $\{\Delta_i\}$ for Δ, P_Δ is the limit group of the direct system

[21] As shown in [24], such a Δ_1 can be effectively computed.

$$\langle \mathrm{SF}(\Delta_i), \zeta_{i+1} \rangle$$

of simplicial groups of Δ_i-linear support functions and injective order preserving homomorphisms $\zeta_i : \mathrm{SF}(\Delta_i) \rightarrowtail \mathrm{SF}(\Delta_{i+1})$ induced by binary starrings $\Delta_{i+1} \preceq_2 \Delta_i$.

Remarks.

1. Amongst other things, simplicial approximations of free ℓ-groups via groups of support functions give rise to a very interesting combinatorics of positive integral matrices. The key point is that the connecting morphisms $\zeta_{i+1} : \mathrm{SF}(\Delta_i) \rightarrowtail \mathrm{SF}(\Delta_{i+1})$ are of a very special kind indeed — geometrically, they are binary starrings. This allows for the definition of a purely matrix-theoretical analogue of binary starring which is the subject of current research, and still far from being well-understood.

2. The above result admits far-reaching generalizations. Techniques already developed in [22] allow to prove that every finitely presented ℓ-group P_Δ is the limit of a direct system of simplicial groups of support functions, in close analogy with what we have just shown. The first author proved that, quite generally, *every* abelian ℓ-group is ultrasimplicial ([18]). While the fan-theoretic interpretation of the latter result is the subject of ongoing research, applications of the ultrasimplicial property to other mathematical realms are described in [19].

3. Direct systems of simplicial groups were originally investigated by functional analysts because of their relevance for the classification of certain C^*-algebras. In general, the limit need not be lattice-ordered — not even when all maps are binary starrings. However, the limit of a direct systems of simplicial groups always is a *dimension group*, namely a directed, unperforated partially ordered abelian group with the Riesz Interpolation Property. See [19] for more details and references.

4. Toric varieties

We briefly introduce the basic correspondence between algebraic varieties and fans. See [23], [15] and [16] for a comprehensive account. Our terminology and notation for commutative algebra follows [5].

We shall work over a fixed algebraically closed ground field $k = \overline{k}$ of characteristic zero, even though it is known that many results remain true in positive characteristic. We continue to use our previous notation — recall that N is a free abelian group of rank n, M is its dual $\mathbb{Z}$-module, $N_\mathbb{R} = \mathbb{R} \otimes N$ and $M_\mathbb{R}$ is the dual of $N_\mathbb{R}$.

A *lattice* in $\mathbb{R}^n$ is a free abelian group $L \subseteq \mathbb{R}^n$ of rank n. A *lattice point* (with respect to L) in $\mathbb{R}^n$ is simply an element of L. In this terminology — a legacy of Minkowski's geometry of numbers — N is a lattice in $N_\mathbb{R}$. Let σ be a cone in $N_\mathbb{R}$, and consider the set $L_\sigma = \sigma \cap N \subseteq N_\mathbb{R}$ of lattice points in σ. Since σ is convex, S is a semigroup under addition. More is true.

Gordan's Lemma. *The lattice points L_σ within a cone σ are a finitely generated semigroup.*

FIGURE 3. A cone $\sigma = \langle (0,1),(1,1)\rangle$ and its dual $\sigma^\vee = \langle (1,0),(-1,1)\rangle$.

While Gordan's Lemma is not difficult to prove (see e.g. [15]), it is an essential result for the definition of affine toric varieties, as we now show.

Again, let $\sigma = \langle \vec{v}_1,\ldots,\vec{v}_m\rangle$ be a cone in $N_\mathbb{R}$, and set $L_\sigma = \sigma \cap N$. By Gordan's Lemma, L_σ is in an additive finitely generated semigroup; let $g_1,\ldots,g_s$ be a set of generators of S. (Each g_i is a vector in $N_\mathbb{R}$, but we drop the vector notation for simplicity.) We switch to multiplicative notation, so that an element of L_σ is a *Laurent monomial* $g_1^{m_1}\cdots g_s^{m_s}$ for (not necessarily unique) natural numbers $m_i \in \mathbb{N}$. (Since $m_i \in \mathbb{N}$, it might seem sufficient to speak of *monomials*. However, the g_i's are not free variables, and because of possible relations amongst them it is actually necessary to deal with Laurent monomials — cfr. Example 4 below.) Consider the semigroup ring $k[L_\sigma]$. Elements of $k[L_\sigma]$ are *Laurent polynomials*, that is, finite sums of Laurent monomials in L_σ with coefficients in k. Clearly, then, $k[L_\sigma]$ is a k-algebra.[22] In fact, it is a finitely generated k-algebra, because L_σ is finitely generated. It is almost immediate that $k[L_\sigma]$ is reduced. (Indeed, if the n-th power of a nonzero Laurent monomial is zero, then the nm_i-th power of some variable g_i appearing in that monomial must be zero; but this implies that g_i is the zero vector, a contradiction. Extension to Laurent polynomials is trivial.) Thus we can define the *affine toric variety*

$$X_\sigma = \operatorname{MaxSpec} k[L_\sigma]\,.$$

Examples.

1. If $\sigma = \langle \vec{e}_1,\ldots,\vec{e}_n\rangle$ is the cone generated by the canonical basis of $N_\mathbb{R}$, $X_\sigma = \operatorname{MaxSpec} k[L_\sigma] = \operatorname{MaxSpec} k[x_1,\ldots,x_n]$ is affine n-space k^n.

2. If $\sigma = \langle (a,b)\rangle \subseteq \mathbb{R}^2$, $(a,b)\neq 0$, $X_\sigma = \operatorname{MaxSpec} k[L_\sigma] = \operatorname{MaxSpec} k[x]$ is the affine line k.

3. If $\sigma = \langle (1,0),(1,2)\rangle \subseteq \mathbb{R}^2$, then $\{(1,0),(1,2)\}$ is *not* a set of generators for L_σ, because e.g. $(1,1)$ is not in the submonoid spanned by $\{(1,0),(1,2)\}$. However, $\{(1,0),(1,1),(1,2)\}$ is a set of generators for L_σ. Hence, $X_\sigma = \operatorname{MaxSpec} k[L_\sigma] = \operatorname{MaxSpec} k[x_1, x_1x_2, x_1x_2^2]$. But $k[x_1,x_1x_2,x_1x_2^2] \cong k[X,Y,Z]/J$, where J is the ideal

[22] Recall that a ring R is a k-*algebra* iff k is a subring of R; that R is *reduced* iff it has no nilpotent elements; and that MaxSpec is a categorical equivalence between finitely generated reduced k-algebras and affine algebraic varieties over k ([5], [12]).

COMBINATORIAL CONVEXITY	ALGEBRAIC GEOMETRY
cone	affine toric variety
face	subvariety
lattice $N \subseteq N_{\mathbb{R}}$	torus = $\operatorname{MaxSpec} k[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$
fan	toric variety
— complete	— compact
— unimodular	— smooth
— simplicial	— quasi-smooth
— induced by a polytope	— projective
barycentric stellar subdivision	equivariant blow up
unimodular map	algebraic isomorphism
map of fans	equivariant toric morphism
cardinality of $\Sigma^{(n)}$, Σ a complete unimodular fan	Euler characteristic $\chi(X_\Sigma)$ of a smooth compact variety

TABLE 1. Fan-Toric Vocabulary.

of $k[X, Y, Z]$ generated by the polynomial $Y^2 - XZ$. Thus, X_σ is the standard cone with vertex at the origin. In particular, X_σ is not "smooth" (at the origin) in any reasonable sense of the word. We shall not define smoothness in this paper; as we already mentioned, an affine toric variety X_σ is smooth iff σ is unimodular.

4. If $\sigma = N_{\mathbb{R}}$, $X_\sigma = \operatorname{MaxSpec} k[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$. The latter is known as the *n-dimensional algebraic torus*, and it plays an important rôle in the theory of toric varieties — see [23]. If σ is not strongly convex, it contains a nontrivial linear subspace $V \subseteq N_{\mathbb{R}}$ of dimension $d \geq 1$. In this case, it is not hard to see that the d-dimensional torus X_V splits off as a direct factor of X_σ, whence the widespread technical device of restricting attention to strongly convex cones.

The next natural step is to construct general toric varieties gluing together affine toric varieties. The data for the gluing morphisms come from a fan Σ. In carrying out this programme, it turns out that it is convenient to consider dual cones of the elements of Σ.

Let $\sigma \subseteq N_{\mathbb{R}}$ be an m-dimensional cone. We set

$$\sigma^\vee = \{f \in M_{\mathbb{R}} \mid f(\vec{u}) \geq 0 \text{ for every } \vec{u} \in \sigma\}$$

It is not hard to check that if $\sigma = \langle \vec{v}_1, \ldots, \vec{v}_r \rangle$, then $\sigma^\vee$ is the intersection of the half-spaces $\{f \in M_{\mathbb{R}} \mid f(\vec{v}_i) \geq 0\} \subseteq M_{\mathbb{R}}$. Thus, by the Fundamental Theorem of Polyhedra, it is a cone in $M_{\mathbb{R}}$, the *dual cone* of σ. The terminology is appropriate in that $\sigma^{\vee\vee} = \sigma$ for any cone σ. Notice that faces τ of σ coincide with sets of the form $\tau = \sigma \cap \ker f$, for some $f \in \sigma^\vee$.

If $\sigma \subseteq N_{\mathbb{R}}$ is an n-dimensional strongly convex cone, its dual $\sigma^\vee$ can be geometrically constructed as follows. For each facet (= maximal proper face) F_i of σ, let $\vec{p}_i$

be the inner normal to F_i. Then $\sigma^\vee$ is the positive linear hull of the finite set $\{\vec{p_i}\}$, that is, $\sigma^\vee = \langle \vec{p}_1, \ldots, \vec{p}_r \rangle$, where r is the number of facets of σ. Further, $\sigma^\vee$ is itself n-dimensional and strongly convex (see Figure 3).

Let $\Sigma \subseteq N_{\mathbb{R}}$ be a fan of strongly convex cones. For any $\sigma \in \Sigma$, we defined $L_\sigma = \sigma \cap N$; now, dually, set $L^\sigma = \sigma^\vee \cap M$. For σ, $\tau \in \Sigma$, consider the affine toric varieties $X_{\sigma^\vee}$ and $X_{\tau^\vee}$ — again, notice that we are passing to duals. If τ is a face of σ, then $L^\sigma \subseteq L^\tau$, so that $k[L^\sigma]$ is a subalgebra of $k[L^\tau]$, and $X_{\sigma^\vee}$ is an open subset of $X_{\tau^\vee}$. Thus, in general, the common face $\sigma \cap \tau$ corresponds to a common open subset $X_{(\sigma\cap\tau)^\vee}$ of $X_{\sigma^\vee}$ and $X_{\tau^\vee}$, along which we can glue these two affine varieties together. The fan Σ, then, encodes a recipe to patch together the affine pieces $X_{\sigma^\vee}$ corresponding to the *dual* cones of cones $\sigma \in \Sigma$, thus yielding (the most general instance of) a *toric variety*, denoted X_Σ. This construction can be carried out without passing to duals, and produces an algebraic variety, say X^Σ. However, the combinatorics of the faces of the cones in Σ would be contravariant with respect to the structure of subvarieties of X^Σ — that is, cones of smaller dimension in Σ would correspond to *larger* subvarieties of X^Σ. Dualization gives a covariant correspondence: cones of smaller dimension in Σ correspond to smaller subvarieties of X_Σ.

Building on this basic construction, it is possible to carry out an extensive translation of algebro-geometric concepts into fan-theoretical language, and vice-versa. This has been done in the last decades. While we refer to [15], [23] and [16] for further information, we collect the very first few entries of the fan-toric dictionary in Table 1.

References

[1] S. Aguzzoli and D. Mundici, An algorithmic desingularization of 3-dimensional toric varieties. *Tohoku Math. J.*, 46:557-572, 1994.

[2] J. W. Alexander. Combinatorial analysis situs. *Trans. Amer. Math. Soc.*, 28:301–329, 1926.

[3] J. W. Alexander. The combinatorial theory of complexes. *Ann. of Math. (2)*, 31:292–320, 1930.

[4] P. S. Alexandrov. Combinatorial topology. Vol. 1, 2 and 3. Translated from the Russian, Reprint of the 1956, 1957 and 1960 translations. Dover Publications, 1998.

[5] M. F. Atiyah and I. G. Macdonald. Introduction to commutative algebra. Addison-Wesley Publishing Co., Reading, Mass., 1969.

[6] K. A. Baker. Free vector lattices. *Canad. J. Math.*, 20:58–66, 1968.

[7] W. M. Beynon. Duality theorems for finitely generated vector lattices. *Proc. London Math. Soc. (3)*, 31:114–128, 1975.

[8] W. M. Beynon. Applications of duality in the theory of finitely generated lattice-ordered abelian groups. *Canad. J. Math.*, 29(2):243–254, 1977.

[9] C. C. Chang. A new proof of the completeness of the Łukasiewicz axioms. *Trans. Amer. Math. Soc.*, 93:74–80, 1959.

[10] R. L. O. Cignoli, I. M. L. D'Ottaviano, and D. Mundici. *Algebraic foundations of many-valued reasoning.* Kluwer Academic Publishers, Dordrecht, 2000.

[11] V. I. Danilov. The geometry of toric varieties. Uspekhi Mat. Nauk, 33:85–134, 1978.

[12] V. I. Danilov. Algebraic varieties and schemes. In I. R. Shafarevich (ed.), *Algebraic Geometry I*, Encyclopaedia of Mathematical Sciences, 23. Springer-Verlag, 1994.
[13] M. R. Darnel. *Theory of lattice-ordered groups.* Marcel Dekker Inc., New York, 1995.
[14] C. De Concini and C. Procesi. Complete symmetric varieties. II. Intersection theory. In *Algebraic groups and related topics (Kyoto/Nagoya, 1983)*, North-Holland, Amsterdam, 1985.
[15] G. Ewald. *Combinatorial convexity and algebraic geometry.* Springer-Verlag, New York, 1996.
[16] W. Fulton. *Introduction to toric varieties.* Ann. of Math. Studies, 131, Princeton University Press, 1993.
[17] A. M. W. Glass, *Partially Ordered Groups.* (1999) World Scientific Publ. Co. Inc., River Edge, NJ.
[18] V. Marra, Every abelian ℓ-group is ultrasimplicial, *Journal of Algebra*, 225:872-884, 2000.
[19] V. Marra and D. Mundici. MV-algebras and abelian ℓ-groups: a fruitful interaction. In J. Martinez (ed.), *Ordered Algebraic Structures*, Kluwer Academic Publishers, *to appear.*
[20] R. McNaughton. A theorem about infinite-valued sentential logic. *J. Symbolic Logic*, 16:1–13, 1951.
[21] D. Mundici. Interpretation of AF C^*-algebras in Łukasiewicz sentential calculus. *J. Funct. Anal.*, 65(1):15–63, 1986.
[22] D. Mundici. Farey stellar subdivisions, ultrasimplicial groups, and K_0 of AF C^*-algebras. *Adv. in Math.*, 68(1):23–39, 1988.
[23] T. Oda. *Convex Bodies and Algebraic Geometry.* Springer-Verlag, 1988.
[24] G. Panti. A geometric proof of the completeness of the Łukasiewicz calculus. *J. Symbolic Logic*, 60(2):563–578, 1995.
[25] G.-C. Rota. Report on the present state of combinatorics. *Discrete Math.*, 153:289–303, 1996.
[26] E. C. Weinberg. Free lattice-ordered abelian groups. *Math. Ann.*, 151:187–199, 1963.
[27] R. Wójcicki. On matrix representations of consequence operations of Łukasiewicz's sentential calculi. *Z. Math. Logik Grundlagen Math.*, 19:239–247, 1973.

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