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A Littlewood-Richardson rule for evaluation representations of $U_q(\widehat{\mathfrak{sl}}_n)$

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Abstract

We give a combinatorial description of the composition factors of the induction product of two evaluation modules of the affine Iwahori-Hecke algebra of type GL_m. Using quantum affine Schur-Weyl duality, this yields a combinatorial description of the composition factors of the tensor product of two evaluation modules of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$.

1 Introduction

1.1 Let H_m denote the Iwahori-Hecke algebra of type A_{m-1} over $\mathbb{C}(t)$. This is a semisimple associative algebra isomorphic to the group algebra $\mathbb{C}(t)[\mathfrak{S}_m]$ of the symmetric group. Hence its simple modules $S(\lambda)$ are parametrized by the partitions λ of m. Consider a decomposition $m = m_1 + m_2$, and two partitions $\lambda^{(1)}$ and $\lambda^{(2)}$ of m_1 and m_2, respectively. Then we have a H_{m_1}-module $S(\lambda^{(1)})$ and a H_{m_2}-module $S(\lambda^{(2)})$, and we can form the induced module

$$S(\lambda^{(1)}) \odot S(\lambda^{(2)}) := \mathrm{Ind}_{H_{m_1}\otimes H_{m_2}}^{H_m} \left(S(\lambda^{(1)}) \otimes S(\lambda^{(2)})\right).$$

Here, $H_{m_1} \otimes H_{m_2}$ is identified to a subalgebra of H_m in the standard way. Using again the isomorphism $H_m \cong \mathbb{C}(t)[\mathfrak{S}_m]$, we see that the multiplicity of a simple H_m-module $S(\mu)$ in $S(\lambda^{(1)}) \odot S(\lambda^{(2)})$ is equal to the classical Littlewood-Richardson coefficient $c^{\mu}_{\lambda^{(1)}\lambda^{(2)}}$ (see *e.g.* [**Mcd**]).

1.2 Let now $\widehat{H}_m$ be the affine Iwahori-Hecke algebra over $\mathbb{C}(t)$ (see 2.1 below). For each invertible $z \in \mathbb{C}(t)$ we have a surjective *evaluation homomorphism* $\tau_z : \widehat{H}_m \to H_m$. Pulling back the simple H_m-module $S(\lambda)$ via τ_z we obtain a simple $\widehat{H}_m$-module $S(\lambda; z)$ called an *evaluation module*. In analogy with 1.1, given two invertible elements z_1 and z_2 of $\mathbb{C}(t)$, we can then form the induced $\widehat{H}_m$-module

$$S(\lambda^{(1)}; z_1) \odot S(\lambda^{(2)}; z_2) := \mathrm{Ind}_{\widehat{H}_{m_1} \otimes \widehat{H}_{m_2}}^{\widehat{H}_m} \left(S(\lambda^{(1)}; z_1) \otimes S(\lambda^{(2)}; z_2) \right).$$

It turns out that if we fix $\lambda^{(1)}, \lambda^{(2)}$ and vary the spectral parameters z_1, z_2, this module is generically irreducible, that is, it is simple except for a finite number of values of the ratio z_1/z_2. In [**LNT**, Theorem 36] a combinatorial description of these special values was given.

In this note we shall make this result more precise by describing all the composition factors of $S(\lambda^{(1)}; z_1) \odot S(\lambda^{(2)}; z_2)$ at these critical values z_1/z_2. We shall also prove that, in contrast with the classical Littlewood-Richardson rule, all the composition factors appear with multiplicity one. The composition factors occuring in a product will be described using the combinatorics of Lusztig's *symbols*, that is, of certain two-row arrays introduced by Lusztig for parametrizing the irreducible complex representations of the classical reductive groups over finite fields [**Lu1**, **Lu2**].

1.3 We will derive our combinatorial formula from some explicit calculations of canonical bases in level 2 representations of the quantum algebra $U_v(\mathfrak{sl}_{n+1})$ performed in [**LM**]. More precisely, by dualizing [**LM**, Theorem 3], we get a formula for the expansion of the product of two quantum flag minors on the dual canonical basis of $U_v(\mathfrak{sl}_{n+1})$ (Theorem 5). Using then Ariki's theorem as in [**LNT**], we obtain immediately the above-mentioned Littlewood-Richardson rule for induction products of two evaluation modules over affine Hecke algebras (Theorem 2).

Finally, by means of the quantum affine analogue of the Schur-Weyl duality developed by Cherednik, Chari-Pressley and Ginzburg-Reshetikhin-Vasserot, we can deduce from this rule a similar one for the tensor product of two evaluation modules over the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_N)$.

2 Composition factors of induced $\widehat{H}_m$-modules

2.1 Let $\widehat{H}_m$ be the affine Hecke algebra of type GL_m over $\mathbb{C}(t)$. It has invertible generators $T_1, \ldots, T_{m-1}, y_1, \ldots, y_m$ subject to the relations

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \qquad (1 \leqslant i \leqslant m-2),$$

$$\begin{aligned}
&T_iT_j = T_jT_i, && (|i-j|>1),\\
&(T_i - t)(T_i+1) = 0, && (1 \leqslant i \leqslant m-1),\\
&y_iy_j = y_jy_i, && (1 \leqslant i,j \leqslant m),\\
&y_jT_i = T_iy_j, && (j \neq i, i+1),\\
&T_iy_iT_i = t\, y_{i+1}, && (1 \leqslant i \leqslant m-1).
\end{aligned}$$

The subalgebra H_m generated by the T_i's is the Iwahori-Hecke algebra of type A_{m-1}.

For any invertible $z \in \mathbb{C}(t)$ we have a unique algebra homomorphism $\tau_z : \widehat{H}_m \to H_m$ such that

$$\tau_z(T_i) = T_i, \quad \tau_z(y_1) = z, \qquad (i = 1, \ldots, m-1).$$

This is called the *evaluation at* z.

We also have an algebra automorphism $\sigma_z : \widehat{H}_m \to \widehat{H}_m$ such that

$$\sigma_z(T_i) = T_i, \quad \sigma_z(y_i) = zy_i, \qquad (i = 1, \ldots, m-1).$$

This is called the *shift by* z.

2.2 As mentioned in the introduction, given two partitions $\lambda^{(1)}$ and $\lambda^{(2)}$, the structure of the induced $\widehat{H}_m$-module

$$S(\lambda^{(1)}; z_1) \odot S(\lambda^{(2)}; z_2)$$

depends essentially on the ratio z_1/z_2. Indeed, by twisting this module with the shift automorphism σ_z we obtain the induced module

$$S(\lambda^{(1)}; zz_1) \odot S(\lambda^{(2)}; zz_2).$$

For example, it is known that if $z_1/z_2 \notin t^{\mathbb{Z}}$ then $S(\lambda^{(1)}; z_1) \odot S(\lambda^{(2)}; z_2)$ is irreducible. Therefore, we can assume without loss of generality that

$$z_i = t^{a_i}, \quad a_i \in \mathbb{Z}, \quad a_i \geqslant \ell(\lambda^{(i)}), \qquad (i = 1, 2), \tag{1}$$

where as usual $\ell(\lambda)$ denotes the length of the partition λ. Since $S(\lambda^{(1)}; z_1) \odot S(\lambda^{(2)}; z_2)$ and $S(\lambda^{(2)}; z_2) \odot S(\lambda^{(1)}; z_1)$ have the same composition factors with the same multiplicities, we can also assume that $a_1 \leqslant a_2$.

2.3 It will be convenient to write partitions in weakly *increasing* order. Given a partition λ and an integer $a \geqslant \ell(\lambda)$ we can make λ into a non-decreasing sequence $(\lambda_1, \ldots, \lambda_a)$

of length a by setting $\lambda_j = 0$ for $j = 1, \dots, a - \ell(\lambda)$. We can then associate to (λ, a) the increasing sequence

$$\beta = (\beta_1, \dots, \beta_a), \quad \beta_j = j + \lambda_j. \tag{2}$$

In this way, given $(\lambda^{(i)}, a_i)$ $(i = 1, 2)$ as in 2.2, we obtain a *symbol*

$$S = \begin{pmatrix} \beta^{(2)} \\ \beta^{(1)} \end{pmatrix} = \begin{pmatrix} \beta_1^{(2)}, \dots, \beta_{a_2}^{(2)} \\ \beta_1^{(1)}, \dots, \beta_{a_1}^{(1)} \end{pmatrix}. \tag{3}$$

For example, the symbol attached to the pairs $((1, 1, 2), 3)$ and $((2, 3), 5)$ is

$$S = \begin{pmatrix} 1 & 2 & 3 & 6 & 8 \\ 2 & 3 & 5 & & \end{pmatrix}.$$

Conversely, given a symbol S, *i.e.* a two-row array as in Eq. (3) with

$$1 \leqslant \beta_1^{(i)} < \cdots < \beta_{a_i}^{(i)} \quad (i = 1, 2),$$

there is a unique pair $(\lambda^{(i)}, a_i)$ $(i = 1, 2)$ whose symbol is S.

2.4 The symbol S of Eq. (3) is said to be *standard* if $\beta_k^{(2)} \leqslant \beta_k^{(1)}$ for $k \leqslant a_1$. In [**LM**, §2.5] we have defined the *pairs* of a standard symbol S, and the set $\mathcal{C}(S)$ of all symbols Σ obtained from S by permuting some of its pairs. As shown in [**LM**, Lemma 9], these notions are equivalent to the notion of admissible involution of Lusztig [**Lu4**].

For the convenience of the reader we shall recall these definitions. Let $S = \binom{\beta}{\gamma}$ be a standard symbol. We define an injection $\psi : \gamma \longrightarrow \beta$ such that $\psi(j) \leqslant j$ for all $j \in \gamma$. To do so it is enough to describe the subsets

$$\gamma^l = \{j \in \gamma \mid \psi(j) = j - l\}, \qquad (0 \leqslant l \leqslant n).$$

We set $\gamma^0 = \gamma \cap \beta$ and for $l \geqslant 1$ we put

$$\gamma^l = \{j \in \gamma - (\gamma^0 \cup \cdots \cup \gamma^{l-1}) \mid j - l \in \beta - \psi(\gamma^0 \cup \cdots \cup \gamma^{l-1})\}.$$

Observe that the standardness of S implies that ψ is well-defined.

Example 1 Take

$$S = \begin{pmatrix} 1 & 3 & 5 & 8 & 9 \\ 3 & 6 & 7 & 10 & \end{pmatrix}.$$

Then

$$\gamma^0 = \{3\},\ \gamma^1 = \{6, 10\},\ \gamma^2 = \cdots = \gamma^5 = \emptyset,\ \gamma^6 = \{7\}.$$

Hence

$$\psi(3) = 3,\ \psi(6) = 5,\ \psi(7) = 1,\ \psi(10) = 9.$$

The pairs $(j, \psi(j))$ with $\psi(j) \neq j$ (that is, $j \notin \beta \cap \gamma$) will be called the pairs of S. Given a standard symbol S with p pairs, we denote by $\mathcal{C}(S)$ the set of all symbols obtained from S by permuting some pairs in S and reordering the rows. We consider S itself as an element of $\mathcal{C}(S)$, hence $\mathcal{C}(S)$ has cardinality 2^p.

2.5 Given a partition λ and an integer a we call *Young diagram of* (λ, a) the Young diagram of λ in which each cell (i, j) is filled with the integer $i - j + a$. For instance, if $\lambda = (2, 3)$ and $a = 5$ then the Young diagram of (λ, a) is

4	5	
5	6	7

The rows of the Young diagram of (λ, a) yield a *multisegment*

$$\mathbf{m}(\lambda, a) := \sum_{1 \leqslant k \leqslant a} [k, k + \lambda_k - 1].$$

This is a formal sum (or multiset) of intervals in $\mathbb{Z}$, in which we discard the empty intervals corresponding to the k's with $\lambda_k = 0$. Thus, continuing with the same example, we have

$$\mathbf{m}((2, 3), 5) = [4, 5] + [5, 7].$$

Similarly, we attach to a pair $(\lambda^{(i)}, a_i)$ $(i = 1, 2)$ or to its symbol S the multisegment

$$\mathbf{m}(S) = \mathbf{m}(\lambda^{(1)}, a_1) + \mathbf{m}(\lambda^{(2)}, a_2).$$

2.6 To each multisegment

$$\mathbf{m} := \sum_k [\alpha_k, \beta_k]$$

is attached an irreducible $\widehat{H}_m$-module $L_{\mathbf{m}}$, where $m = \sum_k (\beta_k + 1 - \alpha_k)$ (see *e.g.* [**LNT**, §2.1]).

2.7 Let us assume that the pair $(\lambda^{(i)}, a_i)$ $(i = 1, 2)$ satisfies the conditions of 2.2. Let Σ denote the symbol attached to this pair. We can now state:

Theorem 2 *The composition factors of* $S(\lambda^{(1)}; t^{a_1}) \odot S(\lambda^{(2)}; t^{a_2})$ *are the modules* $L_{\mathbf{m}(S)}$ *where* S *runs through the set of standard symbols such that* $\Sigma \in \mathcal{C}(S)$. *Each of them occurs with multiplicity one.*

Theorem 2 will be deduced from Theorem 5 below.

Example 3 Let $(\lambda^{(1)}, a_1) = ((1,4),2)$ and $(\lambda^{(2)}, a_2) = ((1,2,3),4)$. The corresponding symbol is

$$\Sigma = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 6 & & \end{pmatrix}.$$

The standard symbols S such that $\Sigma \in \mathcal{C}(S)$ are

$$\begin{pmatrix} 1 & 2 & 5 & 6 \\ 3 & 7 & & \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 5 & 7 \\ 3 & 6 & & \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 5 & 6 \\ 2 & 7 & & \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 6 & & \end{pmatrix}.$$

It follows that the composition factors of $S((1,4);t^2) \odot S((1,2,3);t^4)$ are the $L_{\mathbf{m}}$ where $\mathbf{m}$ is one the following multisegments:

$$\mathbf{n}_1 = [1,2] + [2,6] + [3,4] + [4,5], \quad \mathbf{n}_2 = [1,2] + [2,5] + [3,4] + [4,6],$$

$$\mathbf{n}_3 = [1,1] + [2,2] + [2,6] + [3,4] + [4,5], \quad \mathbf{n}_4 = [1,1] + [2,2] + [2,5] + [3,4] + [4,6].$$

2.8 By restriction to the finite Hecke algebra H_m the irreducible $\widehat{H}_m$-modules $L_{\mathbf{m}(S)}$ decompose into direct sums of Specht modules. The sum of all these Specht modules is given by the (classical) Littlewood-Richardson rule for the product $S(\lambda^{(1)}) \odot S(\lambda^{(2)})$. It would be interesting to find a combinatorial description of the splitting of $S(\lambda^{(1)}) \odot S(\lambda^{(2)})$ thus obtained.

Example 4 Let us continue Example 3. The restrictions to H_{11} of the 4 irreducible $\widehat{H}_{11}$-modules are as follows:

$$\begin{aligned}
L_{\mathbf{n}_1}\downarrow &= S(1,3,7) \oplus S(2,2,7) \oplus S(2,3,6) \oplus S(1,1,3,6) \\
&\quad \oplus S(1,2,2,6) \oplus S(1,2,3,5) \oplus S(2,2,2,5), \\
L_{\mathbf{n}_2}\downarrow &= S(1,4,6) \oplus S(2,3,6) \oplus S(2,4,5) \oplus S(1,1,4,5) \oplus S(1,2,3,5) \\
&\quad \oplus S(1,2,4,4) \oplus S(3,3,5) \oplus S(1,3,3,4) \oplus S(2,2,3,4), \\
L_{\mathbf{n}_3}\downarrow &= S(1,1,2,7) \oplus S(1,2,2,6) \oplus S(1,1,1,2,6) \oplus S(1,1,2,2,5), \\
L_{\mathbf{n}_4}\downarrow &= S(1,1,3,6) \oplus S(1,2,3,5) \oplus S(1,1,1,3,5) \oplus S(1,1,2,3,4).
\end{aligned}$$

This gives a splitting of $S(1,4) \odot S(1,2,3)$.

3 Canonical bases

3.1 Fix $n \geqslant 2$ and let $\mathfrak{g} = \mathfrak{sl}_{n+1}$. We consider the quantum enveloping algebra $U_v(\mathfrak{g})$ over $\mathbb{Q}(v)$ with Chevalley generators e_j, f_j, t_j $(1 \leqslant j \leqslant n)$. The simple roots and the fundamental weights are denoted by α_k and Λ_k $(1 \leqslant k \leqslant n)$ respectively. The irreducible representation of $U_v(\mathfrak{g})$ with highest weight Λ is denoted by $V(\Lambda)$. We denote by $U_v(\mathfrak{n})$ the subalgebra of $U_v(\mathfrak{g})$ generated by e_j $(1 \leqslant j \leqslant n)$.

3.2 Let $\mathbf{B}$ (*resp.* $\mathbf{B}^*$) denote the canonical basis (*resp.* the dual canonical basis) of $U_v(\mathfrak{n})$ ([**Lu3**], [**BZ**]; see also [**LNT**, §3]). The elements of $\mathbf{B}$ and $\mathbf{B}^*$ are naturally labelled by the multisegments $\mathbf{m}$ supported on $[1, n]$. We shall denote them by $b_{\mathbf{m}}$ and $b^*_{\mathbf{m}}$ respectively.

The vectors $b^*_{\mathbf{m}}$ for which $\mathbf{m}$ is of the form

$$\mathbf{m} = \mathbf{m}(\lambda, a)$$

for some partition λ and some integer a are called *quantum flag minors*. Indeed, by [**BZ**], they can be expressed as quantum minors of a triangular matrix whose entries are iterated brackets of the e_i's (see [**LNT**, §5.2]).

3.3 Let $(\lambda^{(i)}, a_i)$ $(i = 1, 2)$ be as in 2.2. We also assume that the multisegments

$$\mathbf{m}_i = \mathbf{m}(\lambda^{(i)}, a_i) \quad (i = 1, 2)$$

are supported on $[1, n]$. Let Σ be the symbol attached to the pair $(\lambda^{(i)}, a_i)$ $(i = 1, 2)$. For a standard symbol S such that $\Sigma \in \mathcal{C}(S)$ we denote by $n(S, \Sigma)$ the number of pairs of S which are permuted to get Σ. Finally, we denote by $N_j(\lambda, a)$ the number of cells of the Young diagram of (λ, a) containing the integer j.

Theorem 5 *We have*

$$b^*_{\mathbf{m}_1} b^*_{\mathbf{m}_2} = v^{-N_{a_1}(\lambda^{(2)}, a_2)} \sum_S v^{n(S, \Sigma)} \, b^*_{\mathbf{m}(S)}$$

where the sum runs through all standard symbols S such that $\Sigma \in \mathcal{C}(S)$.

Example 6 We take $(\lambda^{(1)}, a_1)$ and $(\lambda^{(2)}, a_2)$ as in Example 3. Hence

$$\mathbf{m}_1 = [1, 1] + [2, 5], \qquad \mathbf{m}_2 = [2, 2] + [3, 4] + [4, 6].$$

Then $N_{a_1}(\lambda^{(2)}, a_2) = N_2((1, 2, 3), 4) = 1$, and we obtain, using the notation of Example 3,

$$b^*_{\mathbf{m}_1} b^*_{\mathbf{m}_2} = v^{-1}(v^2 \, b^*_{\mathbf{n}_1} + v \, b^*_{\mathbf{n}_2} + v \, b^*_{\mathbf{n}_3} + b^*_{\mathbf{n}_4}).$$

3.4 *Proof of Theorem 5.* Following [**LNT**, §7.2], we will replace calculations of products of elements of $\mathbf{B}^*$ by calculations of dual canonical bases of finite-dimensional representations of $U_v(\mathfrak{g})$.

3.4.1 Let $U_v(\mathfrak{n}^-)$ denote the subalgebra of $U_v(\mathfrak{g})$ generated by the f_i's, and let $x \mapsto x^\sharp$ denote the algebra isomorphism from $U_v(\mathfrak{n})$ to $U_v(\mathfrak{n}^-)$ defined by $e_i^\sharp = f_i$ $(i = 1, \ldots, n)$. Let Λ be a dominant integral weight and let u_Λ be a highest weight vector of the irreducible module $V(\Lambda)$. Then the map $\pi_\Lambda : x \mapsto x^\sharp u_\Lambda$ projects the canonical basis $\mathbf{B}$ of $U_v(\mathfrak{n})$ to the union of the canonical basis $\mathbf{B}(\Lambda)$ of $V(\Lambda)$ with the set $\{0\}$. The dual map π_Λ^* gives an embedding of the dual canonical basis $\mathbf{B}^*(\Lambda)$ of $V(\Lambda) \simeq V(\Lambda)^*$ into the dual canonical basis $\mathbf{B}^*$ of $U_v(\mathfrak{n}) \simeq U_v(\mathfrak{n})^*$.

3.4.2 In particular the subset of $\mathbf{B}^*$ obtained by embedding the bases $\mathbf{B}^*(\Lambda_a)$ $(1 \leqslant a \leqslant n)$ of the fundamental representations is precisely the subset of quantum flag minors. It is well known that $V(\Lambda_a)$ is a minuscule representation whose bases $\mathbf{B}^*(\Lambda_a)$ and $\mathbf{B}(\Lambda_a)$ coincide. Moreover the elements of these bases are naturally labelled by the pairs (λ, a) whose Young diagram (as defined in 2.5) contains only cells numbered by integers between 1 and n. Denoting them by $b^*_{(\lambda,a)}$ we have

$$\pi^*_{\Lambda_a}(b^*_{(\lambda,a)}) = b^*_{\mathbf{m}(\lambda,a)}$$

Equivalently, we can also label the elements of $\mathbf{B}^*(\Lambda_a)$ by one-row symbols β as in Eq. (2) with $\beta_i \leqslant n+1$.

3.4.3 Similarly, the basis $\mathbf{B}^*(\Lambda_{a_1}) \otimes \mathbf{B}^*(\Lambda_{a_2})$ is naturally labelled by the set of symbols S as in Eq. (3) with $\beta^{(i)}_{a_i} \leqslant n+1$ $(i = 1, 2)$. Using the theory of crystal bases [**K1**, **K2**] one can see that the basis $\mathbf{B}^*(\Lambda_{a_1} + \Lambda_{a_2})$ has a natural labelling by the subset of standard symbols [**LM**, §2.3]. Moreover, denoting by b^*_S the element of $\mathbf{B}^*(\Lambda_{a_1} + \Lambda_{a_2})$ labelled by the standard symbol S we have, using also the notation of 2.5,

$$\pi^*_{\Lambda_{a_1}+\Lambda_{a_2}}(b^*_S) = b^*_{\mathbf{m}(S)}.$$

3.4.4 Let $\iota : V(\Lambda_{a_1} + \Lambda_{a_2}) \longrightarrow V(\Lambda_{a_1}) \otimes V(\Lambda_{a_2})$ be the $U_v(\mathfrak{g})$-module embedding which maps $u_{\Lambda_{a_1}+\Lambda_{a_2}}$ to $u_{\Lambda_{a_1}} \otimes u_{\Lambda_{a_2}}$, and let $\iota^* : V(\Lambda_{a_1}) \otimes V(\Lambda_{a_2}) \longrightarrow V(\Lambda_{a_1} + \Lambda_{a_2})$ be its dual. Let $b_i^* \in \mathbf{B}^*(\Lambda_{a_i})$ $(i = 1, 2)$ and denote by $b^*_{\mathbf{m}_i} = \pi^*_{\Lambda_{a_i}}(b_i^*)$ $(i = 1, 2)$ the corresponding quantum flag minors. It is shown in [**LNT**, §7.2.7] that the image of $b_1^* \otimes b_2^*$ under the composition of maps $\pi^*_{\Lambda_{a_1}+\Lambda_{a_2}} \circ \iota^*$ coincides up to a power of v with the product $b^*_{\mathbf{m}_1} b^*_{\mathbf{m}_2}$. Hence to calculate the $\mathbf{B}^*$-expansion of $b^*_{\mathbf{m}_1} b^*_{\mathbf{m}_2}$ it is enough to calculate the matrix of the map ι^* with respect to the bases $\mathbf{B}^*(\Lambda_{a_1}) \otimes \mathbf{B}^*(\Lambda_{a_1})$ and $\mathbf{B}^*(\Lambda_{a_1} + \Lambda_{a_2})$.

3.4.5 The matrix of ι with respect to the bases $\mathbf{B}(\Lambda_{a_1}+\Lambda_{a_2})$ and $\mathbf{B}(\Lambda_{a_1})\otimes\mathbf{B}(\Lambda_{a_1})$ was calculated in [**LM**, Theorem 3] in terms of Lusztig's symbols. Transposing this matrix we obtain the desired matrix of ι^*. Using 3.4.2 and 3.4.3, we then get the formula of Theorem 5. □

3.5 *Proof of Theorem 2.* By [**LNT**, §3.7] the multisegments $\mathbf{m}$ indexing the composition factors $L_{\mathbf{m}}$ of $S(\lambda^{(1)};t^{a_1})\odot S(\lambda^{(2)};t^{a_2})$ are those occuring in the right-hand side of the formula of Theorem 5. Moreover the composition multiplicities are obtained by specializing v to 1 in the coefficients of this formula. Hence they are all equal to 1. □

4 Tensor products of $U_q(\widehat{\mathfrak{sl}}_N)$-modules

4.1 Let $U_q(\widehat{\mathfrak{sl}}_N)$ be the quantized affine algebra of type $A^{(1)}_{N-1}$ with parameter q a square root of t (see for example [**CP**] for the defining relations of $U_q(\widehat{\mathfrak{sl}}_N)$). The quantum affine Schur-Weyl duality between $\widehat{H}_m$ and $U_q(\widehat{\mathfrak{sl}}_N)$ [**CP**, **Ch**, **GRV**] gives a functor $\mathcal{F}_{m,N}$ from the category of finite-dimensional $\widehat{H}_m$-modules to the category of level 0 finite-dimensional representations of $U_q(\widehat{\mathfrak{sl}}_N)$. If $N \geqslant m$, $\mathcal{F}_{m,N}$ maps the simple modules of $\widehat{H}_m$ to simple modules of $U_q(\widehat{\mathfrak{sl}}_N)$. However, the image of a non-zero simple $\widehat{H}_m$-module may be the zero $U_q(\widehat{\mathfrak{sl}}_N)$-module. More precisely, the simple $\widehat{H}_m$-module $L_{\mathbf{m}}$ is mapped to a non-zero simple $U_q(\widehat{\mathfrak{sl}}_N)$-module if and only if all the segments occuring in $\mathbf{m}$ have length $\leqslant N-1$. In this case the Drinfeld polynomials of $\mathcal{F}_{m,N}(L_{\mathbf{m}})$ are easily calculated from $\mathbf{m}$ (see [**CP**]).

The functor $\mathcal{F}_{m,N}$ transforms induction product into tensor product, that is, for M_1 in $\mathcal{C}_{m_1}$ and M_2 in $\mathcal{C}_{m_2}$ one has

$$\mathcal{F}_{m_1+m_2,N}(M_1\odot M_2)=\mathcal{F}_{m_1,N}(M_1)\otimes\mathcal{F}_{m_2,N}(M_2)\,.$$

4.2 The image under $\mathcal{F}_{m,N}$ of an evaluation module for $\widehat{H}_m$ is an evaluation module for $U_q(\widehat{\mathfrak{sl}}_N)$, and all evaluation modules of $U_q(\widehat{\mathfrak{sl}}_N)$ can be obtained in this way, by varying $m\in\mathbb{N}^*$.

4.3 By application of the Schur functor $\mathcal{F}_{m,N}$ to Theorem 2 we thus obtain a combinatorial description of all composition factors of the tensor product of two evaluation modules of $U_q(\widehat{\mathfrak{sl}}_N)$.

Example 7 We continue Example 3 and Example 6. The image of the $\widehat{H}_5$-module $L_{\mathbf{m}_1}$ under $\mathcal{F}_{5,N}$ is the evaluation module $V(\mathbf{m}_1)$ of $U_q(\widehat{\mathfrak{sl}}_N)$ with Drinfeld polynomials

$$\begin{aligned} P_1(u) &= u - q^{-2}, \\ P_2(u) &= P_3(u) = 1, \\ P_4(u) &= u - q^{-7}, \\ P_k(u) &= 1, \quad (5 \leqslant k \leqslant N-1). \end{aligned}$$

This is a non-zero module if and only if $N \geqslant 5$. Similarly, the image of the $\widehat{H}_6$-module $L_{\mathbf{m}_2}$ under $\mathcal{F}_{6,N}$ is the evaluation module $V(\mathbf{m}_2)$ of $U_q(\widehat{\mathfrak{sl}}_N)$ with Drinfeld polynomials

$$\begin{aligned} P_1(u) &= u - q^{-4}, \\ P_2(u) &= u - q^{-7}, \\ P_3(u) &= u - q^{-10}, \\ P_k(u) &= 1, \quad (4 \leqslant k \leqslant N-1). \end{aligned}$$

This is a non-zero module if and only if $N \geqslant 4$. The images of the $\widehat{H}_{11}$-modules $L_{\mathbf{m}_1}, L_{\mathbf{m}_2}, L_{\mathbf{m}_3}, L_{\mathbf{m}_4}$ under $\mathcal{F}_{11,N}$ are the modules $V(\mathbf{n}_1), V(\mathbf{n}_2), V(\mathbf{n}_3), V(\mathbf{n}_4)$ with respective Drinfeld polynomials

$$\begin{aligned} P_1(u) &= 1, \\ P_2(u) &= (u - q^{-3})(u - q^{-7})(u - q^{-9}), \\ P_3(u) &= P_4(u) = 1, \\ P_5(u) &= u - q^{-8}, \\ P_k(u) &= 1, \quad (6 \leqslant k \leqslant N-1); \end{aligned}$$

$$\begin{aligned} P_1(u) &= 1, \\ P_2(u) &= (u - q^{-3})(u - q^{-7}), \\ P_3(u) &= u - q^{-10}, \\ P_4(u) &= u - q^{-7}, \\ P_k(u) &= 1, \quad (5 \leqslant k \leqslant N-1); \end{aligned}$$

$$\begin{aligned} P_1(u) &= (u - q^{-2})(u - q^{-4}), \\ P_2(u) &= (u - q^{-7})(u - q^{-9}), \\ P_3(u) &= P_4(u) = 1, \\ P_5(u) &= u - q^{-8}, \\ P_k(u) &= 1, \quad (6 \leqslant k \leqslant N-1); \end{aligned}$$

$$
\begin{aligned}
P_1(u) &= (u-q^{-2})(u-q^{-4}),\\
P_2(u) &= u-q^{-7},\\
P_3(u) &= u-q^{-10},\\
P_4(u) &= u-q^{-7},\\
P_k(u) &= 1, \quad (5 \leqslant k \leqslant N-1).
\end{aligned}
$$

The modules $V(\mathbf{n}_1)$ and $V(\mathbf{n}_3)$ are non-zero only if $N \geqslant 6$. Hence $V(\mathbf{m}_1) \otimes V(\mathbf{m}_2)$ has only two composition factors $V(\mathbf{n}_2)$ and $V(\mathbf{n}_4)$ for $N = 5$, and four composition factors $V(\mathbf{n}_1), V(\mathbf{n}_2)$, $V(\mathbf{n}_3), V(\mathbf{n}_4)$ for $N \geqslant 6$.

4.4 We note that our result implies the following

Theorem 8 *All composition factors of the tensor product of two evaluation modules of $U_q(\widehat{\mathfrak{sl}}_N)$ occur with multiplicity one.*

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