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SOME CONJECTURES FOR MACDONALD POLYNOMIALS OF TYPE B, C, D

Michel Lassalle

Centre National de la Recherche Scientifique
Institut Gaspard Monge, Université de Marne-la-Vallée
77454 Marne-la-Vallée Cedex, France

Dedicated to Alain Lascoux on the occasion of his 60th birthday

Abstract. We present conjectures giving formulas for the Macdonald polynomials of type B, C, D which are indexed by a multiple of the first fundamental weight. The transition matrices between two different types are explicitly given.

Introduction

Among symmetric functions, the special importance of Schur functions comes from their intimate connection with representation theory. Actually the irreducible polynomial representations of $GL_n(\mathbb{C})$ are indexed by partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ of length $\leq n$, and their characters are the Schur functions s_λ.

In the eighties, I. G. Macdonald introduced a new family of symmetric functions $P_\lambda(q,t)$. These orthogonal polynomials depend rationally on two parameters q, t and generalize Schur functions, which are obtained for $t = q$ [9,10].

When the indexing partition is reduced to a row (k) (i.e. has length one), the Macdonald polynomial $g_k(q,t)$ of n variables $x = (x_1, \ldots, x_n)$ are given by their generating function

$$\prod_{i=1}^{n} \frac{(tux_i;q)_\infty}{(ux_i;q)_\infty} = \sum_{r\geq 0} u^r g_r(x;q,t),$$

with the standard notation $(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i)$. Of course for $t = q$ the complete functions $s_{(r)} = h_r$ are recovered.

A few years later, generalizing his previous work, Macdonald introduced another class of orthogonal polynomials, which are Laurent polynomials in several variables, and generalize the Weyl characters of compact simple Lie groups [11,12]. In the most simple situation of this new framework, a family $P_\lambda^{(R)}(q,t)$ of polynomials, depending rationally on two parameters q, t, is attached to each root system R.

These orthogonal polynomials are elements of the group algebra of the weight lattice of R, invariant under the action of the Weyl group. They are indexed by the dominant weights of R.

When R is of type A, the orthogonal polynomials $P_\lambda^{(R)}(q,t)$ correspond to the symmetric functions $P_\lambda(q,t)$ previously studied in [9,10]. For $t=q$, they correspond to the Weyl characters $\chi_\lambda^{(R)}$ of compact simple Lie groups.

This paper is only devoted to the Macdonald polynomials which are indexed by a multiple of the first fundamental weight ω_1. Since H. Weyl [15], it is well known that $\chi_{r\omega_1}^{(R)}$ is given by

(i) $h_r(X)+h_{r-1}(X)$, when $R=B_n$,

(ii) $h_r(X)$, when $R=C_n$,

(iii) $h_r(X)-h_{r-2}(X)$, when $R=D_n$,

with $X=(x_1,\ldots,x_n,1/x_1,\ldots,1/x_n)$.

However, as far as the author is aware, no such result is known when $t\neq q$, and no explicit expansion is available for the Macdonald polynomials $P_{r\omega_1}^{(R)}(q,t)$. The purpose of this paper is to present some conjectures generalizing the previous formulas.

Actually this problem can be considered in a more general setting, allowing two distinct parameters t,T, each of which is attached to a length of roots. We give an explicit formula for $P_{r\omega_1}^{(R)}(q,t,T)$ when R is of type B,C,D, together with an explicit formula for the transition matrices between different types. The entries of these transition matrices appear to be fully factorized and reveal deep connections with basic hypergeometric series.

We provide some support for these conjectures by showing that they are verified upon principal specialization. On the other hand, computer calculations show a very strong empirical evidence in their favor.

1. Macdonald polynomials

In this section we introduce our notations, and recall some general facts about Macdonald polynomials. For more details the reader is referred to [11,12,13].

The most general class of Macdonald polynomials is associated with a pair of root systems (R,S), spanning the same vector space and having the same Weyl group, with R reduced. Here we shall only consider the case of a pair (R,R), with R of type B,C,D.

Let V be a finite-dimensional real vector space endowed with a positive definite symmetric bilinear form $\langle u,v\rangle$. For all $v\in V$, we write $|v|=\langle v,v\rangle^{1/2}$, and $v^\vee=2v/|v|^2$.

Let $R \subset V$ be a reduced irreducible root system, W the Weyl group of R, R^+ the set of positive roots, $\{\alpha_1, \ldots, \alpha_n\}$ the basis of simple roots, and $R^\vee = \{\alpha^\vee \mid \alpha \in R\}$ the dual root system.

Let $Q = \sum_{i=1}^n \mathbb{Z}\, \alpha_i$ and $Q^+ = \sum_{i=1}^n \mathbb{N}\, \alpha_i$ be the root lattice of R and its positive octant. Let $P = \{\lambda \in V \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\ \forall \alpha \in R\}$ and $P^+ = \{\lambda \in V \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{N}\ \forall \alpha \in R^+\}$ be the weight lattice of R and the cone of dominant weights.

A basis of Q is formed by the simple roots α_i. A basis of P is formed by the fundamental weights ω_i defined by $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$. A partial order is defined on P by $\lambda \geq \mu$ if and only $\lambda - \mu \in Q^+$.

Let $\mathcal{A}$ denote the group algebra over $\mathbb{R}$ of the free Abelian group P. For each $\lambda \in P$ let e^λ denote the corresponding element of $\mathcal{A}$, subject to the multiplication rule $e^\lambda e^\mu = e^{\lambda+\mu}$. The set $\{e^\lambda, \lambda \in P\}$ forms an $\mathbb{R}$-basis of A.

The Weyl group W acts on P and on $\mathcal{A}$. Let $\mathcal{A}^W$ denote the subspace of W-invariants in A. Such elements are called "symmetric polynomials". There are two important examples of a basis of $\mathcal{A}^W$. The first one is given by the orbit-sums

$$m_\lambda = \sum_{\mu \in W\lambda} e^\mu, \qquad \lambda \in P^+.$$

Another basis is provided by the Weyl characters defined as follows. Let

$$\delta = \prod_{\alpha \in R^+} (e^{\alpha/2} - e^{-\alpha/2}) = e^{-\rho} \prod_{\alpha \in R^+} (e^\alpha - 1),$$

with $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha \in P$. Then $w\delta = \varepsilon(w)\delta$ for any $w \in W$, where $\varepsilon(w) = \det(w) = \pm 1$. For all $\lambda \in P$,

$$\chi_\lambda = \delta^{-1} \sum_{w \in W} \varepsilon(w) e^{w(\lambda+\rho)}$$

is in $\mathcal{A}^W$, and the set $\{\chi_\lambda,\ \lambda \in P^+\}$ forms an $\mathbb{R}$-basis of $\mathcal{A}^W$.

Let $0 < q < 1$. For any indeterminate x and for all $k \in \mathbb{N}$, define

$$(x;q)_\infty = \prod_{i=0}^{\infty} (1 - xq^i), \qquad (x;q)_k = \prod_{i=0}^{k-1} (1 - xq^i).$$

For each $\alpha \in R$ let $t_\alpha = q^{k_\alpha}$ be a positive real number such that $t_\alpha = t_\beta$ if $|\alpha| = |\beta|$. Then we have at most two different values for the t_α's. Define

$$\rho_k = \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha \alpha, \qquad \rho_k^* = \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha \alpha^\vee.$$

If $f = \sum_{\lambda \in P} \ a_\lambda e^\lambda \in \mathcal{A}$, let $\overline{f} = \sum_{\lambda \in P} \ a_\lambda e^{-\lambda}$ and $[\,f\,]_1$ its constant term a_0. The inner product defined on $\mathcal{A}$ by

$$\langle f, g\rangle_{q,t} = \frac{1}{|W|}[f\bar{g}\Delta_{q,t}]_1,$$

with $|W|$ the order of W, and

$$\Delta_{q,t} = \prod_{\alpha \in R} \frac{(e^\alpha;q)_\infty}{(t_\alpha e^\alpha;q)_\infty}$$

is non degenerate and W-invariant.

There exists a unique basis $\{P_\lambda,\ \lambda \in P^+\}$ of $\mathcal{A}^W$, called Macdonald polynomials, such that

(i) $P_\lambda = m_\lambda + \sum_{\mu \in P^+,\ \mu < \lambda} \ a_{\lambda\mu}(q,t)\ m_\mu$
where the coefficients $a_{\lambda\mu}(q,t)$ are rational functions of q and the t_α's,

(ii) $\langle P_\lambda, P_\mu\rangle_{q,t} = 0$ if $\lambda \neq \mu$.

It is clear that the P_λ, if they exist, are unique. Their existence is proved as eigenvectors of an operator $E : \mathcal{A}^W \to \mathcal{A}^W$, selfadjoint with respect to $\langle\ ,\ \rangle_{q,t}$, and having its eigenvalues all distinct. When R is of type B, C, D this operator may be constructed as follows [11,12].

Let π be a minuscule weight of $R^\vee$, i.e. a vector $\pi \in V$ such that $\langle \pi, \alpha\rangle$ takes only values 0 and 1 for $\alpha \in R^+$. Such a vector exists when R is of type B, C, D, and is necessarily a fundamental weight of $R^\vee$. Let

$$\Phi_\pi = \prod_{\alpha \in R^+} \frac{1 - t_\alpha^{\langle \pi,\alpha\rangle} e^\alpha}{1 - e^\alpha},$$

and T_π the translation operator defined on $\mathcal{A}$ by $T_\pi(e^\lambda) = q^{\langle \lambda,\pi\rangle} e^\lambda$ for any $\lambda \in P$. Let E_π the operator defined by

$$E_\pi f = \sum_{w \in W} w(\Phi_\pi \cdot T_\pi f).$$

Macdonald polynomials P_λ are then introduced as eigenvectors of E_π :

$$E_\pi \ P_\lambda = c_\lambda \ P_\lambda \qquad \text{with} \qquad c_\lambda = q^{\langle \pi,\rho_k\rangle} \sum_{w \in W} q^{\langle w\pi, \lambda + \rho_k\rangle}.$$

We may regard any $f = \sum_{\lambda \in P} f_\lambda e^\lambda \in \mathcal{A}$, having only finitely many nonzero coefficients, as a function on V by putting for any $x \in V$,

$$f(x) = \sum_{\lambda \in P} f_\lambda q^{\langle \lambda, x\rangle}.$$

Then Macdonald polynomials satisfy the following Specialization Formula [2]

$$P_\lambda(\rho_k^*) = q^{-\langle\lambda,\rho_k^*\rangle} \prod_{\alpha\in R^+} \frac{(q^{\langle\rho_k,\alpha^\vee\rangle}t_\alpha;q)_{\langle\lambda,\alpha^\vee\rangle}}{(q^{\langle\rho_k,\alpha^\vee\rangle};q)_{\langle\lambda,\alpha^\vee\rangle}}.$$

In the sequel we shall consider R to be of type B_n, C_n or D_n. We shall identify V with $\mathbb{R}^n$ and write $\varepsilon_1,\dots,\varepsilon_n$ for its standard basis. Defining $x_i = e^{\varepsilon_i}$ for $i = 1\dots n$, we shall regard P_λ as a Laurent polynomial of n variables $x_1,\dots,x_n$.

We shall only consider Macdonald polynomials P_λ associated with a weight $\lambda = r\omega_1$, multiple of the first fundamental weight $\omega_1 = \varepsilon_1$.

2. Basic hypergeometric series

We shall need three results about basic hypergeometric series. The author is deeply indebted to Professor Mizan Rahman for communicating their proofs to him. Since these results have intrinsic interest, their proofs are given below in Section 10.

We adopt the notation of [3] and write

$${}_{r+1}\phi_r\left[\begin{matrix} a_1,a_2,\dots,a_{r+1} \\ b_1,b_2,\dots,b_r\end{matrix};q,z\right] = \sum_{i\ge 0} \frac{(a_1;q)_i\dots(a_{r+1};q)_i}{(b_1;q)_i\dots(b_r;q)_i}\,\frac{z^i}{(q;q)_i}.$$

Theorem 1. *We have the following transformation between ${}_2\phi_1$ series*

$$\frac{(u;q)_r}{(q;q)_r}\;{}_2\phi_1\left[\begin{matrix} q^{-r},\ ux \\ q^{1-r}/ux\end{matrix};q,qv/u^2x\right] = \sum_{i=0}^{[r/2]} \frac{(u;q)_{r-2i}}{(q;q)_{r-2i}}\;{}_2\phi_1\left[\begin{matrix} q^{2i-r},\ uy \\ q^{1+2i-r}/uy\end{matrix};q,qv/u^2y\right]$$
$$\times\, y^i v^i\,\frac{(x/y;q)_i}{(q;q)_i}\,\frac{(uq^{r-2i};q)_{2i}}{(uxq^{r-i};q)_i\,(uyq^{r-2i+1};q)_i}.$$

An infinite-dimensional matrix $(f_{ij})_{i,j\in\mathbb{Z}}$ is said to be lower-triangular if $f_{ij} = 0$ unless $i \ge j$. Two infinite-dimensional lower-triangular matrixes $(f_{ij})_{i,j\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ are said to be mutually inverse if $\sum_{i\ge j\ge k} f_{ij}g_{jk} = \delta_{ik}$.

Corollary. *Defining*

$$\mathcal{M}_{r,r-2i}^{(u,v;x,y)} = y^i v^i\,\frac{(x/y;q)_i}{(q;q)_i}\,\frac{(uq^{r-2i};q)_{2i}}{(uxq^{r-i};q)_i\,(uyq^{r-2i+1};q)_i},$$

the infinite matrices $\mathcal{M}(u,v;x,y)$ and $\mathcal{M}(u,v;y,x)$ are mutually inverse.

Michael Schlosser remarked that this corollary is equivalent with Bressoud's matrix inverse [1], which states that, defining

$$\mathcal{A}_{ij}^{(u,v)} = (u/v)^j\,\frac{(u/v;q)_{i-j}}{(q;q)_{i-j}}\,\frac{(u;q)_{i+j}}{(vq;q)_{i+j}}\,\frac{1-vq^{2j}}{1-v},$$

the matrices $\mathcal{A}(u,v)$ and $\mathcal{A}(v,u)$ are mutually inverse. Indeed let d be either 0 or 1. Replacing r by $2r+d$, and i by $r-i$, some factors cancel or can be pulled out of the previous sum, yielding the above form of [1].

Bressoud's matrix inverse was originally derived from the terminating very-well-poised ${}_6\phi_5$ summation [3, (II.21)]. We refer to [4, 8] for some generalizations of [1], as well as more details and references about inversion of infinite matrices.

Theorem 2. *Defining*

$$\frac{(u;q)_r}{(q;q)_r}\ H_r = \sum_{i=0}^{r} a^{r-i}\ \frac{(u;q)_{r-i}}{(q;q)_{r-i}}\ {}_2\phi_1\left[\begin{matrix} q^{i-r},\ v \\ q^{1+i-r}/v \end{matrix}; q, q/v^2a^2\right] \times \frac{(x;q)_i}{(q;q)_i}\ \frac{(uq^{r-i};q)_i}{(vq^{r-i+1};q)_i}\ \frac{(v^2q^{2r-i+1};q)_i}{(xv^2q^{2r-i};q)_i},$$

we have

$$H_r = (-1/v)^r\,\frac{(xv^2;q)_r}{(xv^2;q)_{2r}}\ \frac{(x^2v^2;q^2)_r(v^2q;q^2)_r}{(xv;q)_r}\ {}_4\phi_3\left[\begin{matrix} q^{-r}, xv^2q^r, -av, -1/a \\ vq^{\frac12}, -vq^{\frac12}, -xv \end{matrix}; q, q\right].$$

A converse property may be stated as follows.

Theorem 3. *We have*

$$a^r\,\frac{(u;q)_r}{(q;q)_r}\ {}_2\phi_1\left[\begin{matrix} q^{-r},\ v \\ q^{1-r}/v \end{matrix}; q, q/v^2a^2\right] = \sum_{i=0}^{r} \frac{(u;q)_{r-i}}{(q;q)_{r-i}}\ H_{r-i} \times x^i\ \frac{(1/x;q)_i}{(q;q)_i}\ \frac{(uq^{r-i};q)_i}{(vq^{r-i+1};q)_i}\ \frac{(v^2q^{2r-2i+1};q)_i}{(xv^2q^{2r-2i+1};q)_i}\ \frac{1-v^2q^{2r}}{1-v^2q^{2r-i}}.$$

Observe that, as a consequence of Theorems 2 and 3, we recover the special case $y=1$ of the following matrix inverse.

Lemma 1. *Defining*

$$\mathcal{N}_{ij}^{(u,v;x,y)} = y^{i-j}\ \frac{(x/y;q)_{i-j}}{(q;q)_{i-j}}\ \frac{(uq^j;q)_{i-j}}{(vq^{j+1};q)_{i-j}}\ \frac{(v^2q^{2j+1};q)_{2i-2j}}{(xv^2q^{i+j};q)_{i-j}\ (yv^2q^{2j+1};q)_{i-j}},$$

the infinite matrices $\mathcal{N}(u,v;x,y)$ and $\mathcal{N}(u,v;y,x)$ are mutually inverse.

Proof. Defining

$$\mathcal{B}_{ij}^{(x,y)} = y^{i-j}\ \frac{(x/y;q)_{i-j}}{(q;q)_{i-j}}\ \frac{1}{(xq^{i+j};q)_{i-j}\ (yq^{2j+1};q)_{i-j}},$$

the infinite matrices $\mathcal{B}(x,y)$ and $\mathcal{B}(y,x)$ are mutually inverse, as a consequence of a result of Krattenthaler [4]. If two infinite matrices (f_{ij}) and (g_{kl}) are mutually inverse, for any sequence (d_k), the matrices $(f_{ij}\ d_i/d_j)$ and $(g_{kl}\ d_k/d_l)$ are obviously mutually inverse. Since

$$(v^2q^{2j+1};q)_{2i-2j}\ \frac{(uq^j;q)_{i-j}}{(vq^{j+1};q)_{i-j}} = \frac{(v^2q;q)_{2i}}{(v^2q;q)_{2j}}\ \frac{(u;q)_i}{(u;q)_j}\ \frac{(vq;q)_j}{(vq;q)_i},$$

we apply this property to $\mathcal{B}(xv^2, yv^2)$ and $\mathcal{B}(yv^2, xv^2)$, with

$$d_k = v^{2k}\ (v^2q;q)_{2k}\ \frac{(u;q)_k}{(vq;q)_k}. \qquad \square$$

3. Type C

For the type C root system, the set of positive roots is the union of $R_1 = \{\varepsilon_i \pm \varepsilon_j, 1 \le i < j \le n\}$ and $R_2 = \{2\varepsilon_i, 1 \le i \le n\}$. Elements of each set have the same length, and we write $t_\alpha = t$ for $\alpha \in R_1$ and $t_\alpha = T$ for $\alpha \in R_2$.

The Weyl group W is the semi-direct product of the permutation group S_n by $(\mathbb{Z}/2\mathbb{Z})^n$. It acts on V by signed permutation of components. The fundamental weights are given by $\omega_i = \sum_{j=1}^{i} \varepsilon_j,\ 1 \le i \le n$. The dominant weights $\lambda \in P^+$ can be identified with vectors $\lambda = \sum_{i=1}^{n} \lambda_i \varepsilon_i$, such that $(\lambda_1, \lambda_2, \dots, \lambda_n)$ is a partition. There is only one minuscule weight ω_1. The partial order $\lambda \ge \mu$ is given by

$$\sum_{i=1}^{j} (\lambda_i - \mu_i) \in \mathbb{N}, \quad \text{for} \quad j = 1, \dots, n-1, \qquad \sum_{i=1}^{n} (\lambda_i - \mu_i) \in 2\mathbb{N}.$$

With $t = q^k$ and $T = q^K$, we have

$$\rho_k = \sum_{i=1}^{n} ((n-i)k + K)\, \varepsilon_i, \qquad \rho_k^* = \sum_{i=1}^{n} ((n-i)k + K/2)\, \varepsilon_i.$$

If we write $x_i = e^{\varepsilon_i}$ for $i = 1 \dots n$, and define

$$P^{(C)}_{r\omega_1}(x;q,t,T) = \frac{(q;q)_r}{(t;q)_r}\ g_r^{(C)}(x;q,t,T),$$

the Specialization Formula reads

$$g_r^{(C)}(t^{n-1}T^{\frac12}, t^{n-2}T^{\frac12}, \dots, T^{\frac12};\ q,t,T) = \frac{t^{r(1-n)}}{T^{r/2}}\ \frac{(t^n;q)_r}{(q;q)_r}\ \frac{(T^2t^{2n-2};q)_r}{(Tt^{n-1};q)_r}.$$

Let us introduce the auxiliary quantities $G_r(x;q,t)$ defined by their generating function

$$\prod_{i=1}^{n} \frac{(tux_i;q)_\infty}{(ux_i;q)_\infty}\ \frac{(tu/x_i;q)_\infty}{(u/x_i;q)_\infty} = \sum_{r\ge 0} u^r G_r(x;q,t).$$

In λ-ring notation (see Section 8), they can be written as

$$G_r(x;q,t) = h_r\left[\frac{1-t}{1-q} X^\dagger\right],$$

with $X^\dagger = \sum_{i=1}^{n} (x_i + 1/x_i)$. Their specialization may be given as follows.

Lemma 2. *For any positive integer r we have*

$$G_r(t^{n-1}a,t^{n-2}a,\dots,a;\ q,t)=a^r\ \frac{(t^n;q)_r}{(q;q)_r}\ {}_2\phi_1\left[\begin{matrix}q^{-r},\ t^n\\ t^{-n}q^{1-r}\end{matrix};q,qt^{1-2n}/a^2\right].$$

Proof. Taking into account

$$\prod_{i=1}^n \frac{(t^{n-i+1}ua;q)_\infty}{(t^{n-i}ua;q)_\infty}\ \frac{(t^{i-n+1}u/a;q)_\infty}{(t^{i-n}u/a;q)_\infty}=\frac{(t^nua;q)_\infty}{(ua;q)_\infty}\ \frac{(tu/a;q)_\infty}{(t^{1-n}u/a;q)_\infty},$$

and applying the classical q–binomial formula [3, (II.3)]

$$\frac{(tu;q)_\infty}{(u;q)_\infty}=\sum_{i\geq 0}\frac{(t;q)_i}{(q;q)_i}\ u^i,$$

we have

$$G_r(t^{n-1}a,t^{n-2}a,\dots,a;\ q,t)=\sum_{i=0}^r a^{(r-2i)}t^{i(1-n)}\ \frac{(t^n;q)_i}{(q;q)_i}\ \frac{(t^n;q)_{r-i}}{(q;q)_{r-i}}.\quad\square$$

Conjecture 1. *For any positive integer r we have*

$$g_r^{(C)}(q,t,T)=\sum_{i=0}^{[r/2]}G_{r-2i}(q,t)\ t^i\ \frac{(T/t;q)_i}{(q;q)_i}\ \frac{(t^nq^{r-i};q)_i}{(Tt^{n-1}q^{r-i};q)_i}\ \frac{1-t^nq^{r-2i}}{1-t^nq^{r-i}}.$$

Conversely

$$G_r(q,t)=\sum_{i=0}^{[r/2]}g_{r-2i}^{(C)}(q,t,T)\ T^i\ \frac{(t/T;q)_i}{(q;q)_i}\ \frac{(t^nq^{r-2i};q)_i}{(Tt^{n-1}q^{r-2i+1};q)_i}.$$

In other words, the transition matrix from $g^{(C)}(q,t,T)$ to $G(q,t)$ is $\mathcal{M}(t^n,t;T/t,1)$, and its inverse is $\mathcal{M}(t^n,t;1,T/t)$.

Using λ-ring techniques, we have proved this conjecture for $T=t$.

Theorem 4. *For any positive integer r we have $g_r^{(C)}(q,t,t)=G_r(q,t)$.*

The proof will be given below in Section 9. The Specialization Formula gives some support to Conjecture 1.

Lemma 3. *Conjecture 1 yields the specialization*

$$g_r^{(C)}(t^{n-1}a,t^{n-2}a,\dots,a;\ q,t,T)=a^r\ \frac{(t^n;q)_r}{(q;q)_r}\ {}_2\phi_1\left[\begin{matrix}q^{-r},\ Tt^{n-1}\\ t^{1-n}q^{1-r}/T\end{matrix};q,qt^{2-2n}/Ta^2\right].$$

Proof. A straightforward application of Theorem 1 and Lemma 2 with $u=t^n$, $v=t/a^2$, $x=T/t$, and $y=1$. $\square$

Corollary. *Conjecture 1 is true for* $x_i = t^{n-i}T^{\frac{1}{2}}$ $(1 \le i \le n)$.

Proof. Keeping the same notations, we now have $v = 1/x$. The previous result is a q–Chu–Vandermonde sum [3, (II.7)] given by

$$\begin{aligned} T^{r/2}\frac{(u;q)_r}{(q;q)_r}\,{}_2\phi_1\left[\begin{matrix} q^{-r},\ uv \\ q^{1-r}/uv \end{matrix};q,q/u^2v^2\right] &= T^{r/2}\frac{(u;q)_r}{(q;q)_r}\,\frac{(q^{1-r}/u^2v^2;q)_r}{(q^{1-r}/uv;q)_r} \\ &= T^{r/2}(uv)^{-r}\,\frac{(u;q)_r}{(uv;q)_r}\,\frac{(u^2v^2;q)_r}{(q;q)_r}. \end{aligned}$$

We recover the Specialization Formula. □

4. D versus C

The root system D_n is self-dual: $R = R^\vee$. The set of positive roots is $R^+ = \{\varepsilon_i \pm \varepsilon_j, 1 \le i < j \le n\}$. Roots have all the same length and we write $t_\alpha = t$ for $\alpha \in R^+$.

The Weyl group W is the semi-direct product of the permutation group S_n by $(\mathbb{Z}/2\mathbb{Z})^{n-1}$. It acts on V by signed permutation of components, subject to the condition that the number of minus signs is even. The fundamental weights are given by $\omega_i = \sum_{j=1}^i \varepsilon_j$ for $1 \le i \le n-2$. The "spin weights" ω_{n-1} and ω_n are defined by $\omega_n = \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_n)$ and $\omega_{n-1} = \omega_n - \varepsilon_n$. There are three minuscule weights ω_1, ω_{n-1} and ω_n.

The dominant weights $\lambda \in P^+$ can be identified with vectors $\lambda = \sum_{i=1}^n \lambda_i\varepsilon_i$, whose components are all integers or all half-integers, and subject to the condition $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_{n-1} \ge |\lambda_n|$. The partial order $\lambda \ge \mu$ is given by

$$\sum_{i=1}^{j}(\lambda_i - \mu_i) \in \mathbb{N}, \quad \text{for} \quad j = 1, \ldots, n-2, \qquad \sum_{i=1}^{n-1}(\lambda_i - \mu_i)\ \pm (\lambda_n - \mu_n) \in 2\mathbb{N}.$$

Writing $t = q^k$ we have $\rho_k = \rho_k^* = k\ \sum_{i=1}^{n}(n-i)\varepsilon_i$.

If we define

$$P^{(D)}_{r\omega_1}(x;q,t) = \frac{(q;q)_r}{(t;q)_r}\,g_r^{(D)}(x;q,t),$$

the Specialization Formula reads

$$g_r^{(D)}(t^{n-1},t^{n-2},\ldots,1;\ q,t) = t^{r(1-n)}\ \frac{(t^n;q)_r}{(q;q)_r}\frac{(t^{2n-2};q)_r}{(t^{n-1};q)_r}.$$

We have easily $g_r^{(D)}(q,t) = g_r^{(C)}(q,t,1)$ and Lemma 3 can be written as follows.

Lemma 4. *Conjecture 1 yields the specialization*

$$g_r^{(D)}(t^{n-1}a,t^{n-2}a,\dots,a;\ q,t)=a^r\,\frac{(t^n;q)_r}{(q;q)_r}\ {}_2\phi_1\left[\begin{matrix}q^{-r},\ t^{n-1}\\ t^{1-n}q^{1-r}\end{matrix};q,t^{2-2n}q/a^2\right].$$

As before, for $a=1$ the previous expression is a q–Chu–Vandermonde sum, and we recover the Specialization Formula.

Conjecture 2. *For any positive integer r we have*

$$g_r^{(D)}(q,t)=\sum_{i=0}^{[r/2]} g_{r-2i}^{(C)}(q,t,T)\ T^i\ \frac{(1/T;q)_i}{(q;q)_i}\ \frac{(t^nq^{r-2i};q)_{2i}}{(t^{n-1}q^{r-i};q)_i\ (Tt^{n-1}q^{r-2i+1};q)_i}.$$

Conversely

$$g_r^{(C)}(q,t,T)=\sum_{i=0}^{[r/2]} g_{r-2i}^{(D)}(q,t)\ \frac{(T;q)_i}{(q;q)_i}\ \frac{(t^nq^{r-2i};q)_{2i}}{(Tt^{n-1}q^{r-i};q)_i\ (t^{n-1}q^{r-2i+1};q)_i}.$$

Namely, the transition matrix from $g^{(D)}(q,t)$ to $g^{(C)}(q,t,T)$ is $\mathcal{M}(t^n,t;1/t,T/t)$ and its inverse is $\mathcal{M}(t^n,t;T/t,1/t)$.

Conjectures 1 and 2 are consistent since the former, written for $T=t$ and using Theorem 4, and the latter, written for $T=1$, both yield

Conjecture 3. *For any positive integer r we have*

$$g_r^{(D)}(q,t)=\sum_{i=0}^{[r/2]} G_{r-2i}(q,t)\ t^i\ \frac{(1/t;q)_i}{(q;q)_i}\ \frac{(t^nq^{r-i};q)_i}{(t^{n-1}q^{r-i};q)_i}\ \frac{1-t^nq^{r-2i}}{1-t^nq^{r-i}}.$$

Conversely

$$G_r(q,t)=\sum_{i=0}^{[r/2]} g_{r-2i}^{(D)}(q,t)\ \frac{(t;q)_i}{(q;q)_i}\ \frac{(t^nq^{r-2i};q)_i}{(t^{n-1}q^{r-2i+1};q)_i}.$$

Equivalently, the transition matrix from $g^{(D)}(q,t)$ to $G(q,t)$ is $\mathcal{M}(t^n,t;1/t,1)$, and its inverse is $\mathcal{M}(t^n,t;1,1/t)$.

The corollary of Lemma 3 shows that Conjecture 3 is true for $x_i=t^{n-i}$ $(1\le i\le n)$. Specialization yields another consistency argument.

Lemma 5. *Assuming Lemmas 3 and 4, Conjecture 2 is true for $x_i=t^{n-i}a$, $1\le i\le n$.*

Proof. A straightforward application of Theorem 1 with $u=t^n$, $v=t/a^2$, $x=1/t$, and $y=T/t$. □

5. B versus D

For the type B root system, the set of positive roots is the union of $R_1 = \{\varepsilon_i \pm \varepsilon_j, 1 \le i < j \le n\}$ and $R_2 = \{\varepsilon_i, 1 \le i \le n\}$. Elements of each set have the same length, and we write $t_\alpha = t$ for $\alpha \in R_1$ and $t_\alpha = T$ for $\alpha \in R_2$.

The Weyl group W is the semi-direct product of the permutation group S_n by $(\mathbb{Z}/2\mathbb{Z})^n$. It acts on V by signed permutation of components. The fundamental weights are given by $\omega_i = \sum_{j=1}^i \varepsilon_j$ for $1 \le i \le n-1$, and $\omega_n = \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_n)$. This is the only minuscule weight.

The dominant weights $\lambda \in P^+$ can be identified with vectors $\lambda = \sum_{i=1}^n \lambda_i \varepsilon_i$, whose components are all integers or all half-integers, and subject to the condition $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n \ge 0$. The partial order $\lambda \ge \mu$ is given by $\sum_{i=1}^j (\lambda_i - \mu_i) \in \mathbb{N}$, for $j = 1, \ldots, n$. Writing $t = q^k$ and $T = q^K$, we have

$$\rho_k = \sum_{i=1}^n ((n-i)k + K/2)\,\varepsilon_i, \qquad \rho_k^* = \sum_{i=1}^n ((n-i)k + K)\,\varepsilon_i.$$

If we define

$$P^{(B)}_{r\omega_1}(x; q, t, T) = \frac{(q;q)_r}{(t;q)_r}\, g_r^{(B)}(x; q, t, T),$$

the Specialization Formula reads

$$g_r^{(B)}(t^{n-1}T, t^{n-2}T, \ldots, T;\ q, t, T) = \frac{t^{r(1-n)}}{T^r}\ \frac{(t^n;q)_r}{(q;q)_r}\ \frac{(Tt^{2n-2};q)_r}{(Tt^{2n-2};q)_{2r}}\ \frac{(T^2t^{2n-2};q)_{2r}}{(Tt^{n-1};q)_r}.$$

Conjecture 4. *For any positive integer r we have*

$$g_r^{(B)}(q,t,T) = \sum_{i=0}^r g_{r-i}^{(D)}(q,t)\ \frac{(T;q)_i}{(q;q)_i}\,\frac{(t^nq^{r-i};q)_i}{(t^{n-1}q^{r-i+1};q)_i}\ \frac{(t^{2n-2}q^{2r-i+1};q)_i}{(Tt^{2n-2}q^{2r-i};q)_i}.$$

Conversely

$$g_r^{(D)}(q,t) = \sum_{i=0}^r g_{r-i}^{(B)}(q,t,T)\ T^i\ \frac{(1/T;q)_i}{(q;q)_i}\ \frac{(t^nq^{r-i};q)_i}{(t^{n-1}q^{r-i+1};q)_i} \times \frac{(t^{2n-2}q^{2r-2i+1};q)_i}{(Tt^{2n-2}q^{2r-2i+1};q)_i}\ \frac{1-t^{2n-2}q^{2r}}{1-t^{2n-2}q^{2r-i}}.$$

Namely, the transition matrix from $g^{(B)}(q,t,T)$ to $g^{(D)}(q,t)$ is $\mathcal{N}(t^n, t^{n-1}; T, 1)$, and its inverse is $\mathcal{N}(t^n, t^{n-1}; 1, T)$.

Of course for $T = 1$ we recover $g_r^{(D)}(q,t) = g_r^{(B)}(q,t,1)$.

Lemma 6. *Conjectures 1 and 4 yield the specialization*

$$\begin{aligned} g_r^{(B)}&(t^{n-1}a, t^{n-2}a, \dots, a;\ q,t,T) \\ &= (-1)^r t^{r(1-n)}\ \frac{(t^n;q)_r}{(q;q)_r}\ \frac{(Tt^{2n-2};q)_r}{(Tt^{2n-2};q)_{2r}} \frac{(T^2t^{2n-2};q^2)_r (t^{2n-2}q;q^2)_r}{(Tt^{n-1};q)_r} \\ &\qquad\times {}_4\phi_3 \left[\begin{matrix} q^{-r}, Tt^{2n-2}q^r, -at^{n-1}, -1/a \\ t^{n-1}q^{\frac{1}{2}}, -t^{n-1}q^{\frac{1}{2}}, -Tt^{n-1} \end{matrix} ; q,q \right]. \end{aligned}$$

Proof. A straightforward application of Lemma 4 and Theorem 2 with $u = t^n$, $v = t^{n-1}$ and $x = T$. □

Corollary. *This property is true for $a = T$.*

Proof. Keeping the notations $u = t^n$, $v = t^{n-1}$ and $x = T$, we now have $x = a$. Applying the q–Saalschütz formula [3, (II.12)], we have

$$\begin{aligned} {}_4\phi_3 \left[\begin{matrix} q^{-r}, av^2q^r, -av, -1/a \\ vq^{\frac{1}{2}}, -vq^{\frac{1}{2}}, -av \end{matrix} ; q,q \right] &= {}_3\phi_2 \left[\begin{matrix} q^{-r}, av^2q^r, -1/a \\ vq^{\frac{1}{2}}, -vq^{\frac{1}{2}} \end{matrix} ; q,q \right] \\ &= \frac{(-avq^{\frac{1}{2}}q)_r}{(vq^{\frac{1}{2}};q)_r}\ \frac{(q^{\frac{1}{2}}q^{-r}/av;q)_r}{(-q^{\frac{1}{2}}q^{-r}/v;q)_r}. \end{aligned}$$

This can be written

$$\begin{aligned} {}_4\phi_3 \left[\begin{matrix} q^{-r}, av^2q^r, -av, -1/a \\ vq^{\frac{1}{2}}, -vq^{\frac{1}{2}}, -av \end{matrix} ; q,q \right] &= (-1/a)^r \frac{(avq^{\frac{1}{2}};q)_r}{(vq^{\frac{1}{2}};q)_r}\ \frac{(-avq^{\frac{1}{2}};q)_r}{(-vq^{\frac{1}{2}};q)_r} \\ &= (-1/a)^r \frac{(a^2v^2;q)_{2r}}{(a^2v^2;q^2)_r (v^2q;q^2)_r}. \end{aligned}$$

By substitution we recover the Specialization Formula. □

6. B versus C

The development of $g_r^{(B)}(q,t,U)$ in terms of $g_r^{(C)}(q,t,T)$ (and conversely) can be immediately written by composing the previous conjectures. We did not find a more compact expansion.

Conjecture 5. *For any positive integer r we have*

$$\begin{aligned} g_r^{(B)}(q,t,U) = &\sum_{0 \le i+2j \le r} g_{r-i-2j}^{(C)}(q,t,T)\ T^j\ \frac{(1/T;q)_j}{(q;q)_j}\ \frac{(U;q)_i}{(q;q)_i} \\ &\times \frac{(t^nq^{r-i-2j};q)_{i+2j}}{(t^{n-1}q^{r-i-j};q)_{i+j}\ (Tt^{n-1}q^{r-i-2j+1};q)_j}\ \frac{1-t^{n-1}q^{r-i}}{1-t^{n-1}q^r}\ \frac{(t^{2n-2}q^{2r-i+1};q)_i}{(Ut^{2n-2}q^{2r-i};q)_i}. \end{aligned}$$

Conversely

$$g_r^{(C)}(q,t,T) = \sum_{0\le 2i+j\le r} g_{r-2i-j}^{(B)}(q,t,U)\ U^j\ \frac{(1/U;q)_j}{(q;q)_j}\ \frac{(T;q)_i}{(q;q)_i}$$
$$\frac{(t^{2n-2}q^{2r-4i-2j+1};q)_j}{(Ut^{2n-2}q^{2r-4i-2j+1};q)_j}\ \frac{1-t^{2n-2}q^{2r-4i}}{1-t^{2n-2}q^{2r-4i-j}}\ \frac{(t^nq^{r-2i-j};q)_{2i+j}}{(Tt^{n-1}q^{r-i};q)_i(t^{n-1}q^{r-2i-j+1};q)_{i+j}}.$$

7. Open problems

The statements of Lemmas 3, 4 and 6 can be written in a rather similar form by applying the following property.

Lemma 7. *We have*

$$ {}_2\phi_1\left[\begin{matrix} q^{-r},\ v \\ q^{1-r}/v \end{matrix};q,q/v^2a^2\right] = (av)^{-r}\frac{(v^2;q)_r}{(v;q)_r}\,{}_4\phi_3\left[\begin{matrix} q^{-r},v^2q^r,av,1/a \\ vq^{\frac12},-vq^{\frac12},-v \end{matrix};q,q\right].$$

Proof. A straightforward consequence of [3, (7.4.12) and (7.4.13)]. □

Writing $g_r^{(R)}(a;q,t,T)$ for $g_r^{(R)}(t^{n-1}a,t^{n-2}a,\dots,a;q,t,T)$, we then have

$$\begin{aligned}
g_r^{(C)}(a;q,t,T) &= (-1)^r g_r^{(C)}(T^{\frac12};q,t,T)\\
&\quad\times {}_4\phi_3\left[\begin{matrix} q^{-r},T^2t^{2n-2}q^r,-aT^{\frac12}t^{n-1},-T^{\frac12}/a \\ Tt^{n-1}q^{\frac12},-Tt^{n-1}q^{\frac12},-Tt^{n-1} \end{matrix};q,q\right]\\
g_r^{(D)}(a;q,t) &= (-1)^r g_r^{(D)}(1;q,t)\\
&\quad\times {}_4\phi_3\left[\begin{matrix} q^{-r},t^{2n-2}q^r,-at^{n-1},-1/a \\ t^{n-1}q^{\frac12},-t^{n-1}q^{\frac12},-t^{n-1} \end{matrix};q,q\right]\\
g_r^{(B)}(a;q,t,T) &= (-T)^r g_r^{(B)}(T;q,t,T)\frac{(t^{2n-2}q;q^2)_r}{(T^2t^{2n-2}q;q^2)_r}\\
&\quad\times {}_4\phi_3\left[\begin{matrix} q^{-r},Tt^{2n-2}q^r,-at^{n-1},-1/a \\ t^{n-1}q^{\frac12},-t^{n-1}q^{\frac12},-Tt^{n-1} \end{matrix};q,q\right].
\end{aligned}$$

These formulas seem difficult to unify in a general conjecture written in terms of the root system R.

We are also in lack of a conjecture for the generating function of $g_r^{(R)}$, except when $R=C$ and $T=t$ (see Theorem 4).

In another paper, we shall present conjectures giving for $R\in\{B,C,D\}$,

(i) a "generalized Pieri formula" expanding $g_r^{(R)}g_s^{(R)}$ in terms of the Macdonald polynomials $P_{\lambda_1\omega_1+\lambda_2\omega_2}^{(R)}$,

(ii) conversely, the expansion of any Macdonald polynomial $P_{\lambda_1\omega_1+\lambda_2\omega_2}^{(R)}$ in terms of products $g_r^{(R)}g_s^{(R)}$ ("inverse Pieri formula").

8. λ-rings

This section and the following will be devoted to the proof of Theorem 4. This will be done in the language of λ-rings, which turns out to be the most efficient. Here we only intend to give a short survey of this theory. Details and other applications may be found, for instance, in [6, 7], and in some examples of [10] (see pp. 25, 43, 65 and 79).

The basic idea of the theory of λ-rings is the following. A symmetric function f is usually understood as *evaluated* on a set of variables $A = \{a_1, a_2, a_3, \dots\}$, the value being denoted $f(A)$. When using λ-rings, this interpretation is not the main one. Symmetric functions are first understood as *operators* on polynomials. Thus any symmetric function f is first understood as *acting on* the polynomial P, mapping P to $f[P]$. Of course the standard interpretation may be recovered as a special case. These statements may be made more precise as follows.

Let $A = \{a_1, a_2, a_3, \dots\}$ be a (finite or infinite) set of independent indeterminates, called an alphabet. We introduce the generating functions

$$E_u(A) = \prod_{a\in A}(1+ua), \quad H_u(A) = \prod_{a\in A}\frac{1}{1-ua}, \quad P_u(A) = \sum_{a\in A}\frac{a}{1-ua},$$

whose development defines symmetric functions known as elementary functions $e_k(A)$, complete functions $h_k(A)$, and power sums $p_k(A)$, respectively. Each of these three sets generate the symmetric algebra $\mathbb{S}(A)$.

We define an action, denoted $[\ ]$, of $\mathbb{S}(A)$ on the ring $\mathbb{R}[A]$ of polynomials in A with real coefficients. Since the power sums p_k algebraically generate $\mathbb{S}(A)$, it is enough to define the action of p_k on $\mathbb{R}[A]$. Writing any polynomial as $\sum_{c,P} cP$, with c a real constant and P a monomial in $(a_1, a_2, a_3, \dots)$, we define

$$p_k\left[\sum_{c,P} cP\right] = \sum_{c,P} cP^k.$$

This action extends to $\mathbb{S}[A]$. For instance we obtain

$$E_u\left[\sum_{c,P} cP\right] = \prod_{c,P}(1+uP)^c \quad , \quad H_u\left[\sum_{c,P} cP\right] = \prod_{c,P}(1-uP)^{-c}.$$

More generally, we can define an action of $\mathbb{S}(A)$ on the ring of rational functions, and even on the ring of formal series, by writing

$$p_k\left(\frac{\sum cP}{\sum dQ}\right) = \frac{\sum cP^k}{\sum dQ^k}$$

with c, d real constants and P, Q monomials in $(a_1, a_2, a_3, \dots)$. This action still extends to $\mathbb{S}(A)$.

As an example, being given a (finite or infinite) alphabet $A = \{a_1, a_2, a_3, \dots\}$, let us compute $H_u[h_1(A)]$ and $H_u[h_2(A)]$. We obtain [10, Example 1.5.10, p. 79]

$$\begin{aligned} H_u\big[h_1(A)\big] &= \prod_i (1 - ua_i)^{-1}, \\ H_u\big[h_2(A)\big] &= \prod_i (1 - ua_i^2)^{-1} \prod_{i<j} (1 - ua_i a_j)^{-1}. \end{aligned}$$

In the following we shall write

$$A^\dagger = h_1(A) = \sum_i a_i.$$

Bu definition we have $p_k[A^\dagger] = \sum_i a_i^k$. Thus $p_k[A^\dagger] = p_k(A)$, which yields that for any symmetric function f, we have $f[A^\dagger] = f(A)$. In particular

$$f(1, q, q^2, q^3, \dots, q^{m-1}) = f\left[\frac{1-q^m}{1-q}\right], \quad f(1, q, q^2, q^3, \dots) = f\left[\frac{1}{1-q}\right].$$

The following relations are straightforward consequences of the previous definitions. For any formal series P, Q, we have

$$h_r[P+Q] = \sum_{k=0}^{r} h_{r-k}[P]\; h_k[Q], \quad e_r[P+Q] = \sum_{k=0}^{r} e_{r-k}[P]\; e_k[Q].$$

Or equivalently

$$\begin{aligned} H_u[P+Q] &= H_u[P]\, H_u[Q], \qquad & E_u[P+Q] &= E_u[P]\, E_u[Q] \\ H_u[P-Q] &= H_u[P]\, H_u[Q]^{-1}, & E_u[P-Q] &= E_u[P]\, E_u[Q]^{-1}. \end{aligned}$$

As an application, for a finite alphabet $A = \{a_1, a_2, \dots, a_m\}$ we may write

$$\begin{aligned} H_1\left[\frac{uA^\dagger}{1-q}\right] &= \prod_{i\geq 0} H_1\big[uq^i A^\dagger\big] = \prod_{k=1}^{m} \prod_{i\geq 0} H_1\big[uq^i a_k\big] \\ &= \prod_{k=1}^{m} \prod_{i\geq 0} \frac{1}{1 - uq^i a_k} = \prod_{k=1}^{m} \frac{1}{(ua_k; q)_\infty}, \end{aligned}$$

and

$$H_1\left[\frac{1-t}{1-q} A^\dagger\right] = H_1\left[\frac{A^\dagger}{1-q}\right] \left(H_1\left[\frac{tA^\dagger}{1-q}\right]\right)^{-1}.$$

Finally we obtain

$$H_1\left[\frac{1-t}{1-q}\, A^\dagger\right] = \sum_{r\geq 0} h_r \left[\frac{1-t}{1-q}\, A^\dagger\right] = \prod_{i=1}^{m} \frac{(ta_i; q)_\infty}{(a_i; q)_\infty}.$$

9. Proof of Theorem 4

In this section we assume that R is of type C_n with $T = t$. We define $x_i = e^{\varepsilon_i}$ for $i = 1 \dots n$, and regard elements of $\mathcal{A}$ as Laurent polynomials of n variables $x_1, \dots, x_n$.

The dual root system $R^\vee = B_n$ has one minuscule weight $\pi = \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_n)$. With the notations of Section 1, we have

$$\Phi_\pi = \prod_{i=1}^{n} \frac{1 - tx_i^2}{1 - x_i^2} \prod_{1 \le i < j \le n} \frac{1 - tx_ix_j}{1 - x_ix_j},$$

and the translation operator T_π acts on $\mathcal{A}$ by

$$T_\pi f(x_1, \dots, x_n) = f(q^{\frac{1}{2}}x_1, \dots, q^{\frac{1}{2}}x_n).$$

The Weyl group W is the semi-direct product of the permutation group S_n by $(\mathbb{Z}/2\mathbb{Z})^n$. It acts on V by signed permutation of components. Hence the W-orbit of π is formed by vectors $\frac{1}{2}(\sigma_1\varepsilon_1 + \dots + \sigma_n\varepsilon_n)$ with $\sigma \in (-1, +1)^n$.

The Macdonald operator E_π can be written as

$$E_\pi f = \sum_{\sigma \in (-1,+1)^n} \prod_{i=1}^{n} \frac{1 - tx_i^{2\sigma_i}}{1 - x_i^{2\sigma_i}} \prod_{1 \le i < j \le n} \frac{1 - tx_i^{\sigma_i}x_j^{\sigma_j}}{1 - x_i^{\sigma_i}x_j^{\sigma_j}} \; f(q^{\sigma_1/2}x_1, \dots, q^{\sigma_n/2}x_n).$$

Up to a constant, the Macdonald polynomial $P_{r\omega_1}$ is defined by

$$E_\pi P_{r\omega_1} = \prod_{i=1}^{n-1}(t^i + 1) \; (t^n q^{r/2} + q^{-r/2}) \, P_{r\omega_1}.$$

This normalization constant is given by the condition

$$P_{r\omega_1} = m_{r\omega_1} + \text{lower terms},$$

where the orbit-sum $m_{r\omega_1}$ is given by $m_{r\omega_1}(x) = \sum_{i=1}^{n}(x_i^r + 1/x_i^r)$.

The following notations will be used till the end of this paper. We write

$$X_+ = \{x_1, \dots, x_n\}, \quad X_- = \{1/x_1, \dots, 1/x_n\}, \quad X = X_+ \cup X_-,$$

so that $X^\dagger = \sum_{i=1}^{n}(x_i + 1/x_i)$. Observe that $\Phi_\pi = H_1\big[(1-t)h_2(X_+)\big]$.

We consider the generating series

$$H_1\left[\frac{1-t}{1-q}X^\dagger\right] = \sum_{r \ge 0} u^r \; h_r\left[\frac{1-t}{1-q}X^\dagger\right] = \prod_{i=1}^{n} \frac{(tux_i;q)_\infty}{(ux_i;q)_\infty} \; \frac{(tu/x_i;q)_\infty}{(u/x_i;q)_\infty}.$$

Theorem 4 states that

$$g_r^{(C)}(x;q,t,t) = h_r\left[\frac{1-t}{1-q}X^{\dagger}\right].$$

Since it is well known ([10, p. 314], [7, p. 237]) that for any alphabet $A = \{a_1,\dots,a_m\}$ one has

$$h_r\left[\frac{1-t}{1-q}A^{\dagger}\right] = \frac{(t;q)_r}{(q;q)_r}\ \sum_{i=1}^{m} a_i^r + \text{other terms},$$

we have only to prove

$$E_\pi\ h_r\left[\frac{1-t}{1-q}X^{\dagger}\right] = \prod_{i=1}^{n-1}(t^i+1)\ (t^n q^{r/2}+q^{-r/2})\ h_r\left[\frac{1-t}{1-q}X^{\dagger}\right].$$

This will be established under the following equivalent form.

Theorem 4'. *We have*

$$E_\pi H_1\left[\frac{1-t}{1-q}X^{\dagger}\right] = \prod_{i=1}^{n-1}(t^i+1)\left(t^n H_1\left[\frac{1-t}{1-q}q^{\frac12}X^{\dagger}\right] + H_1\left[\frac{1-t}{1-q}q^{-\frac12}X^{\dagger}\right]\right).$$

In order to evaluate the left-hand side of this identity, we shall need the following trick. For $\sigma=\pm1$ we have

$$xq^{\sigma/2}+\frac{1}{xq^{\sigma/2}} = q^{\frac12}\left(x+\frac{1}{x}\right)+q^{-\frac12}(1-q)x^{-\sigma},$$

which is checked separately for $\sigma=1$ and $\sigma=-1$. Consequently we write

$$\begin{aligned}
H_1\left[\frac{1-t}{1-q}\sum_{i=1}^{n}\left(x_iq^{\sigma_i/2}+\frac{1}{x_iq^{\sigma_i/2}}\right)\right] &= H_1\left[\frac{1-t}{1-q}q^{\frac12}X^{\dagger}+(1-t)q^{-\frac12}\sum_{i=1}^{n}x_i^{-\sigma_i}\right]\\
&= H_1\left[(1-t)q^{-\frac12}\sum_{i=1}^{n}x_i^{-\sigma_i}\right]\ H_1\left[\frac{1-t}{1-q}q^{\frac12}X^{\dagger}\right]\\
&= \prod_{i=1}^{n}\frac{1-tq^{-\frac12}x_i^{-\sigma_i}}{1-q^{-\frac12}x_i^{-\sigma_i}}\ H_1\left[\frac{1-t}{1-q}q^{\frac12}X^{\dagger}\right].
\end{aligned}$$

Similarly, on the right-hand side we get

$$\begin{aligned}
H_1\left[\frac{1-t}{1-q}q^{-\frac12}X^{\dagger}\right] &= H_1\left[\frac{1-t}{1-q}q^{\frac12}X^{\dagger}+(1-t)q^{-\frac12}X^{\dagger}\right]\\
&= H_1\big[(1-t)q^{-\frac12}X^{\dagger}\big]\ H_1\left[\frac{1-t}{1-q}q^{\frac12}X^{\dagger}\right]\\
&= \prod_{i=1}^{n}\frac{1-tq^{-\frac12}x_i}{1-q^{-\frac12}x_i}\ \frac{1-tq^{-\frac12}/x_i}{1-q^{-\frac12}/x_i}\ H_1\left[\frac{1-t}{1-q}q^{\frac12}X^{\dagger}\right].
\end{aligned}$$

By substitution, we see that the statement of Theorem $4'$ is equivalent to the following rational identity, written with $u = q^{-\frac{1}{2}}$,

$$\sum_{\sigma\in(-1,+1)^n} \prod_{i=1}^{n} \frac{1-tx_i^{2\sigma_i}}{1-x_i^{2\sigma_i}} \, \frac{1-tux_i^{-\sigma_i}}{1-ux_i^{-\sigma_i}} \prod_{1\le i<j\le n} \frac{1-tx_i^{\sigma_i}x_j^{\sigma_j}}{1-x_i^{\sigma_i}x_j^{\sigma_j}} = \prod_{i=1}^{n-1}(t^i+1)\left(t^n + \prod_{i=1}^{n} \frac{1-tux_i}{1-ux_i}\, \frac{1-tu/x_i}{1-u/x_i}\right).$$

Writing T_i for the operator $x_i \to 1/x_i$, we are led to prove Theorem 4 under the following equivalent form.

Theorem 5. *We have*

$$(1+T_1)\cdots(1+T_n)\left(\prod_{i=1}^{n} \frac{1-tx_i^2}{1-x_i^2}\, \frac{1-tu/x_i}{1-u/x_i} \prod_{1\le i<j\le n} \frac{1-tx_ix_j}{1-x_ix_j}\right) = \prod_{i=1}^{n-1}(t^i+1)\left(t^n + \prod_{i=1}^{n} \frac{1-tux_i}{1-ux_i}\, \frac{1-tu/x_i}{1-u/x_i}\right).$$

Both sides of this identity are rational functions of u having poles at $u = x_i$ and $u = 1/x_i$, for $i = 1, \ldots, n$. We first prove that their constant terms are equal, i.e. that the statement is true for $u = 0$.

Lemma 8. *We have*

$$(1+T_1)\cdots(1+T_n)\ \Phi_\pi = \prod_{i=1}^{n}(t^i+1)$$

Proof. This is a direct consequence of Weyl's denominator formula

$$\delta = e^{-\rho} \prod_{\alpha\in R^+} (e^\alpha - 1) = \sum_{w\in W} \det(w)\, e^{w\rho},$$

with $\rho = \sum_{i=1}^n (n-i+1)\varepsilon_i$, from which follows

$$\Phi_\pi = \delta^{-1} \sum_{w\in W} \det(w)\, t^{\langle \pi, \rho + w\rho\rangle} e^{w\rho}.$$

Since for any $w \in W$, we have $w\delta = \det(w)\delta$, we obtain

$$(1+T_1)\cdots(1+T_n)\ \Phi_\pi = \sum_{\tau\in W(\pi)} t^{\langle \pi+\tau, \rho\rangle} = \sum_{\sigma\in(-1,+1)^n} \prod_{i=1}^{n} t^{\frac{1}{2}(n-i+1)(1+\sigma_i)}. \quad \square$$

Proof of Theorem 5. It is sufficient to prove that both sides of the identity have the same residue at each of their poles, i.e. at $u = x_i$ and $u = 1/x_i$, $1 \le i \le n$. By symmetry, this has only to be checked for some x_i, say x_n. We shall only do it at $u = x_n$, the proof at $u = 1/x_n$ being similar.

If $A = \{a_1, \dots, a_m\}$ is an arbitrary alphabet, we have

$$\prod_{i=1}^{m} \frac{tu - a_i}{u - a_i} = t^m + (t-1) \sum_{i=1}^{m} \frac{a_i}{u - a_i} \prod_{\substack{j=1 \\ j \neq i}}^{m} \frac{ta_i - a_j}{a_i - a_j}.$$

This decomposition as a sum of partial fractions is actually a Lagrange interpolation (see [5], and also [7, p. 236]). We first apply it to the right-hand side of the identity. Its residue at $u = x_n$ is given by

$$\prod_{i=1}^{n-1} (t^i + 1)\ x_n\ (t-1) \prod_{j=1}^{n-1} \frac{tx_n - x_j}{x_n - x_j} \prod_{j=1}^{n} \frac{tx_n - 1/x_j}{x_n - 1/x_j}.$$

We then apply the Lagrange interpolation to

$$\prod_{i=1}^{n} \frac{tu - x_i^{\sigma_i}}{u - x_i^{\sigma_i}},$$

on the left-hand side of the identity. Only fractions with $\sigma_n = 1$ contribute to the residue at $u = x_n$, which can be written as

$$x_n(t-1) \frac{1 - tx_n^2}{1 - x_n^2}$$
$$\times \sum_{\sigma \in (-1,+1)^{n-1}} \prod_{i=1}^{n-1} \frac{1 - tx_i^{2\sigma_i}}{1 - x_i^{2\sigma_i}} \frac{1 - tx_n x_i^{-\sigma_i}}{1 - x_n x_i^{-\sigma_i}} \frac{1 - tx_i^{\sigma_i} x_n}{1 - x_i^{\sigma_i} x_n} \prod_{1 \le i < j \le n-1} \frac{1 - tx_i^{\sigma_i} x_j^{\sigma_j}}{1 - x_i^{\sigma_i} x_j^{\sigma_j}}.$$

By identification of residues on both sides, and using an obvious induction on n, we are led to prove the identity

$$(1+T_1)\cdots(1+T_n) \left(\prod_{i=1}^{n} \frac{1 - tx_i^2}{1 - x_i^2} \frac{1 - tux_i}{1 - ux_i} \frac{1 - tu/x_i}{1 - u/x_i} \prod_{1 \le i < j \le n} \frac{1 - tx_i x_j}{1 - x_i x_j} \right) =$$
$$\prod_{i=1}^{n} (t^i + 1)\ \frac{1 - tux_i}{1 - ux_i}\ \frac{1 - tu/x_i}{1 - u/x_i}.$$

Since

$$\prod_{i=1}^{n} \frac{1 - tux_i}{1 - ux_i} \frac{1 - tu/x_i}{1 - u/x_i} = H_1\big[(1-t)uX^{\dagger}\big]$$

is obviously invariant under any T_i, the left-hand side may be written

$$\prod_{i=1}^{n} \frac{1 - tux_i}{1 - ux_i} \frac{1 - tu/x_i}{1 - u/x_i}\ (1+T_1)\cdots(1+T_n)\Phi_\pi.$$

We conclude by applying Lemma 8. $\square$

10. Proofs of Theorems 1, 2 and 3

We shall implicitly use many of the formulas about q-shifted factorials, listed in Appendix I of [3]. In particular we shall write

$$(a_1, a_2, \dots, a_r; q)_i = (a_1; q)_i \ (a_2; q)_i \ \dots \ (a_r; q)_i.$$

Theorem 1. *We have the following transformation between ${}_2\phi_1$ series*

$$\frac{(u;q)_r}{(q;q)_r}\ {}_2\phi_1\left[\begin{matrix} q^{-r}, \ ux \\ q^{1-r}/ux \end{matrix}; q, qv/u^2x\right]$$
$$= \sum_{i=0}^{[r/2]} \frac{(u;q)_{r-2i}}{(q;q)_{r-2i}}\ {}_2\phi_1\left[\begin{matrix} q^{2i-r}, \ uy \\ q^{1+2i-r}/uy \end{matrix}; q, qv/u^2y\right]\ y^i v^i\ \frac{(x/y;q)_i\ (uq^{r-2i};q)_{2i}}{(q, uxq^{r-i}, uyq^{r-2i+1};q)_i}.$$

Proof [14]. With $a = v/u$, applying [3, (3.4.7)], we have

$$\frac{(u;q)_r}{(q;q)_r}\ {}_2\phi_1\left[\begin{matrix} q^{-r}, \ ux \\ q^{1-r}/ux \end{matrix}; q, qv/u^2x\right] = a^r\ \frac{(u, 1/a^2;q)_r}{(q, 1/a;q)_r}$$
$$\times {}_8\phi_7\left[\begin{matrix} q^{-r}a, q(q^{-r}a)^{\frac12}, -q(q^{-r}a)^{\frac12}, aux, q^{-\frac{r}{2}}, -q^{-\frac{r}{2}}, q^{\frac12-\frac{r}{2}}, -q^{\frac12-\frac{r}{2}} \\ (q^{-r}a)^{\frac12}, -(q^{-r}a)^{\frac12}, q^{1-r}/ux, q^{1-\frac{r}{2}}a, -q^{1-\frac{r}{2}}a, q^{\frac12-\frac{r}{2}}a, -q^{\frac12-\frac{r}{2}}a \end{matrix}; q, qa/ux\right],$$

or equivalently

$$\frac{(u;q)_r}{(q;q)_r}\ {}_2\phi_1\left[\begin{matrix} q^{-r}, \ ux \\ q^{1-r}/ux \end{matrix}; q, qv/u^2x\right] = a^r\ \frac{(u, 1/a^2;q)_r}{(q, 1/a;q)_r}$$
$$\times \sum_{k=0}^{[r/2]} \frac{1-aq^{2k-r}}{1-aq^{-r}}\ \frac{(q^{-r}a, aux;q)_k}{(q^{1-r}/ux, q;q)_k}\ \frac{(q^{-r};q)_{2k}}{(q^{1-r}a^2;q)_{2k}}\ (qa/ux)^k.$$

This yields

$$\frac{(u;q)_{r-2i}}{(q;q)_{r-2i}}\ {}_2\phi_1\left[\begin{matrix} q^{2i-r}, \ uy \\ q^{1+2i-r}/uy \end{matrix}; q, qv/u^2y\right]$$
$$= (v/u)^r\ \frac{(u, u^2/v^2;q)_r}{(q, u/v;q)_r}\ (q/u)^{2i}\ \frac{(q^{-r}, q^{1-r}v/u;q)_{2i}}{(q^{1-r}/u, q^{1-r}v^2/u^2;q)_{2i}}$$
$$\sum_{k=0}^{[r/2]-i} \frac{1-q^{2i+2k-r}v/u}{1-q^{2i-r}v/u}\ \frac{(q^{2i-r}v/u, vy;q)_k}{(q^{1+2i-r}/uy, q;q)_k}\ \frac{(q^{2i-r};q)_{2k}}{(q^{1+2i-r}v^2/u^2;q)_{2k}}\ (qv/u^2y)^k.$$

If we substitute this value in the right-hand side of the identity, and put $k = j - i$, we obtain

$$(v/u)^r\ \frac{(u, u^2/v^2;q)_r}{(q, u/v;q)_r}\ \sum_{j=0}^{[r/2]} \sum_{i=0}^{j} (q/u)^{2i}\ \frac{(q^{-r}, q^{1-r}v/u;q)_{2i}}{(q^{1-r}/u, q^{1-r}v^2/u^2q)_{2i}}$$
$$\times \frac{1-q^{2j-r}v/u}{1-q^{2i-r}v/u}\frac{(q^{2i-r}v/u, vy;q)_{j-i}}{(q^{1+2i-r}/uy, q;q)_{j-i}}\ \frac{(q^{2i-r};q)_{2j-2i}}{(q^{1+2i-r}v^2/u^2;q)_{2j-2i}}\ (qv/u^2y)^{j-i}$$
$$\times y^i v^i\ \frac{(x/y;q)_i\ (uq^{r-2i};q)_{2i}}{(q, uxq^{r-i}, uyq^{r-2i+1};q)_i}.$$

This may be written as

$$(v/u)^r \frac{(u, u^2/v^2; q)_r}{(q, u/v; q)_r} \times \sum_{j=0}^{[r/2]} \frac{1 - q^{2j-r}v/u}{1 - q^{-r}v/u} \frac{(q^{-r}v/u, vy; q)_j}{(q^{1-r}/uy, q; q)_j} \frac{(q^{-r}; q)_{2j}}{(q^{1-r}v^2/u^2; q)_{2j}} (qv/u^2y)^j$$

$$\times \sum_{i=0}^{j} \frac{(q^{1-r}/uy; q)_{2i}}{(q^{1-r}/u; q)_{2i}} \frac{(q^{-j}, q^{j-r}v/u, x/y; q)_i \, (uq^{r-2i}; q)_{2i}}{(q^{1-j}/vy, q^{1+j-r}/uy, q, uxq^{r-i}, uyq^{r-2i+1}; q)_i} (q^2/vy)^i.$$

The sum over i reads

$$\sum_{i=0}^{j} \frac{1 - uyq^{r-2i}}{1 - uyq^{r-i}} \frac{(q^{-j}, q^{j-r}v/u, q^{1-r}/uy, x/y; q)_i}{(q^{1-j}/vy, q^{1+j-r}/uy, q^{1-r}/ux, q; q)_i} (q^2/vx)^i$$

$$= {}_6\phi_5 \left[\begin{matrix} q^{-r}/uy, q(q^{-r}/uy)^{\frac{1}{2}}, -q(q^{-r}/uy)^{\frac{1}{2}}, x/y, q^{j-r}v/u, q^{-j} \\ (q^{-r}/uy)^{\frac{1}{2}}, -(q^{-r}/uy)^{\frac{1}{2}}, q^{1-r}/ux, q^{1-j}/vy, q^{j+1-r}/uy, \end{matrix} ; q, q/vx \right].$$

By [3, (II.21)] this terminating very-well-poised ${}_6\phi_5$ sum equals

$$\frac{(q^{1-r}/uy, q^{1-j}/vx; q)_j}{(q^{1-r}/ux, q^{1-j}/vy; q)_j} = \frac{(q^{1-r}/uy, vx; q)_j}{(q^{1-r}/ux, vy; q)_j} (y/x)^j.$$

Finally we have proved that the right-hand side of the identity is

$$(v/u)^r \frac{(u, u^2/v^2; q)_r}{(q, u/v; q)_r} \times \sum_{j=0}^{[r/2]} \frac{1 - q^{2j-r}v/u}{1 - q^{-r}v/u} \frac{(q^{-r}v/u, vx; q)_j}{(q^{1-r}/ux, q; q)_j} \frac{(q^{-r}; q)_{2j}}{(q^{1-r}v^2/u^2; q)_{2j}} (qv/u^2x)^j.$$

Hence the statement. □

Lemma 9. *We have*

$${}_4\phi_3 \left[\begin{matrix} q^{-n}, a^2, qa, b \\ a, qa^2/b, q^{1+n}a^2 \end{matrix} ; q, q^{1+n}a/b \right] = \frac{(qa^2, -1; q)_n}{(q^{\frac{1}{2}}a, -q^{\frac{1}{2}}a; q)_n} \, {}_4\phi_3 \left[\begin{matrix} q^{-n}, -qa/b, q^{\frac{1}{2}}a, -q^{\frac{1}{2}}a \\ -q^{1-n}, qa^2/b, -qa \end{matrix} ; q, q \right].$$

Proof [14]. The left-hand side obviously equals

$${}_8\phi_7 \left[\begin{matrix} a^2, qa, -qa, b, -a, q^{\frac{1}{2}}a, -q^{\frac{1}{2}}a, q^{-n} \\ a, -a, qa^2/b, -qa, q^{\frac{1}{2}}a, -q^{\frac{1}{2}}a, q^{1+n}a^2 \end{matrix} ; q, q^{1+n}a/b \right].$$

By applying Watson's transformation formula [3, (III.19)], it can be transformed into the right-hand side. □

Theorem 2. *Defining*

$$\frac{(u;q)_r}{(q;q)_r}\ H_r=\sum_{i=0}^{r}a^{r-i}\frac{(u;q)_{r-i}}{(q;q)_{r-i}}\ {}_2\phi_1\left[\begin{matrix}q^{i-r},\ v\\ q^{1+i-r}/v\end{matrix};q,q/v^2a^2\right]\frac{(x,uq^{r-i},v^2q^{2r-i+1};q)_i}{(q,vq^{r-i+1},xv^2q^{2r-i};q)_i},$$

we have

$$H_r=(-1/v)^r\ \frac{(vq^{\frac12},-vq^{\frac12},-xv,xv^2;q)_r}{(xv^2;q)_{2r}}\ {}_4\phi_3\left[\begin{matrix}q^{-r},xv^2q^r,-av,-1/a\\ vq^{\frac12},-vq^{\frac12},-xv\end{matrix};q,q\right].$$

Proof [14]. Applying Lemma 7 we obtain

$$H_r=\frac{(q;q)_r}{(v;q)_r}\ \sum_{i=0}^{r}\frac{1-vq^{r-i}}{1-vq^r}\ \frac{(v^2;q)_{r-i}}{(q;q)_{r-i}}\ \frac{(x,v^2q^{2r-i+1};q)_i}{(q,xv^2q^{2r-i};q)_i}\ v^{i-r}$$
$$\times\ {}_4\phi_3\left[\begin{matrix}q^{i-r},v^2q^{r-i},av,1/a\\ vq^{\frac12},-vq^{\frac12},-v\end{matrix};q,q\right].$$

Reversing the order of summation, this may be rewritten as

$$H_r=v^{-r}\ \frac{(v^2;q)_r}{(v;q)_r}\ \sum_{i=0}^{r}\frac{(q^{-r},v^2q^r,av,1/a;q)_i}{(q,vq^{\frac12},-vq^{\frac12},-v;q)_i}\ q^i$$
$$\times\ {}_4\phi_3\left[\begin{matrix}q^{-2r}/v^2,q^{1-r}/v,x,q^{i-r}\\ q^{-r}/v,q^{1-2r}/xv^2,q^{1-r-i}/v^2\end{matrix};q,q^{1-i}/xv\right].$$

Using Lemma 9 with $a:=q^{-r}/v$, $b:=x$ and $n:=r-i$, the ${}_4\phi_3$ series equals

$$\frac{(q^{1-2r}/v^2,-1;q)_r}{(q^{\frac12-r}/v,-q^{\frac12-r}/v;q)_r}\ \frac{(vq^{\frac12},-vq^{\frac12};q)_i}{(v^2q^r,-q^{1-r};q)_i}$$
$$\times\ {}_4\phi_3\left[\begin{matrix}q^{i-r},q^{\frac12-r}/v,-q^{\frac12-r}/v,-q^{1-r}/xv\\ q^{1-2r}/xv^2,-q^{1-r}/v,-q^{1-r+i}\end{matrix};q,q\right],$$

which yields

$$H_r=\frac{(v^2,v^2q^r,-1;q)_r}{(v,vq^{\frac12},-vq^{\frac12};q)_r}\ v^{-r}\ q^{-\binom{r}{2}}\ \sum_{i=0}^{r}q^i\ \frac{(q^{-r},av,1/a;q)_i}{(q,-v,-q^{1-r};q)_i}$$
$$\times\ {}_4\phi_3\left[\begin{matrix}q^{i-r},q^{\frac12-r}/v,-q^{\frac12-r}/v,-q^{1-r}/xv\\ q^{1-2r}/xv^2,-q^{1-r}/v,-q^{1-r+i}\end{matrix};q,q\right].$$

Reversing again the order of summation, this may be rewritten as

$$H_r=(-v,-1;q)_r\ v^{-r}\ q^{-\binom{r}{2}}\ \sum_{i=0}^{r}q^i\ \frac{(q^{-r},q^{\frac12-r}/v,-q^{\frac12-r}/v,-q^{1-r}/xv;q)_i}{(q,q^{1-2r}/xv^2,-q^{1-r}/v,-q^{1-r};q)_i}$$
$$\times\ {}_3\phi_2\left[\begin{matrix}q^{i-r},av,1/a\\ -v,-q^{1-r+i}\end{matrix};q,q\right].$$

Since it is a q-Saalschütz sum [3, (II.12)], the ${}_3\phi_2$ series equals

$$\frac{(-av,-1/a;q)_{r-i}}{(-v,-1;q)_{r-i}}=\frac{(-av,-1/a;q)_r}{(-v,-1;q)_r}\;\frac{(-q^{1-r},-q^{1-r}/v;q)_i}{(-q^{1-r}/av,-aq^{1-r};q)_i}.$$

Finally we obtain

$$H_r=(-av,-1/a;q)_r\;v^{-r}\;q^{-\binom{r}{2}}\;{}_4\phi_3\left[\begin{matrix}q^{-r},q^{\frac12-r}/v,-q^{\frac12-r}/v,-q^{1-r}/xv\\ q^{1-2r}/xv^2,-q^{1-r}/av,-aq^{1-r}\end{matrix};q,q\right],$$

and we conclude easily. □

Theorem 3. *We have*

$$a^r\frac{(u;q)_r}{(q;q)_r}\;{}_2\phi_1\left[\begin{matrix}q^{-r},\ v\\ q^{1-r}/v\end{matrix};q,q/v^2a^2\right]$$
$$=\sum_{i=0}^{r}\frac{(u;q)_{r-i}}{(q;q)_{r-i}}\;H_{r-i}\;x^i\;\frac{(1/x,uq^{r-i},v^2q^{2r-2i+1};q)_i}{(q,vq^{r-i+1},xv^2q^{2r-2i+1};q)_i}\;\frac{1-v^2q^{2r}}{1-v^2q^{2r-i}}.$$

Proof [14]. First observe that the right-hand side is

$$\sum_{i=0}^{r}\frac{(u;q)_i}{(q;q)_i}\;\frac{(1/x,uq^i,v^2q^{2i+1};q)_{r-i}}{(q,vq^{i+1},xv^2q^{2i+1};q)_{r-i}}\;\frac{1-v^2q^{2r}}{1-v^2q^{r+i}}\;x^{r-i}H_i$$
$$=x^r\frac{(u,1/x,v^2;q)_r}{(q,vq,xv^2q;q)_r}\frac{1-v^2q^{2r}}{1-v^2}\sum_{i=0}^{r}\frac{(q^{-r},vq,v^2q^r;q)_i}{(q,xq^{1-r},xv^2q^{1+r};q)_i}\;\frac{(xv^2q;q)_{2i}}{(v^2q;q)_{2i}}(-q/v)^i$$
$$\times\;\frac{(vq^{\frac12},-vq^{\frac12},-xv,xv^2;q)_i}{(xv^2;q)_{2i}}\;{}_4\phi_3\left[\begin{matrix}q^{-i},xv^2q^i,-av,-1/a\\ vq^{\frac12},-vq^{\frac12},-xv\end{matrix};q,q\right]$$
$$=x^r\frac{(u,1/x,v^2,-vq;q)_r}{(q,v,-v,xv^2q;q)_r}\sum_{i=0}^{r}\frac{1-xv^2q^{2i}}{1-xv^2}\;\frac{(q^{-r},v^2q^r,xv^2,-xv;q)_i}{(q,xq^{1-r},xv^2q^{1+r},-qv;q)_i}(-q/v)^i$$
$$\times\;{}_4\phi_3\left[\begin{matrix}q^{-i},xv^2q^i,-av,-1/a\\ vq^{\frac12},-vq^{\frac12},-xv\end{matrix};q,q\right].$$

Expanding the ${}_4\phi_3$ series over the index $j\le i$, putting $i=j+k$, and summing over k, we obtain

$$x^r\frac{(u,1/x,v^2,-vq;q)_r}{(q,v,-v,xv^2q;q)_r}\sum_{j=0}^{r}\frac{(xv^2q;q)_{2j}}{(v^2q;q)_{2j}}\;\frac{(vq,-av,-1/a,v^2q^r,q^{-r};q)_j}{(q,xq^{1-r},xv^2q^{1+r};q)_j}(q/v)^j\;q^{-\binom{j}{2}}$$
$$\times\;{}_6\phi_5\left[\begin{matrix}xv^2q^{2j},q(xv^2q^{2j})^{\frac12},-q(xv^2q^{2j})^{\frac12},-xvq^j,v^2q^{r+j},q^{j-r}\\ (xv^2q^{2j})^{\frac12},-(xv^2q^{2j})^{\frac12},-vq^{j+1},xq^{j-r+1},xv^2q^{j+r+1}\end{matrix};q,-q^{1-j}/v\right].$$

By [3, (II.21)] this terminating very-well-poised ${}_6\phi_5$ sum equals

$$\frac{(xv^2q^{2j+1},-q^{1-r}/v;q)_{r-j}}{(-vq^{j+1},xq^{j-r+1};q)_{r-j}}=\left(\frac{-1}{xv}\right)^r v^jq^{\binom{j}{2}}\frac{(xv^2q,-v;q)_r}{(1/x,-vq;q)_r}\frac{(-vq,xv^2q^{1+r},xq^{1-r};q)_j}{(-v;q)_j(xv^2q;q)_{2j}}.$$

Thus, on simplification, the right-hand side may be written

$$\frac{(u,v^2;q)_r}{(q,v;q)_r}\,(-v)^{-r}\;{}_4\phi_3\left[\begin{matrix} q^{-r},v^2q^r,-av,-1/a \\ vq^{\frac12},-vq^{\frac12},-v \end{matrix};q,q\right].$$

Applying Lemma 7 with $-a$ substituted to a, we obtain the left-hand side. □

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Institut Gaspard Monge, Université de Marne-la-Vallée, 77454 Marne-la-Vallée Cedex, France

E-mail address: `lassalle@univ-mlv.fr`

URL: `http://igm.univ-mlv.fr/~lassalle`