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ON DEGREES IN THE HASSE DIAGRAM OF THE STRONG BRUHAT ORDER

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Abstract. For a permutation π in the symmetric group S_n let the *total degree* be its valency in the Hasse diagram of the strong Bruhat order on S_n, and let the *down degree* be the number of permutations which are covered by π in the strong Bruhat order. The maxima of the total degree and the down degree and their values at a random permutation are computed. Proofs involve variants of a classical theorem of Turán from extremal graph theory.

1. The Down, Up and Total Degrees

Definition 1.1. For a permutation $\pi \in S_n$ let the *down degree* $d_-(\pi)$ be the number of permutations in S_n which are covered by π in the strong Bruhat order. Let the *up degree* $d_+(\pi)$ be the number of permutations which cover π in this order. The *total degree* of π is the sum

$$d(\pi) := d_-(\pi) + d_+(\pi),$$

i.e., the valency of π in the Hasse diagram of the strong Bruhat order.

Explicitly, for $1 \le a < b \le n$ let $t_{a,b} = t_{b,a} \in S_n$ be the transposition interchanging a and b, and for $\pi \in S_n$ let

$$\ell(\pi) := \min\{k \mid \pi = s_{i_1} s_{i_2} \cdots s_{i_k}\}$$

be the *length* of π with respect to the standard Coxeter generators $s_i = t_{i,i+1}$ $(1 \le i < n)$ of S_n. Then

$$\begin{aligned} d_-(\pi) &= \#\{t_{a,b} \mid \ell(t_{a,b}\pi) = \ell(\pi) - 1\}, \\ d_+(\pi) &= \#\{t_{a,b} \mid \ell(t_{a,b}\pi) = \ell(\pi) + 1\}, \\ d(\pi) = d_-(\pi) + d_+(\pi) &= \#\{t_{a,b} \mid \ell(t_{a,b}\pi) = \ell(\pi) \pm 1\}. \end{aligned}$$

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For the general definitions and other properties of the weak and strong Bruhat orders see, e.g., [9, Ex. 3.75] and [2, §§2.1, 3.1].

We shall describe $\pi \in S_n$ by its sequence of values $[\pi(1), \ldots, \pi(n)]$.

Observation 1.2. *π covers σ in the strong Bruhat order on S_n if and only if there exist $1 \le i < k \le n$ such that*

(1) *$b := \pi(i) > \pi(k) =: a$.*
(2) *$\sigma = t_{a,b}\pi$, i.e., $\pi = [\ldots, b, \ldots, a, \ldots]$ and $\sigma = [\ldots, a, \ldots, b, \ldots]$.*
(3) *There is no $i < j < k$ such that $a < \pi(j) < b$.*

Corollary 1.3. *For every $\pi \in S_n$*

$$d_-(\pi) = d_-(\pi^{-1}).$$

Example 1.4. In S_3, $d_-[123] = 0$, $d_-[132] = d_-[213] = 1$, and $d_-[321] = d_-[231] = d_-[312] = 2$. On the other hand, $d[321] = d[123] = 2$ and $d[213] = d[132] = d[312] = d[231] = 3$.

Remark 1.5. The classical descent number of a permutation π in the symmetric group S_n is the number of permutations in S_n which are covered by π in the (right) *weak* Bruhat order. Thus, the down degree may be considered as a "strong descent number".

Definition 1.6. For $\pi \in S_n$ denote

$$D_-(\pi) := \{t_{a,b} \,|\, \ell(t_{a,b}\pi) = \ell(\pi) - 1\},$$

the *strong descent set* of π.

Example 1.7. The strong descent set of $\pi = [7, 9, 5, 2, 3, 8, 4, 1, 6]$ is

$$D_-(\pi) = \{t_{1,2}, t_{1,3}, t_{1,4}, t_{2,5}, t_{3,5}, t_{4,5}, t_{4,8}, t_{5,7}, t_{5,9}, t_{6,7}, t_{6,8}, t_{8,9}\}.$$

Remark 1.8. Generalized pattern avoidance, involving strong descent sets, was applied by Woo and Yong [11] to determine which Schubert varieties are Gorenstein.

Proposition 1.9. *The strong descent set $D_-(\pi)$ uniquely determines the permutation π.*

Proof. By induction on n. The claim clearly holds for $n = 1$.

Let π be a permutation in S_n, and let $\bar{\pi} \in S_{n-1}$ be the permutation obtained by deleting the value n from π. Note that, by Observation 1.2,

$$D_-(\bar{\pi}) = D_-(\pi) \setminus \{t_{a,n} \,|\, 1 \le a < n\}.$$

By the induction hypothesis $\bar{\pi}$ is uniquely determined by this set. Hence it suffices to determine the position of n in π.

Now, if $j := \pi^{-1}(n) < n$ then clearly $t_{\pi(j+1),n} \in D_-(\pi)$. Moreover, by Observation 1.2, $t_{a,n} \in D_-(\pi) \implies a \geq \pi(j+1)$. Thus $D_-(\pi)$ determines

$$\bar{\pi}(j) = \pi(j+1) = \min\{a \,|\, t_{a,n} \in D_-(\pi)\},$$

and therefore determines j. Note that this set of a's is empty if and only if $j = n$. This completes the proof. □

2. Maximal Down Degree

In this section we compute the maximal value of the down degree on S_n and find all the permutations achieving the maximum. We prove

Proposition 2.1. *For every positive integer n*

$$\max\{d_-(\pi) |\ \pi \in S_n\} = \lfloor n^2/4 \rfloor.$$

Remark 2.2. The same number appears as the order dimension of the strong Bruhat poset [7]. An upper bound on the maximal down degree for finite Coxeter groups appears in [4, Prop. 3.4].

For the proof of Proposition 2.1 we need a classical theorem of Turán.

Definition 2.3. Let $r \leq n$ be positive integers. The *Turán graph* $T_r(n)$ is the complete r-partite graph with n vertices and all parts as equal in size as possible, i.e., each size is either $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$. Denote by $t_r(n)$ the number of edges of $T_r(n)$.

Theorem 2.4 (Turán's Theorem; [10], [3, IV, Theorem 8])**.**

1. *Every graph of order n with more than $t_r(n)$ edges contains a complete subgraph of order $r+1$.*
2. *$T_r(n)$ is the unique graph of order n with $t_r(n)$ edges that does not contain a complete subgraph of order $r+1$.*

We shall apply the special case $r = 2$ (due to Mantel) of Turán's theorem to the following graph.

Definition 2.5. The *strong descent graph* of $\pi \in S_n$, denoted $\Gamma_-(\pi)$, is the undirected graph whose set of vertices is $\{1, \ldots, n\}$ and whose set of edges is

$$\{\{a, b\} \,|\, t_{a,b} \in D_-(\pi)\}.$$

By definition, the number of edges in $\Gamma_-(\pi)$ equals $d_-(\pi)$.

Remark 2.6. Permutations for which the strong descent graph is connected are called *indecomposable*. Their enumeration was studied in [5]; see [6, pp. 7–8]. The number of components in $\Gamma_-(\pi)$ is equal to the number of *global descents* in πw_0 (where $w_0 := [n, n-1, \ldots, 1]$), which were introduced and studied in [1, Corollaries 6.3 and 6.4].

Lemma 2.7. *For every $\pi \in S_n$, the strong descent graph $\Gamma_-(\pi)$ is triangle-free.*

Proof. Assume that $\Gamma_-(\pi)$ contains a triangle. Then there exist $1 \le a < b < c \le n$ such that $t_{a,b}, t_{a,c}, t_{b,c} \in D_-(\pi)$. By Observation 1.2,

$$t_{a,b}, t_{b,c} \in D_-(\pi) \Longrightarrow \pi^{-1}(c) < \pi^{-1}(b) < \pi^{-1}(a) \Longrightarrow t_{a,c} \notin D_-(\pi).$$

This is a contradiction. □

Proof of Proposition 2.1. By Theorem 2.4(1) together with Lemma 2.7, for every $\pi \in S_n$

$$d_-(\pi) \le t_2(n) = \lfloor n^2/4 \rfloor.$$

Equality holds since

$$d_-([\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \ldots, n, 1, 2, \ldots, \lfloor n/2 \rfloor]) = \lfloor n^2/4 \rfloor.$$

□

Next we classify (and enumerate) the permutations which achieve the maximal down degree.

Lemma 2.8. *Let $\pi \in S_n$ be a permutation with maximal down degree. Then π has no decreasing subsequence of length 4.*

Proof. Assume that $\pi = [\ldots d \ldots c \ldots b \ldots a \ldots]$ with $d > c > b > a$ and $\pi^{-1}(a) - \pi^{-1}(d)$ minimal. Then $t_{a,b}, t_{b,c}, t_{c,d} \in D_-(\pi)$ but, by Observation 1.2, $t_{a,d} \notin D_-(\pi)$. It follows that $\Gamma_-(\pi)$ is not a complete bipartite graph, since $\{a, b\}$, $\{b, c\}$, and $\{c, d\}$ are edges but $\{a, d\}$ is not. By Lemma 2.7, combined with Theorem 2.4(2), the number of edges in $\Gamma_-(\pi)$ is less than $\lfloor n^2/4 \rfloor$. □

Proposition 2.9. *For every positive integer n*

$$\#\{\pi \in S_n \mid d_-(\pi) = \lfloor n^2/4 \rfloor\} = \begin{cases} n, & \text{if } n \text{ is odd;} \\ n/2, & \text{if } n \text{ is even.} \end{cases}$$

Each such permutation has the form

$$\pi = [t + m + 1, t + m + 2, \ldots, n, t + 1, t + 2, \ldots, t + m, 1, 2, \ldots, t],$$

where $m \in \{\lfloor n/2 \rfloor, \lceil n/2 \rceil\}$ and $1 \le t \le n - m$. Note that $t = n - m$ (for m) gives the same permutation as $t = 0$ (for $n - m$ instead of m).

Proof. It is easy to verify the claim for $n \le 3$. Assume $n \ge 4$.

Let $\pi \in S_n$ with $d_-(\pi) = \lfloor n^2/4 \rfloor$. By Theorem 2.4(2), $\Gamma_-(\pi)$ is isomorphic to the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. Since $n \ge 4$, each side of the graph contains at least two vertices. Let $1 = a < b$ be two vertices on one side, and $c < d$ two vertices on the other side of the graph. Since $t_{b,c}, t_{b,d} \in D_-(\pi)$, there are three possible cases:

(1) $b < c$, and then $\pi = [\ldots c \ldots d \ldots b \ldots]$
(since $\pi = [\ldots d \ldots c \ldots b \ldots]$ contradicts $t_{b,d} \in D_-(\pi)$).
(2) $c < b < d$, and then $\pi = [\ldots d \ldots b \ldots c \ldots]$.
(3) $d < b$, and then $\pi = [\ldots b \ldots c \ldots d \ldots]$
(since $\pi = [\ldots b \ldots d \ldots c \ldots]$ contradicts $t_{b,c} \in D_-(\pi)$).

The same also holds for $a = 1$ instead of b, but then cases 2 and 3 are impossible since $a = 1 < c$. Thus necessarily c appears before d in π, and case 2 is therefore impossible for *any* b on the same side as $a = 1$. In other words: no vertex on the same side as $a = 1$ is intermediate, either in position (in π) or in value, to c and d.

Assume now that n is even. The vertices not on the side of 1 form (in π) a block of length $n/2$ of numbers which are consecutive in value as well in position. They also form an increasing subsequence of π, since $\Gamma_-(\pi)$ is bipartite. The numbers preceding them are all larger in value, and are increasing; the numbers succeeding them are all smaller in value, are increasing, and contain 1. It is easy to check that each permutation π of this form has maximal $d_-(\pi)$. Finally, π is completely determined by the length $1 \le t \le n/2$ of the last increasing subsequence.

For n odd one obtains a similar classification, except that the length of the side not containing 1 is either $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$. This completes the proof. □

3. Maximal Total Degree

Obviously, the maximal value of the total degree $d = d_- + d_+$ cannot exceed $\binom{n}{2}$, the total number of transpositions in S_n. This is slightly better than the bound $2\lfloor n^2/4 \rfloor$ obtainable from Proposition 2.1. The actual maximal value is smaller.

Theorem 3.1. *For $n \ge 2$, the maximal total degree in the Hasse diagram of the strong Bruhat order on S_n is*

$$\lfloor n^2/4 \rfloor + n - 2.$$

In order to prove this result, associate with each permutation $\pi \in S_n$ a graph $\Gamma(\pi)$, whose set of vertices is $\{1, \ldots, n\}$ and whose set of edges is

$$\{\{a,b\} \mid \ell(t_{a,b}\pi) - \ell(\pi) = \pm 1\}.$$

This graph has many properties; e.g., it is K_5-free and is the edge-disjoint union of two triangle-free graphs on the same set of vertices. However, these properties are not strong enough to imply the above result. A property which does imply it is the following bound on the minimal degree.

Lemma 3.2. *There exists a vertex in* $\Gamma(\pi)$ *with degree at most* $\lfloor n/2 \rfloor + 1$.

Proof. Assume, on the contrary, that each vertex in $\Gamma(\pi)$ has at least $\lfloor n/2 \rfloor + 2$ neighbors. This applies, in particular, to the vertex $\pi(1)$. Being the first value of π, the neighborhood of $\pi(1)$ in $\Gamma(\pi)$, viewed as a subsequence of $[\pi(2), \ldots, \pi(n)]$, consists of a shuffle of a decreasing sequence of numbers larger than $\pi(1)$ and an increasing sequence of numbers smaller than $\pi(1)$. Let a be the rightmost neighbor of $\pi(1)$. The intersection of the neighborhood of a with the neighborhood of $\pi(1)$ is of cardinality at most two. Thus the degree of a is at most

$$n - (\lfloor n/2 \rfloor + 2) + 2 = \lceil n/2 \rceil \le \lfloor n/2 \rfloor + 1,$$

which is a contradiction. □

Proof of Theorem 3.1. First note that, by definition, the total degree of $\pi \in S_n$ in the Hasse diagram of the strong Bruhat order is equal to the number of edges in $\Gamma(\pi)$. We will prove that this number $e(\Gamma(\pi)) \le \lfloor n^2/4 \rfloor + n - 2$, by induction on n.

The claim is clearly true for $n = 2$. Assume that the claim holds for $n - 1$, and let $\pi \in S_n$. Let a be a vertex of $\Gamma(\pi)$ with minimal degree, and let $\bar{\pi} \in S_{n-1}$ be the permutation obtained from π by deleting the value a (and decreasing by 1 all the values larger than a). Then

$$e(\Gamma(\bar{\pi})) \ge e(\Gamma(\pi) \setminus a),$$

where the latter is the number of edges in $\Gamma(\pi)$ which are not incident with the vertex a. By the induction hypothesis and Lemma 3.2,

$$\begin{aligned} e(\Gamma(\pi)) &= e(\Gamma(\pi) \setminus a) + d(a) \le e(\Gamma(\bar{\pi})) + d(a) \\ &\le \lfloor (n-1)^2/4 \rfloor + (n-1) - 2 + \lfloor n/2 \rfloor + 1 \\ &= \lfloor n^2/4 \rfloor + n - 2. \end{aligned}$$

Equality holds since, letting $m := \lfloor n/2 \rfloor$,

$$e(\Gamma([m+1, m+2, \ldots, n, 1, 2, \ldots, m])) = \lfloor n^2/4 \rfloor + n - 2.$$

□

Theorem 3.3.

$$\#\{\pi \in S_n \,|\, d(\pi) = \lfloor n^2/4 \rfloor + n - 2\} = \begin{cases} 2, & \textit{if } n = 2; \\ 4, & \textit{if } n = 3 \textit{ or } n = 4; \\ 8, & \textit{if } n \geq 6 \textit{ is even;} \\ 16, & \textit{if } n \geq 5 \textit{ is odd.} \end{cases}$$

The extremal permutations have one of the following forms:

$$\pi_0 := [m+1, m+2, \ldots, n, 1, 2, \ldots, m] \qquad (m \in \{\lfloor n/2 \rfloor, \lceil n/2 \rceil\}),$$

and the permutations obtained from π_0 by one or more of the following operations:

$$\pi \mapsto \pi^r := [\pi(n), \pi(n-1), \ldots, \pi(2), \pi(1)] \qquad \text{(reversing } \pi),$$
$$\pi \mapsto \pi^s := \pi \cdot t_{1,n} \qquad \text{(interchanging } \pi(1) \text{ and } \pi(n)),$$
$$\pi \mapsto \pi^t := t_{1,n} \cdot \pi \qquad \text{(interchanging 1 and } n \text{ in } \pi).$$

Proof. It is not difficult to see that all the specified permutations are indeed extremal, and their number is as claimed (for all $n \geq 2$).

The claim that there are no other extremal permutations will be proved by induction on n. For small values of n (say $n \leq 4$) this may be verified directly. Assume now that the claim holds for some $n \geq 4$, and let $\pi \in S_{n+1}$ be extremal. Following the proof of Lemma 3.2, let a be a vertex of $\Gamma(\pi)$ with degree at most $\lfloor (n+1)/2 \rfloor + 1$, which is either $\pi(1)$ or its rightmost neighbor. As in the proof of Theorem 3.1, let $\bar{\pi} \in S_n$ be the permutation obtained from π by deleting the value a (and decreasing by 1 all the values larger than a). All the inequalities in the proof of Theorem 3.1 must hold as equalities, namely: $e(\Gamma(\pi) \setminus a) = e(\Gamma(\bar{\pi}))$, $d(a) = \lfloor (n+1)/2 \rfloor + 1$, and $\bar{\pi}$ is extremal in S_n. By the induction hypothesis, $\bar{\pi}$ must have one of the prescribed forms. In all of them, $\{\bar{\pi}(1), \bar{\pi}(n)\} = \{m, m+1\}$ is an edge of $\Gamma(\bar{\pi})$. Therefore the corresponding edge $\{\pi(1), \pi(n+1)\}$ (or $\{\pi(2), \pi(n+1)\}$ if $a = \pi(1)$, or $\{\pi(1), \pi(n)\}$ if $a = \pi(n+1)$) is an edge of $\Gamma(\pi) \setminus a$, namely of $\Gamma(\pi)$. If $a \neq \pi(1), \pi(n+1)$ then $\pi(n+1)$ is the rightmost neighbor of $\pi(1)$, contradicting the choice of a. If $a = \pi(n+1)$ we may use the operation $\pi \mapsto \pi^r$. Thus we may assume from now on that $a = \pi(1)$.

Let $N(a)$ denote the set of neighbors of a in $\Gamma(\pi)$. Assume first that

$$\bar{\pi} = \pi_0 = [m+1, m+2, \ldots, n, 1, 2, \ldots, m] \qquad (m \in \{\lfloor n/2 \rfloor, \lceil n/2 \rceil\}).$$

Noting that $\lceil n/2 \rceil = \lfloor (n+1)/2 \rfloor$ and keeping in mind the decrease in certain values during the transition $\pi \mapsto \bar{\pi}$, we have the following cases:

(1) $a > m+1$: in this case $1, \ldots, m \notin N(a)$, so that

$$d(a) \leq n - m \leq \lceil n/2 \rceil = \lfloor (n+1)/2 \rfloor < \lfloor (n+1)/2 \rfloor + 1.$$

Thus π is not extremal.

(2) $a < m$: in this case $m+3, \ldots, n+1, m+1 \notin N(a)$, so that

$$d(a) \le 1 + (m-1) \le \lceil n/2 \rceil < \lfloor (n+1)/2 \rfloor + 1.$$

Again, π is not extremal.

(3) $a \in \{m, m+1\}$: in this case

$$d(a) = 1 + m \le \lfloor (n+1)/2 \rfloor + 1,$$

with equality iff $m = \lfloor (n+1)/2 \rfloor$. This gives $\pi \in S_{n+1}$ of the required form (either π_0 or π_0^s).

A similar analysis for $\bar{\pi} = \pi_0^s$ gives extremal permutations only for $a \in \{m+1, m+2\}$ and $d(a) = 3$, so that $n = 4$ and $\bar{\pi} = [2413] \in S_4$. The permutations obtained are $\pi = [32514]$ and $\pi = [42513]$, which are $\pi_0^{rt}, \pi_0^{rst} \in S_5$, respectively.

The other possible values of $\bar{\pi}$ are obtained by the $\pi \mapsto \pi^r$ and $\pi \mapsto \pi^t$ operations from the ones above, and yield analogous results. $\square$

4. Expectation

In this subsection we prove an exact formula for the expectation of the down degree of a permutation in S_n.

Theorem 4.1. *For every positive integer n, the expected down degree of a random permutation in S_n is*

$$E_{\pi \in S_n}[d_-(\pi)] = \sum_{i=2}^{n} \sum_{j=2}^{i} \sum_{k=2}^{j} \frac{1}{i \cdot (k-1)} = (n+1) \sum_{i=1}^{n} \frac{1}{i} - 2n.$$

It follows that

Corollary 4.2. *As $n \to \infty$,*

$$E_{\pi \in S_n}[d_-(\pi)] = n \ln n + O(n)$$

and

$$E_{\pi \in S_n}[d(\pi)] = 2n \ln n + O(n).$$

To prove Theorem 4.1 we need some notation. For $\pi \in S_n$ and $2 \le i \le n$ let $\pi_{|i}$ be the permutation obtained from π by omitting all letters which are larger than or equal to i. For example, if $\pi = [6,1,4,8,3,2,5,9,7]$ then $\pi_{|9} = [6,1,4,8,3,2,5,7]$, $\pi_{|7} = [6,1,4,3,2,5]$, and $\pi_{|4} = [1,3,2]$.

Also, denote by $\pi^{|j}$ the suffix of length j of π. For example, if $\pi = [6,1,4,8,3,2,5,9,7]$ then $\pi^{|3} = [5,9,7]$ and $\pi_{|4}^{|2} = [3,2]$.

Let $l.t.r.m.(\pi)$ be the number of left-to-right maxima in π:

$$l.t.r.m.(\pi) := \#\{i \mid \pi(i) = \max_{1\le j\le i} \pi(j)\}$$

Lemma 4.3. *For every $\pi \in S_n$, if $\pi^{-1}_{|i+1}(i) = j$ then*

$$d_-(\pi_{|i+1}) - d_-(\pi_{|i}) = l.t.r.m.(\pi^{|i-j}_{|i}).$$

Proof of Theorem 4.1. Clearly, for every $\pi \in S_n$

$$d_-(\pi) = \sum_{i=2}^{n} \left[d_-(\pi_{|i+1}) - d_-(\pi_{|i})\right].$$

Thus, by Lemma 4.3,

$$d_-(\pi) = \sum_{i=2}^{n} l.t.r.m.(\pi^{|i-j_i}_{|i}),$$

where j_i is the position of i in $\pi_{|i+1}$, i.e., $j_i := \pi^{-1}_{|i+1}(i)$.

Define a random variable X to be the down degree $d_-(\pi)$ of a random (uniformly distributed) permutation $\pi \in S_n$. Then, for each $2 \le i \le n$, $\pi_{|i+1}$ is a random (uniformly distributed) permutation in S_i, and therefore $j = \pi^{-1}_{|i+1}(i)$ is uniformly distributed in $\{1, \dots, i\}$ and $\pi^{|i-j}_{|i+1}$ is essentially a random (uniformly distributed) permutation in S_{i-j} (after monotonically renaming its values). Therefore, by linearity of the expectation,

$$E[X] = \sum_{i=2}^{n} \frac{1}{i} \sum_{j=1}^{i} E[X_{i-j}] = \sum_{i=2}^{n} \frac{1}{i} \sum_{t=0}^{i-1} E[X_t], \tag{1}$$

where $X_t := l.t.r.m.(\sigma)$ for a random $\sigma \in S_t$.

Recall from [9, Corollary 1.3.8] that

$$\sum_{\sigma\in S_t} q^{l.t.r.m.(\sigma)} = \prod_{k=1}^{t} (q + k - 1).$$

It follows that, for $t \ge 1$,

$$\begin{aligned} E[X_t] &= \frac{1}{t!} \sum_{\sigma\in S_t} l.t.r.m.(\sigma) = \frac{1}{t!} \left(\frac{d}{dq} \sum_{\sigma\in S_t} q^{l.t.r.m.(\sigma)} \right)\Bigg|_{q=1} \\ &= \frac{1}{t!} \left(\frac{d}{dq} \prod_{k=1}^{t} (q + k - 1) \right)\Bigg|_{q=1} = \frac{1}{t!} \sum_{r=1}^{t} \prod_{\substack{1\le k\le t \\ k\ne r}} k = \sum_{r=1}^{t} \frac{1}{r}. \end{aligned}$$

Of course, $E[X_0] = 0$. Substituting these values into (1) gives

$$E[X] = \sum_{i=2}^{n} \sum_{t=1}^{i-1} \sum_{r=1}^{t} \frac{1}{i \cdot r}$$

and this is equivalent (with $j = t+1$ and $k = r+1$) to the first formula in the statement of the theorem.

The second formula may be obtained through the following manipulations:

$$\begin{aligned}
E[X] &= \sum_{i=2}^{n} \sum_{j=2}^{i} \sum_{k=2}^{j} \frac{1}{i \cdot (k-1)} = \sum_{2 \le k \le j \le i \le n} \frac{1}{i \cdot (k-1)} \\
&= \sum_{2 \le k \le i \le n} \frac{i-k+1}{i \cdot (k-1)} = \sum_{2 \le k \le i \le n} \left(\frac{1}{k-1} - \frac{1}{i} \right) \\
&= \sum_{2 \le k \le n} \frac{n-k+1}{k-1} - \sum_{2 \le i \le n} \frac{i-1}{i} \\
&= n \sum_{k=2}^{n} \frac{1}{k-1} - (n-1) - (n-1) + \sum_{i=2}^{n} \frac{1}{i} \\
&= n \sum_{i=1}^{n} \frac{1}{i} - 2n + \sum_{i=1}^{n} \frac{1}{i}.
\end{aligned}$$

□

Proof of Corollary 4.2. Notice that

$$\sum_{i=1}^{n} \frac{1}{i} = \ln n + O(1).$$

(The next term in the asymptotic expansion is Euler's constant.) Substitute into Theorem 4.1 to obtain the desired result. □

5. Generalized Down Degrees

Definition 5.1. For $\pi \in S_n$ and $1 \le r < n$ let

$$D_-^{(r)}(\pi) := \{t_{a,b} |\ \ell(\pi) > \ell(t_{a,b}\pi) > \ell(\pi) - 2r\}$$

the *r-th strong descent set* of π.

Define the *r-th down degree* as

$$d_-^{(r)}(\pi) := \# D_-^{(r)}(\pi).$$

Example 5.2. The first strong descent set and down degree are those studied in the previous section; namely, $D_-^{(1)}(\pi) = D_-(\pi)$ and $d_-^{(1)}(\pi) = d_-(\pi)$.

The $(n-1)$-th strong descent set is the set of *inversions*:

$$D_-^{(n-1)}(\pi) = \{t_{a,b} \mid a < b,\, \pi^{-1}(a) > \pi^{-1}(b)\}.$$

Thus

$$d_-^{(n-1)}(\pi) = inv(\pi),$$

the inversion number of π.

Observation 5.3. *For every $\pi \in S_n$ and $1 \le a < b \le n$, $t_{a,b} \in D_-^{(r)}(\pi)$ if and only if $\pi = [\ldots, b, \ldots, a, \ldots]$ and there are less than r letters between the positions of b and a in π whose value is between a and b.*

Example 5.4. Let $\pi = [7, 9, 5, 2, 3, 8, 4, 1, 6]$. Then

$$D_-^{(1)}(\pi) = \{t_{6,7}, t_{6,8}, t_{1,4}, t_{1,3}, t_{1,2}, t_{4,8}, t_{4,5}, t_{8,9}, t_{3,5}, t_{2,5}, t_{5,9}, t_{5,7}\}$$

and

$$D_-^{(2)}(\pi) = D_-^{(1)}(\pi) \cup \{t_{6,9}, t_{1,8}, t_{4,9}, t_{4,7}, t_{3,9}, t_{3,7}, t_{2,9}, t_{2,7}\}.$$

Corollary 5.5. *For every $\pi \in S_n$ and $1 \le r < n$*

$$d_-^{(r)}(\pi) = d_-^{(r)}(\pi^{-1}).$$

Proof. By Observation 5.3, $t_{a,b} \in D_-^{(r)}(\pi)$ if and only if $t_{\pi^{-1}(a),\pi^{-1}(b)} \in D_-^{(r)}(\pi^{-1})$. $\square$

Definition 5.6. The *r-th strong descent graph* of $\pi \in S_n$, denoted $\Gamma_-^{(r)}(\pi)$, is the graph whose set of vertices is $\{1, \ldots, n\}$ and whose set of edges is

$$\{\{a, b\} \mid t_{a,b} \in D_-^{(r)}(\pi)\}.$$

The following lemma generalizes Lemma 2.7.

Lemma 5.7. *For every $\pi \in S_n$, the graph $\Gamma_-^{(r)}(\pi)$ contains no subgraph isomorphic to the complete graph K_{r+2}.*

Proof. Assume that there is a subgraph in $\Gamma_-^{(r)}(\pi)$ isomorphic to K_{r+2}. Then there exists a decreasing subsequence

$$n \ge a_1 > a_2 > \cdots > a_{r+2} \ge 1$$

such that for all $1 \le i < j \le r+2$, t_{a_i,a_j} are r-th strong descents of π. In particular, for every $1 \le i < r+2$, $t_{a_i,a_{i+1}}$ are r-th strong descents of π. This implies that, for every $1 \le i < r+2$, a_{i+1} appears to the right of a_i in π. Then, by Observation 5.3, $t_{a_1,a_{r+2}}$ is not an r-th strong descent. Contradiction. $\square$

Corollary 5.8. *For every* $1 \le r < n$*,*

$$max\{d_-^{(r)}(\pi) \mid \pi \in S_n\} \le t_{r+1}(n) \le \binom{r+1}{2}\left(\frac{n}{r+1}\right)^2.$$

Proof. Combining Turán's Theorem together with Lemma 5.7. □

Note that for $r = 1$ and $r = n - 1$ equality holds in Corollary 5.8.

Remark 5.9. For every $\pi \in S_n$ let $\bar{\pi}$ be the permutation obtained from π by omitting the value n. If j is the position of n in π then

$$d_-^{(r)}(\pi) - d_-^{(r)}(\bar{\pi})$$

equals the number of $(r-1)$-th almost left-to-right minima in the $(j-1)$-th suffix of $\bar{\pi}$, see e.g. [8]. This observation may be applied to calculate the expectation of $d_-^{(r)}(\pi)$.

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