Séminaire Lotharingien de Combinatoire **53** (2006), Article B53h

MATRIX REALIZATIONS OF PAIRS OF YOUNG TABLEAUX, KEYS AND SHUFFLES

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Abstract. A key $\mathcal{H}$ is a semi-standard tableau of partition shape whose evaluation is a permutation of the shape. We give a necessary and sufficient condition that the Knuth class of a key equals the set of shuffles of its columns. In particular, on a three-letter alphabet the Knuth class of a key equals the set of shuffles of its columns, and on a four-letter alphabet, the Knuth class of a key is either the set of shuffles of its columns or the set of shuffles of its distinct columns with a single word taking appropriate multiplicities. For some instances of $\mathcal{H}$ this result has been already applied to exhibit a matrix realization, over a local principal ideal domain, of a pair of tableaux $(\mathcal{T}, \mathcal{H})$, where $\mathcal{T}$ is a skew-tableau whose word is in the Knuth class of $\mathcal{H}$. Generalized Lascoux–Schützenberger operators, based on nonstandard matching of parentheses, arise in the matrix realization, over local principal ideal domain, of a pair $(\mathcal{T}, \mathcal{H})$ on a two-letter alphabet, and they are used to show that, over a t-letter alphabet, the pair $(\mathcal{T}, \mathcal{H})$ has a matrix realization only if the word of $\mathcal{T}$ is in the Knuth class of $\mathcal{H}$.

1. Introduction

Given an n by n non-singular matrix A, with entries in a local principal ideal domain with prime p, by Gaußian elimination, one can reduce A to a diagonal matrix with diagonal entries $p^{\lambda_1}, \ldots, p^{\lambda_n}$, for unique nonnegative integers $\lambda_1 \geq \ldots \geq \lambda_n$, called the Smith normal form of A. The sequence $p^{\lambda_1}, \ldots, p^{\lambda_n}$ defines the invariant factors of A, and $(\lambda_1, \ldots, \lambda_n)$ the invariant partition of A. It is known that α, β, γ are invariant partitions of non-singular matrices A, B, and C such that $AB = C$ if and only if there exists a Littlewood–Richardson tableau $\mathcal{T}$ of type (α, β, γ), that is, a skew-tableau of shape γ/α whose word is in the Knuth class of the key $\mathcal{H}$ of evaluation β (Yamanouchi tableau β). This matrix problem is equivalent to the existence of p-modules $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ with invariant partitions α, β, γ such that $\mathcal{B} \subseteq \mathcal{C}$ and $\mathcal{C}/\mathcal{B} \cong \mathcal{A}$ [18]. (Interestingly, the eigenvalues of a sum of Hermitian matrices $A + B = C$ are characterized by the same condition [10, 27].) This theory was developed, with different approaches, by several authors, such as P. Hall, J. A. Green, T. Klein, I. Gohberg, M. A. Kaashoek, R. C. Thompson [14, 15, 18, 13, 24, 30, 5, 29]. (For an overview see [10, 11, 12].) One can solve this problem by introducing the notion of a matrix realization of a pair $(\mathcal{T}, \mathcal{H})$ where $\mathcal{T}$ is a skew-tableau with the same

[1]Work supported by CMUC/FCT.

evaluation as the key $\mathcal{H}$ [5, 2, 1] [see Section 5, Definition 5.1] which is equivalent, in the module setting, to the existence of a chain of p-submodules $(0) = \mathcal{B}_t \subseteq \cdots \subseteq \mathcal{B}_1 \subseteq \mathcal{B}_0 = \mathcal{B} \subseteq \mathcal{C}$ such that the sequence of invariant partitions of $\mathcal{C}/\mathcal{B}_k$, $0 \leq k \leq t$, defines a Littlewood–Richardson tableau of shape γ/α and evaluation given by the weight of the invariant partitions of $\mathcal{B}_{k-1}/\mathcal{B}_k$, $1 \leq k \leq t$ [18]. Within this notion it is natural to ask under which conditions does there exist a matrix realization of a pair $(\mathcal{T}, \mathcal{H}_\sigma)$, where $\mathcal{H}_\sigma$ is a key associated with $\sigma \in \mathcal{S}_t$ [8, 9, 22, 23]. In the case of the reverse permutation in $\mathcal{S}_t$ [4], or any permutation in $\mathcal{S}_3$ [6], or any adjacent transposition [3, 25], it has been shown that $(\mathcal{T}, \mathcal{H}_\sigma)$ has a matrix realization if and only if the word of $\mathcal{T}$ is in the plactic class of $\mathcal{H}_\sigma$. For these permutations, the elements of the plactic class of $\mathcal{H}_\sigma$ are shuffles of the columns of $\mathcal{H}_\sigma$ and this property has been used to exhibit a matrix realization $(\mathcal{T}, \mathcal{H}_\sigma)$. Here, in Theorem 5.3, we show that, for any $\sigma \in \mathcal{S}_t$, $t \geq 1$, $(\mathcal{T}, \mathcal{H}_\sigma)$ has a matrix realization only if the word of $\mathcal{T}$ is in the plactic class of $\mathcal{H}_\sigma$.

Due to the embedding of the symmetric group in the set of tableaux, originally defined by Ehresmann in [8], the symmetric group acts on the set of keys $\mathcal{H}_\sigma$, $\sigma \in \mathcal{S}_t$, in the obvious way. This action coincides with the one defined by the operations on the free algebra, described by A. Lascoux and M. P. Schützenberger in the plactic monoid [21, 20], based on the standard matching of parentheses, a particular parentheses matching on words in a two-letter alphabet. As these operations preserve the Q-symbol, they are bijections between the plactic classes $\mathcal{H}_\sigma$ and $\mathcal{H}_{s_i\sigma}$, with s_i the adjacent transposition of the integers i, $i+1$. Matrix realizations of pairs $(\mathcal{T}, \mathcal{H})$, over a two-letter alphabet, give rise to operations based on other parentheses matching than the standard one, as shown in Example 5.1 (see also [6]). Corollary 3.13 characterizes the operations, based on more general parentheses matching, which transform a word in the plactic class of $\mathcal{H}_\sigma$ into one in the plactic class of $\mathcal{H}_{s_i\sigma}$. The proof of Theorem 5.3 is based on these operations and their characterization.

Two columns commute (in the plactic sense) if and only if they are comparable for the inclusion order. In fact, the words of the plactic class of a key over a three-letter alphabet are shuffles of their columns [6]. In the case of a four-letter alphabet this property does not remain true. Nevertheless, by Greene's theorem [17], shuffling together the columns of a key always leads to a word in the plactic class of this key. We characterize the keys for which the plactic class may be described by shuffling together their columns. The keys associated with the identity and the reverse permutations in $\mathcal{S}_t$, $t \geq 1$, are simple examples of those keys. Finally, for $\sigma \in \mathcal{S}_4$, we show that we may describe the plactic class of any associated key, in terms of shuffling, by adding, in those cases where the columns of the key are not enough, one single word 434121.

The paper is organized as follows. In the next section we collect some notation and basic notions necessary in the sequel. The relationship between shuffling and Knuth operations on words is discussed. The following question is raised: *if the columns $u_1, \ldots, u_k$ are pairwise comparable for the inclusion order, under which conditions is the set of all shuffles of $u_1, \ldots, u_k$, denoted by $Sh(u_1, \ldots, u_k)$, the plactic class of $u_1 \ldots u_k$?* Indeed, the containment of $Sh(u_1, \ldots, u_k)$ in the plactic class of $u_1 \ldots u_k$ follows from Greene's theorem [17]. It remains to analyze whether the reverse inclusion holds.

In Section 3, the key tableaux, that is, the tableaux with pairwise comparable columns for the inclusion order, are considered. They can be seen as the tableaux whose evaluation is a permutation of the shape or as the image of the embedding of the symmetric group in the set of tableaux, originally defined by Ehresmann [8]. σ-Yamanouchi words are introduced as words congruent to a key of the permutation σ. These words are directly related with the action of the symmetric group, defined by the operations on all words described by A. Lascoux and M. P. Schützenberger in the plactic monoid [21, 20], based on the standard parentheses matching. Operations on words based on more general parentheses matchings are considered. A criterion characterizing those which transform a σ-Yamanouchi word into a $s_i\sigma$-Yamanouchi word is given.

In Section 4, the answer to the question raised in Section 2 is given by imposing conditions on the key $u_1 \dots u_k$ such that the plactic class of $u_1 \dots u_k$ is contained in $Sh(u_1, \dots, u_k)$. In the case of $\mathcal{S}_4$, a full description of the plactic classes of the associated keys, in terms of shuffling, is given. In Section 5, a matrix interpretation and application of the generalized Lascoux–Schützenberger operations, based on nonstandard matching of parentheses, is considered. Finally, in the Appendix, the permutations in $\mathcal{S}_5$ and $\mathcal{S}_6$ giving a positive answer to the question raised in Section 2 are listed.

2. Words, shuffles and Knuth congruence

2.1. **Words and tableaux.** Let $\mathbb{N}$ be the set of positive integers with the usual order "$\leq$". Given $i, j \in \mathbb{N}$, where $i \leq j$, $[i, j]$ is an interval in $\mathbb{N}$ with the usual order. If $t \in \mathbb{N}$, $[t]$ denotes the set $\{1, \dots, t\}$.

Let $A = \{a, b, \dots, f\}$, $a < b < \cdots < f$, be a finite subset of $\mathbb{N}$. We denote by A^* the free monoid in the alphabet A; that is, A^* is the collection of all finite words over the alphabet A, with the concatenation operation. The neutral element is the empty word, denoted by ϵ.

Given a word $w = x_1 \cdots x_k$ over the alphabet A, we call k the length of w and denote it by $|w|$. Furthermore, we denote by $|w|_x$ the multiplicity of the letter $x \in A$ in w. The sequence $(|w|_a, \dots, |w|_f)$ is called the *evaluation* of w in the alphabet A. We have $|w| = |w|_a + \cdots + |w|_f$, and the length of ϵ is zero. Given B a subalphabet of A, $w_{|B}$ denotes the word obtained by erasing the letters not in B.

If the letters in w are in strictly decreasing order, that is, $x_i > x_{i+1}$ for all i, w is called a *column* in A^*. A column shall be identified with the set consisting of its entries. Given two finite subsets $P = \{p_1, p_2, \dots\}$, $Q = \{q_1, q_2, \dots\}$ of $\mathbb{N}$ with $|P| \leq |Q|$, we write $P \leq Q$ if $p_i \leq q_i$ for $i = 1, 2, \dots, |P|$.

A partition $\lambda = (\lambda_1, \dots, \lambda_k, \dots)$ is a weakly decreasing (finite or infinite) sequence of nonnegative integers, with only a finite number of nonzero entries. The number of nonzero entries of λ is called the *length* of λ. A partition λ is identified with its Young diagram, a left-justified arrangement of boxes, or dots, with λ_i boxes (dots) in the i-th row, where rows are arranged from bottom to top. (We adopt the French notation.) The conjugate of partition λ is the partition λ', the transpose of the Young diagram λ. It is convenient not to distinguish between two partitions which only differ by a string of zeros at the end. Sometimes we write $\lambda = (1^{p_1}, 2^{p_2}, \cdots)$ to indicate that i

appears p_i times as a part of λ. For instance, the partition $(3,2,2,1) = (3,2^2,1,0)$ corresponds to the Young diagram

$$\begin{array}{ccc} \bullet & & \\ \bullet & \bullet & \\ \bullet & \bullet & \\ \bullet & \bullet & \bullet \end{array},$$

and its conjugate partition is $(4,3,1)$. Clearly, the lengths of the columns in weakly decreasing order define the transpose Young diagram.

A Young tableau $\mathcal{T}$ of shape λ is a filling of the Young diagram of λ with positive integers, weakly increasing across each row and strictly increasing up each column [9, 20]. An example of a Young tableau of shape $(3,2,2,1)$ is given by

$$\mathcal{T} = \begin{array}{ccc} 5 & & \\ 4 & 5 & \\ 2 & 2 & \\ 1 & 1 & 2 \end{array}.$$

The word of a Young tableau is the word obtained by reading its columns from top to bottom, starting on the left and moving to the right. A Young tableau shall be identified with its word. In the example above, we have $\mathcal{T} = 5421\,521\,2$ (the empty spaces only indicate the end of a column and the starting of a new one). The evaluation of a Young tableau $\mathcal{T}$ is the evaluation of its word. For instance, the evaluation of the Young tableau above is $(2,3,1,2)$.

A skew-diagram is the diagram obtained by removing a smaller Young diagram from a larger one that contains it. If λ and μ are partitions with $\mu \subseteq \lambda$, that is, $\mu_i \leq \lambda_i$ for all i, we define a skew-tableau of shape λ/μ as a filling of the skew-diagram λ/μ, that is, weakly increasing across each row and strictly increasing up each column [9]. A Young tableau of shape λ may be seen as a skew-tableau of shape $\lambda/(0)$, where (0) denotes the empty partition. An example of a skew-tableau of shape $(4,4,2,1)/(3,1)$ is given by

$$\mathcal{T} = \begin{array}{cccc} 4 & & & \\ 2 & 2 & & \\ \bullet & 1 & 3 & 3 \\ \bullet & \bullet & \bullet & 2 \end{array}.$$

The word $w(\mathcal{T})$ of a skew-tableau $\mathcal{T}$ is the word obtained by reading its columns from top to bottom, starting on the left and moving to the right. In the example above we have $w(\mathcal{T}) = 42\,21\,3\,32$. As for Young tableaux, the evaluation of a skew-tableau $\mathcal{T}$ is the evaluation of its word. For instance, the evaluation of the skew-tableau $\mathcal{T}$ above is $(1,3,2,1)$.

A skew-tableau $\mathcal{T}$ of shape λ/μ and evaluation $(m_1,\dots,m_t)$ may also be represented by a nested sequence of partitions [24] $\mathcal{T} = (\lambda^0,\lambda^1,\dots,\lambda^t)$, where $\mu = \lambda^0 \subseteq \lambda^1 \subseteq \cdots \subseteq \lambda^t = \lambda$, such that for $k = 1,\dots,t$, all the boxes of the skew diagram λ^k/λ^{k-1} are filled with k, with $m_k = |\lambda^k| - |\lambda^{k-1}|$. In the example above, $\mathcal{T} = (\lambda^0,\lambda^1,\lambda^2,\lambda^3,\lambda^4)$, where $\lambda^0 = \mu$, $\lambda^1 = (3,2)$, $\lambda^2 = (4,2,2)$, $\lambda^3 = (4,4,2)$ and $\lambda^4 = \lambda$.

2.2. **Shuffles and Knuth congruence.** A word w contains u as a subword if w, as a sequence of letters, contains u as a subsequence. A word w is a shuffle of the words u and v if u and v can be embedded as subwords of w that occupy complementary sets of positions within w. A shuffle w of the words $u_1, u_2, \dots, u_q$ is the empty word ϵ for $q = 0$, the word u_1 for $q = 1$, and is, otherwise, a shuffle of u_1 with some shuffle of the words $u_2, \dots, u_q$. Let $Sh(u_1, u_2, \dots, u_q)$ denote the set of shuffles of $u_1, u_2, \dots, u_q$. By abuse of notation we write $w = sh(u_1, u_2, \dots, u_q)$ to mean that w is some shuffle of $u_1, u_2, \dots, u_q$.

For example, it is simple to check that 4423142121 is a shuffle of $4321, 421$ and 421. When there is no danger of confusion, to avoid cumbersome notation, we shall write $sh((u_1)^{l_1}, \dots, (u_q)^{l_q})$ to designate a shuffle of l_i words u_i, for $i = 1, \dots, q$. Thus, we have $4423142121 \in Sh(4321, (421)^2)$.

Knuth's congruence $\equiv$ [19] on words over the alphabet A is the congruence generated by the so-called elementary transformations, where x, y, z are letters and u, v are words in A:

$$uxzxv \equiv uzxxv, \quad uzzxv \equiv uzxzv, \quad x < z, \tag{1}$$

$$uxzyv \equiv uzxyv, \quad x < y < z, \tag{2}$$

$$uyzxv \equiv uyxzv, \quad x < y < z. \tag{3}$$

The relations (1), (2), (3), also called *plactic relations*, are the algebraic version of the plactic congruence [9, 19, 20]. C. Schensted [9, 20, 28] has described an algorithm, known as *Schensted's insertion algorithm*, which associates to each word w a tableau $P(w)$. Two words w, w' are plactic equivalent if and only if $P(w) = P(w')$ [9, 19, 20]. The set of all tableaux is a *section* of the plactic congruence. This means that every word can be obtained by a finite sequence of elementary Knuth transformations from a tableau.

Let $u_1, \dots, u_k$ in A^* be columns in decreasing order of length. If $\mathcal{T} = u_1 \dots u_k$ is a tableau, then $\mathcal{T}$ is the unique tableau of $Sh(u_1, \dots, u_k)$ only if the columns of the tableau $\mathcal{T}$ are pairwise comparable for the inclusion order; that is $\{u_k\} \subseteq \cdots \subseteq \{u_1\}$. For instance, $Sh(41, 3) = \{3\,41, 41\,3, 431\}$ has two tableaux $41\,3$ and 431.

An elementary Knuth transformation (1), (2), or (3) applied to a shuffle of columns, say $u_1, \dots, u_k$, involves at least two of these columns. If an elementary Knuth transformation (2) or (3) involves three distinct letters $x < y < z$ of $sh(u_1, \dots, u_k)$, each one belonging to a different column u_i, then the output word is still a shuffle of $u_1, \dots, u_k$.

Proposition 2.1. *Let $u_1, \dots, u_k$, $k \geq 3$, be columns in A^*, and x, y, z, u and v as in* (2), (3) *such that each letter x, y and z appears in a distinct column. Then*

$$uxzyv \in Sh(u_1, \dots, u_k) \Leftrightarrow uzxyv \in Sh(u_1, \dots, u_k);$$

$$uyzxv \in Sh(u_1, \dots, u_k) \Leftrightarrow uyxzv \in Sh(u_1, \dots, u_k).$$

For instance, in the alphabet [5], consider the word $5\overline{2}4\underline{4}\overline{1}2\underline{2}1\underline{1} \in Sh(5421, 421, 21)$, where the underlined letters define the word 421, the overlined letters define the word 21, and the remaining letters define the word 5421. The application of the elementary Knuth transformation $\underline{4}\,\overline{1}\,2 \equiv \overline{1}\,\underline{4}\,2$ to $5\,\overline{2}\,4\,\underline{4}\,\overline{1}\,2\,\underline{2}\,1\,\underline{1}$ gives the word $w = 5\,\overline{2}\,4\,\overline{1}\,\underline{4}\,2\,\underline{2}\,1\,\underline{1}$, which is still a shuffle of 5421, 421 and 21.

If the Knuth transformation involves only two distinct letters of $sh(u_1\dots,u_q)$, it is also clear that the output word is still a shuffle of $u_1\dots,u_q$.

Proposition 2.2. *Let $u_1\dots,u_k$, $k\geq 2$, be columns in A^*, and x, z, u and v as in* (1)*. Then*

$$uzxxv\in Sh(u_1,\dots,u_k)\Leftrightarrow uxzxv\in Sh(u_1,\dots,u_k);$$

$$uzzxv\in Sh(u_1,\dots,u_k)\Leftrightarrow uzxzv\in Sh(u_1,\dots,u_k).$$

Corollary 2.3. *Let $u_1\dots,u_k$, $k\geq 2$, be columns in a two-letter alphabet $A=\{a,b\}$, and $w\in Sh(u_1,\dots,u_k)$ of evaluation (p,q). Then*
 (a) *$P(w)\in Sh(u_1,\dots,u_k)$.*
 (b) *$Sh(u_1,\dots,u_k)$ is either*
  (i) *the plactic class of $P(w)=(ba)^q1^p$ or $P(w)=(ba)^pb^q$, if the columns u_i are pairwise comparable for the inclusion order, for $i=1,\dots,k$; or*
  (ii) *the union of the plactic classes of $P(w)=(ba)^ra^sb^v$, $r+s=p$, $r+v=q$, and $P(w)=(ba)^{p+q}$, otherwise.*

Let $x<y<z\in A$. The columns $\overline{z}\,\overline{y}$ and $\underline{y}\,\underline{x}$, in the elementary Knuth transformations $x\,\overline{z}\,\overline{y}\equiv\overline{z}\,x\,\overline{y}$ (2), and $\underline{y}\,z\,\underline{x}\equiv\underline{y}\,\underline{x}\,z$ (3), respectively, are not broken by these transformations and, therefore, the shuffle of $\overline{z}\,\overline{y}$ and x, and $\underline{y}\,\underline{x}$ and z is preserved. The only column that is broken by these Knuth transformations is zx which is transformed into xz. Consider again the columns 41 and 3. We have $Sh(41,3)=\{3\,41,41\,3,431\}$, but $3\,41\equiv 31\,4\notin Sh(41,3)$ and $41\,3\equiv 143\notin Sh(3,41)$. Therefore, considering, for example, the tableau 4321 41, the Knuth transformation $341\equiv 314$ implies $43\underline{4}\,\underline{1}21=sh(4321,41)\equiv 43\underline{1}\,\underline{4}21$, but $43\underline{1}\,\underline{4}21$ can not be obtained by a shuffle of the columns 4321 and 41.

Supposing that $u_1,\dots,u_k$ are columns pairwise comparable in the inclusion order, we raise the question: *Under which conditions $Sh(u_1,\dots,u_k)$ equals the plactic class of the tableau $\mathcal{T}=u_1\cdots u_k$?*

We start by noticing that the containment $Sh(u_1,\dots,u_k)$ in the Knuth class of $\mathcal{T}$ follows from Greene's theorem [17]. Since $u_1\supseteq\cdots\supseteq u_k$, the maximum of the sums of the lengths of j decreasing and disjoint subwords of $w\in Sh(u_1,\dots,u_k)$ is $|u_1|+\cdots+|u_j|$, for all $j\geq 1$. It follows from Greene's theorem that the conjugate shape of $P(w)$ is $(|u_1|,\dots,|u_k|)$, which means that $P(w)=u_1\cdots u_k$. But as we have seen above, in general, we do not have equality. In Section 4 we determine the conditions for which equality holds.

3. Keys and σ-Yamanouchi words

3.1. **Parentheses matching operations.** Given a set I, let $\mathcal{S}_I$ be the set of all bijections on I, and $\mathcal{S}_t:=\mathcal{S}_{[t]}$ the symmetric group of order t. The symmetric group $\mathcal{S}_t$, $t\geq 1$, is generated by the simple transpositions $s_i=(i\,i+1)$, $i=1,\dots,t-1$, which satisfy the Moore-Coxeter relations:

$$\text{(I) } s_i^2=id,\ \ \text{(II) } s_is_j=s_js_i,\ \text{ if } |i-j|\neq 1,\ \text{ and (III) } s_is_{i+1}s_i=s_{i+1}s_is_{i+1},$$

where id denotes the identity.

Let w be a word over the alphabet $[t]$ and $i, i+1 \in [t]$. An operation θ_i on w [6] consists of (a) a longest matching on $w_{|\{i,i+1\}}$ between letters $i+1$ and letters i to their right, by putting a left parenthesis on the left of each letter $i+1$, and a right parenthesis on the right of each letter i, such that the unmatched right and left parentheses indicate a subword of the form $i^s(i+1)^r$; (b) this subword will be replaced in $w_{|\{i,i+1\}}$ with $i^r(i+1)^s$. By abuse of notation, we write $\theta_i(w)$ to refer to any word which is related with w in the defined way.

Let $w = 31314221412$ be a word over the alphabet $[4]$. For example, inserting parentheses on the right of the letters 1 and on the left of the letters 2 of the word $w_{|\{1,2\}} = 1122211$, we get 1)1)(2(2(21)1). We may match the two left most letters 2 with the letters 1 to their right. The unmatched letters indicate the subword $w' = 112 = 1^2 2^1$. Thus, $\theta_1(w_{|\{1,2\}}) = \underline{1}\,\underline{2}\,2\,2\underline{2}\,1\,1$, where the underlined word is the subword w' replaced with $1^1 2^2$. Finally, $\theta_1(w) = 3\,1\,3\,2\,4\,2\,2\,2\,4\,1\,1$.

A. Lascoux and M. P. Schützenberger [21, 20] have introduced the following involutions θ_i^*, for $i = 1, \ldots, t-1$, on all words over the alphabet $[t]$, based on the standard matching of parentheses on words over a two-letter alphabet, a particular matching of parentheses. Let w be a word over the alphabet $[t]$. To compute $\theta_i^*(w)$, first extract from w the subword v containing the letters i and $i+1$ only. Second, bracket every factor $i+1\; i$ of v. The letters which are not bracketed constitute a subword v_1 of v. Then bracket every factor $i+1\; i$ of v_1. There remains a subword v_2. Continue this procedure until it stops, giving a word v_k of type $i^r\,(i+1)^s$. Then, replace it with the word $i^s\,(i+1)^r$ and, after this, recover all the removed letters of w, including the ones different from i and $i+1$. The operations θ_i^*, $1 \le i \le t-1$, satisfy the Moore-Coxeter relations [21, 20] and define an action of $\mathcal{S}_t$ over $[t]^*$.

Let $w = 31314221412$ as above. To compute $\theta_1^*(w)$, we get $v = 112(21)12$, $v_1 = 11(21)2$ and $v_2 = 112 = 1^2 2^1$. Thus, $\theta_1^*(w) = 3\,\underline{1}\,3\,\underline{2}\,4\,2\,2\,1\,4\,1\,\underline{2}$, where the underlined word is the subword v_2 replaced with $1^1 2^2$.

The operations θ_i^*, $1 \le i \le t-1$, are compatible with the plactic equivalence and preserve the Q-symbol.

Proposition 3.1. [21, 20] *Let w, w' be words in $[t]^*$, and let $i \in [t]$. Then, $w \equiv w'$ if and only if $\theta_i^*(w) \equiv \theta_i^*(w')$. In particular, $P(\theta_i^*(w)) = \theta_i^*(P(w))$.*

Note that in the examples above we have $\theta_1(w) \not\equiv \theta_1^*(w)$. In the case of words w congruent to a tableau whose columns are comparable in the inclusion order, we will give a criterion such that $\theta_i(w) \equiv \theta_i^*(w)$.

3.2. **Keys and σ-Yamanouchi words.** By definition, a *key* is a tableau such that its columns are pairwise comparable in the inclusion order [23]. Equivalently, a key is a tableau whose evaluation is a permutation of its shape. For instance, over the alphabet $[6]$, $65431\,641\,41$ is a key of shape $(3,3,2,1,1)$ and evaluation $(3,0,1,3,1,2)$.

Let $(l_t, \ldots, l_2, l_1)$ be a sequence of nonnegative integers. Then, $m = (l_1 + \cdots + l_t, \ldots, l_{t-1} + l_t, l_t)$ is a partition and $(t^{l_t}, \ldots, 2^{l_2}, 1^{l_1})$ its conjugate. For instance, $(1, 1, \ldots, 1)$ defines the self-conjugate partition $(t, t-1, \ldots, 1)$.

Let $\sigma \in \mathcal{S}_t$ written as a word $a_1 \cdots a_t$ in $[t]^*$. For $k = 1, \ldots, t$, denote by $r_{\sigma,k}$ the column with underlying set $\{a_1, \ldots, a_k\}$. In particular, when $\sigma = 12\ldots t$, we get $r_k = k \ldots 21$. Clearly, $\{r_t\} \supseteq \{r_{\sigma,t-1}\} \supseteq \ldots \supseteq \{r_{\sigma,1}\}$.

Definition 3.1. *Key of a permutation* [23]. To each pair consisting of a permutation $\sigma \in \mathcal{S}_t$ and a sequence of nonnegative integers $(l_t, \dots, l_1)$, Ehresmann [8] associated a key of shape m, here denoted by $\mathcal{H}_{\sigma(l_t,\dots,l_1)}$, by taking the sequence $(r_t)^{l_t}, (r_{\sigma,t-1})^{l_{t-1}}, \dots, (r_{\sigma,1})^{l_1}$ of left reordered factors of σ; that is,

$$\mathcal{H}_{\sigma(l_t,\dots,l_1)} := (r_t)^{l_t}\,(r_{\sigma,t-1})^{l_{t-1}} \cdots (r_{\sigma,1})^{l_1},$$

is the key of σ of shape m. In particular, $\mathcal{H}_{(l_t,\dots,l_1)} = (t \cdots 21)^{l_t} \cdots (21)^{l_2}\ (1)^{l_1}$.

For $i = 1, \dots, t$, the letter a_i appears only in the columns $r_t, \dots, r_{\sigma,i}$. Hence, the multiplicity of a_i in $\mathcal{H}_{\sigma(l_t,\dots,l_1)} = (r_t)^{l_t}\,(r_{\sigma,t-1})^{l_{t-1}} \dots (r_{\sigma,1})^{l_1}$ is $\sum_{k=i}^{t} l_k$, for $i = 1, \dots, t$. We put $\sigma m = (m_1, \dots, m_t)$, where $m_i = \sum_{k=\sigma^{-1}(i)}^{t} l_k$, $i = 1, \dots, t$. Hence $\mathcal{H}_{\sigma(l_t,\dots,l_1)} := (r_t)^{l_t}\,(r_{\sigma,t-1})^{l_{t-1}} \dots (r_{\sigma,1})^{l_1}$ is also the key of σ with evaluation σm; equivalently, the unique tableau of evaluation σm and shape m.

Note that $\mathcal{H}_{\sigma(l_j e_j)} = (r_{\sigma,j})^{l_j}$, and $\mathcal{H}_{\sigma(l_t,\dots,l_1)} = (\mathcal{H}_{\sigma(l_t e_t)})^{l_t} \dots (\mathcal{H}_{\sigma(l_1 e_1)})^{l_1}$, with $e_j = (\delta_{i,j})$, $j = 1, \dots, t$.

On the other hand, if $\mathcal{T} = q_t \dots q_2\, q_1$ is a key, with $q_t = t \dots 2\,1$ and $|q_{t-1}| > \dots > |q_1|$, we get that column q_k is such that $\{q_k\} = \{q_t\} \setminus \{a_t, \dots, a_{k+1}\}$ with $\{a_t, \dots, a_{k+1}\} \subseteq \{1, \dots, t\}$, $1 \le k \le t-1$. Putting $\sigma := a_1 \dots a_t$ this shows that $\mathcal{H}_{\sigma(1,\dots,1)} = \mathcal{T}$. Therefore, given a sequence $(l_t, \dots, l_1)$ of positive integers,

$$\sigma \longrightarrow \mathcal{H}_{\sigma(l_t,\dots,l_2,l_1)} = (r_t)^{l_t}\,(r_{\sigma,t-1})^{l_{t-1}} \dots (r_{\sigma,1})^{l_1},$$

defines an embedding of $\mathcal{S}_t$ into the set of tableaux of conjugate shape $(t^{l_t}, \dots, 2^{l_2}, 1^{l_1})$.

For example, with $\sigma = 3124 \in \mathcal{S}_4$, we have $r_4 = 4321, r_{s_2,3} = 321, r_{s_2,2} = 31$, $r_{s_2,1} = 3$, $\mathcal{H}_{\sigma(1,1,1,1)} = \begin{matrix} 4 & & & \\ 3 & 3 & & \\ 2 & 2 & 3 & \\ 1 & 1 & 1 & 3 \end{matrix}$, and $\mathcal{H}_{\sigma(1,1,2,0)} = (4321)^1(321)^1(31)^2(3)^0 = \begin{matrix} 4 & & & \\ 3 & 3 & & \\ 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{matrix}$.

With $s_2 = 1324$, we have $r_4 = 4321, r_{s_2,3} = 321, r_{s_2,2} = 31$, $r_{s_2,1} = 1$, and

$$\mathcal{H}_{s_2(1,1,2,0)} = (4321)^1(321)^1(31)^2(1)^0 = \begin{matrix} 4 & & & \\ 3 & 3 & & \\ 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{matrix} = \mathcal{H}_{\sigma(1,1,2,0)}.$$

Let $I := [t] \setminus \{i\}$, with $i \in [t]$, and let $\sigma_{|I} := a_1 \dots a_{t|I} \in \mathcal{S}_I$. If $\sigma^{-1}(i) = p$, then letter $a_p = i$ appears only in columns $r_t \dots, r_{\sigma,p}$. Hence, when we erase letter i in column $r_{\sigma,p}$, we obtain column $r_{\sigma,p-1}$, and we have

$$\mathcal{H}_{\sigma(l_t,\dots,l_1)_{|I}} = \mathcal{H}_{(\sigma_{|I})(l_t,\dots,l_{p+1},l_p+l_{p-1},\dots,l_1)}.$$

With $\sigma = s_4 = 12354$, we have $r_5 = 54321, r_{s_4,4} = 5321, r_{s_4,3} = 321$, $r_{s_4,2} = 21$, $r_{s_4,1} = 1$ and

$$\mathcal{H}_{s_4(0,1,1,2,1)} = (54321)^0(5321)^1(321)^1(21)^2(1)^1 = \begin{matrix} 5 & & & & \\ 3 & 3 & & & \\ 2 & 2 & 2 & 2 & \\ 1 & 1 & 1 & 1 & 1 \end{matrix};$$

and with $I = [5] \setminus \{2\}$,

$$(\mathcal{H}_{s_4(0,1,1,2,1)})_{|I} = \mathcal{H}_{(s_{4|I})(0,1,1,2+1)} = (5431)^0(531)^1(31)^1(1)^3 = \begin{array}{ccccc} 5 & & & & \\ 3 & 3 & & & \\ 1 & 1 & 1 & 1 & 1 \end{array}.$$

Proposition 3.2. *Let $\theta^* \in <\theta_1^* \dots \theta_{t-1}^*>$, and let σ be a permutation in $\mathcal{S}_t$ with the same reduced decomposition. Then*
(a) $\theta^*(r_k) = r_{\sigma,k}$, $1 \le k \le t$.
(b) $\theta^*(\mathcal{H}_{(l_t,\dots,l_1)}) = \mathcal{H}_{\sigma(l_t,\dots,l_1)} = (\theta^*(r_t))^{l_t}\ (\theta^*(r_{t-1}))^{l_{t-1}} \cdots (\theta^*(r_1))^{l_1}$.

Proof: Follows from Definition 3.1. □

When there is no danger of confusion, we will drop the "$(l_t,\dots,l_1)$" in the notation $\mathcal{H}_{\sigma(l_t,\dots,l_1)}$, to denote a key of σ.

A word w over the alphabet $[t]$ is said to be *Yamanouchi* [20] if any right factor v of w satisfies $|v|_1 \ge |v|_2 \ge \cdots \ge |v|_t$. This is equivalent to saying that $w \in Sh((r_t)^{l_t}, \cdots, (r_1)^{l_1})$, where $(|v|_1, |v|_2, \cdots, |v|_t) = (l_1 + \cdots + l_t, \dots, l_{t-1} + l_t, l_t)$. Thus, for each $(l_t, \dots, l_2, l_1)$, the key $\mathcal{H}_{(l_t,\dots,l_2,l_1)}$ is a Yamanouchi tableau. Clearly, any shuffle of a Yamanouchi word is still a Yamanouchi word.

Proposition 3.3. [20, Lemma 5.4.7] *The set of Yamanouchi words with evaluation m forms a single plactic class, whose representative word is the tableau $\mathcal{H}_{(l_t,\dots,l_2,l_1)}$.*

Thus, the set of words obtained by shuffling the columns of $\mathcal{H}_{(l_t,\dots,l_2,l_1)}$ and the one obtained by applying a finite sequence of Knuth transformations on the tableau $\mathcal{H}_{(l_t,\dots,l_2,l_1)}$ are the same.

The dual of a tableau $\mathcal{T}$ is defined as the tableau with same shape and reverse evaluation, obtained by applying the Schützenberger involution to $\mathcal{T}$. Thus the dual of $\mathcal{H}$ is $\mathcal{H}_{op}$. On the other hand, if $w \equiv \mathcal{H}$ then $w^* \equiv \mathcal{H}^* \equiv \mathcal{H}_{op}$, where w^* denotes the dual word of w. The characterization of Yamanouchi words given by Proposition 3.3 leads to the following definition.

Definition 3.2. Let $t \ge 1$ and $\sigma \in \mathcal{S}_t$. A word w over the alphabet $[t]$ is said to be σ-Yamanouchi if $w \equiv \mathcal{H}_\sigma$. In particular, when σ is the identity, w is a Yamanouchi word, and when σ is the reverse permutation, w is a dual Yamanouchi word.

Since the operations θ_i^* are compatible with the plactic equivalence, we may characterize σ-Yamanouchi words using the operations θ_i^* [21, 20] as well.

Proposition 3.4. *Let $t \ge 1$ and $\sigma \in \mathcal{S}_t$. Let w be a word over the alphabet $[t]$. Then, w is a σ-Yamanouchi word if and only if $\theta_{i_r}^* \cdots \theta_{i_1}^*(w)$ is a Yamanouchi word, where $s_{i_1} \cdots s_{i_r}$ is a reduced decomposition of σ.*

Proof: We have $w \equiv \mathcal{H}_\sigma$ if and only if $\theta^*(w) \equiv \theta^*(\mathcal{H}_\sigma) = \mathcal{H}$, where $\theta^* = \theta_{i_r}^* \cdots \theta_{i_1}^*$. □

As in the case of Yamanouchi words, we find that a shuffle of σ-Yamanouchi words is still a σ-Yamanouchi word.

Proposition 3.5. *Let $\sigma \in \mathcal{S}_t$. If w and w' are σ-Yamanouchi words over the alphabet $[t]$, then any word in $Sh(w, w')$ is also a σ-Yamanouchi word.*

Proof: Let $u \in Sh(w, w')$, and assume that $w \equiv \mathcal{H}(\sigma, (l_t, \ldots, l_1))$ and $w' \equiv \mathcal{H}(\sigma, (l'_t, \ldots, l'_1))$. By Greene's theorem [17], we find that the conjugate shape of $P(u)$ is $(t^{l_t+l'_t}, \ldots, 1^{l_1+l'_1})$, that is, $P(u) = \mathcal{H}(\sigma, (l_t + l'_t, \ldots, l_1 + l'_1))$. □

By induction, we may easily extend Proposition 3.5 to a shuffle of k σ-Yamanouchi words, for every $k \in \mathbb{N}$.

Corollary 3.6. *Let $i \in \{1, \ldots, t-1\}$, $w = sh((r_t))^{l_t}, \ldots, (r_1)^{l_1})$ and let $\widetilde{\theta}_i(w) = sh((\theta_i^*(r_t))^{l_t}, \ldots, (\theta_i^*(r_1))^{l_1})$. Then $\widetilde{\theta}_i(w)$ is a s_i-Yamanouchi word.*

Proposition 3.7. [23, 20] *If B is an interval of A, then*

$$w \equiv w' \quad \textit{implies} \quad w_{|B} \equiv w'_{|B}.$$

Corollary 3.8. *Let w be a word over the alphabet A, and let $A' = A \setminus \{f\}$. Then, $P(w_{|A'}) = P(w)_{|A'}$.*

Proof: From the previous proposition, we have $w_{|A'} \equiv P(w)_{|A'}$. Thus $P(w_{|A'})$ is obtained from $P(w)$ by removing the letters f. □

If w is a σ-Yamanouchi word then the word $w_{|\{i,i+1\}} \equiv \mathcal{H}_{\sigma|\{i,i+1\}}$ and, thus, $w_{|\{i,i+1\}}$ is either a Yamanouchi or dual Yamanouchi word for $1 \leq i \leq t-1$. (Consider a shift of the alphabet $\{1, 2\}$.) Moreover, if w has evaluation $(m_1, \ldots, m_t)$, then $w = sh(u, t^{m_t})$ with $u \equiv \mathcal{H}_{\sigma(l_t,\ldots,l_2,l_1)|[t-1]}$ and $w_{|\{t-1,t\}}$ a Yamanouchi or dual Yamanouchi word.

The word $3121 \equiv 321\,1$ is a s_1-Yamanouchi word and 434 a dual Yamanouchi word. Nevertheless, considering the words in $\{w \in Sh(3121, 44) : w_{|\{3,4\}} = 434\} = \{\underline{4}3\underline{4}121 \equiv 4321\,41, \underline{4}31\underline{4}21 \equiv 4321\,41, \underline{4}312\underline{4}1 \equiv 4321\,1\,4, \underline{4}3121\underline{4} \equiv 4321\,1\,4\}$ we find that $\underline{4}312\underline{4}1$, $\underline{4}3121\underline{4}$ are not σ-Yamanouchi words, for any $\sigma \in \{4213; 2413; 2143; 2134\} \subseteq \mathcal{S}_4$.

This leads to the following question: *given $I = [t]\setminus\{i\}$, with $i \in [t]$, and $u \in I^*$ congruent to the key of evaluation $(m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_t)$, under which conditions can u be embedded in a word congruent to the key of evaluation $(m_1, \ldots, m_t)$?*

The answer is given by the following proposition.

Proposition 3.9. *Let $w \in [t]^*$ and $\sigma \in \mathcal{S}_t$. Given $i \in [t]$, let $I = [t] \setminus \{i\}$, and suppose $w_{|I} \equiv \mathcal{H}_{\sigma|I}$. Then, w is a σ-Yamanouchi word if and only if $w_{|\{j,j+1\}}$ is either a Yamanouchi or a dual Yamanouchi word, for $j = i-1, i$, and $\theta_j^*(w)_{|[j]} \equiv \mathcal{H}_{s_j\sigma|[j]}$, for $j = i-1, \ldots, t-1$.*

Proof: The conditions are clearly necessary. Suppose now that $w_{|_{\{j,j+1\}}}$ is either a Yamanouchi or a dual Yamanouchi word, for $j = i-1, i$, and $\theta_j^*(w)_{|[j]} \equiv \mathcal{H}_{s_j\sigma|[j]}$, for $j = i-1, \ldots, t-1$.

We start with the case $i = t$, and thus, $I = [t-1]$. Consider $(m_1, \ldots, m_t)$, the evaluation of w, and assume without loss of generality that $m_{t-1} \geq m_t$. From the equality $P(w_{|_{[t-1]}}) = P(w)_{|[t-1]} = \mathcal{H}_{\sigma|[t-1]}$, we find that the letters $t-1$ of w are in the first m_{t-1} columns of $P(w)$.

Since $w_{|\{t-1,t\}} \equiv P(w)_{|\{t-1,t\}}$ is a Yamanouchi word, the m_t letters t of w are displayed in the first m_{t-1} columns of $P(w)$. On the other hand, the tableau $P(\theta_{t-1}^*(w)_{|[t-1]})$ is obtained by erasing the m_{t-1} letters t in the first m_{t-1} columns of the tableau $P(\theta_{t-1}^*(w)) = \theta_{t-1}^*(P(w))$. So if the letters t in tableau $P(w)$ are not in the first m_t columns, the letters $t-1$ are not in the first m_t columns of $P(\theta^*_{t-1}(w)_{|[t-1]}) = \mathcal{H}_{s_{t-1}\sigma_{|[t-1]}}$. This is absurd.

Assume now that $i \neq t$. Then, $w_{|[i-1]} \equiv \mathcal{H}_{\sigma|[i-1]}$, and since $w_{|\{i-1,i\}}$ is either a Yamanouchi or a dual Yamanouchi word, and $\theta^*_{i-1}(w)_{|[i-1]} \equiv \mathcal{H}_{s_{i-1}\sigma_{|[i-1]}}$, by the case $i = t$ we find that

$$w_{|[i]} \equiv \mathcal{H}_{\sigma|[i]}.$$

Next, note that $w_{|\{i,i+1\}}$ is either a Yamanouchi or a dual Yamanouchi word, and that $\theta^*_i(w)_{|[i]} \equiv \mathcal{H}_{s_i\sigma|[i]}$. Thus, again by the case $i = t$, we must have

$$w_{|[i+1]} \equiv \mathcal{H}_{\sigma|[i+1]}.$$

Noticing that, since $w_{|I} \equiv \mathcal{H}_{\sigma|I}$, the word $w_{|\{j,j+1\}}$ is either Yamanouchi or dual Yamanouchi, for $j = i+1, \dots, t-1$, we may repeat the process described above for $j = i+1, \dots, t-1$, obtaining $w_{|[t]} \equiv \mathcal{H}_{\sigma|[t]}$. $\square$

For instance, in the alphabet $[5]$, let $I := [5] \setminus \{3\}$, $\sigma = 51324 \in \mathcal{S}_5$, and consider the key $\mathcal{H}_{\sigma(1,0,2,0,1)} = 543215315315$, associated with σ and $(1,0,2,0,1)$, and the word $w_{|I} := 555415211 \equiv \mathcal{H}_{\sigma(1,0,2,0,1)_{|I}}$. Define $w = 553541352131$. Since $w_{|\{2,3\}} = 3323$ is dual Yamanouchi, $w_{|\{3,4\}} = 3433$ is Yamanouchi, $\theta^*_2(w)_{|[2]} = 212121 \equiv \mathcal{H}_{s_2\sigma|[2]}$, $\theta^*_3(w)_{|[3]} = 13211 \equiv \mathcal{H}_{s_3\sigma|[3]}$, and $\theta^*_4(w)_{|[4]} = 44341342131 \equiv \mathcal{H}_{s_4\sigma|[4]}$, by the previous theorem the word w is σ-Yamanouchi, and has $w_{|I}$ as a subword. Consider now the word $w' = 533554135211$, which also has $w_{|I}$ as a subword. Although $w'_{|\{2,3\}} = 3332$ is dual Yamanouchi, $w'_{|\{3,4\}} = 3343$ is Yamanouchi, $\theta^*_2(w)_{|[2]} = 221211 \equiv \mathcal{H}_{s_2\sigma|[2]}$, and $\theta^*_3(w)_{|[3]} = 13211 \equiv \mathcal{H}_{s_3\sigma|[3]}$, w' is not a σ-Yamanouchi word since $\theta^*_4(w')_{|[4]} = 43344134211$ is not in the plactic class of $\mathcal{H}_{s_4\sigma|[4]}$.

Given the word $w = w_1w_2\cdots w_k \in A^*$ and $P = \{i_1, \dots, i_l\} \subseteq [k]$, we define the restriction of w to the set P by $w|P := w_{i_1}\cdots w_{i_l}$. If $w, w' \in A^*$ have lengths k and k', respectively, and $P \subseteq [k+k']$ has cardinality k, we denote by $sh_P(w,w')$ the shuffle of w and w' satisfying $sh_P(w,w')|P = w$ [16]. For instance, with $P = \{1,4\}$, we have $sh_P(81,321) = 83211$. Clearly, different sets P and Q may give the same word $sh_P(w,w') = sh_Q(w,w')$. In the example above, we also have $sh_P(81,321) = sh_Q(81,321)$, with $Q = \{1,5\}$.

Corollary 3.10. *Let $w = sh_P(t^r,u) \equiv \mathcal{H}_\sigma$ with $u \in [t-1]^*$. If $w' = sh_Q(t^r,u)$ with $Q \leq P$, then $w' \equiv \mathcal{H}_\sigma$.*

Proof: By induction on t. If $t = 1,2$, the claim is obvious. Let $t \geq 3$ and write

$$\begin{aligned}\theta^*_{t-1}(w)_{|[t-1]} &= sh_{\widetilde{P}}((t-1)^r, w_{|[t-2]}) \equiv \mathcal{H}_{s_{t-1}\sigma_{|[t-1]}},\\ \theta^*_{t-1}(w')_{|[t-1]} &= sh_{\widetilde{Q}}((t-1)^r, w_{|[t-2]}).\end{aligned}$$

Clearly, we must have $\widetilde{Q} \leq \widetilde{P}$. Then, by induction, it follows that $\theta^*_{t-1}(w')_{|[t-1]} \equiv \mathcal{H}_{s_{t-1}\sigma_{|[t-1]}}$, and by the previous proposition, we find that $w' \equiv \mathcal{H}_\sigma$. $\square$

An operation θ_i may not act on the set $\{\mathcal{H}_\sigma : \sigma \in \mathcal{S}_t\}$. For example, consider $\mathcal{H} = 4321\,21$, and $\theta_2(\mathcal{H}) = \theta_2(4321\,21) = 433121 \notin \{\mathcal{H}_\sigma : \sigma \in \mathcal{S}_t\}$. Although $433121 \equiv 4321\,31 = \mathcal{H}_{s_2}$, we may have even worse $w = 314321 \equiv \mathcal{H}_{s_2s_1}$ and $\theta_2(w) =$

$314221 \equiv 321\,41\,2 \notin \{\mathcal{H}_\sigma : \sigma \in \mathcal{S}_t\}$. Nevertheless, it is possible to give a criterion on θ_i such that if $w \equiv \mathcal{H}_\sigma$ then $\theta_i(w) \equiv \mathcal{H}_{s_i\sigma}$.

The criterion given by the previous proposition can be generalized to the operations θ_i.

Lemma 3.11. *Let $w \in [t]^*$ and $w_{|\{t-1,t\}}$ a Yamanouchi or dual Yamanouchi word. Let $u = \theta_{t-1}(w)$, $z = \theta^*_{t-1}(w)$ and $v = w_{|[t-2]}$. Then*

$$w_{|[t-1]} = sh_P(u_{|\{t-1\}}, v) \ \text{ and } \ w_{|[t-1]} = sh_Q(z_{|\{t-1\}}, v), \quad \text{with } Q \leq P.$$

Proof: It is enough to consider the cases $w_{|\{t-1,t\}} = t\,t-1\,t-1$ and $w_{|\{t-1,t\}} = t\,t\,t-1$. Therefore, if $\theta_{t-1} \neq \theta^*_{t-1}$, we have

$$\begin{aligned}\theta_{t-1}(t\,t-1\,t-1) &= t\,t\,\underline{t-1},\\ \theta^*_{t-1}(t\,t-1\,t-1) &= t\,\underline{t-1}\,t,\end{aligned}$$

and

$$\begin{aligned}\theta_{t-1}(t\,t\,t-1) &= t\,\underline{t-1}\,\underline{t-1},\\ \theta^*_{t-1}(t\,t\,t-1) &= \underline{t-1}\,t\,\underline{t-1}.\end{aligned}$$

□

Theorem 3.12. *Let $w \in [t]^*$ and $\sigma \in \mathcal{S}_t$. Given $i \in [t]$, let $I = [t] \setminus \{i\}$, and suppose that $w_{|I} \equiv \mathcal{H}_{\sigma|I}$. Then, w is a σ-Yamanouchi word if and only if $w_{|\{j,j+1\}}$ is either a Yamanouchi or a dual Yamanouchi word, for $j = i-1, i$, and $\theta_j(w)_{|[j]} \equiv \mathcal{H}_{s_j\sigma_{|[j]}}$, for some operation θ_j, $j = i-1, \dots, t-1$.*

Proof: According to the previous lemma and corollary, if $\theta_j(w)_{|[j]} \equiv \mathcal{H}_{s_j\sigma_{|[j]}}$, then $\theta^*_j(w)_{|[j]} \equiv \mathcal{H}_{s_j\sigma_{|[j]}}$, for $j = i-1, \dots, t-1$. By the previous proposition, it follows that $w \equiv \mathcal{H}_\sigma$. □

Corollary 3.13. *Let $w \in [t]^*$, $t \geq 2$, and $\sigma \in \mathcal{S}_t$. If $w \equiv \mathcal{H}_\sigma$, then $\theta_i(w) \equiv \mathcal{H}_{s_i\sigma}$ if and only if $(\theta_i w)_{|\{i+1,i+2\}}$ is either a Yamanouchi or a dual Yamanouchi word, $\theta_i(w)_{|[i]} \equiv \mathcal{H}_{s_i\sigma_{|[i]}}$, and $\theta_j(\theta_i(w))_{|[j]} \equiv \mathcal{H}_{s_js_i\sigma_{|[j]}}$, for some operation θ_j, $j = i+1, \dots, t-1$.*

Proof: Taking $\theta_j = \theta^*_j$, $j = i+1, \dots, t-1$, we find that the conditions are clearly necessary. Assume now that $w_{|\{i+1,i+2\}}$ is either a Yamanouchi or a dual Yamanouchi word, $\theta_i(w)_{|[i]} \equiv \mathcal{H}_{s_i\sigma_{|[i]}}$, and $\theta_j(\theta_i(w))_{|[j]} \equiv \mathcal{H}_{s_js_i\sigma_{|[j]}}$, for $j = i+1, \dots, t-1$.

Since $\theta_i(w)_{|[i]} \equiv \mathcal{H}_{s_i\sigma_{|[i]}}$, $\theta_i w_{|\{i,i+1\}}$ is either a Yamanouchi or a dual Yamanouchi word, and $\theta_i(\theta_i(w)_{|[i]}) = w_{|[i]} \equiv \mathcal{H}_{\sigma_{|[i]}}$, by the previous theorem we must have

$$\theta_i(w)_{|[i+1]} \equiv \mathcal{H}_{s_i\sigma_{|[i+1]}}.$$

Now, since $(\theta_i w)_{|\{i+1,i+2\}}$ is either a Yamanouchi or a dual Yamanouchi word, and there is an operation θ_{i+1} such that $\theta_{i+1}(\theta_i(w))_{|[i+1]} \equiv \mathcal{H}_{s_{i+1}s_i\sigma_{|[i+1]}}$, again by the previous theorem, we find that

$$\theta_i(w)_{|[i+2]} \equiv \mathcal{H}_{s_i\sigma_{|[i+2]}}.$$

Finally, note that, since $\theta_i(w)_{|\{j,j+1\}} = w_{|\{j,j+1\}}$, the word $\theta_i(w)_{|\{j,j+1\}}$ is either a Yamanouchi or a dual Yamanouchi word, for $j = i+2, \ldots, t-1$. Therefore, repeating the process above for $j = i+2, \ldots, t-1$, we obtain $\theta_i(w)_{|[t]} \equiv \mathcal{H}_{s_i\sigma_{|[t]}}$. $\square$

Remark 3.1. In particular, it follows from the previous theorem and the corresponding statement for dual words, that $\theta_1(w) \equiv \mathcal{H}_{s_1\sigma}$ if and only if $\theta_1(w)_{|[2,t]} \equiv \mathcal{H}_{s_1\sigma_{|[2,t]}}$.

Consider $w = 43143213321$ a σ-Yamanouchi word, where $\sigma = 3124$, and the keys $\mathcal{H}_{\sigma(2,0,1,1)} = 4321\,4321\,31\,3$ and $\mathcal{H}_{s_2\sigma(2,0,1,1)} = 4321\,4321\,21\,2$. We have $\theta_2^*(w) = 42143212321 \equiv \mathcal{H}_{s_2\sigma}$, but $\theta_2(w) = 43142213221$ is not a $s_2\sigma$-Yamanouchi word. Note that although $\theta_2(w)_{|\{3,4\}} = 4343$ is Yamanouchi, $\theta_2(w)_{|[2]} = 1221221$ is not in the plactic class of $\mathcal{H}_{s_2\sigma_{|[2]}}$. The word $\theta_2'(w) = 42143213221$ is a $s_2\sigma$-Yamanouchi word, since $\theta_2'(w)_{|\{3,4\}} = 4433$ is Yamanouchi, $\theta_2'(w)_{|[2]} = 2121221 \equiv \mathcal{H}_{s_2\sigma_{|[2]}}$, and $\theta_3^*(\theta_2'(w))_{|[3]} = \theta_2'(w)_{|[3]} = 213213221 \equiv \mathcal{H}_{s_3s_2\sigma_{|[3]}}$.

4. σ-Yamanouchi words as shuffles of columns of a key

It has been shown in [6] that, when $t \leq 3$, a word in $[t]^*$ of evaluation m is σ-Yamanouchi if and only if it is a shuffle of the columns of $\mathcal{H}_{\sigma(l_t,\ldots,l_1)}$. For $t \geq 4$ this is no longer true in general. For example, consider the Yamanouchi tableau $4321\,21$ and $421321 = sh(4321, 21)$. We have $\theta_3^*\theta_2^*(421321) = \theta_3^*(431321) = 431421 \equiv sh(4321, 41)$, but 431421 is not a shuffle of the columns of s_3s_2-Yamanouchi tableau $4321\,41$. We have already seen that shuffling together the columns of $\mathcal{H}_\sigma$ has the same effect as performing Knuth transformations on the tableau $\mathcal{H}_\sigma$. The reciprocal is not true in general. In what follows, we identify the keys for which Knuth transformations on the key tableau and shuffling together its columns lead to the same words.

Let w be a word obtained by applying an elementary Knuth transformation to $sh(u_1, \ldots, u_k)$, $k \geq 2$, where $u_1, \ldots, u_k$ are columns pairwise comparable in the inclusion order. From Propositions 2.1 and 2.2 and the discussion therein, we may assume that w is obtained by applying an elementary Knuth transformation to a shuffle of two of these columns, u and v say, where the transformation involves three distinct letters $x < y < z$, with zx a factor of v and y a letter of u. Since the letter y is in u, but not in v, and u and v are comparable in the inclusion order, we have $\{v\} \subseteq \{u\}$.

Lemma 4.1. *Let u_1, u_2 be columns in A^*, and $x, z \in A$ such that zu_1u_2x is a column. Then,*

(i) $sh(u_2, u_1u_2x) \in Sh(u_2x, u_1u_2)$.

(ii) $sh(u_1, zu_1u_2) \in Sh(zu_1, u_1u_2)$.

Proof: (i) Write $u_2 = a_1 \cdots a_r$, and $sh(u_2, u_1u_2x) = c_1 \cdots c_l$. For each $j = 1, \ldots, r$, let $p_j := \min\{i : c_i = a_j\}$, and let $p' \in [l]$ such that $c_{p'} = z$. Then, it is clear that

$$sh(u_2, u_1u_2x) = sh_P(u_2x, u_1u_2),$$

where $P := \{p_1, \ldots, p_r, p'\}$.

(ii) Write $u_1 = a_1 \cdots a_r$, and $sh(u_1, zu_1u_2) = c_1 \cdots c_l$. For each $j = 1, \ldots, r$, let $p_j := \max\{i : c_i = a_j\}$, and let $p' \in [l]$ such that $c_{p'} = x$. Then, we have

$$sh(u_1, zu_1u_2) = sh_P(zu_1, u_1u_2),$$

where $P := \{p_1, \dots, p_r, p'\}$. $\square$

Lemma 4.2. *Let u and v be columns in A^* with $u = u_1u_2zyxu_3u_4$ and $v = u_2zxu_3$, where x, y, z are letters. Then, $Sh(u, v)$ is closed under Knuth transformations, that is, $Sh(u, v)$ is the Knuth class of the two-column key $u\,v$.*

Proof: As before, the only cases to consider are the application of the elementary Knuth transformations $zxy \equiv xzy$ and $yzx \equiv yxz$, respectively, to $sh(u, v) = w_1\, \underline{z}\, \underline{x}\, \overline{y}\, w_2$ and $sh(u, v) = w_1\, \overline{y}\, \underline{z}\, \underline{x}\, w_2$, where y is a letter of u, zx a factor of v, $w_1 = sh(u_2, u_1u_2z)$ and $w_2 = sh(xu_3u_4, u_3)$.

In the case of the elementary Knuth transformation $zxy \equiv xzy$, we have

$$sh(u, v) = w_1\underline{z}\,\underline{x}\,\overline{y}w_2 \equiv w_1\underline{x}\,\underline{z}\,\overline{y}w_2, \tag{4}$$

where $w_1 = sh(u_2, u_1u_2z)$, $w_2 = sh(xu_3u_4, u_3)$. By Lemma 4.1 (i), we must have $w_1 = sh_P(u_2z, u_1u_2)$, for some set P. Thus, we have for the right-hand side of (4) that

$$w_1xzyw_2 = sh(u_2z\underline{x}u_3, u_1u_2\underline{z}\,\overline{y}\,xu_3u_4) \in Sh(u, v).$$

The case of the elementary Knuth transformation $yzx \equiv yxz$ is analogous to the previous one. $\square$

As an illustration of the lemma above, consider, in the alphabet $[6]$, the columns $u = 654321$ and $v = 542$, and let $sh(u, v) = \underline{5}\; 6\; 5\; 4\; \underline{4}\; \underline{2}\; 3\; 2\; 1$, where the underlined letters define the word v, and the remaining letters define u. Applying the Knuth transformation $423 \equiv 243$, to $sh(u, v)$, we get the word $\underline{5}\; 6\; 5\; \underline{4}\; \underline{2}\; 4\; 3\; 2\; 1$, which is also a shuffle of u and v.

Next, we identify keys for which Knuth transformations on the key tableau are equivalent to shuffling together its columns. First, however, we need the following definition.

Definition 4.1. Let $w \in A^*$ be a column. We say that w has a *gap* of size $q \geq 0$, with respect to the alphabet A, if there exists a factor $j + k\, j$, $k \geq 1$, of w such that $|[j + 1, j + k - 1] \cap A| = q$.

For instance, the column 41 has a gap of size 2 with respect to the alphabet $[5]$, but has only a gap of size 1 with respect to the alphabet $\{1, 2, 4, 5\}$. The column 531 has two gaps of size 1 with respect to the alphabet $[6]$, but has no gap if we consider the alphabet to be $\{1, 3, 5, 6\}$. In this case, 531 is an interval of the ordered alphabet $\{1, 3, 5, 6\}$.

Theorem 4.3. *Let $\mathcal{H}$ be a key with first column A. Then, the Knuth class of $\mathcal{H}$ is equal to the set of all shuffles of its columns if and only if each of its column is either an interval of A or is obtained from an interval of A by removing a single letter.*

Proof: The *only if* part. Assume that each column of $\mathcal{H}$ is either an interval of A or is obtained from an interval of A by removing a single letter, and let $w \equiv \mathcal{H}$. We may assume, without loss of generality, that w is obtained by performing a single elementary Knuth transformation $xzy \equiv zxy$, or $yzx \equiv yxz$, with $x < y < z$, on a shuffle of two columns of $\mathcal{H}$, say u and v, such that zx is a factor of v and y is a letter of u. Since $\{v\} \subseteq \{u\}$, we must have $u = u_1u_2zyxu_3u_4$, and $v = u_2zxu_3$, for some columns u_i, $i = 1, 2, 3, 4$. Since, by Lemma 4.2, the set $Sh(u, v)$ is closed under Knuth transformations, we find that w is still a shuffle of the columns of $\mathcal{H}$.

The *if part*. Suppose that the key $\mathcal{H} = Av_2 \cdots v_k$ is such that the column v_i has (i) a gap of size at least two or (ii) two or more gaps, both with respect to the ambient alphabet A, for some $2 \le i \le k$. Without loss of generality, we may assume that in these cases, v_i has one of the following forms:

(i) $v_i = u_2 d a u_3$, where $u_2 = s(s-1)\cdots d+1$, $u_3 = (a-1)\cdots(r+1)r$, with $s, r, d, a \in [t]$ such that $d - a = 3$;

(ii) $v_i = u_2 f d u_3 c a u_4$, where $u_2 = s(s-1)\cdots(f+1)$, $u_3 = (d-1)\cdots(c+1)$, and $u_4 = (a-1)\cdots(r+1)r$, with $f - d, c - a = 2$.

Let $w_1 = v_2 \cdots v_{i-1} v_{i+1} \cdots v_k$. In case (i), write $A = u_1 u_2 dcbau_3 u_4$ and consider the word

$$\mathcal{H} \equiv w_1 \underline{u}_2\, u_1\, u_2\, d\, c\, (\underline{d}\, \underline{a}\, b)\, a\, u_3\, u_4\, \underline{u}_3, \tag{5}$$

where the underlined letters define column v_i and the remaining define column A. Applying the transformation $dab \equiv adb$ on (5), we get

$$(5) \equiv w_1 \overline{u}_2\, u_1\, u_2\, d\, c\, (\overline{a}\, \overline{d}\, b)\, a\, u_3\, u_4\, \overline{u}_3. \tag{6}$$

Clearly, the right-hand side of (6) is in the plactic class of $\mathcal{H}$, but is not a shuffle of its columns.

In case (ii), write $A = u_1 u_2 fedu_3 cbau_4 u_5$, and consider the following word

$$\mathcal{H} \equiv w_1 u_1\, u_2\, f\, e\, d\, u_3\, c\, \underline{u}_2\, \underline{f}\, \underline{d}\, \underline{u}_3 (\underline{c}\, \underline{a}\, b)\, \underline{u}_4\, a\, u_4\, u_5, \tag{7}$$

where the underlined letters define column v_i. Applying the transformation $cab \equiv acb$ on (7), we get

$$(7) \equiv w_1 u_1\, u_2\, f\, e\, d\, u_3\, c\, \overline{u}_2\, \overline{f}\, \overline{d}\, \overline{u}_3 (\overline{a}\, \overline{c}\, b)\, \overline{u}_4\, a\, u_4\, u_5. \tag{8}$$

As in case (i), it is clear that the right-hand side of (8) is in the plactic classe of $\mathcal{H}$, but is not a shuffle of its columns. $\square$

Corollary 4.4. [6] *If $\sigma = id$ or the reverse permutation $t\, t-1 \cdots 2\, 1$, a word over the alphabet $[t]$ is a σ-Yamanouchi word if and only if it is a shuffle of the columns of $\mathcal{H}_\sigma$.*

Corollary 4.5. (*a*) *If $\mathcal{H}$ is a key over a three-letter alphabet, then the Knuth class of $\mathcal{H}$ equals the set of shuffles of its columns.*

(*b*) [6] *If $\sigma \in \mathcal{S}_t$, $t = 2, 3$, a word over the alphabet $[t]$ is a σ-Yamanouchi word if and only if it is a shuffle of the columns of $\mathcal{H}_\sigma$.*

As keys are characterized by their evaluation, we may consider the planar representation of the evaluation to check whether the condition of the previous theorem is satisfied. Let $\mathcal{H}$ be a key of evaluation $(m_1, \ldots, m_t)$ and consider the planar representation obtained by drawing m_i bullets in row i, for $i = 1, \ldots, t$. After deleting empty rows we are in the ambient alphabet of the first column of $\mathcal{H}$ and the condition stated in Theorem 4.3 says that the plactic class of $\mathcal{H}$ is the set of all shuffles of its columns if and only if each column has, at most, a single gap of size 1 with respect to the ambient alphabet given by the first column. For instance, the planar representation of the evaluation $(4,0,1,2,6,3,5)$ of the key $\mathcal{H}(\sigma,(0,1,\ldots,1))$, with $\sigma = 5716432 \in \mathcal{S}_6$, is

$$\begin{array}{c|cccccc}
7 & \bullet & \bullet & \bullet & \bullet & \bullet & \\
6 & \bullet & \bullet & \bullet & & & \\
5 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
4 & \bullet & \bullet & & & & \\
3 & \bullet & & & & & \\
1 & \bullet & \bullet & \bullet & \bullet & & \\
\hline
 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}.$$

The conjugate shape of $\mathcal{H}(\sigma,(0,1,\ldots,1))$ is $(7^0,6,5,4,3,2,1)$, the number of bullets in each column of the planar representation of the evaluation. The third column of this representation has a gap of size 2, and the fourth column has two gaps with respect to the first column. Thus, the plactic class of $\mathcal{H}_{\sigma,(0,1,\ldots,1)}$ is not the set of all shuffles of its columns. On the other hand, the plactic class of $\mathcal{H}_{\sigma,(0,1,1,0,0,1,1)}$ is the shuffle of its columns, since the columns of the planar representation of the evaluation $(2,0,1,2,4,2,3)$,

$$\begin{array}{c|cccc}
7 & \bullet & \bullet & \bullet & \\
6 & \bullet & \bullet & & \\
5 & \bullet & \bullet & \bullet & \bullet \\
4 & \bullet & \bullet & & \\
3 & \bullet & & & \\
1 & \bullet & \bullet & & \\
\hline
 & 2 & 3 & 6 & 7
\end{array}$$

has, at most, one gap of size 1. Each column is either an interval of $A = \{1,3,4,5,6,7\}$ or is obtained from an interval of A by removing one letter.

As we have seen in the example above, in $\mathcal{S}_t$, $t > 3$, there are permutations for which the associated keys do not satisfy the conditions of Theorem 4.3. For $s_3s_2 = 1423 \in \mathcal{S}_4$ we have $\mathcal{H}_{s_3s_2} = (4321)^{l_4}(421)^{l_3}(41)^{l_2}1^{l_1}$ and column 41 has a gap of length 2 with respect to the column 4321. The word $w = 431421$ is a s_3s_2-Yamanouchi word and is not a shuffle of the columns of the tableau $\mathcal{H}_{s_3s_2} = 4321\,41$. It is easy to check that in $\mathcal{S}_4$, the only permutations for which there are associated keys that fail to satisfy the conditions of Theorem 4.3 are 1423, 1432, 4123 and 4132.

Corollary 4.6. (a) *Let $\mathcal{H}$ be a key with first column $A = \{a,b,c,d\}$. Then the plactic class of $\mathcal{H}$ is the shuffle of its columns if and only if $\mathcal{H}$ does not contain the column $d\,a$.*

(b) *Let $\sigma \in \mathcal{S}_4$, and let (l_4,l_3,l_2,l_1) be a sequence of nonnegative integers with $l_4, l_2 > 0$. The plactic class of $\mathcal{H}_{\sigma(l_4,l_3,l_2,l_1)}$ is $Sh((r_4)^{l_4},\ (r_{\sigma,3})^{l_3},\ (r_{\sigma,2})^{l_2}, (r_{\sigma,1})^{l_1})$ if and only if σ is in $\mathcal{S}_4 \setminus \{1423,\ 1432,\ 4123, 4132\}$.*

In the Appendix, the permutations of $\mathcal{S}_5$ and $\mathcal{S}_6$, such that the set $Sh((r_t)^{l_t}$, $(r_{\sigma,t-1})^{l_{t-1}}, \ldots, (r_{\sigma,2})^{l_2}$, $(r_{\sigma,1})^{l_1})$, with $l_i > 0$, $i = 1,\ldots,t$, $t = 5,6$, is not the whole plactic class of $\mathcal{H}_\sigma$, are listed.

For $t > 3$ the columns of $\mathcal{H}_\sigma$ are not enough to characterize the σ-Yamanouchi words in terms of shuffling them together. In the case of $\mathcal{S}_4$, the next theorem shows that it is necessary and sufficient to include the word 431421 in the set of distinct columns of $\mathcal{H}_{\sigma(l_4,\dots,l_1)}$, $l_4, l_2 > 0$, to characterize by shuffle operations the σ-Yamanouchi words over the alphabet $[4]$, for any $\sigma \in \{1423,\ 1432,\ 4123, 4132\}$.

Theorem 4.7. *Let $\sigma \in \mathcal{S}_4$, and let $(l_4,\dots,l_1)$ be a sequence of nonnegative integers with $l_4 > 0$. Then, the plactic class of $\mathcal{H}_{\sigma(l_4,\dots,l_1)}$ is*

$$Sh((r_4)^{l_4}, (r_{\sigma,3})^{l_3}, (r_{\sigma,2})^{l_2}, (r_{\sigma,1})^{l_1}), \quad \textit{if } l_2 = 0 \textit{ or } \sigma \in \mathcal{S}_4 \setminus \{1423, 1432, 4123, 4132\}, \tag{9}$$

and, otherwise,

$$Sh((r_4)^{l_4}, (r_{\sigma,3})^{l_3}, (r_{\sigma,2})^{l_2}, (r_{\sigma,1})^{l_1}) \cup Sh((431421)^{n_5}, (r_4)^{n_4}, (r_{\sigma,3})^{n_3}, (r_{\sigma,2})^{n_2}, (r_{\sigma,1})^{n_1}), \tag{10}$$

where $\sum_{j=1}^{4} l_j |r_{\sigma,j}|_i = n_5 |431421|_i + \sum_{j=1}^{4} n_j |r_{\sigma,j}|_i$, for $i = 1,\ 2,\ 3, 4$.

Proof: Assume $l_2 > 0$, otherwise the conditions of Theorem 4.3 hold. When $\sigma \in \mathcal{S}_4 \setminus \{1423, 1432, 4123, 4132\}$, we have already proved in Corollary 4.6 that the plactic class of $\mathcal{H}_{\sigma(l_4,\dots,l_1)}$ is the set

$$Sh(\mathcal{H}_{\sigma(l_4,\dots,l_1)}) = Sh((r_4)^{l_4}, (r_{\sigma,3})^{l_3}, (r_{\sigma,2})^{l_2}, (r_{\sigma,1})^{l_1}).$$

Assume now that $\sigma \in \{1423, 1432, 4123, 4132\}$, noticing that $r_{\sigma,2} = 41$. As discussed before, the only case to consider, when analyzing the effect of a single Knuth transformation on a shuffle of the columns of $\mathcal{H}_{\sigma(l_t,\dots,l_1)}$, is when the Knuth transformation $zxy \equiv xzy$ or $yzx \equiv yxz$ involves three distinct letters, $z > y > x$, with zx a factor of a column, and y a letter of another column of $\mathcal{H}_\sigma$. Thus, using Lemma 4.2, we find that any word obtained by application of a single Knuth transformation on a shuffle of two columns of $\mathcal{H}_{\sigma(l_4,\dots,l_1)}$, other than $sh(r_4, r_{\sigma,2}) = 43\underline{4}\underline{1}21$, is still a shuffle of the columns of $\mathcal{H}_{\sigma(l_4,\dots,l_1)}$.

In the case of the shuffle $sh(r_4, r_{\sigma,2}) = 43\underline{41}21$, the application of the transformation $341 \equiv 314$ or $412 \equiv 142$ gives the word 431421, which is not a shuffle of the columns of $\mathcal{H}_{\sigma(l_4,\dots,l_1)}$.

Now, an exhaustive analysis of the effect of a single Knuth transformation on all possible shuffles between any two words from the set $\{431421, r_4, r_{\sigma,3}, r_{\sigma,2}, r_{\sigma,1}\}$, shows that the resulting word is still a shuffle of two, or more, words of this set.

Thus, if $w \equiv \mathcal{H}_{\sigma(l_4,\dots,l_1)}$, w is obtained by a finite number of Knuth transformations on $\mathcal{H}_{\sigma(l_4,\dots,l_1)}$. Hence, it must be a shuffle of the words $431421, r_4, r_{\sigma,3}, r_{\sigma,2}$, and $r_{\sigma,1}$, with appropriate multiplicities. $\square$

5. Matrix realizations of pairs of tableaux

Let $\mathcal{R}_p$ be a local principal ideal domain with maximal ideal (p). The matrices under consideration have entries in $\mathcal{R}_p$. Let $\mathcal{U}_n$ be the group of $n \times n$ unimodular matrices over $\mathcal{R}_p$. Given $n \times n$ matrices A and B, we say that B is *left equivalent* to A (written $B \sim_L A$) if $B = UA$ for some unimodular matrix U; B is *right equivalent* to A (written $B \sim_R A$) if $B = AV$ for some unimodular matrix V; and B is *equivalent* to A (written $B \sim A$) if $B = UAV$ for some unimodular matrices U, V. The relations $\sim_L$, $\sim_R$ and $\sim$ are equivalence relations on the set of all $n \times n$ matrices over $\mathcal{R}_p$.

Let A be an $n\times n$ non-singular matrix. By the Smith normal form theorem [7, 26], there exist nonnegative integers $\lambda_1, ..., \lambda_n$ with $\lambda_1 \geq ... \geq \lambda_n$ such that A is equivalent to the diagonal matrix

$$\mathrm{diag}(p^{\lambda_1}, ..., p^{\lambda_n}).$$

The sequence $\lambda = (\lambda_1, ..., \lambda_n)$ of exponents of the p-powers in the Smith normal form of A is a partition of length $\leq n$ uniquely determined by the matrix A. The partition λ is a full invariant of the equivalence class containing A and we call λ the *invariant partition* of A. More generally, if we are given a sequence $(f_1, ..., f_n)$ of nonnegative integers, the following notation for p-powered diagonal matrices will be used:

$$\mathrm{diag}_p(f_1, ..., f_n) := \mathrm{diag}(p^{f_1}, ..., p^{f_n}).$$

Given $J \subseteq [n]$, we write $\chi^J = (f_1, ..., f_n)$ with $f_i = 1$ if $i \in J$ and 0 otherwise, thus, we put $D_J := \mathrm{diag}_p(\chi^J)$. Given a partition λ of length $\leq n$, we write $\Delta_\lambda = \mathrm{diag}_p(\lambda)$. If $\lambda = (0)$, $\Delta_{(0)} = I_n$ the identity matrix of order n.

Let $\sigma \in \mathcal{S}_t$, $t \geq 1$, and let m be a partition of length t such that $\sigma m = (m_1, \ldots, m_t)$. In what follows, $\mathcal{T}$ will denote a skew-tableau of evaluation $(m_1, ..., m_t)$ and shape λ/μ where the length of λ is $\leq n$. Next we define the matrix realization of a pair of Young tableaux $(\mathcal{T}, \mathcal{F})$, with $\mathcal{F}$ a tableau of evaluation $(m_1, ..., m_t)$ and shape ν, following [5, 6, 4]. (Note that in this paper the tableaux are strictly increasing up along columns.)

Definition 5.1. Let $\mathcal{T} = (\lambda^0, \lambda^1, ..., \lambda^t)$ and $\mathcal{F} = (0, \mu^1, ..., \mu^t)$ be tableaux, both of evaluation $(m_1, ..., m_t)$. We say that a sequence of $n \times n$ non-singular matrices $A_0, B_1, ..., B_t$ is a *matrix realization* of the pair of tableaux $(\mathcal{T}, \mathcal{F})$ (or realizes $(\mathcal{T}, \mathcal{F})$) if:

I. For each $r \in \{1, ..., t\}$, the matrix B_r has invariant partition $(1^{m_r}, 0^{n-m_r})$.
II. For each $r \in \{0, 1, ..., t\}$, the matrix $A_r := A_0B_1...B_r$ has invariant partition the conjugate of λ^r.
III. For each $r \in \{1, ..., t\}$, the matrix $B_1...B_r$ has invariant partition the conjugate of μ^r.

$(\mathcal{T}, \mathcal{F})$ is called an *admissible pair* of tableaux.

Conditions (I) and (II) alone are trivially feasible. But, in conjunction with condition (III), they impose a non-trivial restriction on the concept of matrix realization. Here, we restrict ourselves to pairs $(\mathcal{T}, \mathcal{H}_{\sigma(l_t,...,l_1)})$. The next theorem, proved in [6, 4], shows that, without loss of generality, we may consider matrix realizations of $(\mathcal{T}, \mathcal{H}_{\sigma(l_t,...,l_1)})$ with a particular simple form.

Theorem 5.1. *The following conditions are equivalent:*

(a) $(\mathcal{T}, \mathcal{H}_{\sigma(l_t,...,l_1)})$ *is an admissible pair.*
(b) *There exists* $U \in \mathcal{U}_n$ *such that* $\Delta_\lambda U, D_{[m_1]}, ..., D_{[m_t]}$ *realizes* $(\mathcal{T}, \mathcal{H}_{\sigma(l_t,...,l_1)})$.

The characterization of σ-Yamanouchi words as shuffles of the columns $\mathcal{H}_\sigma$ has been used to determine necessary and sufficient conditions for the admissibility of a pair of Young tableaux $(\mathcal{T}, \mathcal{H}_{\sigma(l_t,...,l_1)})$, when σ is the identity or the reverse permutation in $\mathcal{S}_t$, $t \geq 1$ [4, 5], or any permutation in $\mathcal{S}_3$ [6]. In these cases, the pair $(\mathcal{T}, \mathcal{H}_{\sigma(l_t,...,l_1)})$ is admissible only if the word of $\mathcal{T}$ is in the Knuth class of $\mathcal{H}_{\sigma(l_t,...,l_1)}$. The next theorem extends the necessary condition of this result to any $\sigma \in \mathcal{S}_t$, $t \geq 1$, and verifies the recursive criterion for σ-Yamanouchi words given in Theorem 3.12. The proof of this theorem needs the following proposition, proved in [6].

Proposition 5.2. [6] *Let (m_1, m_2) be a partition. Let w and w' be the words of the tableaux realized by the sequences $\Delta_\lambda U, D_{[m_1]}, D_{[m_2]}$ and $\Delta_\lambda U, D_{[m_2]}, D_{[m_1]}$, respectively. Then, there exists an operation θ_1 such that $w' = \theta_1(w)$.*

The following example shows that the operation θ_1 in the previous proposition is not necessarily θ_1^*.

Example 5.1. Let $U = P_{4321}T_{14}(p)$, where P_{4321} is the permutation matrix associated with $4321 \in \mathcal{S}_4$ and $T_{14}(p)$ is the elementary matrix obtained from the identity by placing the prime p in position $(1,4)$. It is a simple task to check that, with $\lambda = (2,1)$ and $U = P_{4321}T_{14}(p)$, the sequences $\Delta_\lambda U, D_{[3]}, D_{[2]}$ and $\Delta_\lambda U, D_{[2]}, D_{[3]}$ are matrix realizations for the pairs $(\mathcal{T}, \mathcal{H}(id, (2,1)))$ and $(\mathcal{T}', \mathcal{H}(s_1, (2,1)))$, where $\mathcal{T} = \begin{array}{cccc} 2 & & & \\ \bullet & 1 & 2 & \\ \bullet & \bullet & 1 & 1 \end{array}$ and $\mathcal{T}' = \begin{array}{cccc} 2 & & & \\ \bullet & 2 & 2 & \\ \bullet & \bullet & 1 & 1 \end{array}$. The words $w(\mathcal{T}) = 21211$ and $w(\mathcal{T}') = 22211$ satisfy $\theta_1(w(\mathcal{T})) = w(\mathcal{T}')$, where θ_1 is the operation based on a parentheses matching defined as (21(21)1).

Thus, the matrix setting generates parentheses matching operations, different from the standard ones defined by Lascoux and Schützenberger. However, if we choose $U' = P_{3241}T_{24}(p)$, the sequences $\Delta_\lambda U', D_{[3]}, D_{[2]}$ and $\Delta_\lambda U', D_{[2]}, D_{[3]}$ are matrix realizations for the pairs $(\mathcal{T}, \mathcal{H}(id, (2,1)))$ and $(\mathcal{T}'', \mathcal{H}(s_1, (2,1)))$, where $\mathcal{T}'' = \begin{array}{cccc} 2 & & & \\ \bullet & 1 & 2 & \\ \bullet & \bullet & 1 & 2 \end{array}$.
In this case, we have $\theta_1^*(w(\mathcal{T})) = w(\mathcal{T}'') = 21212 \neq \theta_1(\mathcal{T}) = 22211$.

Theorem 5.3. *Let $\sigma \in \mathcal{S}_t$, and let $(l_t, \cdots, l_1)$ be a sequence of nonnegative integers. The pair $(\mathcal{T}, \mathcal{H}_{\sigma(l_t,\ldots,l_1)})$ is admissible only if $w(\mathcal{T}) \equiv \mathcal{H}_{\sigma(l_t,\ldots,l_1)}$.*

Proof: By induction on $t \geq 1$. When $t = 1$ there is nothing to prove, and the case $t = 2$ has been proved in [5, 4]. Assume the claim for $t - 1 \geq 2$. Thanks to Theorem 5.1 we may assume the existence of an unimodular matrix $U \in \mathcal{U}_n$ such that $\Delta_\lambda, UD_{[m_1]}, \ldots, D_{[m_t]}$ realizes $(\mathcal{T}, \mathcal{H}_{\sigma(l_t,\ldots,l_1)})$. Put $w := w(\mathcal{T})$. By the inductive step, the word $w_{|[t-1]}$ of the tableau realized by the sequence

$$\Delta_\lambda, UD_{[m_1]}, \ldots, D_{[m_{t-1}]}$$

satisfies $P(w_{|[t-1]}) = \mathcal{H}_{\sigma|[t-1]}$.

We consider the case $m_{t-1} \geq m_t$, the other one is similar. There exists an unimodular matrix $U' \in \mathcal{U}_n$ such that

$$\Delta_\lambda UD_{[m_1]} \cdots D_{[m_{t-1}]}D_{[m_t]} \sim_L \Delta' U' D_{[m_{t-1}]}D_{[m_t]},$$

where $\Delta' = \operatorname{diag}_p(\lambda + \chi^{J_1} + \cdots + \chi^{J_{t-2}})$. Since $m_{t-1} \geq m_t$, by the case $t = 2$, the sequence $\Delta' U', D_{[m_{t-1}]}, D_{[m_t]}$ realizes a tableau whose word $w_{|\{t-1,t\}}$ is a Yamanouchi word. Finally, consider the sequence

$$\Delta_\lambda U, D_{[m_1]}, \cdots, D_{[m_{t-2}]}, D_{[m_t]},$$

and let w' be the word of the corresponding tableau. According to the previous proposition, we must have $w' = (\theta_{t-1}(w))_{|[t-1]}$, for some operation θ_{t-1}, and by the inductive step, $P(w') = \mathcal{H}_{s_{t-1}\sigma_{|[t-1]}}$. By Theorem 3.12, we find that w is a σ-Yamanouchi word. □

6. Appendix

Below we list the permutations σ in $\mathcal{S}_5$ and $\mathcal{S}_6$ for which the set $Sh((r_t)^{l_t}, (r_{\sigma,t-1})^{l_{t-1}}, \ldots, (r_{\sigma,2})^{l_2}, (r_{\sigma,1})^{l_1})$, with $l_i > 0$, $i = 1, \ldots, t$, $t = 5, 6$, is not the whole plactic class of $\mathcal{H}_\sigma$.

$\mathcal{S}_5$		
$w4w'$,	$w \in \mathcal{S}_{\{2,5\}}$,	$w' \in \mathcal{S}_{\{1,3\}}$;
$w3w'$,	$w \in \mathcal{S}_{\{2,5\}}$,	$w' \in \mathcal{S}_{\{1,4\}}$;
ww',	$w \in \mathcal{S}_{\{1,4,5\}}$,	$w' \in \mathcal{S}_{\{2,3\}}$;
ww',	$w \in \mathcal{S}_{\{1,3,5\}}$,	$w' \in \mathcal{S}_{\{2,4\}}$;
$w3w'$,	$w \in \mathcal{S}_{\{1,4\}}$,	$w' \in \mathcal{S}_{\{2,5\}}$;
$w2w'$,	$w \in \mathcal{S}_{\{1,4\}}$,	$w' \in \mathcal{S}_{\{3,5\}}$;
ww',	$w \in \mathcal{S}_{\{1,2,5\}}$,	$w' \in \mathcal{S}_{\{3,4\}}$.

$\mathcal{S}_6$			
$ww'w''$,	$w \in \mathcal{S}_{\{3,6\}}$,	$w' \in \mathcal{S}_{\{4,5\}}$,	$w'' \in \mathcal{S}_{\{1,2\}}$;
$ww'w''$,	$w \in \mathcal{S}_{\{2,5\}}$,	$w' \in \mathcal{S}_{\{3,4\}}$,	$w'' \in \mathcal{S}_{\{1,6\}}$;
$ww'w''$,	$w \in \mathcal{S}_{\{1,4\}}$,	$w' \in \mathcal{S}_{\{2,3\}}$,	$w'' \in \mathcal{S}_{\{5,6\}}$;
$w4w'$,	$w \in \mathcal{S}_{\{2,5,6\}}$,	$w' \in \mathcal{S}_{\{1,3\}}$;	
$w5w'$,	$w \in \mathcal{S}_{\{2,4,6\}}$,	$w' \in \mathcal{S}_{\{1,3\}}$;	
$w46w'$,	$w \in \mathcal{S}_{\{2,5\}}$,	$w' \in \mathcal{S}_{\{1,3\}}$;	
$w52w'$,	$w \in \mathcal{S}_{\{3,6\}}$,	$w' \in \mathcal{S}_{\{1,4\}}$;	
$w3w'$,	$w \in \mathcal{S}_{\{2,5,6\}}$,	$w' \in \mathcal{S}_{\{1,4\}}$;	
$w5w'$,	$w \in \mathcal{S}_{\{2,3,6\}}$,	$w' \in \mathcal{S}_{\{1,4\}}$;	
$w36w'$,	$w \in \mathcal{S}_{\{2,5\}}$,	$w' \in \mathcal{S}_{\{1,4\}}$;	
$w42w'$,	$w \in \mathcal{S}_{\{3,6\}}$,	$w' \in \mathcal{S}_{\{1,5\}}$;	
$w3w'$,	$w \in \mathcal{S}_{\{2,4,6\}}$,	$w' \in \mathcal{S}_{\{1,5\}}$;	
$w4w'$,	$w \in \mathcal{S}_{\{2,3,6\}}$,	$w' \in \mathcal{S}_{\{1,5\}}$;	
ww',	$w \in \mathcal{S}_{\{1,4,5,6\}}$,	$w' \in \mathcal{S}_{\{2,3\}}$;	
ww',	$w \in \mathcal{S}_{\{1,3,5,6\}}$,	$w' \in \mathcal{S}_{\{2,4\}}$;	
ww',	$w \in \mathcal{S}_{\{1,3,4,6\}}$,	$w' \in \mathcal{S}_{\{2,5\}}$;	
$w3w'$,	$w \in \mathcal{S}_{\{1,4,5\}}$,	$w' \in \mathcal{S}_{\{2,6\}}$;	
$w4w'$,	$w \in \mathcal{S}_{\{1,3,5\}}$,	$w' \in \mathcal{S}_{\{2,6\}}$;	
$w35w'$,	$w \in \mathcal{S}_{\{1,4\}}$,	$w' \in \mathcal{S}_{\{2,6\}}$;	
ww',	$w \in \mathcal{S}_{\{1,2,5,6\}}$,	$w' \in \mathcal{S}_{\{3,4\}}$;	
ww',	$w \in \mathcal{S}_{\{1,2,4,6\}}$,	$w' \in \mathcal{S}_{\{3,5\}}$;	
$w41w'$,	$w \in \mathcal{S}_{\{2,5\}}$,	$w' \in \mathcal{S}_{\{3,6\}}$;	
$w2w'$,	$w \in \mathcal{S}_{\{1,4,5\}}$,	$w' \in \mathcal{S}_{\{3,6\}}$;	

$$
\begin{array}{lll}
w4w', & w \in \mathcal{S}_{\{1,2,5\}}, & w' \in \mathcal{S}_{\{3,6\}}; \\
w25w', & w \in \mathcal{S}_{\{1,4\}}, & w' \in \mathcal{S}_{\{3,6\}}; \\
ww', & w \in \mathcal{S}_{\{1,2,3,6\}}, & w' \in \mathcal{S}_{\{4,5\}}; \\
w31w', & w \in \mathcal{S}_{\{2,5\}}, & w' \in \mathcal{S}_{\{4,6\}}; \\
w2w', & w \in \mathcal{S}_{\{1,3,5\}}, & w' \in \mathcal{S}_{\{4,6\}}; \\
w3w', & w \in \mathcal{S}_{\{1,2,5\}}, & w' \in \mathcal{S}_{\{4,6\}}.
\end{array}
$$

There are a total of 52 permutations in $\mathcal{S}_5$ and 488 permutations in $\mathcal{S}_6$ that fail to satisfy the conditions of Theorem 4.3.

Acknowledgments. We thank the referee for helpful comments and useful suggestions. In particular, we are grateful for pointing out to us Greene's theorem in [17].

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