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MAGIC SQUARES, ROOK POLYNOMIALS AND PERMUTATIONS

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Abstract. We study in this paper the set of magic squares and their relation with some restricted permutations.

Résumé. Nous étudions dans cet article l'ensemble des carrés magiques et leur relation avec des permutations spéciales.

1. Introduction

The *oldest magic square* $\begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix}$ first appeared in ancient Chinese literature under the name **Lo Shu** two thousands years BC. The reader is likely to have encountered such objects, which following Ehrhart [2] are referred to as *historical magic squares*. These are square matrices of order n whose entries are nonnegative integers $\{1, \ldots, n^2\}$ and whose rows and columns and the main diagonals sum up to the same number, which is called *the magic sum.* MacMahon [7] and Stanley [10] defined magic squares in modern combinatorics as square matrices of order n whose entries are nonnegative integers and whose rows and columns sum up to the same number, which is called *the line sum.* In this paper we will study the magic squares following the next definition.

Definition 1.1. *A magic square is a square matrix of order n, whose entries are nonnegative integers and the sum of each row, each column and both the main diagonals adds up to the same number, which is called the magic sum.*

Example 1.2. $\begin{pmatrix} 3 & 6 & 0 \\ 0 & 3 & 6 \\ 6 & 0 & 3 \end{pmatrix}$ *is a magic square of order* 3 *and whose magic sum is equal to* 9.

MacMahon [7] has already enumerated the number of all magic squares of order 3 in 1915, and it was not until 2002 that Ahmed et al. [1] could find the number of magic squares of order 4 for a given magic sum. The number of magic squares of order $n \geq 5$ with magic sum $s \geq 2$ is a challenge! We will introduce notions on magic permutations, which are generators of all magic squares. In 1879, Hertzsprung [5] defined the number of magic permutations as well as the number of permutations without fixed points and without reflected

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points, well before the development of rook theory ([3], [4], [8], [11]) as a method for enumeration of permutations with restricted positions. Riordan [8] in 1958 and Simpson [9] in 1995 recalled these recurrence relations. We will use weighted rook polynomials to generalize the results on generalized restricted permutations and we will give an unexpected relation which relates derangements and restricted permutations. We will denote by MS_n the set of magic squares of order n.

2. Magic permutations

We will recall the following definitions:

Definition 2.1. *A permutation σ of order n is a bijection over n objects.*

We will denote by $[n]$ the set $\{1, \ldots, n\}$, and by $\mathfrak{S}_n$ the set of all permutations over $[n]$.

Definition 2.2. *We say that an integer i is a fixed point for the permutation σ if $\sigma(i) = i$.*

Definition 2.3. *We say that an integer i is a reflected point for the permutation σ if $\sigma(i) = n - i + 1$.*

We will denote by $\mathrm{Fix}(\sigma)$ the set of the fixed points of the permutation σ, and by $\mathrm{Rfl}(\sigma)$ the set of its reflected points.

Definition 2.4. *We say that an integer i is a pivot point if i is a fixed reflected point.*

Remark 2.5. *If n is even, no permutation of length n has a pivot point.*

Remark 2.6. *The only pivot point of a permutation of length n is the integer $\dfrac{n+1}{2}$ if n is odd.*

Example 2.7. *For the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 6 & 5 & 4 & 2 & 7 & 3 \end{pmatrix}$, we have*

$$\mathrm{Fix}(\sigma) = \{1,\, 4\} \ \textit{and}\ \mathrm{Rfl}(\sigma) = \{2,\, 3,\, 4\}\,.$$

We will write a permutation σ of length n as a square matrix of order n such that the i-th column is represented by the vector column $e_{\sigma(i)}$ where

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix} \text{ and } e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Example 2.8. *If we consider the permutation in Example 2.7, we have:*

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Remark 2.9. *The reflected points and the fixed points of a permutation σ are shown in the matrix representation of the permutation σ as the occurrence of the integer* 1 *on the main diagonals.*

Definition 2.10. *A magic permutation is a permutation σ whose matrix representation is a magic square of magic sum* 1*.*

Proposition 2.11. *A permutation σ is magic if σ has one fixed point and one reflected point.*

Example 2.12. *The following permutations σ_1 and σ_2 of length* 9 *are magic:*

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 2 & 5 & 6 & 8 & 1 & 4 & 9 & 7 \end{pmatrix}$$

and

$$\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 9 & 7 & 5 & 1 & 8 & 3 & 6 \end{pmatrix}$$

Proposition 2.13. *There does not exist a magic permutation of length n for $n = 2,\ 3$.*

Proposition 2.14. *If a permutation σ is magic, then:*

(1) *σ^{-1} is magic,*
(2) *the reflected permutation σ' of the permutation σ, defined by $\sigma'(i) = n - \sigma(i) + 1$, is magic.*

Proof. Notice that a fixed point of the permutation σ remains a fixed point for σ^{-1} and becomes a reflected point for the reflected permutation σ' and vice-versa. Notice also that if the integer i is a reflected point for the permutation σ, then the integer i is a fixed point for the reflected permutation σ' and the integer $n - i + 1$ is a reflected point for σ^{-1} and vice-versa. □

If we denote by a_n and x_n the number of magic permutations and the number of permutations without fixed points and without reflected points of length n respectively, we can find in the following table the first values of these numbers:

n	0	1	2	3	4	5	6	7	8
a_n	0	1	0	0	8	20	96	656	5568
x_n	1	0	0	0	4	16	80	672	4752

Hertzsprung [5] established the following theorem:

Theorem 2.15. *The numbers a_n and x_n satisfy the following recurrences:*

$$\begin{aligned} a_{2n} &= n(x_{2n} - (2n-3)x_{2n-1}) \\ a_{2n+1} &= (2n+1)x_{2n} + 3nx_{2n-1} - 2n(n-1)x_{2n-2}, \\ x_n &= (n-1)x_{n-1} + 2(n-d)x_{n-e}, \end{aligned}$$

where $(d,e) = (2,4)$ if n is even, $(1,2)$ if n is odd.

We will generalize the theory of rook polynomials to enumerate some restricted permutations.

3. Rook polynomials

We will study in this section the number of permutations $\sigma \in \mathfrak{S}_n$ where for each i, certain values of $\sigma(i)$ are disallowed (namely, $\sigma(i) \neq i$ and $\sigma(i) \neq n-i+1$). We have a board $\mathfrak{B} \subset [n] \times [n]$. Each square s on $\mathfrak{B}$ has a weight ω_s. We define the rook numbers (actually polynomials) of $\mathfrak{B}$ by

$$r_k = \sum_{|A|=k} \prod_{s \in A} \omega_s$$

where the sum is over all subsets $A \subset \mathfrak{B}$ of cardinality k with no two squares on the same row or column. We define the generalized hit numbers h_i:

$$h_k = \sum_{\pi} \omega(\pi)$$

where the sum is over all permutations π of $[n]$ with k hits (values of i such that $(i, \pi(i)) \in \mathfrak{B}$) and the weight $\omega(\pi)$ of the permutation π is the products $\prod_{i=1}^{n} \omega_{(i,\pi(i))}$ where $\omega_{(i,\pi(i))}$ is the weight of the square $(i, \pi(i))$ if $(i, \pi(i)) \in \mathfrak{B}$ and is 1 otherwise. The generalized hit polynomial is

$$H = \sum_k h_k.$$

We can find a relation between the hit polynomial H and the rook numbers r_i just as in the usual case. We claim that

$$H^+ = \sum_k r_k (n-k)! \qquad (\star)$$

where H^+ is the result of replacing each weight ω_s for the square $s \in \mathfrak{B}$ with $\omega_s + 1$. To see this, note that $r_k(n-k)!$ counts pairs (A, π) where A is a rook placement in $\mathfrak{B}$ of size k, and π extends A to a permutation of $[n]$. If we fix π and sum over all possible rook placement A, we are summing over all of the subsets of the hits of A and this gives H^+. If we replace each weight ω_s in $(\star)$ by $\omega_s - 1$ we get

$$H = \sum_k r_k^-(n-k)!$$

where r_k^- is the result of replacing each ω_s with $\omega_s - 1$.

Theorem 3.1 (The first main result)**.** *Let $n = 2m$ and let $\mathfrak{B}$ the following board with weights as indicated. (This is the case $n = 6$)*

α					β
	α			β	
		α	β		
		β	α		
	β			α	
β					α

By permuting the rows and columns we get

α	β				
β	α				
		α	β		
		β	α		
				α	β
				β	α

We see that

$$\sum_k r_k X^k = \left[1 + (2\alpha + 2\beta)X + (\alpha^2 + \beta^2)X^2\right]^m,$$

so

$$\sum_k r_k^- X^k = \left[1 + (2\alpha + 2\beta - 4)X + ((\alpha-1)^2 + (\beta-1)^2)X^2\right]^m, \qquad (\star\star)$$

$\sum_k r_k^- x^k = [1 + (2\alpha + 2\beta - 4)x + ((\alpha-1)^2 + (\beta-1)^2)x^2]^m$ *and therefore we obtain the hit polynomial $H = \sum_k r_k^-(n-k)!$ with r_k^- as above.*

Theorem 3.2. *The number x_{2m} of permutations in the symmetric group $\mathfrak{S}_{2m}$ without reflected points and without fixed points is given by the formula*

$$x_{2m} = \sum_k r_k^-(2m-k)!$$

where the numbers r_k^- are the coefficients of the polynomial

$$\sum_k r_k^- X^k = (1 - 4X + 2X^2)^m.$$

Proof. To count permutations with no reflected points and no fixed points, we set $\alpha = \beta = 0$ in the equation $(\star\star)$ to get

$$\sum_k r_k^- X^k = (1 - 4X + 2X^2)^m,$$

and we obtain the result. □

Theorem 3.3. *The number a_{2m} of permutations of the symmetric group $\mathfrak{S}_{2m}$ having one reflected point and one fixed point is given by the formula:*

$$a_{2m} = \sum_k r_k^- (2m - k)!$$

where the numbers r_k^- are the coefficients of the polynomial

$$4m(m-1)X^2(1-X)^2(1-4X+2X^2)^{m-2}.$$

Proof. To count permutations with one reflected point and one fixed point we look at the coefficient of $\alpha\beta$ in the equation $(\star\star)$. So we want the coefficient of $\alpha\beta$ in

$$\left[1 + (2\alpha + 2\beta - 4)X + ((\alpha - 1)^2 + (\beta - 1)^2)X^2\right]^m$$

which is easily computed to be

$$4m(m-1)X^2(1-X)^2(1-4X+2X^2)^{m-2}.$$

And this gives the result. □

Theorem 3.4. *The number of permutations of the symmetric group $\mathfrak{S}_{2m}$ with no reflected points and one fixed point is given by the formula:*

$$\sum_k r_k^- (2m - k)!$$

where the numbers r_k^- are the coefficients of the polynomial

$$2mX(1-X)(1-4X+2X^2)^{m-1}.$$

Proof. To count permutations with no reflected points and one fixed point we set $\beta = 0$ in the equation $(\star\star)$ and look at the coefficient of α. So we want the coefficient of α in

$$\left[1 + (2\alpha - 4)X + ((\alpha - 1)^2 + 1)X^2\right]^m$$

which is easily computed to be

$$2mX(1-X)(1-4X+2X^2)^{m-1}$$

and this gives the result. □

Theorem 3.5. *The number of permutations in the symmetric group $\mathfrak{S}_{2m}$ with two fixed points without reflected points is given by the formula:*

$$\sum_k r_k^-(2m-k)!$$

where the numbers r_k^- are the coefficients of the polynomial

$$(2m^2 - m - 4m^2X + 2m^2X^2)X^2(1 - 4X + 2X^2)^{m-2}.$$

Proof. To count permutations with two fixed points without reflected points, we look at the coefficient of α^2 in the equation ($\star\star$) which is easily computed to be

$$\begin{aligned} 2m(m-1)X^2(1-X)^2(1-4X+2X^2)^{m-2} + mX^2(1-4X+2X^2)^{m-1} \\ = (2m^2 - m - 4m^2X + 2m^2X^2)X^2(1-4X+2X^2)^{m-2}. \end{aligned}$$

□

Theorem 3.6 (The second main result)**.** *For n odd we take the following board where we have a separate weight for the middle square*

α				β
	α		β	
		γ		
	β		α	
β				α

By permuting the rows and columns we get

α	β			
β	α			
		α	β	
		β	α	
				γ

We see that

$$\sum_k r_k X^k = (1+\gamma X)\left[1 + (2\alpha + 2\beta)X + (\alpha^2 + \beta^2)X^2\right]^m,$$

so

$$\sum_k r_k^- X^k = (1+(\gamma-1)X)\left[1 + (2\alpha + 2\beta - 4)X + ((\alpha-1)^2 + (\beta-1)^2)X^2\right]^m, \qquad (\star\star\star)$$

and therefore $H = \sum_k r_k^-(n-k)!$ with r_k^- as above and $n = 2m+1$.

Theorem 3.7. *The number x_{2m+1} of permutations in the symmetric group $\mathfrak{S}_{2m+1}$ without fixed points and without reflected points is given by the formula*

$$\sum_k r_k^-(2m+1-k)!$$

where the numbers r_k^- *are the coefficients of the polynomial*

$$(1-X)(1-4X+2X^2)^m.$$

Proof. To count permutations with no reflected points and no fixed points, we set $\alpha=\beta=\gamma=0$ in the equation $(\star\star\star)$ to get

$$\sum_k r_k^- X^k = (1-X)(1-4X+2X^2)^m$$

and

$$x_{2m+1} = \sum_k r_k^- (2m+1-k)!.$$

□

Theorem 3.8. *The number of permutations in the symmetric group* $\mathfrak{S}_{2m+1}$ *with no reflected points and one fixed point is given by the formula:*

$$\sum_k r_k^- (2m+1-k)!$$

where the numbers r_k^- *are the coefficients of the polynomial*

$$2mX(1-X)(1-4X+2X^2)^{m-1}.$$

Proof. To count permutations with no reflected points and one fixed point we set $\beta=\gamma=0$ in the equation $(\star\star\star)$ and look at the coefficient of α. So we want the coefficient of α in

$$(1-X)\left[1+(2\alpha-4)X+((\alpha-1)^2+1)X^2\right]^m$$

which is easily computed to be

$$2mX(1-X)(1-4X+2X^2)^{m-1}.$$

□

Theorem 3.9. *The number of permutations in the symmetric group* $\mathfrak{S}_{2m+1}$ *with no reflected points and two fixed points is given by the formula:*

$$\sum_k r_k^- (2m+1-k)!$$

where the numbers r_k^- *are the coefficients of the polynomial*

$$2m(m-1)X^2(1-X)^3(1-4X+2X^2)^{m-2}+mX^2(1-X)(1-4X+2X^2)^{m-1}.$$

Proof. To count permutations with no reflected points and two fixed points, we set $\beta=\gamma=0$ in the equation $(\star\star\star)$ and we look at the coefficient of α^2 which is easily computed to be

$$2m(m-1)X^2(1-X)^3(1-4X+2X^2)^{m-2}+mX^2(1-X)(1-4X+2X^2)^{m-1}.$$

□

Theorem 3.10. *The number of permutations in the symmetric group $\mathfrak{S}_{2m+1}$ having a pivot point and no other fixed or reflected points is equal to the number x_{2m}.*

Proof. If the pivot point is removed from such a permutation, what remains is a permutation in $\mathfrak{S}_{2m}$ with no fixed or reflected points. □

Corollary 3.11. *The number a_{2m+1} of permutations in the symmetric group $\mathfrak{S}_{2m+1}$ with one reflected point and one fixed point is given by the formula:*

$$a_{2m+1} = x_{2m} + \sum_k r_k^-(2m+1-k)!$$

where the numbers r_k^- are the coefficients of the polynomial

$$4m(m-1)X^2(1-X)^3(1-4X+2X^2)^{m-2}.$$

Proof. First, to count permutations with one fixed point and one reflected point without pivot points, we set $\gamma = 0$ in the equation $(\star\star\star)$ and look at the coefficient of $\alpha\beta$ in

$$(1-X)\left[1+(2\alpha+2\beta-4)X+((\alpha-1)^2+(\beta-1)^2)X^2\right]^m$$

to get

$$\sum_k r_k^- X^k = 4m(m-1)X^2(1-X)^3(1-4X+2X^2)^{m-2}$$

and

$$a_{2m+1} = \sum_k r_k^-(2m+1-k)! + x_{2m}.$$

□

Remark 3.12. *To count derangements, that is, permutations without fixed points, we set $\alpha = \gamma = 0$ and $\beta = 1$ in the equation $(\star\star\star)$ to get the usual hit polynomial*

$$(1-X)^n.$$

Definition 3.13. *We say that a subset F of the set $[n]$ is:*

(1) *semi-reflected if there exists at least one element $i \in F$ such that $n-i+1 \in F$.*
(2) *self-reflected if $n-i+1 \in F$, for all elements i in the subset F.*

The proof of the following lemmas is a simple exercise of combinatorics.

Lemma 3.14. *The number of pairs of disjoint subsets F and R of the set $[2n]$ such that $\#(F \cup R) = 2k$ and both F and R are self-reflected is equal to $2^k \binom{n}{k}$.*

Lemma 3.15. *The number of pairs of disjoint subsets F and R of the set $[2n]$ or $[2n+1]$ such that $\#(F \cup R) = n$ and both F and R are not semi-reflected is equal to 2^{2n}.*

Theorem 3.16. *The number of permutations of length $2n$ whose set of all fixed or reflected points is of cardinality $2k$, and is a self-reflected set, is equal to $\binom{n}{k} 2^k x_{2(n-k)}$.*

Proof. We consider the following board with weights as indicated. We illustrate it with the case for $2n = 6$.

α_1					β_1
	α_2			β_2	
		α_3	β_3		
		β_4	α_4		
	β_5			α_5	
β_6					α_6

By permuting the rows and columns we get

α_1	β_1				
β_6	α_6				
		α_2	β_2		
		β_5	α_5		
				α_3	β_3
				β_4	α_4

We see that

$$\sum_k r_k X^k = \prod_{i=1}^{n} [1 + (\alpha_i + \alpha_{2n-i+1} + \beta_i + \beta_{2n-i+1})X + (\alpha_i \alpha_{2n-i+1} + \beta_i \beta_{2n-i+1})X^2],$$

so

$$\sum_k r_k^- X^k = \prod_{i=1}^{n} [1 + (\alpha_i + \alpha_{2n-i+1} + \beta_i + \beta_{2n-i+1} - 4)X + ((\alpha_i - 1)(\alpha_{2n-i+1} - 1) + (\beta_i - 1)(\beta_{2n-i+1} - 1))X^2],$$

and therefore $H = \sum_k r_k^- (n-k)!$ with r_k^- as above.
To count exactly once each permutation of length $2n$ in which the set of fixed or reflected points is self-reflected and has cardinality $2k$, we look first at the coefficient of $\prod_{s=1}^{p} \mu_{i_s} \prod_{s=p+1}^{k} \nu_{i_s}$ where $\mu_{i_s} = \alpha_{i_s} \alpha_{2n-i_s+1}$ and $\nu_{i_s} = \beta_{i_s} \beta_{2n-i_s+1}$. We should make $\{i_1, \ldots, i_p\}$ and $\{i_{p+1}, \ldots, i_k\}$ disjoint subsets of the set $[n]$ and moreover we should take $i_1 < \ldots < i_p$ and $i_{p+1} < \ldots < i_k$, and by Lemma 3.14 the number of possible choices for the set $\{i_1, \ldots, i_p\}$ and $\{i_{p+1}, \ldots, i_k\}$ is equal

to $2^k\binom{n}{k}$. We can compute by induction the coefficient of $\prod_{s=1}^{p}\mu_{i_s}\prod_{s=p+1}^{k}\nu_{i_s}$ which is equal to $X^{2k}(1-4X+2X^2)^{2(n-k)}$ and this gives the result. □

Theorem 3.17. *The number of permutations of* $[2n+1]$*, in which the set of fixed or reflected points has cardinality* $2k+1$ *and is self-reflected, is equal to* $\binom{n}{k}2^k x_{2(n-k)}$.

Proof. Notice that if the cardinality of a self-reflected subset of the set $[2n+1]$ is odd, this subset contains the integer $n+1$. Since the integer $n+1$ must be fixed point, we can remove it to get one of the permutations counted by Theorem 3.16. □

4. Derangements

We will conclude this paper with an unexpected relation which relates derangements and restricted permutations.

Theorem 4.1. *The number of permutations of* $[2n]$*, in which the set of all fixed or reflected points has cardinality* n *and is not semi-reflected, is equal to* $2^{2n}d_n$.

Proof. We consider again a board as in the proof of Theorem 3.16. To count exactly once each permutation of length $2n$ in which the set of all fixed or reflected points is not semi-reflected and has cardinality n, we look first at the coefficient of $\prod_{s=1}^{p}\alpha_{i_s}\prod_{s=p+1}^{n}\beta_{i_s}$ from

$$\prod_{i=1}^{n}[1+(\alpha_i+\alpha_{2n-i+1}+\beta_i+\beta_{2n-i+1}-4)X$$
$$+((\alpha_i-1)(\alpha_{2n-i+1}-1)+(\beta_i-1)(\beta_{2n-i+1}-1))X^2]$$

such that no occurrences of the form $\alpha_\ell\alpha_{2n-\ell+1}$ or $\beta_\ell\beta_{2n-\ell+1}$ appear in the products. By induction we deduce this coefficient which is equal to

$$X^n(1-X)^n.$$

For the sequence $\{i_1,\dots,i_n\}$, we should make $\{i_1,\dots,i_p\}$ and $\{i_{p+1},\dots,i_n\}$ disjoint subsets of the set $[2n]$ and moreover we should take $i_1<\dots<i_p$ and $i_{p+1}<\dots<i_n$. By Lemma 3.15 the number of possible choices for the set $\{i_1,\dots,i_p\}$ and $\{i_{p+1},\dots,i_n\}$ is equal to 2^{2n} and this gives the result. □

Theorem 4.2. *The number of permutations of* $[2n+1]$*, in which the set of all fixed or reflected points is not semi-reflected and has cardinality* n*, is equal to* $2^{2n}d_{n+1}$.

Proof. Notice that if the cardinality of a non semi-reflected subset of the set $[2n+1]$ is equal to n, this subset does not contain the integer $n+1$. Since the integer $n+1$ is not a fixed point, we can add it to get one of the permutations counted by Theorem 4.1. □

Theorem 4.3. *The number of permutations of* $[2n+1]$*, in which the set of all fixed or reflected points contains the integer* $n+1$ *and has cardinality* $n+1$ *and is not semi-reflected if the element* $n+1$ *is deleted, is equal to* $2^{2n}d_n$.

Proof. Notice that if the cardinality of a non semi-reflected subset of the set $[2n+1]$ is equal to $n+1$, this subset contains the integer $n+1$. Since the integer $n+1$ must be fixed point, we can remove it to get one of the permutations counted by Theorem 4.1. □

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