Séminaire Lotharingien de Combinatoire **54** (2006), Article B54e

ON PARTITIONS AVOIDING 3-CROSSINGS

MIREILLE BOUSQUET-MÉLOU AND GUOCE XIN

To Xavier Viennot, on the occasion of his 60th birthday

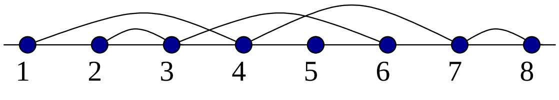
Abstract. A partition on $[n]$ has a crossing if there exists $i_1 < i_2 < j_1 < j_2$ such that i_1 and j_1 are in the same block, i_2 and j_2 are in the same block, but i_1 and i_2 are not in the same block. Recently, Chen et al. refined this classical notion by introducing *k-crossings*, for any integer k. In this new terminology, a classical crossing is a 2-crossing. The number of partitions of $[n]$ avoiding 2-crossings is well-known to be the nth Catalan number $C_n = \binom{2n}{n}/(n+1)$. This raises the question of counting k-noncrossing partitions for $k \geq 3$. We prove that the sequence counting 3-noncrossing partitions is P-recursive, that is, satisfies a linear recurrence relation with polynomial coefficients. We give explicitly such a recursion. However, we conjecture that k-noncrossing partitions are not P-recursive, for $k \geq 4$.

We obtain similar results for partitions avoiding *enhanced* 3-crossings.

1. Introduction

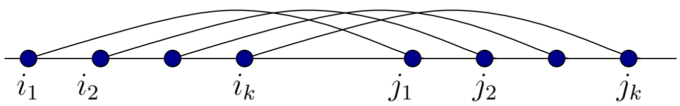
A partition of $[n] := \{1, 2, \ldots, n\}$ is a collection of nonempty and mutually disjoint subsets of $[n]$, called *blocks*, whose union is $[n]$. The number of partitions of $[n]$ is the *Bell number* B_n. A well-known refinement of B_n is given by the *Stirling number* (of the second kind) $S(n, k)$. It counts partitions of $[n]$ having exactly k blocks.

Recently another refinement of the Bell numbers by *crossings* and *nestings* has attracted some interest [8, 15, 13]. This refinement is based on the standard representation of a partition P of $[n]$ by a graph, whose vertex set $[n]$ is identified with the points $i \equiv (i, 0)$ on the plane, for $1 \leq i \leq n$, and whose edge set consists of arcs connecting the elements that occur *consecutively* in the same block (when each block is totally ordered). For example, the standard representation of $1478 - 236 - 5$ is given by the following graph.

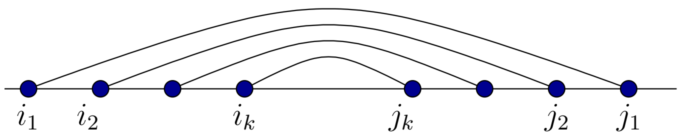


Then crossings and nestings have a natural definition. A *k-crossing* of P is a collection of k edges (i_1, j_1), (i_2, j_2), ..., (i_k, j_k) such that $i_1 < i_2 < \cdots < i_k < j_1 < j_2 < \cdots < j_k$. This means a subgraph of P as drawn as follows.

Key words and phrases. Partitions, Crossings, D-finite series.



A different notion of k-crossing is obtained by representing each block by a complete graph [14, p. 85]. A *k-nesting* of P is a collection of k edges (i_1, j_1), (i_2, j_2), ..., (i_k, j_k) such that $i_1 < i_2 < \cdots < i_k < j_k < j_{k-1} < \cdots < j_1$, as represented below.



A partition is *k-noncrossing* if it has no k-crossings, and *k-nonnesting* if it has no k-nestings.

A variation of k-crossings (nestings), called *enhanced* k-crossings (nestings), was also studied in [8]. One first adds a loop to every isolated point in the standard representation of partitions. Then by allowing $i_k = j_1$ in a k-crossing, we get an enhanced k-crossing; in particular, a partition avoiding enhanced 2-crossings has parts of size 1 and 2 only. Similarly, by allowing $i_k = j_k$ in the definition of a k-nesting, we get an enhanced k-nesting.

Chen et al. gave in [8] a bijection between partitions of $[n]$ and certain "vacillating" tableaux of length $2n$. Through this bijection, a partition is k-noncrossing if and only if the corresponding tableau has height less than k, and k-nonnesting if the tableau has width less than k. A simple symmetry on tableaux then entails that k-noncrossing partitions of $[n]$ are equinumerous with k-nonnesting partitions of $[n]$, for all values of k and n. A second bijection relates partitions to certain "hesitating" tableaux, in such a way the size of the largest *enhanced* crossing (nesting) becomes the height (width) of the tableau. This implies that partitions of $[n]$ avoiding enhanced k-crossings are equinumerous with partitions of $[n]$ avoiding enhanced k-nestings.

The number $C_2(n)$ of 2-noncrossing partitions of $[n]$ (usually called *noncrossing partitions* [16, 22]) is well-known to be the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. For $k > 2$, the number of k-noncrossing partitions of $[n]$ is not known, to the extent of our knowledge. However, for any k, the number of k-noncrossing *matchings* of $[n]$ (that is, partitions in which all blocks have size 2) is known to form a P-recursive sequence, that is, to satisfy a linear recurrence relation with polynomial coefficients [11, 8]. In this paper, we enumerate 3-noncrossing partitions of $[n]$ (equivalently, 3-nonnesting partitions of $[n]$). We obtain for $C_3(n)$ a (not so simple) closed form expression (Proposition 7) and a linear recurrence relation with polynomial coefficients.

Proposition 1. *The number $C_3(n)$ of 3-noncrossing partitions is given by $C_3(0) = C_3(1) = 1$, and for $n \geq 0$,*

$$9n(n+3)C_3(n) - 2\left(5n^2+32n+42\right)C_3(n+1) + (n+7)(n+6)C_3(n+2) = 0. \quad (1)$$

Equivalently, the associated generating function $\mathcal{C}(t) = \sum_{n\geq 0} C_3(n)t^n$ satisfies

$$t^2(1-9t)(1-t)\frac{d^2}{dt^2}\mathcal{C}(t) + 2t\left(5-27t+18t^2\right)\frac{d}{dt}\mathcal{C}(t) + 10(2-3t)\mathcal{C}(t) = 20. \quad (2)$$

Finally, as n tends to infinity,

$$C_3(n) \sim \frac{3^9 \cdot 5}{2^5}\frac{\sqrt{3}}{\pi}\frac{9^n}{n^7}.$$

The first few values of the sequence $C_3(n)$, for $n \geq 0$, are

$$1, 1, 2, 5, 15, 52, 202, 859, 3930, 19095, 97566, \ldots.$$

A standard study [12, 24] of the above differential equation, which can be done automatically using the Maple package DEtools, suggests that $C_3(n) \sim \kappa\, 9^n/n^7$ for some positive constant κ. However, one needs the explicit expression of $C_3(n)$ given in Section 2.5 to prove this statement and find the value of κ. The above asymptotic behaviour is confirmed experimentally by the computation of the first values of $C_3(n)$ (a few thousand values can be computed rapidly using the package Gfun of Maple [21]). For instance, when $n = 50000$, then $n^7 C_3(n)/9^n \simeq 1694.9$, while $\kappa \simeq 1695.6$.

As discussed in the last section of the paper, the above result might remain isolated, as there is no (numerical) evidence that the generating function of 4-noncrossing partitions should be P-recursive.

We obtain a similar result for partitions avoiding *enhanced* 3-crossings (or enhanced 3-nestings). The number of partitions of $[n]$ avoiding enhanced 2-crossings is easily seen to be the nth *Motzkin number* [23, Exercise 6.38].

Proposition 2. *The number $E_3(n)$ of partitions of $[n]$ having no enhanced 3-noncrossing is given by $E_3(0) = E_3(1) = 1$, and for $n \geq 0$,*

$$8(n+3)(n+1)E_3(n) + \left(7n^2+53n+88\right)E_3(n+1) - (n+8)(n+7)E_3(n+2) = 0.$$

Equivalently, the associated generating function $\mathcal{E}(t) = \sum_{n\geq 0} E_3(n)t^n$ satisfies

$$t^2(1+t)(1-8t)\frac{d^2}{dt^2}\mathcal{E}(t) + 2t\left(6-23t-20t^2\right)\frac{d}{dt}\mathcal{E}(t) + 6\left(5-7t-4t^2\right)\mathcal{E}(t) = 30.$$

Finally, as n tends to infinity,

$$E_3(n) \sim \frac{2^{16} \cdot 5}{3^3}\frac{\sqrt{3}}{\pi}\frac{8^n}{n^7}.$$

The first few values of the sequence $E_3(n)$, for $n \geq 0$, are

$$1, 1, 2, 5, 15, 51, 191, 772, 3320, 15032, 71084, \dots.$$

Observe that $C_3(5)$ and $E_3(5)$ differ by 1: this difference comes from the partition $135-24$ which has an enhanced 3-crossing but no 3-crossing (equivalently, from the partition $15-24-3$ which has an enhanced 3-nesting but no 3-nesting). Again, the study of the differential equation suggests that $E_3(n) \sim \kappa \frac{8^n}{n^7}$, for some positive constant κ, but we need the explicit expression (33) of $E_3(n)$ to prove this statement and find the value of κ. Numerically, we have found that for $n = 50000$, $n^7 E_3(n)/8^n \simeq 6687.3$, while $\kappa \simeq 6691.1$.

The starting point of our proof of Proposition 1 is the above mentioned bijection between partitions avoiding $k+1$-crossings and vacillating tableaux of height at most k. As described in [8], these tableaux can be easily encoded by certain k-dimensional lattice walks. Let D be a subset of $\mathbb{Z}^k$. A *D-vacillating* lattice walk of length n is a sequence of lattice points $p_0, p_1, \dots, p_n$ in D such that for all i,

(*i*) $p_{2i+1} = p_{2i}$ or $p_{2i+1} = p_{2i} - e_j$ for some coordinate vector $e_j = (0, \dots, 0, 1, 0, \dots, 0)$,
(*ii*) $p_{2i} = p_{2i-1}$ or $p_{2i} = p_{2i-1} + e_j$ for some e_j.

We will be interested in two different domains D of $\mathbb{Z}^k$: the domain $Q_k = \mathbb{N}^k$ of points with non-negative integer coordinates and the Weyl chamber of type C_k (with a slight change of coordinates) $W_k = \{ (a_1, a_2, \dots, a_k) \in \mathbb{Z}^k : a_1 > a_2 > \cdots > a_k \geq 0 \}$. Vacillating walks in W_k are related to $k+1$-noncrossing partitions as follows.

Theorem 3 (**Chen et al.** [8]). *Let $C_k(n)$ denote the number of k-noncrossing partitions of $[n]$. Then $C_{k+1}(n)$ equals the number of W_k-vacillating lattice walks of length $2n$ starting and ending at $(k-1, k-2, \dots, 0)$.*

The proof of Proposition 1 goes as follows: using the reflection principle, we first reduce the enumeration of vacillating walks in the Weyl chamber W_k to that of vacillating walks in the non-negative domain Q_k. This reduction is valid for any k. We then focus on the case $k = 2$. We write a functional equation satisfied by a 3-variable series that counts vacillating walks in Q_2. This equation is based on a simple recursive construction of the walks. It is solved using a 2-dimensional version of the so-called *kernel method.* This gives the generating function $\mathcal{C}(t) = \sum C_3(n)t^n$ as the constant term in a certain algebraic series. We then use the Lagrange inversion formula to find an explicit expression of $C_3(n)$, and apply the *creative telescoping* of [20] to obtain the recurrence relation. We finally derive from the expression of $C_3(n)$ the asymptotic behaviour of these numbers. The proof of Proposition 2, given in Section 3, is similar.

2. Partitions with no 3-crossing

2.1. The reflection principle

Let $\delta = (k-1, k-2, \ldots, 0)$ and let λ and μ be two lattice points in W_k. Denote by $w_k(\lambda, \mu, n)$ (respectively $q_k(\lambda, \mu, n)$) the number of W_k-vacillating (respectively Q_k-vacillating) lattice walks of length n starting at λ and ending at μ. Thus Theorem 3 states that $C_{k+1}(n)$ equals $w_k(\delta, \delta, 2n)$. The reflection principle, in the vein of [10, 25], gives the following:

Proposition 4. *For any starting and ending points λ and μ in W_k, the number of W_k-vacillating walks going from λ to μ can be expressed in terms of the number of Q_k-vacillating walks as follows:*

$$w_k(\lambda, \mu, n) = \sum_{\pi \in \mathfrak{S}_k} (-1)^\pi q_k(\lambda, \pi(\mu), n), \tag{3}$$

where $(-1)^\pi$ is the sign of π and $\pi(\mu_1, \mu_2, \ldots, \mu_k) = (\mu_{\pi(1)}, \mu_{\pi(2)}, \ldots, \mu_{\pi(k)})$.

Proof. Consider the set of hyperplanes $\mathcal{H} = \{\, x_i = x_j : 1 \le i < j \le k \,\}$. The reflection of the point $(a_1, \ldots, a_k)$ with respect to the hyperplane $x_i = x_j$ is simply obtained by exchanging the coordinates a_i and a_j. In particular, the set of (positive) steps taken by vacillating walks, being $\{e_1, \ldots, e_k\}$, is invariant under such reflections. The same holds for the negative steps, and of course for the "stay" step, 0. This implies that reflecting a Q_k-vacillating walk with respect to $x_i = x_j$ gives another Q_k-vacillating walk. Note that this is not true when reflecting with respect to $x_i = 0$ (since e_i is transformed into $-e_i$).

Define a total ordering on $\mathcal{H}$. Take a Q_k-vacillating walk w of length n going from λ to $\pi(\mu)$, and assume it touches at least one hyperplane in $\mathcal{H}$. Let m be the first time it touches a hyperplane. Let $x_i = x_j$ be the smallest hyperplane it touches at time m. Reflect all steps of w after time m across $x_i = x_j$; the resulting walk w' is a Q_k-vacillating walk going from λ to $(i,j)(\pi(\mu))$, where (i,j) denotes the transposition that exchanges i and j. Moreover, w' also (first) touches $\mathcal{H}$ at time m, and the smallest hyperplane it touches at this time is $x_i = x_j$.

The above transformation is a sign-reversing involution on the set of Q_k-vacillating paths that go from λ to $\pi(\mu)$, for some permutation π, and hit one of the hyperplanes of $\mathcal{H}$. In the right-hand side of (3), the contributions of these walks cancel out. One is left with the walks that stay within the Weyl chamber, and this happens only when π is the identity. The proposition follows. ∎

This proposition, combined with Theorem 3, gives the number of $(k+1)$-noncrossing partitions of n as a linear combination of the numbers $q_k(\delta, \pi(\delta), 2n)$. Hence, in order to count $(k+1)$-noncrossing partitions, it suffices to find a formula for $q_k(\delta, \mu, 2n)$, for certain ending points μ. This is what we do below for $k = 2$.

2.2. **A functional equation**

Let us specialize Theorem 3 and Proposition 4 to $k = 2$. This gives

$$\begin{aligned} C_3(n) &= w_2((1,0),(1,0),2n) \\ &= q_2((1,0),(1,0),2n) - q_2((1,0),(0,1),2n). \end{aligned} \tag{4}$$

From now on, a lattice walk always means a *Q_2-vacillating lattice walk starting at* $(1,0)$, unless specified otherwise.

Let $a_{i,j}(n) := q_2((1,0),(i,j),n)$ be the number of lattice walks of length n ending at (i,j). Let

$$F_e(x,y;t) = \sum_{i,j,n\geq 0} a_{i,j}(2n)x^i y^j t^{2n}$$

and

$$F_o(x,y;t) = \sum_{i,j,n\geq 0} a_{i,j}(2n+1)x^i y^j t^{2n+1}$$

be respectively the generating functions of lattice walks of even and odd length. These series are power series in t with coefficients in $\mathbb{Q}[x,y]$. We will often work in a slightly larger ring, namely the ring $\mathbb{Q}[x,1/x,y,1/y][[t]]$ of power series in t whose coefficients are Laurent polynomials in x and y.

Now we find functional equations for $F_e(x,y;t)$ and $F_o(x,y;t)$. By appending to an even length walk a west step $(-1,0)$, or a south step $(0,-1)$, or a stay step $(0,0)$, we obtain either an odd length walk (in Q_2), or an even length walk ending on the x-axis followed by a south step, or an even length walk ending on the y-axis followed by a west step. This correspondence is easily seen to be a bijection, and gives the following functional equation:

$$F_e(x,y;t)(1+\bar{x}+\bar{y})t = F_o(x,y;t) + \bar{y}tH_e(x;t) + \bar{x}tV_e(y;t), \tag{5}$$

where $\bar{x} = 1/x$, $\bar{y} = 1/y$, and $H_e(x;t)$ (respectively $V_e(y;t)$) is the generating function of even lattice walks ending on the x-axis (respectively on the y-axis).

Similarly, by adding to an odd length walk an east step $(1,0)$, or a north step $(0,1)$, or a stay step, we obtain an even length walk of positive length. The above correspondence is a bijection and gives another functional equation:

$$F_o(x,y;t)(1+x+y)t = F_e(x,y;t) - x. \tag{6}$$

Solving equations (5) and (6) for $F_e(x,y;t)$ and $F_o(x,y;t)$ gives

$$F_e(x,y;t) = \frac{x - t^2\bar{y}(1+x+y)H_e(x;t) - t^2\bar{x}(1+x+y)V_e(y;t)}{1-t^2(1+x+y)(1+\bar{x}+\bar{y})},$$

$$F_o(x,y;t) = t\,\frac{x(1+\bar{x}+\bar{y}) - \bar{x}V_e(y;t) - \bar{y}H_e(x;t)}{1-t^2(1+x+y)(1+\bar{x}+\bar{y})}.$$

Since we are mostly interested in even length walks ending at $(1,0)$ and at $(0,1)$, it suffices to determine the series $H_e(x;t)$ and $V_e(y;t)$, and hence to solve one of the above two functional equations. We choose the one for $F_o(x,y;t)$, which is simpler.

Proposition 5. *The generating function $F_o(x,y;t)$ of Q_2-vacillating lattice walks of odd length starting from $(1,0)$ is related to the generating functions of even lattice walks of the same type ending on the x- or y-axis by*

$$K(x,y;t)F(x,y;t) = xy + x^2y + x^2 - xH(x;t) - yV(y;t), \tag{7}$$

where $F_o(x,y;t) = tF(x,y;t^2)$, $V(y;t) = V_e(y;t^2)$, $H(x;t) = H_e(x;t^2)$, and $K(x,y;t)$ is the kernel *of the equation:*

$$K(x,y;t) = xy - t(1+x+y)(x+y+xy). \tag{8}$$

From now on, we will very often omit the variable t in our notation. For instance, we will write $H(x)$ instead of $H(x;t)$. Observe that (7) defines $F(x,y)$ uniquely as a series in t with coefficients in $\mathbb{Q}[x,y]$. Indeed, setting $y=0$ shows that $H(x) = x+t(1+x)F(x,0)$ while setting $x=0$ gives $V(y) = t(1+y)F(0,y)$.

2.3. The kernel method

We are going to apply to (7) the *obstinate kernel method* that has already been used in [4, 5] to solve similar equations. The classical kernel method consists in coupling the variables x and y so as to cancel the kernel $K(x,y)$. This gives some "missing" information about the series $V(y)$ and $H(x)$ (see for instance [6, 1]). In its obstinate version, the kernel method is combined with a procedure that constructs and exploits several (related) couplings (x,y). This procedure is essentially borrowed from [9], where similar functional equations occur in a probabilistic context.

Let us start with the standard kernel method. First fix x, and consider the kernel (8) as a quadratic polynomial in y. Only one of its roots, denoted Y_0 below, is a power series in t:

$$\begin{aligned} Y_0 &= \frac{1-(\bar{x}+3+x)t - \sqrt{(1-(1+x+\bar{x})t)^2 - 4t}}{2\,(1+\bar{x})\,t} \\ &= (1+x)t + \frac{(1+x)(1+3x+x^2)}{x}t^2 + \cdots \end{aligned}$$

The coefficients of this series are Laurent polynomials in x, as is easily seen from the equation

$$Y_0 = t(1+x+Y_0)\left(1+(1+\bar{x})Y_0\right). \tag{9}$$

Setting $y = Y_0$ in (7) gives a valid identity between series of $\mathbb{Q}[x,\bar{x}][[t]]$, namely

$$xH(x) + Y_0V(Y_0) = xY_0 + x^2Y_0 + x^2.$$

The second root of the kernel is $Y_1 = x/Y_0 = O(t^{-1})$, so that the expression $F(x,Y_1)$ is not well-defined.

Now let $(X,Y) \neq (0,0)$ be a pair of Laurent series in t with coefficients in a field $\mathbb{K}$ such that $K(X,Y)=0$. Recall that K is quadratic in x and y. In particular, the equation $K(x,Y)=0$ admits a second solution X'. Define $\Phi(X,Y) = (X',Y)$. Similarly, define $\Psi(X,Y) = (X,Y')$, where Y' is the second solution of $K(X,y)=0$. Note that Φ and Ψ are involutions. Moreover, with the kernel given by (8), one has $Y' = X/Y$ and $X' = Y/X$. Let us examine the action of Φ and Ψ on the pair (x, Y_0): we obtain an orbit of cardinality 6 (Figure 1).

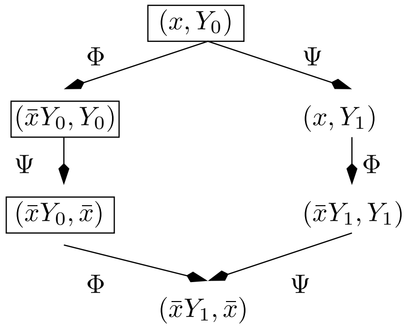


FIGURE 1. The orbit of (x, Y_0) under the action of Φ and Ψ.

The 6 pairs of power series given in Figure 1 cancel the kernel, and we have framed the ones that can be legally substituted for (x, y) in the main functional equation (7). Denoting $Y \equiv Y_0$, we thus obtain *three* equations relating the unknown series $H(x)$ and $V(y)$:

$$\begin{aligned} xH(x) + YV(Y) &= xY + x^2Y + x^2, &(10)\\ \bar{x}YH(\bar{x}Y) + YV(Y) &= \bar{x}Y^2 + \bar{x}^2Y^3 + \bar{x}^2Y^2, &(11)\\ \bar{x}YH(\bar{x}Y) + \bar{x}V(\bar{x}) &= \bar{x}^2Y + \bar{x}^3Y^2 + \bar{x}^2Y^2. &(12)\end{aligned}$$

2.4. **Positive and negative parts of power series**

A simple linear combination of the above three equations (namely, (10)−(11)+(12)) allows us to eliminate the terms $V(Y)$ and $H(\bar{x}Y)$. We are left with:

$$xH(x) + \bar{x}V(\bar{x}) = x^2 + (\bar{x}^2 + x + x^2)Y + (\bar{x}^3 - \bar{x})Y^2 - \bar{x}^2Y^3.$$

Since $xH(x)$ contains only positive powers of x and $\bar{x}V(\bar{x})$ contains only negative powers of x, we have characterized the series $H(x)$ and $V(y)$.

Proposition 6. *The series $H(x)$ and $V(y)$ counting Q_2-vacillating walks of even length starting at $(1, 0)$ and ending on the x-axis and on the y-axis satisfy*

$$xH(x) = \underset{x}{\mathrm{PT}}\left(x^2 + (\bar{x}^2 + x + x^2)Y + (\bar{x}^3 - \bar{x})Y^2 - \bar{x}^2Y^3\right),$$
$$\bar{x}V(\bar{x}) = \underset{x}{\mathrm{NT}}\left(x^2 + (\bar{x}^2 + x + x^2)Y + (\bar{x}^3 - \bar{x})Y^2 - \bar{x}^2Y^3\right),$$

where the operator PT_x (respectively NT_x) extracts positive (respectively negative) powers of x in series of $\mathbb{Q}[x, \bar{x}][[t]]$.

One may then go back to (5–6) to obtain expressions of the series $F_e(x, y; t)$ and $F_o(x, y; t)$. However, our main concern in this note is the number $C_3(n)$

of 3-noncrossing partitions of $[n]$. Going back to (4), we find that $C_3(n)$ is determined by the following three equations:

$$\begin{aligned} C_3(n) &= q_2((1,0),(1,0),2n) - q_2((1,0),(0,1),2n), \\ q_2((1,0),(1,0),2n) &= [xt^n]\ H(x) = [x^2t^n]\ xH(x), \\ q_2((1,0),(0,1),2n) &= [yt^n]\ V(y) = [\bar{x}^2t^n]\ \bar{x}V(\bar{x}). \end{aligned}$$

Using Proposition 6, we obtain the generating function $\mathcal{C}(t)$ of 3-noncrossing partitions:

$$\mathcal{C}(t) = \underset{x}{\mathrm{CT}}\ (\bar{x}^2 - x^2)\left(x^2 + (\bar{x}^2 + x + x^2)Y + (\bar{x}^3 - \bar{x})Y^2 - \bar{x}^2Y^3\right), \tag{13}$$

where the operator CT_x extracts the constant term in x of series in $\mathbb{Q}[x,\bar{x}][[t]]$. Observe that $Y(x) = xY(\bar{x})$. This implies that for all $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}$,

$$[x^\ell]Y(x)^k = [x^{\ell-k}]Y(\bar{x})^k = [x^{k-\ell}]Y(x)^k, \quad \text{that is,} \quad \underset{x}{\mathrm{CT}}(x^{-\ell}Y^k) = \underset{x}{\mathrm{CT}}(x^{\ell-k}Y^k).$$

This allows us to rewrite (13) using "only" six terms:

$$\mathcal{C}(t) = 1 + \underset{x}{\mathrm{CT}}\ \left((\bar{x}^1 - x^4)Y + (\bar{x}^5 - \bar{x})Y^2 - (\bar{x}^4 - x^0)Y^3\right).$$

The above equation says that $\mathcal{C}(t)$ is the constant term of an algebraic function. By a very general theory [17], $\mathcal{C}(t)$ is *D-finite*. That is, it satisfies a linear differential equation with polynomial coefficients. In the next section, we show that $\mathcal{C}(t)$ satisfies the equation (2) (or, equivalently, the P-recurrence (1)). Note that this recurrence can be easily *guessed* using the Maple package Gfun: indeed, the first 15 values of $C_3(n)$ already yield the correct recursion.

2.5. **The Lagrange inversion formula and creative telescoping**

From now on, several routes lead to the recurrence relation of Proposition 1, depending on how much software one is willing to use. We present here the one that we believe to be the shortest. Starting from (9), the Lagrange inversion formula gives [23, Thm. 5.4.2]:

$$[t^n]\,\underset{x}{\mathrm{CT}}\ (x^\ell Y^k) = \sum_{j\in\mathbb{Z}} a_n(\ell,k,j) \quad \text{with} \quad a_n(\ell,k,j) = \frac{k}{n}\binom{n}{j}\binom{n}{j+k}\binom{2j+k}{j-\ell}. \tag{14}$$

By convention, the binomial coefficient $\binom{a}{b}$ is zero unless $0 \le b \le a$. Hence for $n \ge 1$,

$$C_3(n) = \sum_{j\in\mathbb{Z}} \big(a_n(-1,1,j) - a_n(4,1,j) + a_n(-5,2,j-1) - a_n(-1,2,j-1) \\ - a_n(-4,3,j-1) + a_n(0,3,j-1)\big). \tag{15}$$

Of course, we could replace all occurrences of $j-1$ by j in the above expression, but this results in a bigger final formula.

Proposition 7. *For $n \geq 1$, the number of 3-noncrossing partitions of $[n]$ is*

$$C_3(n) = \sum_{j=1}^{n} \frac{4(n-1)!\, n!\, (2j)!}{(j-1)!\, j!\, (j+1)!\, (j+4)!\, (n-j)!\, (n-j+2)!} P(j,n)$$

with

$$P(j,n) = 24 + 18\,n + (5 - 13\,n)\,j + (11\,n + 20)\,j^2 + (10\,n - 2)\,j^3 + (4\,n - 11)\,j^4 - 6\,j^5.$$

Proof of the recurrence relation of Proposition 1. We finally apply to the above expression Zeilberger's algorithm for creative telescoping [20, Ch. 6] (we used the Maple package Ekhad, available from http://www.math.rutgers.edu/∼zeilberg/programsAB.html). This gives a recurrence relation for the sequence $C_3(n)$: for $n \geq 2$,

$$\begin{aligned} &9n(n-2)(n-3)\left(4n^2+15n+17\right)C_3(n-3) \\ &\quad -(n-2)\left(76n^4+373n^3+572n^2+203n-144\right)C_3(n-2) \\ &\quad +(n+3)\left(44n^4+189n^3+227n^2+30n-160\right)C_3(n-1) \\ &\quad -(n+5)(n+4)(n+3)\left(4n^2+7n+6\right)C_3(n) = 0. \end{aligned}$$

The initial conditions are $C_3(0) = C_3(1) = 1$. (There is no need to define $C_3(-1)$, because of the factor $(n-2)$ in the coefficient of $C_3(n-3)$.) It is then very simple to check that the sequence defined by these two initial conditions and the three term recursion of Proposition 1 satisfies also the above four term recursion. This can be rephrased by saying that (1) is a right factor of the four term recursion obtained via creative telescoping. More precisely, applying to (1) the operator

$$(n+1)\left(4\,n^2+39\,n+98\right) - (n+6)\left(4\,n^2+31\,n+63\right)N,$$

where N is the shift operator replacing n by $n+1$, gives the four term recursion (with n replaced by $n+3$). ■

It is worth noting that though the sequence $\{C_3(n) : n \in \mathbb{N}\} = \{w_2((1,0),(1,0),2n) : n \in \mathbb{N}\}$ satisfies a simple recurrence of order 2 (given by (1)), experimenting with the package Gfun suggests that the sequences $\{q_2((1,0),(1,0),2n) : n \in \mathbb{N}\}$ and $\{q_2((1,0),(0,1),2n) : n \in \mathbb{N}\}$ satisfy more complicated recurrences of order 3.

2.6. Asymptotics

Finally, we will derive from the expression (15) of $C_3(n)$ the asymptotic behaviour of this sequence, as stated in Proposition 1. For any fixed values of k and ℓ, with $k \geq 1$, we consider the numbers $a_n(\ell,k,j)$ defined by (14). For the sake of simplicity, we denote them by $a_n(j)$, and introduce the numbers

$$b_n(j) = \binom{n}{j}\binom{n}{j+k}\binom{2j+k}{j-\ell},$$

so that $a_n(j) = k\, b_n(j)/n$. Let $B_n = \sum_j b_n(j)$. The sum runs over $j \in [\ell_+, n-k]$, where $\ell_+ = \max(\ell, 0)$. Below, we often consider j as a *real* variable running in that interval. The number $b_n(j)$ is then defined in terms of the Gamma function rather than in terms of factorials. We will show that B_n admits, for any N, an expansion of the form

$$B_n = 9^n \left(\sum_{i=1}^{N} \frac{c_i}{n^i} + O(n^{-N-1}) \right), \tag{16}$$

where the coefficients c_i depend on k and ℓ, and explain how to obtain these coefficients. We follow the standards steps for estimating sums of positive terms that are described, for instance, in [2, Section 3]. We begin with a unimodality property of the numbers $b_n(j)$.

Lemma 8 (**Unimodality**). *For n large enough, the sequence $b_n(j)$, for $j \in [\ell_+, n-k]$, is unimodal, and its maximum is reached in the interval $[2n/3 - k/2 - 1/2, 2n/3 - k/2 - 1/3]$.*

Proof. One has

$$\begin{aligned} q_n(j) &:= \frac{b_n(j)}{b_n(j+1)} \\ &= \frac{1}{(n-j)(n-j-k)} \frac{(j+1)(j+k+1)}{2j+k+1} \frac{(j-\ell+1)(j+k+\ell+1)}{2j+k+2}. \end{aligned}$$

Each of these three factors is easily seen to be an increasing function of j on the relevant interval. Moreover, for n large enough, $q_n(\ell_+) < 1$ while $q_n(n-k-1) > 1$. Let j_0 be the smallest value of j such that $q_n(j) \geq 1$. Then $b_n(j) < b_n(j+1)$ for $j < j_0$ and $b_n(j) \geq b_n(j+1)$ for $j \geq j_0$. We have thus proved unimodality. Solving $q_n(x) = 1$ for x helps to locate the mode:

$$x = \frac{2n}{3} - \frac{5+6k}{12} + O(1/n).$$

It is then easy to check that $q_n(2n/3 - k/2 - 1/2) < 1$ and $q_n(2n/3 - k/2 - 1/3) > 1$ for n large enough. ∎

The second step of the proof reduces the range of summation.

Lemma 9 (**A smaller range**). *Let $\epsilon \in (0, 1/6)$. Then for all m,*

$$B_n = \sum_{|j-2n/3| \leq n^{1/2+\epsilon}} b_n(j) + o(9^n n^{-m}).$$

Proof. Let $j = 2n/3 \pm n^{1/2+\epsilon}$. The Stirling formula gives

$$b_n(j) = \left(\frac{3}{2}\right)^{5/2} \frac{9^n}{(\pi n)^{3/2}} e^{-9n^{2\epsilon}/2} (1 + o(1)) = o(9^n n^{-m}) \tag{17}$$

for all m (the details of the calculation are given in greater detail below, in the proof of Lemma 10). Thus by Lemma 8,

$$\sum_{|j-2n/3|>n^{1/2+\epsilon}} b_n(j) \leq n\left(b_n(2n/3-n^{1/2+\epsilon})+b_n(2n/3+n^{1/2+\epsilon})\right) = o(9^n n^{-m}).$$

The result follows. ■

A first order estimate of $b_n(j)$, generalizing (17), suffices to obtain, upon summing over j, an estimate of B_n of the form (16) with $N=1$. However, numerous cancellations occur when summing the 6 terms $a_n(\ell,k,j)$ in the expression (15) of $C_3(n)$. This explains why we have to work out a longer expansion of the numbers $b_j(n)$ and B_n.

Lemma 10 (Expansion of $b_j(n)$). *Let $\epsilon \in (0,1/6)$. Write $j = 2n/3+r$ with $r = s\sqrt{n}$ and $|s| \leq n^\epsilon$. Then for all $N \geq 1$,*

$$\begin{aligned} b_n(j) &= \binom{n}{j}\binom{n}{j+k}\binom{2j+k}{j-\ell} \\ &= \left(\frac{3}{2}\right)^{5/2} \frac{9^n}{(\pi n)^{3/2}} e^{-9r^2/(2n)} \left(\sum_{i=0}^{N-1} \frac{c_i(s)}{n^{i/2}} + O(n^{N(3\epsilon-1/2)})\right) \end{aligned}$$

where $c_i(s)$ is a polynomial in k, ℓ and s, of degree $3i$ in s. Moreover, c_i is an even [odd] function of s if i is even [odd]. In particular, $c_0(s)=1$. This expansion is uniform in s.

Proof. Note that we simply want to *prove the existence* of an expansion of the above form. The coefficients can be obtained routinely using (preferably) a computer algebra system. In what follows, $(c_i)_{i\geq 0}$ stands for a sequence of real numbers such that $c_0=1$. The actual value of c_i may change from one formula to another. Similarly, $(c_i(s))_{i\geq 0}$ denotes a sequence of polynomials in s such that $c_0(s)=1$, having the parity property stated in the lemma.

We start from the Stirling expansion of the Gamma function: for all $N \geq 1$,

$$\Gamma(n+1) = n^n\sqrt{2\pi n}\, e^{-n}\left(\sum_{i=0}^{N-1}\frac{c_i}{n^i} + O(n^{-N})\right). \tag{18}$$

This gives, for $j=2n/3+r$, with $r=s\sqrt{n}$, and for any N,

$$\Gamma(j+1) = 2j^j\sqrt{\frac{\pi n}{3}}e^{-j}\left(\sum_{i=0}^{N-1}\frac{c_i(s)}{n^{i/2}} + O(n^{N(\epsilon-1/2)})\right) \tag{19}$$

for some polynomials $c_i(s)$ of degree i in s. This estimate is uniform in s. Similarly,

$$\Gamma(n-j+1) = (n-j)^{n-j}\sqrt{\frac{2\pi n}{3}}\, e^{j-n}\left(\sum_{i=0}^{N-1}\frac{c_i(s)}{n^{i/2}} + O(n^{N(\epsilon-1/2)})\right), \tag{20}$$

for some polynomials $c_i(s)$ of degree i in s. Now

$$\begin{aligned}\log \frac{n^n}{j^j(n-j)^{n-j}} &= \log \frac{3^n}{2^j} - \frac{9r^2}{4n} - \sum_{i\geq 2} \frac{3^i\, r^{i+1}}{i(i+1)n^i}\left(1-(-1/2)^i\right) \\ &= \log \frac{3^n}{2^j} - \frac{9r^2}{4n} - \sum_{i=1}^{N-1} \frac{c_i(s)}{n^{i/2}} + O(n^{2\epsilon+N(\epsilon-1/2)}),\end{aligned}$$

for some polynomials $c_i(s)$ of degree $i+2$ in s. Observe that $(i+2)/(i/2) \leq 6$ for $i \geq 1$. Hence

$$\begin{aligned}\frac{n^n}{j^j(n-j)^{n-j}} &= \frac{3^n}{2^j} e^{-9r^2/(4n)} \exp\left(-\sum_{i=1}^{N-1} \frac{c_i(s)}{n^{i/2}} + O(n^{2\epsilon+N(\epsilon-1/2)})\right) \\ &= \frac{3^n}{2^j} e^{-9r^2/(4n)} \left(\sum_{i=0}^{N-1} \frac{c_i(s)}{n^{i/2}} + O(n^{N(3\epsilon-1/2)})\right), \end{aligned} \tag{21}$$

for polynomials $c_i(s)$ of degree $3i$ in s. Putting together (18–21), one obtains, uniformly in s:

$$\binom{n}{j} = \frac{3}{2\sqrt{\pi n}} \frac{3^n}{2^j} e^{-9r^2/(4n)} \left(\sum_{i=0}^{N-1} \frac{c_i(s)}{n^{i/2}} + O(n^{N(3\epsilon-1/2)})\right), \tag{22}$$

for polynomials $c_i(s)$ of degree $3i$ in s.

Similarly,

$$\binom{n}{j+k} = \frac{3}{2\sqrt{\pi n}} \frac{3^n}{2^{j+k}} e^{-9r^2/(4n)} \left(\sum_{i=0}^{N-1} \frac{c_i(s)}{n^{i/2}} + O(n^{N(3\epsilon-1/2)})\right), \tag{23}$$

with the same degree condition on the polynomials $c_i(s)$. Finally,

$$\binom{2j+k}{j-l} = \sqrt{\frac{3}{2\pi n}} 2^{2j+k} \left(\sum_{i=0}^{N-1} \frac{c_i(s)}{n^{i/2}} + O(n^{N(\epsilon-1/2)})\right) \tag{24}$$

for polynomials $c_i(s)$ of degree i in s. Putting together (22–24), we obtain the estimate of $b_n(j)$ given in the lemma. ∎

It remains to sum our estimates of $b_n(j)$ for values of j such that $|j-2n/3| \leq n^{1/2+\epsilon}$.

Proposition 11 (**Expansion of B_n**)**.** *For all $N \geq 1$,*

$$\begin{aligned}B_n &= \sum_j \binom{n}{j}\binom{n}{j+k}\binom{2j+k}{j-\ell} \\ &= \left(\frac{3}{2}\right)^{5/2} \frac{9^n}{\pi^{3/2} n} \left(\sum_{i=0}^{N-1} \frac{1}{n^i} \int_{\mathbb{R}} c_{2i}(s) e^{-9s^2/2} ds + O(n^{-N})\right)\end{aligned}$$

where the polynomials $c_i(s)$*, depending on* k *and* ℓ*, are those of Lemma* 10. *In particular, as* $c_0(s) = 1$*,*

$$B_n(j) = \frac{3^{3/2}}{4\pi}\frac{9^n}{n}\left(1 + O(1/n)\right).$$

Proof. We start from Lemma 9, and combine it with the uniform expansion of Lemma 10. We need to estimate sums of the following type, for $i \in \mathbb{N}$:

$$\sum_{|j-2n/3|\le n^{1/2+\epsilon}} \left(\frac{j-2n/3}{\sqrt{n}}\right)^i e^{-9(j-2n/3)^2/(2n)}.$$

Using the Euler-MacLaurin summation formula [19, Eq. (5.62)], one obtains, for all $m \ge 1$,

$$\frac{1}{\sqrt{n}} \sum_{|j-2n/3|\le n^{1/2+\epsilon}} \left(\frac{j-2n/3}{\sqrt{n}}\right)^i e^{-9(j-2n/3)^2/(2n)} = \int_{\mathbb{R}} s^i e^{-9s^2/2} ds + o(n^{-m}).$$

The above integral vanishes if i is odd, and can be expressed in terms of the Gamma function otherwise. This gives the estimate of Proposition 11, but with a rest of order $n^{N(6\epsilon-1)}$. However, this expansion is valid for all N, and for all $\epsilon > 0$. From this, the rest can be seen to be of order n^{-N}. ∎

With the strategy described above (and Maple...), we have obtained the expansion of B_n to the order n^{-6}. Given that $a_n(j) \equiv a_n(\ell, k, j) = k\, b_n(j)/n$, this gives the expansion of $\sum_j a_n(\ell, k, j)$ to the order n^{-7}:

$$\sum_j a_n(\ell, k, j) = \frac{3^{3/2}}{4\pi}\frac{9^n}{n^2}\left(\sum_{i=0}^{5} \frac{c_i}{(4n)^i} + O(1/n^6)\right).$$

The coefficients of this expansion are too big to be reported here for generic values of k and ℓ (apart from $c_0 = k$). We simply give the values of c_i for the 6 pairs (ℓ, k) that are involved in the expression (15) of $C_3(n)$:

(ℓ, k)	c_0	c_1	c_2	c_3	c_4	c_5	
$(-1, 1)$	1	-7	37	-184	871	-4087	+
$(4, 1)$	1	-127	8317	-381904	14034391	-444126847	−
$(-5, 2)$	2	-230	13682	-573416	19338062	-564941270	+
$(-1, 2)$	2	-38	434	-4136	36302	-305558	−
$(-4, 3)$	3	-237	9831	-293664	7227813	-157405197	−
$(0, 3)$	3	-165	4863	-106104	1959573	-32693205	+

Each pair (ℓ, k) contributes in the expression of $C_3(n)$ with a weight ± 1, depending on the sign indicated on the corresponding line of the above table. One observes that the first 5 terms cancel, which leaves

$$C_3(n) = \frac{3^{3/2}}{4\pi}\frac{9^n}{n^2}\left(\frac{4199040}{(4n)^5} + O(1/n^6)\right).$$

This completes the proof of Proposition 1. ∎

2.7. Other starting points

So far, we have focused on the enumeration of Q_2-vacillating walks starting at $(1,0)$. However, our approach works just as well for other starting points, or for combinations of starting points. Let $A(x,y)$ be the generating function of starting points. For instance, $A(x,y) = x^i y^j$ corresponds to starting at (i,j), while $A(x,y) = 1/(1-x)$ corresponds to starting anywhere on the x-axis.

Then a calculation similar to that of Section 2.2 gives the following functional equation

$$K(x,y;t)F^A(x,y;t) = (x+y+xy)A(x,y) - xH^A(x;t) - yV^A(y;t),$$

where F^A, H^A, and V^A are the analogues of F, H and V. Specializing $A(x,y)$ to x gives back (7). The kernel method of Section 2.3 now gives

$$\begin{aligned} xH^A(x) + \bar{x}V^A(\bar{x}) = (x+Y+xY)\,A\,(x,Y) - \left(\bar{x}Y + Y + \bar{x}Y^2\right) A\,(\bar{x}Y, Y) \\ + \left(\bar{x}Y + \bar{x} + \bar{x}^2 Y\right) A\,(\bar{x}Y, \bar{x})\,. \end{aligned}$$

Thus if A is a rational function (and in particular if $A(x,y) = x^i y^j$), then $H^A(x)$, $V^A(x)$, and $F^A(x,y)$ are all D-finite.

3. Partitions with no enhanced 3-crossing

Our approach for counting 3-noncrossing partitions can be easily adapted to the enumeration of partitions avoiding *enhanced* 3-crossings. As discussed in [8], partitions of $[n]$ avoiding enhanced $k+1$-crossings are in bijection with *hesitating* tableaux of height at most k. In turn, these hesitating tableaux are in one-to-one correspondence with certain W_k-*hesitating* lattice walks. A hesitating lattice walk satisfies the following walking rules: when pairing every two steps from the beginning, each pair of steps has one of the following three types: i) a stay step followed by an e_i step, ii) a $-e_i$ step followed by a stay step, iii) an e_i step followed by a $-e_j$ step.

Then partitions of $[n]$ avoiding enhanced $k+1$-crossings are in bijection with W_k-hesitating walks of length $2n$ starting and ending at $(k-1,\dots,2,1,0)$. As before, W_k denotes the Weyl chamber $\{\,(a_1,a_2,\dots,a_k) \in \mathbb{Z}^k : a_1 > a_2 > \cdots > a_k \geq 0\,\}$.

For convenience, all notations we used for vacillating walks will be recycled for hesitating walks. Thus $\delta = (k-1,k-2,\dots,0)$, and for two lattice points λ and μ in W_k, we denote by $w_k(\lambda,\mu,n)$ (respectively $q_k(\lambda,\mu,n)$) the number of W_k-hesitating (respectively Q_k-hesitating) lattice walks of length n starting at λ and ending at μ. A careful investigation shows that the reflection principle “works” for hesitating lattice walks.

Proposition 12. *For any starting and ending points λ and μ in W_k, the number of W_k-hesitating walks going from λ to μ can be expressed in terms of the number of Q_k-hesitating walks as follows:*

$$w_k(\lambda,\mu,n) = \sum_{\pi \in \mathfrak{S}_k} (-1)^{\pi} q_k(\lambda, \pi(\mu), n),$$

where $(-1)^\pi$ *is the sign of* π *and* $\pi(\mu_1, \mu_2, \ldots, \mu_k) = (\mu_{\pi(1)}, \mu_{\pi(2)}, \ldots, \mu_{\pi(k)})$.

Proof. The key property here is that the set of pairs of steps that are allowed for a hesitating walk is left invariant when reflecting either one, or both steps with respect to the hyperplane $x_i = x_j$. This is clearly seen by abbreviating the three types of step pairs as $(0,+), (-,0)$ and $(+,-)$, and recalling that the above reflection exchanges the ith and jth coordinates (equivalently, the unit vectors e_i and e_j). The rest of the argument copies the proof of Proposition 4. ∎

Let us now focus on the case $k = 2$. The connection between W_2-hesitating walks and partitions avoiding enhanced 3-crossings entails

$$E_3(n) = w_2((1,0),(1,0),2n) = q_2((1,0),(1,0),2n) - q_2((1,0),(0,1),2n). \quad (25)$$

Let us write a functional equation counting Q_2-hesitating walks that start from $(1,0)$. Let $a_{i,j}(n) := q_2((1,0),(i,j),2n)$ be the number of such walks having length $2n$ and ending at (i,j). Let

$$F(x,y;t) = \sum_{i,j,n} a_{i,j}(2n) x^i y^j t^n$$

be the associated generating function. Then by appending to an (even length) walk an allowed pair of steps, we obtain the following functional equation:

$$\begin{aligned}(x + y + \bar{x} + \bar{y} + (x+y)(\bar{x}+\bar{y}))\, tF(x,y;t) \\ = F(x,y;t) - x + H(x;t)(\bar{y} + x\bar{y})t + V(y;t)(\bar{x} + \bar{x}y)t,\end{aligned}$$

where $H(x;t)$ (respectively $V(y;t)$) is the generating function of even lattice walks ending on the x-axis (respectively y-axis). This functional equation can be rewritten as

$$K(x,y;t)F(x,y;t) = x^2y - x(1+x)tH(x;t) - y(1+y)tV(y;t), \quad (26)$$

where $K(x,y;t)$ is the kernel given by

$$K(x,y;t) = xy - t(1+x)(1+y)(x+y).$$

From now on, we will very often omit the variable t in our notation.

Let us now solve (26). First fix x, and consider the kernel as a quadratic polynomial in y. Only one of its roots, denoted Y below, is a formal series in t:

$$Y = \frac{1 - t\bar{x}(1+x)^2 - \sqrt{\left(1 - t\bar{x}(1+x)^2\right)^2 - 4t\bar{x}(1+x)^2}}{2\,(1+\bar{x})\,t} = O(t).$$

The coefficients of this series are Laurent polynomials in x, as is easily seen from the equation

$$Y = t(1+\bar{x})(1+Y)(x+Y). \quad (27)$$

The second root of the kernel is $Y_1 = x/Y = O(t^{-1})$, and the expression $F(x, Y_1)$ is not well-defined.

Observe that the new kernel only differs from (8) by a term txy. Hence, as in the case of vacillating walks, the product of the roots is x, and the kernel is symmetric in x and y. This implies that the diagram of the roots, obtained by taking iteratively conjugates, is still given by Figure 1. Again, the 3 pairs of power series that are framed can be legally substituted for (x, y) in the functional equation (26). We thus obtain:

$$\begin{aligned} x(1+x)tH(x) + Y(1+Y)tV(Y) &= x^2Y, && (28)\\ \bar{x}Y(1+\bar{x}Y)tH(\bar{x}Y) + Y(1+Y)tV(Y) &= \bar{x}^2Y^3, && (29)\\ \bar{x}Y(1+\bar{x}Y)tH(\bar{x}Y) + \bar{x}(1+\bar{x})tV(\bar{x}) &= \bar{x}^3Y^2. && (30) \end{aligned}$$

Now (28)−(29)+(30) gives

$$x(1+x)tH(x) + \bar{x}(1+\bar{x})tV(\bar{x}) = x^2Y - \bar{x}^2Y^3 + \bar{x}^3Y^2.$$

By (27), the series $Y/t/(1+x)$ is a formal series in t with coefficients in $\mathbb{Q}[x, \bar{x}]$. Thus we can divide the above identity by $t(1+x)$, and then extract the positive and negative parts.

Proposition 13. *The series $H(x)$ and $V(y)$, which count Q_2-hesitating walks of even length ending on the x-axis and on the y-axis, satisfy*

$$xH(x) = \operatorname*{PT}_x \frac{Y}{t(1+x)}(x^2 - \bar{x}^2Y^2 + \bar{x}^3Y),$$
$$\bar{x}^2V(\bar{x}) = \operatorname*{NT}_x \frac{Y}{t(1+x)}(x^2 - \bar{x}^2Y^2 + \bar{x}^3Y).$$

Let us now return to the number $E_3(n)$ of partitions of $[n]$ avoiding enhanced 3-crossings, given by (25). The generating function $\mathcal{E}(t)$ of these numbers is:

$$\mathcal{E}(t) = \operatorname*{CT}_x \frac{Y(\bar{x}^2 - x^3)}{t(1+x)}(x^2 - \bar{x}^2Y^2 + \bar{x}^3Y). \tag{31}$$

Observe that, again, $Y(x) = xY(\bar{x})$. Therefore, for all $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}$,

$$\operatorname*{CT}_x \left(\frac{\bar{x}^\ell Y^k}{t(1+x)}\right) = \operatorname*{CT}_x \left(\frac{x^{\ell-k+1}Y^k}{t(1+x)}\right).$$

This allows us to rewrite (31) with only non-negative powers of x:

$$\mathcal{E}(t) = \operatorname*{CT}_x \frac{Y}{t(1+x)}\left(1 - x^5 - (x^2 - x)Y^2 + (x^4 - 1)Y\right). \tag{32}$$

The above equation shows that $\mathcal{E}(t)$ is the constant term of an algebraic function. It is thus *D-finite*. Let us now compute a linear differential equation it satisfies (equivalently, a P-recursion for its coefficients).

Starting from (27), the Lagrange inversion formula gives

$$[t^n] \operatorname*{CT}_x \frac{x^\ell Y^k}{t(1+x)} = \sum_{j\in\mathbb{Z}} a_n(\ell, k, j)$$

with

$$a_n(\ell,k,j) = \frac{k}{n+1}\binom{n+1}{j}\binom{n+1}{j+k}\binom{n}{j-\ell}.$$

From (32), we obtain

$$E_3(n) = \sum_{j\in\mathbb{Z}} \big(a_n(0,1,j) - a_n(5,1,j) - a_n(2,3,j) + a_n(1,3,j) \\ + a_n(4,2,j) - a_n(0,2,j)\big)\,. \quad (33)$$

This gives an explicit (but not so simple) expression of $E_3(n)$, to which we apply Zeilberger's algorithm for creative telescoping. This proves that, for $n \geq 1$,

$$8n\,(n-1)\,(n-2)\,E_3(n-3) + 3\left(5n^2+17n+8\right)(n-1)\,E_3(n-2) \\ + 3\,(n+1)\,(2n+5)\,(n+4)\,E_3(n-1) - (n+6)\,(n+5)\,(n+4)\,E_3(n) = 0,$$

with initial condition $E_3(0) = 1$. It is then straightforward to check that the sequence defined in Proposition 2 also satisfies the above P-recursion. More precisely, applying the operator $(n+2)+(n+7)N$ to the recursion of Proposition 2 gives the above four term recursion.

The study of the aymptotic behaviour of $E_3(n)$ parallels what we did for $C_3(n)$. The maximum of $a_n(\ell,k,j)$ is now reached for $j \sim n/2$ rather than $2n/3$. Using the same notations as in Section 2.6, we obtain

$$B_n(j) \sim \frac{8^{n+1}}{\sqrt{3}\pi n},$$

but again, numerous cancellations occur when we sum the 6 required estimates, so as to obtain the estimate of Proposition 2.

4. **Final comments**

It is natural to ask whether for any k, the sequence $C_k(n)$ that counts k-noncrossing partitions of $[n]$ is P-recursive. Our opinion is that this is unlikely, at least for $k = 4$. This is based on the following observations:

(1) We have written a functional equation for Q_3-vacillating walks, with kernel

$$K(x,y,z;t) = 1 - t(1+x+y+z)(1+\bar{x}+\bar{y}+\bar{z}).$$

Using this equation, we have computed the first 100 numbers in the sequence $C_4(n)$. This is sufficient for the Maple package Gfun to discover a P-recursion of order 8 with coefficients of degree 8, if it exists. But no such recursion has been found.

(2) Let us solve the above kernel in z. The two roots Z_0 and Z_1 are related by

$$Z_0 Z_1 = \frac{1+x+y}{1+\bar{x}+\bar{y}}. \quad (34)$$

Since the kernel is symmetric in x, y and z, the diagram of the roots, obtained by taking conjugates, is generated by the transformations Φ_i for $i = 1, 2, 3$, where

$$\Phi_3(x, y, z) = \left(x, y, \bar{z}\, \frac{1 + x + y}{1 + \bar{x} + \bar{y}} \right)$$

and Φ_1 and Φ_2 are defined similarly. But these transformations now generate an *infinite* diagram. There exist in the literature a few signs indicating that a finite (respectively infinite) diagram is related to a D-finite (respectively non-D-finite) generating function. First of all, a number of equations with a finite diagram have been solved, and shown to have a D-finite solution [3, 4, 5, 18]. Furthermore, the only problem with an infinite diagram that has been thoroughly studied has been proved to be non-D-finite [7]. Finally, the conjectural link between finite diagrams and D-finite series is confirmed by a systematic numerical study of walks in the quarter plane [18].

The above paragraphs can be copied verbatim for W_3-hesitating walks and partitions avoiding enhanced 4-crossings. The kernel is now

$$K(x, y, z) = 1 - t(x + y + z + \bar{x} + \bar{y} + \bar{z} + (x + y + z)(\bar{x} + \bar{y} + \bar{z})),$$

but the roots Z_0 and Z_1 are still related by (34).

As recalled in the introduction, the sequence $M_k(n)$ that counts k-noncrossing *matchings* of $[n]$ (that is, partitions in which all blocks have size 2) is D-finite for all k. More precisely, the associated exponential generating function,

$$\mathcal{M}_k(t) = \sum_n M_k(n) \frac{t^{2n}}{(2n)!}$$

is given by [11]:

$$\mathcal{M}_k(t) = \det \left(I_{i-j}(2t) - I_{i+j}(2t) \right)_{1 \le i,j \le k-1},$$

where

$$I_n(2t) = \sum_{j \ge 0} \frac{t^{n+2j}}{j!(n+j)!}$$

is the hyperbolic Bessel function of the first kind of order n. The existence of such a closed form implies that $\mathcal{M}_k(t)$ is D-finite [17]. The specialization to matchings of the bijection between partitions and vacillating walks results in a bijection between $k+1$-noncrossing matchings and *oscillating tableaux* of height at most k, or, equivalently, W_k-oscillating walks. These walks can take any (positive or negative) unit step $\pm e_i$, without any parity restriction. The kernel of the equation ruling the enumeration of such walks is simply

$$K(x_1, \ldots, x_k) = 1 - t(x_1 + \cdots + x_k + \bar{x}_1 + \cdots + \bar{x}_k).$$

The diagram of the roots, generated by the Φ_i, for $1 \leq i \leq k$, where $\Phi_i(x_1, \ldots, x_k) = (x_1, \ldots, x_{i-1}, \bar{x}_i, x_{i+1}, \ldots, x_k)$, is now finite (the group of transformations Φ_i being itself isomorphic to $(\mathbb{Z}/2\mathbb{Z})^k$). This, again, confirms the possible connection between finite diagrams and D-finite series.

Acknowledgements: We are grateful to Anders Björner and Richard Stanley for inviting us to the "Algebraic Combinatorics" program at the Institut Mittag-Leffler in Spring 2005, during which this work was done. We also thank Christian Krattenthaler for explaining us how the asymptotics of our sequences could be worked out completely. MBM was partially supported by the European Commission's IHRP Programme, grant HPRN-CT-2001-00272, "Algebraic Combinatorics in Europe". GX would like to thank Richard Stanley and Rosena Du for introducing him to this topic.

References

[1] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, and D. Gouyou-Beauchamps. Generating functions for generating trees. *Discrete Math.*, 246(1-3):29–55, 2002.

[2] E. A. Bender. Asymptotic methods in enumeration. *SIAM Rev.*, 16:485–515, 1974.

[3] M. Bousquet-Mélou. Counting walks in the quarter plane. In *Mathematics and computer science* 2*, (Versailles,* 2002*)*, Trends Math., pages 49–67. Birkhäuser, Basel, 2002.

[4] M. Bousquet-Mélou. Four classes of pattern-avoiding permutations under one roof: generating trees with two labels. *Electron. J. Combinatorics*, 9(2):Research Paper 19, 2003.

[5] M. Bousquet-Mélou. Walks in the quarter plane: Kreweras' algebraic model. *Ann. Appl. Probab.*, 15(2):1451–1491, 2005.

[6] M. Bousquet-Mélou and M. Petkovšek. Linear recurrences with constant coefficients: the multivariate case. *Discrete Math.*, 225(1-3):51–75, 2000.

[7] M. Bousquet-Mélou and M. Petkovšek. Walks confined in a quadrant are not always D-finite. *Theoret. Comput. Sci.*, 307:257–276, 2003.

[8] W. Chen, E. Deng, R. Du, R. Stanley, and C. Yan. Crossings and nestings of matchings and partitions. *Trans. Amer. Math. Soc.*, to appear. Arχiv math.CO/0501230.

[9] G. Fayolle, R. Iasnogorodski, and V. Malyshev. *Random walks in the quarter-plane: Algebraic methods, boundary value problems and applications*, volume 40 of *Applications of Mathematics*. Springer-Verlag, Berlin, 1999.

[10] I. M. Gessel and D. Zeilberger. Random walk in a Weyl chamber. *Proc. Amer. Math. Soc.*, 115(1):27–31, 1992.

[11] D. J. Grabiner and P. Magyar. Random walks in Weyl chambers and the decomposition of tensor powers. *J. Algebraic Combin.*, 2(3):239–260, 1993.

[12] E. L. Ince. *Ordinary Differential Equations*. Dover Publications, New York, 1944.

[13] A. Kasraoui and J. Zeng. Distribution of crossings, nestings and alignments of two edges in matchings and partitions. Arχiv math.CO/0601081.

[14] M. Klazar. Bell numbers, their relatives, and algebraic differential equations. *J. Combin. Theory Ser. A*, 102(1):63–87, 2003.

[15] C. Krattenthaler. Growth diagrams, and increasing and decreasing chains in fillings of Ferrers shapes. *Adv. Appl. Math.*, to appear. Arχiv:math.CO/0510676.

[16] G. Kreweras. Sur les partitions non croisées d'un cycle. *Discrete Math.*, 1(4):333–350, 1972.

[17] L. Lipshitz. D-finite power series. *J. Algebra*, 122:353–373, 1989.

[18] M. Mishna. Classifying lattice walks in the quarter plane. In preparation.

[19] A. M. Odlyzko. Asymptotic enumeration methods. In *Handbook of combinatorics, Vol. 1, 2*, pages 1063–1229. Elsevier, Amsterdam, 1995.
[20] M. Petkovšek, H. S. Wilf, and D. Zeilberger. $A = B$. A K Peters Ltd., Wellesley, MA, 1996.
[21] B. Salvy and P. Zimmermann. Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable. *ACM Transactions on Mathematical Software*, 20(2):163–177, 1994.
[22] R. Simion. Noncrossing partitions. *Discrete Math.*, 217(1-3):367–409, 2000.
[23] R. P. Stanley. *Enumerative combinatorics* 2, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999.
[24] J. Wimp and D. Zeilberger. Resurrecting the asymptotics of linear recurrences. *J. Math. Anal. Appl.*, 111(1):162–176, 1985.
[25] D. Zeilberger. André's reflection proof generalized to the many-candidate ballot problem. *Discrete Math.*, 44(3):325–326, 1983.

CNRS, LaBRI, Université Bordeaux 1, 351 cours de la Libération, 33405 Talence Cedex, France
E-mail address: mireille.bousquet@labri.fr

941 Patterson Office Tower, Department of Mathematics, University of Kentucky, Lexington, KY 40506–0027, USA
E-mail address: guoce.xin@gmail.com