ADDING ±1 TO THE ARGUMENT OF A HALL–LITTLEWOOD POLYNOMIAL

ALAIN LASCOUX

Dédié à Xavier Viennot

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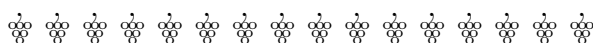
ABSTRACT. Shifting by ±1 power sums: $p_i \to p_i \pm 1$ induces a transformation on symmetric functions that we detail in the case of Hall–Littlewood polynomials. By iteration, this gives a description of these polynomials in terms of plane partitions, as well as some generating functions. We recover in particular an identity of Warnaar related to Rogers–Ramanujan identities.

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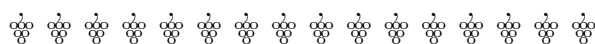
1. INTRODUCTION

To free analysis from infinitesimal quantities, and to quieten down the metaphysical anguish of Bishop Berkeley[1], Lagrange [3] proposed to replace differential calculus by the study of the behaviour of a function[2] fx under the addition of an "increment" y to x:

$$fx \to f(x + y) .$$

This works perfectly well, at least under the mild proviso of analyticity in the neighbourhood of x.

One can adopt the same strategy in the realm of symmetric polynomials, adding a letter x to a set of indeterminates (we say *alphabet A*). However, there exists a canonical involution on symmetric functions, and

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[1]Il s'est élevé un Docteur ennemi de la Science qui a déclaré la guerre aux Mathématiciens; ce Docteur monte en Chaire pour apprendre aux fidèles que la Géométrie est contraire à la religion; il leur dit d'être en garde contre les Géomètres, ce sont, selon lui, des gens aveugles et indociles qui ne savent ni raisonner, ni croire; des visionnaires qui se refusent aux choses simples et qui donnent tête baissée dans les merveilles.... , préface de Buffon [9]. Other (anonymous) defenses of infinitesimals have been published in England. See, for example, *A defence of free-thinking in Mathematics*, London (1735), and *Geometry no friend to infidelity*, London (1734).

[2]Lagrange, in harmony with D. Knuth, writes fx and $f(x+y)$, forgetting parentheses when the argument is a single letter. We shall follow Lagrange, when no ambiguity is to be feared.

this entails that there exists in fact two versions of the "addition" of x. In terms of λ-rings, one has to study both transformations

$$fA \rightarrow f(A + x) \,, \qquad fA \rightarrow f(A - x) \,,$$

which may look rather different when considering explicit polynomials.

Of course, when restricting to homogeneous polynomials, one does not lose any information by specializing x to 1, i.e., by studying instead

$$fA \rightarrow f(A + 1) \,, \qquad fA \rightarrow f(A - 1) \,.$$

We consider these two operations in the case of Hall–Littlewood polynomials in Theorem 3.1 and Theorem 4.2.

An interesting outcome is a short proof of an identity of Warnaar (5.2) concerning a generating function of Hall–Littlewood polynomials related to the Rogers–Ramanujan identities.

Recall that the ring of symmetric polynomials $\mathfrak{Sym}(X)$ in an infinite set of indeterminates $X = \{x_1, x_2, \ldots\}$, with coefficients in $\mathbb{Q}[[t]]$, t another indeterminate, admits a linear basis, the *Schur functions* $S_\lambda X$, indexed by all partitions[3] λ.

One can take, for a symmetric function, more general arguments than a set of indeterminates, and we shall need to use, given two sets of indeterminates X, Y, and a symmetric function f, the functions $f(X \pm Y)$, $f(XY)$, $f(X(1 - t))$, $f(X \pm 1)$. Since $\mathfrak{Sym}(X)$ is a ring of polynomials in the power sums p_i, the above symmetric functions are induced from the case where f is a power sum, setting

$$p_i(X \pm Y) = p_i X \pm p_i Y \,, \ p_i(XY) = p_i X p_i Y \,,$$
$$p_i(X(1 - t)) = (1 - t^i) p_i X \,, \ p_i(X \pm 1) = p_i X \pm 1 \,.$$

For more informations about the flexibility of arguments of symmetric functions, and the use of λ-rings, see [5].

We shall also need the generating function of complete functions:

$$\sigma_1 X := \prod_{x \in X} (1 - x)^{-1} = \sum_i S_i X \,.$$

Schur functions occur naturally when decomposing the Cauchy kernel $\sigma_1(XY)$. The Hall–Littlewood polynomials, our present concern, are associated to the kernel $\sigma_1(XY(1 - t)) := \prod_{x \in X, y \in Y} (1 - txy)(1 - xy)^{-1}$.

2. Hall–Littlewood Polynomials

From now on, we fix a positive integer n. $\mathfrak{Part}$ will be the set of partitions of length not more than n, considered as elements of $\mathbb{N}^n$. Schur functions may be defined as determinants of complete functions, and

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[3]We follow Macdonald's conventions [8]. Partitions are weakly decreasing sequences of non-negative integers. One identifies two such sequences if they only differ by trailing 0's on the right.

this allows to extend their indexation to any $v \in \mathbb{Z}^n$, n arbitrary. This amounts to introducing the relations[4]

$$S_v = -S_{\ldots,v_{i+1}-1,v_i+1,\ldots} \ , \ \ S_{v_1,\ldots,v_n} = 0 \text{ if } v_n < 0 \ . \tag{1}$$

A more powerful point of view than using straightening relations is to introduce *symmetrizing operators*, which can be defined as products of isobaric divided differences π_i, $i = 1, 2, \ldots$ (operators act on their left):

$$f \to f\,\pi_i := \frac{x_i f - x_{i+1} f^{s_i}}{x_i - x_{i+1}}$$

where s_i acts on functions of X by transposition of x_i, x_{i+1}.

The operator $x^v \to S_v(x_1, \ldots, x_n)$, $v \in \mathbb{N}^n$, can be expressed as a product, denoted π_ω, of π_i. It can also be written as a summation over the symmetric group $\mathfrak{S}_n$:

$$f\,\pi_\omega = \sum_{\sigma \in \mathfrak{S}_n} \left(f \prod_{1 \leq i < j \leq n} (1 - x_j/x_i)^{-1} \right)^\sigma .$$

We refer to Chapter 7 of [5] for some of its properties.

In particular, for any i, one has $\pi_i\,\pi_\omega = \pi_\omega$, and the reordering (1) of the indexation of Schur functions comes from the relation $x_{i+1}\,\pi_i = 0$. Indeed, if $v \in \mathbb{Z}^n$ is such that $v_i = v_{i+1}$, and $\alpha, \beta \in \mathbb{Z}$, then

$$x^v \left(x_i^\alpha x_{i+1}^\beta + x_i^\beta x_{i+1}^\alpha \right) x_{i+1}\,\pi_i = x_{i+1}\,\pi_i\,x^v \left(x_i^\alpha x_{i+1}^\beta + x_i^\beta x_{i+1}^\alpha \right) = 0 \, ,$$

because symmetric functions in x_i, x_{i+1} commute with π_i.

Some care is needed when taking exponents or indices in $\mathbb{Z}^n$, instead of $\mathbb{N}^n$ (this corresponds to the difference between using characters of the symmetric group, or of the linear group).

We first extend the natural order on partitions to elements of $\mathbb{Z}^n$ by

$$v \geq u \quad \text{if and only if for all } k > 0, \quad \sum_{i=k}^{n} (v_i - u_i) \geq 0 \, .$$

The operator π_ω commutes with multiplication by any power of $x_1 \cdots x_n$. In the case of a positive power, one has $(x_1 \cdots x_n)^k S_\lambda = S_{\lambda+[k,\ldots,k]}$. But in the case of a negative power, we also have to obey the rule that $S_v = 0$ when $v_n = 0$. The solution is to combine the symmetrization with a truncation operator:

$$x^v \to x^v \quad \text{if} \quad v \geq 0 \quad , \quad x^v \to 0 \quad \text{otherwise.}$$

Denote by $⊎$ the operator "truncation followed by π_ω." Now, one has $x^v ⊎ = S_v$, for all $v \in \mathbb{Z}^n$, and moreover, one can compute the image of a Laurent series with a finite number of terms of exponents ≥ 0.

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[4]For example, $S_{-2,2,0,0} = -S_{1,-1,0,0} = S_{1,-1,0,0,0} = 0$ and $S_{-3,2,1,1} = -S_{1,-2,1,1} = S_{1,0,-1,1} = -S_{1,0,0,0} = -S_1$.

Introducing an extra indeterminate t, and given any $u \in \mathbb{Z}^n$, one defines the *modified Hall–Littlewood polynomial* Q'_u by

$$Q'_u = x^u \prod_{1 \le i < j \le n} (1 - tx_i/x_j)^{-1} \ \biguplus \ . \tag{2}$$

This is, in fact, a finite sum of Schur functions, because we first eliminate in the expansion of $x^u \prod(1 - tx_i/x_j)^{-1}$ all the monomials which are not ≥ 0.

The set $\{Q'_\lambda : \lambda \in \mathfrak{Part}\}$ is a basis of $\mathfrak{Sym}$, which specializes to the basis of Schur functions for $t = 0$. Any Q'_u can be expressed in terms of the Q'_λ thanks to the following relations:

$$Q'_{\dots,\alpha,\beta+1,\dots} + Q'_{\dots,\beta,\alpha+1,\dots} - t\, Q'_{\dots,\beta+1,\alpha,\dots} - t\, Q'_{\dots,\alpha+1,\beta,\dots} = 0\,. \tag{3}$$

These relations still result from $x_{i+1}\pi_i = 0$, because, when v is such that $v_i = v_{i+1}$, then

$$x^v \left(x_i^\alpha x_{i+1}^\beta + x_i^\beta x_{i+1}^\alpha \right) \frac{(1 - tx_i/x_{i+1})}{\prod_{j<h}(1 - tx_j/x_h)} \, x_{i+1}\pi_i = 0\,,$$

the factor on the left of x_{i+1} being symmetrical in x_i, x_{i+1}, and therefore commuting with π_i.

Relations (3), together with $Q'_{\dots,u_n} = 0$ if $u_n < 0$, suffice to express any Q'_u in terms of the $Q'_\lambda : \lambda \in \mathfrak{Part}$.

The other types of Hall–Littlewood polynomials are

$$Q_\lambda X := Q'_\lambda\big(X\,(1 - t)\big)\,, \tag{4}$$

$$P_\lambda X := b_\lambda^{-1}\, Q_\lambda X\,, \tag{5}$$

with $b_\lambda = \prod_i \prod_{j=1}^{m_i}(1 - t^j)$, for $\lambda = 1^{m_1}2^{m_2}3^{m_3}\cdots$.

These functions can also be defined by symmetrization, but problems arise when taking general weights instead of only *dominant weights* (i.e., partitions), see the terminal note.

We give in Corollary 3.2 a description of the functions Q'_λ in terms of *plane partitions*. Recall that the Q'_λ admit another combinatorial description, this time in terms of tableaux:

$$Q'_\mu = \sum_T t^{\mathfrak{ch}T}\, S_{\mathfrak{sh}T}\,, \tag{6}$$

where the sum is over all tableaux of weight μ, $\mathfrak{sh}T$ being the shape of T, and $\mathfrak{ch}T$ being the *charge* of T (cf. [8, III.6])[5].

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[5]The charge is a *rank function* on the set of all tableaux. More generally, it can be defined as a function on words in letters $1, 2, 3, \dots$, satisfying the following relations in the case of a dominant evaluation, i.e., when $\mu_1 \ge \mu_2 \ge \mu_3 \ge \cdots$:

$$\mathfrak{ch}\left(\cdots 3^{\mu_3} 2^{\mu_2} 1^{\mu_1}\right) = 0\,, \quad \mathfrak{ch}(w\,i) = \mathfrak{ch}(iw) + 1 \text{ if } i > 1\,, \tag{7}$$

and the invariance with respect to the plactic relations [6]:

$$w \equiv w' \ \Rightarrow \ \mathfrak{ch}w = \mathfrak{ch}w'\,. \tag{8}$$

3. Adding 1

Adding ± 1 to the argument of a symmetric function is, as we already said, induced by the transformation

$$p_i \to p_i \pm 1 \,,\; i = 1, 2, \ldots$$

of the powers sums p_i.

In terms of Schur functions, for a partition $\lambda \in \mathbb{N}^n$, this amounts to

$$S_\lambda(X - 1) = \sum_{v \in \{0,1\}^n} (-1)^{|v|} S_{\lambda - v} X \,, \tag{9}$$

$$S_\lambda(X + 1) = \sum_{u \in \mathbb{N}^n} S_{\lambda + u} X \,. \tag{10}$$

Both summations can be reduced to a summation over partitions, erasing vertical or horizontal strips from the diagram of λ [8, I.5].

One can rewrite (9,10) as

$$S_\lambda(X - 1) = x^\lambda \prod_{1 \le i \le n} (1 - 1/x_i) \, \pi_\omega \,, \tag{11}$$

$$S_\lambda(X + 1) = x^\lambda \prod_{1 \le i \le n} \frac{1}{1 - 1/x_i} \, \pi_\omega \,, \tag{12}$$

and therefore,

$$Q'_\lambda(X - 1) = x^\lambda \frac{\prod_{1 \le i \le n}(1 - 1/x_i)}{\prod_{1 \le i < j \le n}(1 - tx_i/x_j)} \pi_\omega = \sum_{v \in \{0,1\}^n} (-1)^{|v|} Q'_{\lambda - v} X \,, \tag{13}$$

$$Q'_\lambda(X + 1) = x^\lambda \frac{1}{\prod_{1 \le i \le n}(1 - 1/x_i)} \frac{1}{\prod_{1 \le i < j \le n}(1 - tx_i/x_j)} \pi_\omega \,. \tag{14}$$

The first summation is easy to reduce, using the reordering

$$\sum_v Q'_{k^m - v} = \begin{bmatrix} m \\ \alpha \end{bmatrix} Q'_{k^{m-\alpha},(k-1)^\alpha} \,, \tag{15}$$

where v runs over all permutations of $[1^\alpha, 0^{m-\alpha}]$, and where $\begin{bmatrix} m \\ \alpha \end{bmatrix}$ denotes the t-binomial coefficient $(1 - t^m) \cdots (1 - t^{m-\alpha+1})/(1 - t) \cdots (1 - t^\alpha)$.

Iterating (15), one gets

$$Q'_\lambda(X - 1) = \sum_\mu \prod_i (-1)^{\alpha_i} \begin{bmatrix} m_i \\ \alpha_i \end{bmatrix} Q'_\mu \,, \tag{16}$$

where the sum is over all partitions $\mu = 0^{\alpha_1} 1^{m_1 - \alpha_1} 1^{\alpha_2} 2^{m_2 - \alpha_2} 2^{\alpha_3} 3^{m_3 - \alpha_3} \cdots$ differing from $\lambda = 1^{m_1} 2^{m_2} 3^{m_3} \cdots$ by a vertical strip.

The second summation is a little more complicated to transform, and contrary to the case of Schur functions, will not restrict to erasing strips.

Let us first introduce *skew Hall–Littlewood polynomials* [8, III.5] $Q'_{\lambda/\mu}$, by

$$Q'_\lambda(X+Y) = \sum_{\mu\in\mathfrak{Part}} Q'_{\lambda/\mu}X\, Q'_\mu Y\,, \qquad (17)$$

i.e. $Q'_{\lambda/\mu}X$ is defined as the coefficient of $Q'_\mu Y$ in the expansion of the function Q'_λ evaluated in $X+Y$. Note that this definition makes sense for $\lambda\in\mathbb{Z}^n$, and not only for $\lambda\in\mathfrak{Part}$.

For any pair of partitions λ,μ, let us define

$$\mathbf{n}(\lambda/\mu) = \sum_i (\lambda_i^\sim - \mu_i^\sim)(\lambda_i^\sim - \mu_i^\sim - 1)/2\,,$$

where $\lambda^\sim$ and $\mu^\sim$ are the partitions conjugate to λ, μ respectively. In the case where $\mu=0$, then one writes $\mathbf{n}(\lambda)$ instead of $\mathbf{n}(\lambda/0)$. Moreover, $\mathbf{n}(\lambda) = 0\lambda_1 + 1\lambda_2 + 2\lambda_3 + \cdots$.

The main result of this section is the following evaluation of $Q'_{\lambda/\mu}1$.

Theorem 3.1. *Given a partition λ, then*

$$Q'_\lambda(X+1) = \sum_{\mu\subseteq\lambda} Q'_\mu X\, \aleph(\lambda/\mu)\,,$$

with

$$Q'_{\lambda/\mu}1 = \aleph(\lambda/\mu) := b_\mu^{-1}\, t^{\mathbf{n}(\lambda/\mu)}\, (1-t^{\nu_1-0})(1-t^{\nu_2-1})\cdots(1-t^{\nu_r-r+1})\,, \quad (18)$$

$\nu_1,\ldots,\nu_r$ *being the parts of index $\mu_1,\ldots,\mu_r$, $r=\ell(\mu)$, of the partition conjugate to λ.*

One can visualize the value of $Q'_{\lambda/\mu}1 = \aleph(\lambda/\mu)$ in the Cartesian plane as follows: colour in black the right-most box of each row of the diagram of μ, write $0,1,2,\ldots$ in the successive boxes of the same column of λ/μ. Each column with α black boxes, and β boxes above, gives a contribution $t^{\binom{\beta}{2}}\begin{bmatrix}\alpha+\beta\\\alpha\end{bmatrix}$ to $Q'_{\lambda/\mu}(1)$, which is the product over all columns of these contributions. For example, for $\lambda=[4432221]$, $\mu=[2211]$, then $\lambda^\sim=[7632]$, $\nu=[6677]$, $\mathbf{n}(\lambda/\mu)=13$, $b_\mu=(1-t)^2(1-t^2)^2$,

$$Q'_{4432221/2211}1 = t^{13}(1-t^6)(1-t^{6-1})(1-t^{7-2})(1-t^{7-3})(1-t)^{-2}(1-t^2)^{-2}\,,$$

$$\Rightarrow\quad Q'_{4432221/2211}1 = t^3\begin{bmatrix}5\\2\end{bmatrix}\, t^6\begin{bmatrix}6\\2\end{bmatrix}\, t^3\begin{bmatrix}3\\0\end{bmatrix}\, t^1\begin{bmatrix}2\\0\end{bmatrix}\,.$$

Proof of the theorem. We shall evaluate $\sum_u Q'_{\lambda - u}X$ by decomposing the set $u \in \mathbb{N}^n$ into two subsets, according to whether $u_1 = 0$ or not.

Let a, j, k be such that

$$\lambda_1 = a = \cdots = \lambda_k > \lambda_{k+1} \ , \ \ \mu_1 = a = \cdots = \mu_j > \mu_{j+1} \,,$$

and write $\lambda = a^k \zeta$, $\mu = a^j \eta$.

In terms of these parameters, using (15), one wants to show that

$$Q'_{\lambda/\mu} 1 = t^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix} Q'_{[a^{k-j}\zeta]/\eta}(1) \,. \tag{19}$$

The sub-sum $\sum_{u:\, u_1=0} Q'_{\lambda - u}X$ is equal to

$$Q'_a X \odot Q'_{a^{k-1}\zeta}(X + 1) \,,$$

where, by definition, $Q'_a \odot Q'_\nu$ is the concatenation $Q'_{a,\nu}$, and extending by linearity.

By induction on $\ell(\lambda)$, the coefficient of $Q'_\mu X$ in $Q'_a X \odot Q'_{a^{k-1}\zeta}(X + 1)$ is equal to

$$t^{\binom{k-j}{2}} \begin{bmatrix} k - 1 \\ j - 1 \end{bmatrix} Q'_{[a^{k-j}\zeta]/\eta} 1 \ .$$

The terms $Q'_{\lambda - u}X$, $u_1 \geq 1$, can be rewritten

$$\sum_{u \in \mathbb{N}^n} Q_{\lambda^- - u} X Q'_{\lambda^-}(X + 1) = t^{k-1} Q'_{a^{k-1}, a-1, \zeta}(X + 1) \,,$$

with $\lambda^- = [a{-}1, \lambda_2, \lambda_3, \dots]$.

By induction on $|\lambda|$, the coefficient of $Q'_\mu X$ in this function is equal to

$$t^{k-1} t^{\binom{k-1-j}{2}} \begin{bmatrix} k - 1 \\ j \end{bmatrix} Q'_{[a^{k-j}\zeta]/\eta} 1 \ .$$

The identity

$$\begin{bmatrix} k - 1 \\ j - 1 \end{bmatrix} + t^{k-1+\binom{k-1-j}{2}-\binom{k-j}{2}} \begin{bmatrix} k - 1 \\ j \end{bmatrix} = \begin{bmatrix} k - 1 \\ j - 1 \end{bmatrix} + t^j \begin{bmatrix} k - 1 \\ j \end{bmatrix} = \begin{bmatrix} k \\ j \end{bmatrix}$$

allows to sum up the two contributions and finishes the proof. $\square$

For example, $Q'_{221}(X + 1)$ is obtained by the following enumeration:

$$= t^4 Q'_0 + t^2(1 + t + t^2)Q'_1 + t(1 + t)Q'_2 + t(1 + t + t^2)Q'_{11} + (1 + t)^2 Q'_{21}$$
$$+ t Q'_{111} + Q'_{22} + (1 + t)Q'_{211} + Q'_{221} \,.$$

Recall that a *plane partition* of shape a partition λ is a filling of the diagram of λ with positive integers such that numbers weakly increase when faring towards the origin. In other words, the successive domains occupied by the letters $1, 2, , 3, \ldots$ are skew partitions $\lambda/\lambda^1, \lambda^1/\lambda^2, \lambda^2/\lambda^3, \ldots$. Define the *weight* of a plane partition $\boxplus$ to be the product

$$\aleph_x(\boxplus) = \aleph(\lambda/\lambda^1)x_1^{|\lambda/\lambda^1|}\aleph(\lambda^1/\lambda^2)x_2^{|\lambda^1/\lambda^2|}\aleph(\lambda^2/\lambda^3)x_3^{|\lambda^2/\lambda^3|}\cdots.$$

Iteration of Theorem 3.1 leads to the following description of Q'_λ.

Corollary 3.2. *Let λ be a partition, n be a positive integer. Then*

$$Q'_\lambda(x_1 + \cdots + x_n) = \sum_\boxplus \aleph_x(\boxplus),$$

where the sum is over all plane partitions in $1, \ldots, n$ of shape λ.

For example, for $\lambda = [21]$, one has

$$Q'_{21} = \sum_i t\,\boxed{\begin{array}{c}i\\ \hline i \mid i\end{array}} + \sum_{i<j}(1{+}t)\,\boxed{\begin{array}{c}i\\ \hline j \mid i\end{array}} + t\,\boxed{\begin{array}{c}j\\ \hline j \mid i\end{array}} + \boxed{\begin{array}{c}i\\ \hline j \mid j\end{array}}$$

$$+ \sum_{i<j<k}\boxed{\begin{array}{c}i\\ \hline k \mid j\end{array}} + (1{+}t)\boxed{\begin{array}{c}j\\ \hline k \mid i\end{array}}$$

$$= \sum_i t\,x_i^3 + \sum_{i<j}(1{+}t)\,x_i^2 x_j + t\,x_i x_j^2 + x_i x_j^2 + \sum_{i<j<k}(2+t)x_i x_j x_k.$$

On the other hand, the interpretation in terms of tableaux and charge reads

$$Q'_{21} = S_{21} + tS_3$$

$$= \sum_{i<j}\boxed{\begin{array}{c}j\\ \hline i \mid i\end{array}} + \boxed{\begin{array}{c}j\\ \hline i \mid j\end{array}} + \sum_{i<j<k}\boxed{\begin{array}{c}j\\ \hline i \mid k\end{array}} + \boxed{\begin{array}{c}k\\ \hline i \mid j\end{array}} + \sum_{i\le j\le k} t\,\boxed{i \mid j \mid k}.$$

4. ARGUMENT $1 - X$

Instead of describing $Q'_\lambda(X - 1)$, we prefer taking $Q'_\lambda(1 - X)$. Addition and subtraction of alphabets involve rather different properties of symmetric functions. In the case of Hall–Littlewood polynomials, with a hypothesis on λ, we shall find a connection between $Q'_\lambda(1 - X)$ and resultants.

Recall that, given two finite alphabets $\mathbb{A}, \mathbb{B}$, of respective cardinalities α, β, then the *resultant* $\prod_{a\in\mathbb{A}, b\in\mathbb{B}}(a - b)$ is equal to the Schur function $S_{\beta^\alpha}(\mathbb{A} - \mathbb{B})$. More generally, the resultant appears as a factor of some Schur functions, thanks to a proposition due to Berele and Regev [5, 1.4.3].
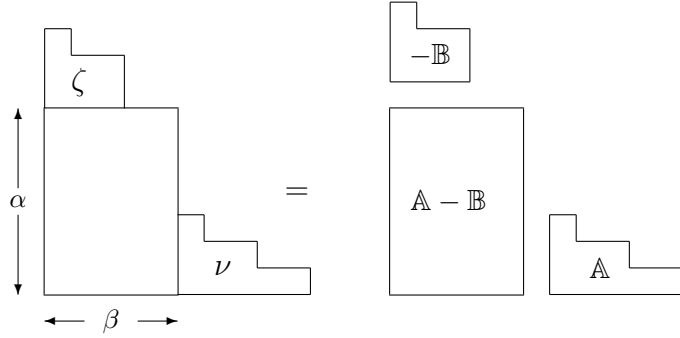
Proposition 4.1. *Let* $\mathbb{A}$, $\mathbb{B}$, *be of cardinalities* α, β, $p \in \mathbb{N}$, $\zeta \in \mathbb{N}^p$, $\nu \in \mathbb{N}^\alpha$. *Then, writing* $[\beta + \nu_1, \ldots, \beta + \nu_\alpha, \zeta_1, \ldots, \zeta_p] = [\beta^\alpha + \nu, \zeta]$, *one has*

$$S_{\beta^\alpha + \nu, \zeta}(\mathbb{A} - \mathbb{B}) = S_\zeta(-\mathbb{B}) \, S_\nu(\mathbb{A}) \prod_{a \in \mathbb{A}, b \in \mathbb{B}} (a - b) \, . \tag{20}$$

Moreover,

$$\nu \in \mathfrak{Part}, \ \nu \supseteq (\beta + 1)^{\alpha+1} \ \Rightarrow S_\nu(\mathbb{A} - \mathbb{B}) = 0 \, . \tag{21}$$

Pictorially, the relation is

The next theorem shows that $Q'_{n^k}(1 - X)$ is also a resultant, and more generally, factors out a resultant from $Q'_\lambda(1 - X)$ (it is more convenient to take a power of t instead of 1, to simplify exponents).

Theorem 4.2. *Let* $n, r \in \mathbb{N}$, X *be an alphabet of cardinality* n, λ *be a partition that is written, with* ζ *such that* $\zeta_1 < n$,

$$\lambda = [n + \nu_1, \ldots, n + \nu_k, \zeta_1, \zeta_2, \ldots] = [n^k + \nu, \zeta] \, .$$

Then

$$Q'_\lambda(t^r - X) = t^{\mathbf{n}(\nu) + r|\nu|} \prod_{i=r}^{k+r-1} \prod_{j=1}^{n} (t^i - x_j) \, Q'_\zeta(t^{k+r} - X) \, . \tag{22}$$

Proof. $Q'_\lambda(1 - X) = \sum_{\mu \subseteq \lambda} Q'_\mu X \, Q'_{\lambda/\mu} t^r$. The terms $Q'_\mu X$ vanish if $\mu_1 > n$. Eq. (18) shows that $Q'_{\lambda/\mu} t^r$, when $\mu \le n$, is equal to

$$Q'_{[n^k, \zeta]/\mu} t^r \, t^{\mathbf{n}(\nu) + r|\nu|} \, .$$

Thus we need only treat the case where $\nu = 0$, and we shall do it by induction on k.

The function $Q'_{[n^{k-1}, \zeta]/\mu}(t^{r+1} - X)$ can be written as as sum of Schur functions , enumerating all tableaux of commutative evaluation

$$2^n \cdots k^n (k+1)^{\zeta_1} (k+2)^{\zeta_2} \cdots \, ,$$

of shape not containing $[n + 1, n + 1]$ (otherwise the Schur function vanishes, according to (21)). Given any such tableau T, of shape ρ, given any horizontal strip ρ/ξ, then there exists a unique pair (u, T) such that $T \equiv uT'$, with T' of shape ξ and u a tableau of row shape. If $\xi_1 \le n$, then

$T'' = T' 1^n u$ is a tableau of charge $\mathfrak{ch}T + \ell(u)$, which contributes to $Q'_{[n^k,\varsigma]}$, and all tableaux of weight $[n^k, \varsigma]$, of shape not containing $[n{+}1, n{+}1]$ are obtained in this way.

The contribution of T to $Q'_{[n^{k-1},\varsigma]/\mu}(t^{r+1} - X)$ is, writing $R(y, X)$ for $\prod_{1 \leq i \leq n}(y - x_j)$,

$$t^{\mathfrak{ch}T} S_\rho(t^{r+1} - X) = t^{\mathfrak{ch}T} S_{\rho_2,\rho_3,\dots}(-X)\, R(t^{r+1}, X)\, t^{(r+1)(\rho_1 - n)}\,,$$

thanks to the factorization (20). Each of its successors $T'' = T'1^n u$ contributes to $Q'_{[n^k,\varsigma]}(t^r - X)$ the term

$$t^{\mathfrak{ch}T''} S_{n+\ell(u),\xi}(t^r - X) = t^{\mathfrak{ch}T} t^{\ell(u)} S_\xi(-X) R(t^r, X) t^{r\ell(u)}\,.$$

Summing over all the tableaux T'' whose predecessor is T, one gets the contribution[6]

$$t^{\mathfrak{ch}T}\, R(t^r, X)\, S_\rho(t^{r+1} - X)\,,$$

and this shows that the contribution of T has been multiplied by a factor independent of T.

Summation over all tableaux T of weight $[n^{k-1}, \varsigma]$ gives the equality

$$Q'_{[n^k,\varsigma]}(t^r - X) = R(t^r, X) Q'_{[n^{k-1},\varsigma]}(t^{r+1} - X)$$

and finishes the proof. $\square$

For example, for $n = 2$, let us illustrate, on a single tableau, the inductive step from $Q'_{22211}(t - X)$ to $Q'_{222211}(1 - X)$. The tableau

$$\begin{array}{|c|c|c|c|}
\hline
6 & & & \\
\hline
5 & & & \\
\hline
3 & 4 & & \\
\hline
2 & 2 & 3 & 4 \\
\hline
\end{array}$$

has charge 5 and gives the contribution

$$t^5 S_{4211}(t - X) = t^5 S_{211}(-X)\, S_4(t - X) = t^5 S_{211}(-X)\, R(t, X)\, t^2\,.$$

Its different factorizations, the issuing new tableaux and their contributions are

---

[6]Indeed, when y a single letter, $S_\rho(y - X)$ is equal to the sum $\sum_\xi y^{|\rho/\xi|} S_\xi(-X)$.

$$
\begin{array}{c}
\boxed{6} \\
\boxed{2}\,\boxed{3}\;\boxed{5} \\
\boxed{4}\,\boxed{4} \\
\boxed{2}\,\boxed{3}
\end{array}
\quad\longrightarrow\quad
\begin{array}{c}
\boxed{6} \\
\boxed{5} \\
\boxed{4}\,\boxed{4} \\
\boxed{2}\,\boxed{3} \\
\boxed{1}\,\boxed{1}\,\boxed{2}\,\boxed{3}
\end{array}
\qquad t^7 S_{2211}(-X)\,R(1,X)
$$

$$
\begin{array}{c}
\boxed{5} \\
\boxed{2}\,\boxed{3}\,\boxed{6}\;\boxed{4}\,\boxed{4} \\
\boxed{2}\,\boxed{3}
\end{array}
\quad\longrightarrow\quad
\begin{array}{c}
\boxed{5} \\
\boxed{4}\,\boxed{4} \\
\boxed{2}\,\boxed{3} \\
\boxed{1}\,\boxed{1}\,\boxed{2}\,\boxed{3}\,\boxed{6}
\end{array}
\qquad t^8 S_{221}(-X)\,R(1,X)
$$

$$
\begin{array}{c}
\boxed{6} \\
\boxed{2}\,\boxed{3}\,\boxed{4}\;\boxed{5} \\
\boxed{4} \\
\boxed{2}\,\boxed{3}
\end{array}
\quad\longrightarrow\quad
\begin{array}{c}
\boxed{6} \\
\boxed{5} \\
\boxed{4} \\
\boxed{2}\,\boxed{3} \\
\boxed{1}\,\boxed{1}\,\boxed{2}\,\boxed{3}\,\boxed{4}
\end{array}
\qquad t^8 S_{2111}(-X)\,R(1,X)
$$

$$
\begin{array}{c}
\boxed{5} \\
\boxed{2}\,\boxed{3}\,\boxed{4}\,\boxed{6}\;\boxed{4} \\
\boxed{2}\,\boxed{3}
\end{array}
\quad\longrightarrow\quad
\begin{array}{c}
\boxed{5} \\
\boxed{4} \\
\boxed{2}\,\boxed{3} \\
\boxed{1}\,\boxed{1}\,\boxed{2}\,\boxed{3}\,\boxed{4}\,\boxed{6}
\end{array}
\qquad t^9 S_{211}(-X)\,R(1,X)\,.
$$

The sum of these contributions is

$$
t^7 R(1,X)\left(S_{2211}(-X) + t\,S_{221}(-X) + t\,S_{2111}(-X) + t^2\,S_{211}(-X)\right)
$$
$$
= t^7 R(1,X)\,S_{2211}(t-X) = t^7 R(1,X)R(t,X)S_{211}(t-X)\,.
$$

In the special case $X = \{x\}$ of cardinality 1, and $r = 0$, one recovers that [8, p.226]

$$
Q'_\lambda(1-x) = Q_\lambda\left(\frac{1-x}{1-t}\right) = t^{\mathbf{n}(\lambda)}(1-x)(1-xt^{-1})\cdots(1-xt^{1-\ell(\lambda)})\,,\quad (23)
$$

an identity which allows to describe the *principal specializations* $Q'_\lambda(1 - t^N) = Q_\lambda((1-t^N)/(1-t))$ when taking $y = t^N$.

In the case of an alphabet of cardinality 2, $X = \{x_1, x_2\}$, the remaining partition ζ is such that $\zeta = 1^\beta$ for some $\beta \in \mathbb{N}$, and therefore, the factor $Q'_\zeta(t^j - X)$ is equal to

$$
Q_{1^\beta}\left(\frac{t^j - X}{1-t}\right) = b_{1^\beta}\, S_{1^\beta}\left(\frac{t^j - x_1 - x_2}{1-t}\right)\,.
$$

In total,

$$
Q'_{2^k+\nu,\,1^\beta}(1 - x_1 - x_2)
$$
$$
= t^{\mathbf{n}(\nu)}\,(1-t)\cdots(1-t^\beta)\prod_{i=0}^{k-1}(t^i-x_1)(t^i-x_2)\,S_{1^\beta}\left(\frac{t^k - x_1 - x_2}{1-t}\right)\,.\quad (24)
$$

5. Generating Functions

The results of the preceding sections may be used to compute generating series of Hall–Littlewood polynomials. For example, given the series $\sum_{\mu\in\mathfrak{Part}} c_\mu P_\mu(X)$, with some arbitrary coefficients c_μ, suppose that one wants to evaluate the product

$$\sigma_1(-X) \sum c_\mu P_\mu X = \prod_{x\in X}(1-x) \sum c_\mu P_\mu X \ .$$

To do so, one first extends the family c_μ to a family $c_v \ : \ v \in \mathbb{Z}^n$ by imposing the relations (3). It amounts introducing a second alphabet Y, and putting $c_\mu = Q_\mu Y$. Since $\sigma_1(XY(1\text{–}t)) = \sum Q_\mu Y P_\mu X$, then

$$\sigma_1(-X) \sum_\mu c_\mu P_\mu(X) = \sigma_1\big(-X + XY(1\text{–}t)\big)$$

$$= \prod_{x\in X}(1\text{–}x) \prod_{x\in X, y\in Y} \frac{1-txy}{1-xy}$$

$$= \sigma_1\left(X(Y - \frac{1}{1-t})(1-t) \right)$$

$$= \sum_{\lambda\in\mathfrak{Part}} P_\lambda X \, Q_\lambda \left(Y - \frac{1}{1\text{–}t} \right) \ .$$

Knowing the expansion of $Q_\lambda(Y - 1/(1\text{–}t))$, or equivalently, of $Q'_\lambda(Y' - 1)$, with $Y' = Y(1\text{–}t)$, given in (13), one concludes

$$\sigma_1(-X) \sum_\mu c_\mu P_\mu X = \sum_\lambda \sum_{v\in\{0,1\}^n} (-1)^{|v|} c_{\lambda-v} P_\lambda X \ . \tag{25}$$

Let us give another similar example.

Proposition 5.1. *Given two alphabets X, Y, we have*

$$\sigma_1\left(X + XY(1\text{–}t) \right) = \prod_{x\in X} \frac{1}{1-x} \prod_{y\in Y} \frac{1-txy}{1-xy}$$

$$= \sum_{\lambda\in\mathfrak{Part}} \sum_{\mu\subseteq\lambda} P_\lambda X P_\mu Y \, b_\mu Q'_{\lambda/\mu} 1 \,, \tag{26}$$

the value of $b_\mu Q'_{\lambda/\mu} 1$ being given in (18).

Proof. One writes $X + XY(1\text{–}t) = X(Y + (1\text{–}t)^{-1})(1\text{–}t)$, and therefore one obtains

$$\sigma_1\left(X + XY(1\text{–}t) \right) = \sum_\lambda P_\lambda X Q_\lambda(Y + (1\text{–}t)^{-1})$$

$$= \sum_{\lambda,\mu} P_\lambda X P_\mu b_\mu Y Q_{\lambda/\mu}((1\text{–}t)^{-1}) \,,$$

which is the required formula. $\qquad\square$

For example, the coefficient of $P_{42}X$, in terms of the $P_\mu = P_\mu Y$, is

$$t^2 P_0 + t(1-t^2)P_1 + (1-t^2)P_2 + t(1-t)(1-t^2)P_{11}$$
$$+ (1-t)P_3 + (1-t)(1-t^2)P_{21} + (1-t)P_4 + (1-t)^2 P_{31} + (1-t)(1-t^2)P_{22}$$
$$+ (1-t)^2 P_{41} + (1-t)^2 P_{32} + (1-t)^2 P_{42}\,.$$

Using the explicit values (24), one obtains as a corollary for $X = (t - x_1 - x_2)(1 - t)^{-1}$, $Y = (1 - y)(1 - t)^{-1}$, the expansion of

$$\sigma_1\left(\frac{(t - x_1 - x_2)(2 - y)}{1 - t}\right)$$

$$= \sigma_1\left(2\frac{t}{1 - t} + \frac{y(x_1 + x_2)}{1 - t} - \frac{yt}{1 - t} - 2\frac{x_1 + x_2}{1 - t}\right)$$

$$= \prod_{i=0}^{\infty} \frac{1}{(1 - t^{i+1})^2}\frac{1}{(1 - t^i y x_1)(1 - t^i y x_2)}$$

$$\cdot (1 - yt^{i+1})\big((1 - t^i x_1)(1 - t^i x_2)\big)^2\,.$$

Warnaar evaluates a similar generating series, with simpler coefficients. Let, for any pair of partitions,

$$\theta(\lambda, \mu) = t^{\mathbf{n}(\lambda/\mu) - |\mu|}\,.$$

Then Warnaar's identity [10, Th. 1.1] is the following.

Theorem 5.2. *For any pair of alphabets X, Y, one has*

$$\sigma_1\left(X + Y + (\frac{1}{t} - 1)XY\right) = \sum_{\lambda, \mu \in \mathfrak{Part}} \theta(\lambda, \mu)P_\lambda X P_\mu Y\,. \qquad (27)$$

Proof. Writing the left-hand side $\sigma_1 X\,\sigma_1\left(Y(1 + (t^{-1} - 1)X)\right)$, and using that $\sigma_1 XY((1-t) = \sum Q_\mu X P_\mu Y$, one rewrites the identity to prove as

$$Q_\lambda\left(\frac{1}{1 - t} + t^{-1}X\right) \overset{?}{=} \frac{1}{\sigma_1 X}\sum_\mu \theta(\lambda, \mu)P_\mu X\,,$$

or

$$\sum_\mu t^{-|\mu|}Q_\mu X\,Q_{\lambda/\mu}\left(\frac{1}{1 - t}\right) = \sum_\mu t^{-|\mu|}Q_\mu X\,Q'_{\lambda/\mu}1$$

$$\overset{?}{=} \sigma_1(-X)\sum_\mu \theta(\lambda, \mu)P_\mu X\,. \qquad (28)$$

Thanks to (25), the right-hand side can be written

$$\sum_{v \in \{0,1\}^n} \theta(\lambda, \mu - v)P_\mu X = \sum_{v \in \{0,1\}^n} \theta(\lambda, \mu - v)b_\mu^{-1}Q_\mu X\,.$$

Therefore, (28) is a consequence of (18) and is, in fact, equivalent to (26). $\square$

Let us mention that Warnaar used another expression of $\theta(\lambda, \mu)$, putting

$$\theta(\lambda, \mu) = t^{\mathbf{n}(\lambda) + \mathbf{n}(\mu) - (\lambda^\sim, \mu^\sim)} \,, \tag{29}$$

where $(\lambda^\sim, \mu^\sim) = \sum \lambda_i^\sim \mu_i^\sim$.

6. Scalar Product

The combinatorics of Hall–Littlewood polynomials fundamentally reduces to the fact that $\{P_\lambda\}$ is an orthogonal basis of the ring of polynomials, and that it can be extended to a family satisfying the *straightening relations* (3). Given these two ingredients, we can forget Hall and Littlewood altogether.

Thus, let us consider the ring of Laurent series in $x_1, \ldots, x_n$, with a finite number of terms with exponent $\geq [0, \ldots, 0]$, modulo the ideal generated by the relations

$$(x^v + x^{v s_i})(x_{i+1} - t x_i) \simeq 0 \,, \; i = 1, \ldots, n{-}1, \; v \in \mathbb{Z}^n, \; x^v \simeq 0 \text{ if } v_n < 0 \,. \tag{30}$$

Any element of this ring can be written uniquely as a linear combination of dominant monomials $x^\lambda : \lambda_1 \geq \cdots \geq \lambda_n \geq 0$.

We define a scalar product by its restriction to dominant monomials:

$$((x^\lambda, x^\mu)) = b_\lambda \, \delta_{\lambda, \mu} \,, \; \lambda, \mu \in \mathfrak{Part} \,. \tag{31}$$

We may observe that the scalar product on Laurent polynomials used in [2],

$$(f, g)_t := CT \left(f(x_1, \ldots, x_n) \, g \left(\frac{1}{x_n}, \ldots, \frac{1}{x_1} \right) \prod_{1 \leq i < j \leq n} \frac{1 - x_i/x_j}{1 - t x_i/x_j} \right) \,, \tag{32}$$

where CT means "constant term", is such that

$$(Q_\lambda, x^\mu) = b_\lambda \, \delta_{\lambda, \mu} \,.$$

This indicates that the present study is related to more general constructions about non-symmetric Hall–Littlewood polynomials.

In the present set-up, to recover Warnaar's generating function, we essentially need to interpret the function $\theta(\lambda, \mu)$, $\lambda, \mu \in \mathfrak{Part}$, as a scalar product. To do so, let us also write $\theta(x^\lambda, x^\mu)$ for $\theta(\lambda, \mu)$, and extend the definition of θ to all monomials by linearity using relations (30).

Proposition 6.1. *For any pair $\lambda, \mu \in \mathfrak{Part}$, we have*

$$\left(\left(\frac{x^\lambda t^{-|\lambda|}}{\prod_{i=1}^n (1 - t/x_i)} , \frac{x^\mu}{\prod_{i=1}^n (1 - 1/x_i)} \right) \right) = \theta(\lambda, \mu) \,. \tag{33}$$

Proof. Multiplying[7] x^μ by $\prod_{i=1}^n(1 - 1/x_i)$, let us prove the equivalent statement that

$$\left(\left(\frac{x^\lambda t^{-|\lambda|}}{\prod_{i=1}^n(1 - t/x_i)}, x^\mu\right)\right) = \sum_{v \in \{0,1\}^n} (-1)^{|v|} \theta(\lambda, \mu - v). \qquad (34)$$

Let a, k be such that $\mu_1 = a = \cdots = \mu_k > \mu_{k+1}$, let $r = \lambda_a^\sim$. Then, for any $j : 0 \le j \le k$, one has

$$\theta(\lambda, \mu - [1^j, 0^{n-j}]) = \theta(\lambda, \mu)t^{-0-1-\cdots-(j-1)}t^{jr}.$$

Summing over all vectors $u \in \{0, 1\}^n$, $u_i = 0$ for $i > k$, and taking into account reordering, one gets

$$\sum_u (-1)^{|u|}\theta(\lambda, \mu - u) = \theta(\lambda, \mu)\sum_{j=0}^k (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} t^{-\binom{j}{2}}t^{jr}$$

$$= \theta(\lambda, \mu)(1 - t^{r-0})(1 - t^{r-1})\cdots(1 - t^{r-k+1}).$$

By induction $[\mu_{k+1}, \mu_{k+2}, \ldots] \to \mu = [a^k, \mu_{k+1}, \ldots]$, this proves that

$$\sum_{v \in \{0,1\}^n} (-1)^{|v|}\theta(\lambda, \mu - v) = \theta(\lambda, \mu)(1 - t^{\nu_1})(1 - t^{\nu_2 - 1})(1 - t^{\nu_3 - 2})\cdots, \qquad (35)$$

$\nu_1, \nu_2, \nu_3, \ldots$ being the parts of $\lambda^\sim$ of index $\mu_1, \mu_2, \mu_3, \ldots$. Notice that the product is null if $\mu \not\subseteq \lambda$.

Using (18), one rewrites the right hand side of (35) as

$$\theta(\lambda, \mu)t^{-\mathbf{n}(\lambda/\mu)} b_\mu Q'_{\lambda/\mu} 1. \qquad (36)$$

On the other hand,

$$Q'_\lambda(1 + t^{-1}X) = \sum_{u \in \mathbb{N}^n} Q'_{\lambda - u}(t^{-1}X) = \sum t^{-|\mu|}Q'_\mu X Q'_{\lambda/\mu} 1.$$

This implies that

$$t^{-|\lambda|}\prod_{i=1}^n(1 - t/x_i)^{-1} \simeq \sum_{u \in \mathbb{N}^n} x^{\lambda - u}t^{|u| - |\lambda|}$$

is congruent to $\sum_\mu t^{-|\mu|}x^\mu Q'_{\lambda/\mu} 1$. Therefore,

$$\left(\left(\frac{x^\lambda t^{-|\lambda|}}{\prod_{i=1}^n(1 - t/x_i)}, x^\mu\right)\right) = t^{-|\mu|}b_\mu Q'_{\lambda/\mu} 1,$$

and this is the required identity (34), using that $\theta(\lambda, \mu) = t^{\mathbf{n}(\lambda/\mu) - |\mu|}$. $\square$

---

[7]Multiplying by x^{-u}, $u \in \mathbb{N}^n$, is compatible with relation (30): $x^v = 0$ if $v_n < 0$.

7. Note

The functions Q'_λ have an interpretation in terms of the cohomology of flag manifolds [1], [8, III.7]. They describe graded multiplicities of representations of the symmetric group. The functions Q_λ also have an interpretation, as Euler–Poincaré characteristic of line bundles over the flag manifold [4]. In combinatorial terms, this gives the following definition.

Let π_ω be the symmetrizing operator defined before. Given a partition $\lambda \in \mathbb{N}^n$, let m_0 be the multiplicity of the part 0 ($=n - \ell(\lambda)$). Then [8, III.2]

$$Q_\lambda(X) = \frac{(1-t)^n}{(1-t)\cdots(1-t^{m_0})}\, x^\lambda \prod_{i<j}(1 - tx_j/x_i)\, \pi_\omega\,. \qquad (37)$$

However, the fact that the normalization factor depends on λ prevents us from using (37), which is equivalent to formula (2.14) of [8], as a definition for Q_v when v is not a partition (contrary to what is stated p. 214 of [8]).

Indeed, let $n = 2$. Then

$$x^{02}(1 - tx_2/x_1)\pi_\omega = \frac{x^{02}(1 - tx_2/x_1)}{1 - x_2/x_1} + \frac{x^{20}(1 - tx_1/x_2)}{1 - x_1/x_2}$$
$$= t(x^{20} + x^{11} + x^{02}) - x^{11}\,.$$

On the other hand, the relations (3) impose[8]

$$Q'_{02} = t\, Q'_{20} + (t-1)Q'_{11}\,,$$

and the same relation must be valid for Q_{02}, resulting from the transformation $X \to X(1-t)$. But

$$tQ_{20}(X) + (t-1)Q_{11}(X) = (t - t^2)\,(x^{20} + x^{11} + x^{02}) + (t-1)(1 - t + t^3)x^{11}$$

is not proportional to the image of x^{02}.

The confusing fact is that

$$(x^{02} - tx^{20} + (1-t)x^{11})\,(1 - tx_2/x_1)\,\pi_\omega = 0\,,$$

that is, the images of monomials under the operator $(1 - tx_2/x_1)\pi_\omega$ satisfy the same relations as the functions Q_v. However, because of the different normalization factors, this does not imply that the image of x^{02} be proportional to Q_{02}.

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[8]Macdonald [8, p. 214] writes $Q_{16} = tQ_{61} + (t^2 - 1)Q_{52} + (t^3 - t)Q_{43}$, $Q_{15} = tQ_{51} + (t^2 - 1)Q_{42} + (t^2 - t)Q_{33}$. Read $t^m - t^{m-1}$ instead of $t^{m+1} - t^m$.

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Alain Lascoux

CNRS, IGM, Université Paris-Est

77454 Marne-la-Vallée Cedex, France

Email: Alain.Lascoux@univ-mlv.fr