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GRAPH WEIGHTS ARISING FROM MAYER'S THEORY OF CLUSTER INTEGRALS

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A notre très bon ami Xavier Viennot

Abstract. We study graph weights (i.e., graph invariants) which arise naturally in Mayer's theory of cluster integrals in the context of a non-ideal gas. Various choices of the interaction potential between two particles yield various graph weights $w(g)$. For example, in the case of the Gaussian interaction, the so-called Second Mayer weight $w(c)$ of a connected graph c is closely related to the graph complexity, i.e., the number of spanning trees, of c. We give special attention to the Second Mayer weight $w(c)$ which arises from the hard-core continuum gas in one dimension. This weight is a signed volume of a convex polytope $\mathcal{P}(c)$ naturally associated with c. Among our results are the values $w(c)$ for all 2-connected graphs c of size at most 6, in Appendix B, and explicit formulas for three infinite families: complete graphs, (unoriented) cycles and complete graphs minus an edge.

Introduction.

Graph weights can be defined as functions on (simple, finite) graphs taking scalar or polynomial values and which are invariant under isomorphisms, i.e., under vertex relabellings. Since most graphical concepts share this invariance property, examples of graph weights abound. For a graph g, $\alpha(g)$ = the independence number of g and $\chi_g(\lambda)$ = the chromatic polynomial of g are examples of graph weights. Another example is given by the *graph complexity* of G, denoted by $\gamma(g)$, which is defined as the number of maximal spanning forests of g.

In the present paper, we study graph weights which arise naturally in Mayer's theory of cluster integrals in the context of a non-ideal gas in a vessel $V \subseteq \mathbb{R}^d$. Various choices of the dimension d and of the interaction potential $\varphi(r)$ between two particles at distance r yield various graph weights $w(g)$. In the thermodynamic limit, the gas pressure is closely related to the exponential generating function $C_w(z)$ of connected graphs c weighted by the so-called Second Mayer weight $w(c)$. There has been continued interest from the physicists for Mayer and virial expansions. See for example Clisby and McCoy [6] and the references therein. While physicists are interested in summing the weights of all connected or 2-connected graphs of a given order, the present paper focuses on individual graph contributions and their combinatorial significance.

In the first section, we review Mayer's theory following the lines of Uhlenbeck and Ford [31] and Leroux [16]. See also Thompson [30]. Special emphasis is put on the existence conditions of the thermodynamic limit which defines the second Mayer weight $w(c)$ of a connected graph c and on its property of block multiplicativity. A complete proof for the existence of the thermodynamic limit, not found in [31] and [16] is given in Appendix A. Also reviewed are the functional relations between weighted connected graphs and 2-connected graphs.

The example of the Gaussian interaction, is studied in Section 2. In this case, the weight $w(c)$ involves three block multiplicative parameters of the connected graph c, namely the number of vertices minus 1, the number of edges $e(c)$ and the graph complexity $\gamma(c)$ which is the number of spanning trees of c. The proof of this known result is reviewed here. It uses the multidimensional Gaussian integral and the Theorem of Kirchhoff relating $\gamma(c)$ to the determinant of a matrix K associated to c. Moreover, we give an extension of the Gaussian weight which refines $\gamma(c)$ in order to reflect the degree sequence of the spanning subtrees of c.

The last section is devoted to the Second Mayer weight $w(c)$ which arises from the hard-core continuum gas in one dimension. This weight is a signed volume of a convex polytope $\mathcal{P}(c)$ naturally associated with c. It is known that in this case, the generating function $C_w(z)$ (the pressure) is equal to the Lambert function $L(z)$ which is defined by the functional equation $L(z)\exp(L(z)) = z$. We give a combinatorial proof of this result. This implies that the total weight of all connected graphs over a set of size N is $(-N)^{N-1}$ and that of all 2-connected graphs is $-N(N-2)!$. Naturally, it raises the question of finding a combinatorial explanation of these formulas and of computing or understanding better the weight of individual connected graphs and perhaps expressing it in terms of other graph invariants. We have only partial answers to these questions to offer. First, we have computed the volume $\mathrm{Vol}(\mathcal{P}(c))$ of all 2-connected graphs c of size at most 6, together with the Ehrhart polynomial, using the fact that the vertices of the polytope $\mathcal{P}(c)$ have integer coordinates. This data is given in Appendix B. The weight of any connected graph c whose blocks have size at most 6 can then be deduced by block multiplicativity. Secondly, we have found explicit formulas for three infinite families of connected graphs namely complete graphs, (unoriented) cycles and complete graphs minus an edge. An alternate useful tool, a decomposition of the polytope $\mathcal{P}(c)$ into a certain number of $(N-1)$-dimensional simplexes, of volume $1/(N-1)!$ is exploited in the final subsection.

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1. A review of Mayer's theory of cluster integrals

1.1. **The grand canonical partition function.** In the context of a non-ideal gas with N particles in a vessel V included in $\mathbb{R}^d$, we represent the particles' positions by vectors $\vec{x}_1, \ldots, \vec{x}_N$. If we consider that the system is free from external influences, the *partition function* is defined as

$$Z(V,T,N) = \frac{1}{N!\lambda^{dN}} \int_V \cdots \int_V \exp\left(-\beta \sum_{i<j} \varphi(|\vec{x}_i - \vec{x}_j|)\right) d\vec{x}_1 \ldots d\vec{x}_N, \tag{1}$$

where λ and β depend on the temperature T and where the interaction between two particles at distance r is expressed by a potential function $\varphi(r)$ as illustrated in Figure 1a).

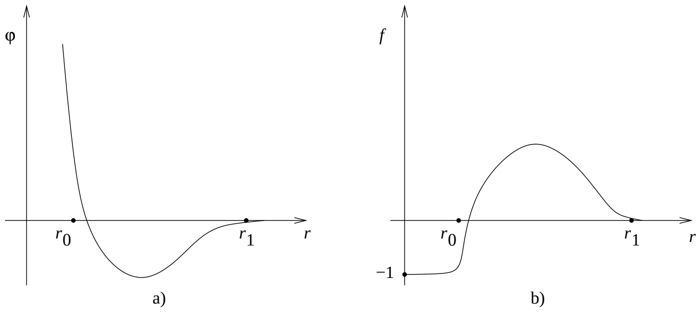

Figure 1. a) the function $\varphi(r)$ b) the function $f(r)$

We also define the *grand canonical partition function* as the generating series of the partition functions:

$$Z_{\mathrm{gr}}(V,T,z) = \sum_{n=0}^{\infty} Z(V,T,N)(\lambda^d z)^N, \tag{2}$$

where z is called the *fugacity* or the *activity* of the system. The generating function identities that we consider are in the sense of formal power series in the activity z. For issues of convergence, see Ruelle [24]. The system's macroscopic parameters can be described using the grand canonical partition function. In particular, the pressure P, the mean number of particles $\bar{N}$ and the density ρ can be written as

$$\frac{P}{kT} = \frac{1}{V}\log Z_{\mathrm{gr}}(V,T,z), \;\; \bar{N} = z\frac{\partial}{\partial z}\log Z_{\mathrm{gr}}(V,T,z) \text{ and } \rho := \frac{\bar{N}}{V}, \tag{3}$$

where V is also used as the volume of the vessel.

1.2. **Mayer's idea.** In order to study these functions, Mayer (1940) sets

$$1 + f_{ij} = \exp\left(-\beta\varphi(|\overrightarrow{x_i} - \overrightarrow{x_j}|)\right), \tag{4}$$

where $f_{ij} = f(|\overrightarrow{x_i} - \overrightarrow{x_j}|)$. The general form of Mayer's function

$$f(r) = \exp(-\beta\varphi(r)) - 1, \tag{5}$$

compared to the potential function $\varphi(r)$, is shown in Figure 1. Since

$$\prod_{1\le i<j\le N}(1+f_{ij}) = \sum_{g\in\mathcal{G}[N]}\prod_{\{i,j\}\in g} f_{ij}, \tag{6}$$

where $\mathcal{G}[N]$ denotes the set of all (simple) graphs over the set of vertices $[N] = \{1,2,\ldots,N\}$, the partition function $Z(V,T,N)$ becomes

$$\begin{aligned}
Z(V,T,N) &= \frac{1}{N!\lambda^{dN}}\int_{V^N}\exp\left(-\beta\sum_{i<j}\varphi(|\overrightarrow{x_i}-\overrightarrow{x_j}|)\right)d\overrightarrow{x_1}\cdots d\overrightarrow{x_N}\\
&= \frac{1}{N!\lambda^{dN}}\int_{V^N}\prod_{1\le i<j\le N}(1+f_{ij})\,d\overrightarrow{x_1}\cdots d\overrightarrow{x_N}\\
&= \frac{1}{N!\lambda^{dN}}\sum_{g\in\mathcal{G}[N]}\int_{V^N}\prod_{\{i,j\}\in g} f_{ij}\; d\overrightarrow{x_1}\cdots d\overrightarrow{x_N}\\
&= \frac{1}{N!\lambda^{dN}}\sum_{g\in\mathcal{G}[N]} W(g),
\end{aligned}\tag{7}$$

where the weight $W(g)$ of a graph g is given by the integral

$$W(g) = \int_{V^N}\prod_{\{i,j\}\in g} f_{ij}d\overrightarrow{x_1}\cdots d\overrightarrow{x_N}. \tag{8}$$

This is the *First Mayer weight* of a graph g. It should be clear that $W(g)$ is invariant under graph isomorphisms, since any relabelling defined by a vertex bijection also induces a change of variables whose Jacobian is the determinant of the corresponding permutation matrix and which transforms one integral into the other.

In terms of $W(g)$, the grand-canonical function becomes

$$\begin{aligned}
Z_{\mathrm{gr}}(V,T,z) &= \sum_{N=0}^{\infty} Z(V,T,N)(\lambda^d z)^N\\
&= \sum_{N=0}^{\infty}\frac{1}{N!\lambda^{dN}}\sum_{g\in\mathcal{G}[N]} W(g)(\lambda^d z)^N\\
&= \sum_{N=0}^{\infty}\frac{1}{N!}\sum_{g\in\mathcal{G}[N]} W(g)z^N\\
&= \mathcal{G}_W(z),
\end{aligned}\tag{9}$$

the exponential generating series of graphs weighted by the function W.

Proposition 1. *$W(g)$, the First Mayer Weight of a simple graph g, is multiplicative on the connected components of g. In other words, for $c_1, c_2, \ldots, c_m$ the m connected components of g, we have*

$$W(g) = W(c_1)W(c_2)\ldots W(c_m).$$

Proof. Since the connected components of g are disjoint, we can label the vertices in such a way that 1 to k_1 are in the component c_1, k_1+1 to k_2 in c_2, and so on until $k_{m-1}+1$ to $k_m = N$ in c_m. We then have

$$\begin{aligned}
W(g) &= \int_{V^N} \prod_{\{i,j\}\in g} f_{ij} d\vec{x}_1 \ldots d\vec{x}_N \\
&= \int_{V^N} \prod_{\{i,j\}\in c_1} f_{ij} \prod_{\{i,j\}\in c_2} f_{ij} \cdots \prod_{\{i,j\}\in c_m} f_{ij} d\vec{x}_1 \ldots d\vec{x}_N \\
&= \int_{V^{k_1}} \prod_{\{i,j\}\in c_1} f_{ij} d\vec{x}_1 \ldots d\vec{x}_{k_1} \times \int_{V^{k_2-k_1}} \prod_{\{i,j\}\in c_2} f_{ij} d\vec{x}_{k_1+1} \ldots d\vec{x}_{k_2} \times \\
&\qquad \cdots \times \int_{V^{k_m-k_{m-1}}} \prod_{\{i,j\}\in c_m} f_{ij} d\vec{x}_{k_{m-1}+1} \ldots d\vec{x}_N \\
&= W(c_1)\cdot W(c_2)\cdot \ldots \cdot W(c_m). \qquad (10)
\end{aligned}$$

□

Since W is multiplicative on connected components, the exponential formula can be used:

$$\mathcal{G}_W(z) = \exp(\mathcal{C}_W(z)), \qquad (11)$$

where $\mathcal{C}$ denotes the species (class) of connected graphs, so that

$$\begin{aligned}
\log \mathcal{G}_W(z) &= \mathcal{C}_W(z) \\
&= \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{c\in\mathcal{C}[N]} W(c) z^N. \qquad (12)
\end{aligned}$$

Corollary 2. *The pressure of the system can be expressed in terms of the exponential generating function of connected graphs weighted by W. More precisely, we have*

$$\frac{P}{kT} = \frac{1}{V}\log Z_{\mathrm{gr}}(V,T,z) = \frac{1}{V}\mathcal{C}_W(z).$$

□

1.3. **The thermodynamic limit** $w(c)$**.** Let c be a connected graph over $\{1,2,\ldots,N\}$. The *Second Mayer weight* $w(c)$ is defined as the limit

$$\begin{aligned}
w(c) &= \lim_{V\to\infty} \frac{1}{V} W(c) \\
&= \lim_{V\to\infty} \frac{1}{V} \int_{V^N} \prod_{\{i,j\}\in c} f_{ij}\, d\vec{x}_1 \ldots d\vec{x}_N. \qquad (13)
\end{aligned}$$

Here, V going to infinity has the following meaning. The vessel $V \in \mathbb{R}^d$ must contain a ball $B(0,R)$ centered at the origin, with radius $R \in]0,\infty)$. V goes to infinity means that R goes to infinity. Let us recall that the symbol V denotes both the vessel and the volume.

The following proposition gives us the conditions on Mayer's function f for $w(c)$ to exist.

Proposition 3. *If the function $f : [0, \infty) \to \mathbb{R}$ is integrable and bounded and if*

$$\int_0^\infty r^{d-1}|f(r)|\, dr < \infty, \tag{14}$$

for example if $|f(r)| = \mathrm{O}(1/r^{d+\epsilon})$, $r \to \infty$, for some $\epsilon > 0$, then for any fixed $\vec{x}_N \in \mathbb{R}^d$, the function $F_{\vec{x}_N} : \mathbb{R}^{d\cdot(N-1)} \to \mathbb{R}$, defined by

$$F_{\vec{x}_N}(\vec{x}_1, \ldots, \vec{x}_{N-1}) = \prod_{\{i,j\}\in c} f(|\vec{x}_i - \vec{x}_j|) = \prod_{\{i,j\}\in c} f_{ij} \tag{15}$$

is integrable over $(\mathbb{R}^d)^{N-1}$ and its integral is independent of $\vec{x}_N$. Moreover the limit (13) *exists and is equal to*

$$w(c) = \int_{(\mathbb{R}^d)^{N-1}} \prod_{\{i,j\}\in c;\ \vec{x}_N=\vec{0}} f_{ij}\, d\vec{x}_1 \ldots d\vec{x}_{N-1}. \tag{16}$$

Proof. See Appendix A. □

We will often use equation (16) as an alternate definition for $w(c)$. It also follows from the proof of Proposition 3 that the rooting at the vertex N, for which $\vec{x}_N$ is set equal to 0 in (16), can be replaced by any other vertex J. For example, with $J = 1$, the formula becomes

$$w(c) = \int_{(\mathbb{R}^d)^{N-1}} \prod_{\{i,j\}\in c;\ \vec{x}_1=\vec{0}} f_{ij}\, d\vec{x}_2 \ldots d\vec{x}_N. \tag{17}$$

In the thermodynamic limit, the pressure is given by

$$\begin{aligned}
\frac{P}{kT} &= \lim_{V\to\infty} \frac{1}{V} \log Z_{\mathrm{gr}}(V, T, z) \\
&= \lim_{V\to\infty} \frac{1}{V} \mathcal{C}_W(z) \\
&= \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{c\in\mathcal{C}[N]} \lim_{V\to\infty} \frac{1}{V} W(c) z^N \\
&= \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{c\in\mathcal{C}[N]} w(c) z^N.
\end{aligned}$$

In other words,

$$\frac{P}{kT} = \mathcal{C}_w(z). \tag{18}$$

Proposition 4. *In the thermodynamic limit, the pressure of the system is given directly in terms of the exponential generating function of connected graphs weighted by the Second Mayer Weight $w(c)$, according to formula* (18).

A *2-connected graph* is defined as a connected graph that stays connected if any one of its vertices is removed. This includes the complete graph on two vertices K_2. A *block* in a connected graph is a maximal 2-connected subgraph. A *cutpoint* in a connected graph is a vertex whose removal disconnects the graph. Any connected graph c can be uniquely decomposed into 2-connected graphs (the blocks of c) which are linked together by the cutpoints of c in a tree-like fashion (see [1], p.266). This decomposition partitions the edges of c. A *leaf-block* b is a block of c which is linked to the rest of the graph by only one cutpoint p. In that case, the graph $c \setminus b$, which is obtained from c by removing all edges and all vertices of b except the cutpoint p, remains connected.

Proposition 5. *The second Mayer weight w is* block-multiplicative. *More precisely, for any connected graph c whose blocks are $b_1, b_2, \dots, b_m$, we have*

$$w(c) = w(b_1)w(b_2)\dots w(b_m). \tag{19}$$

Proof. The proof is by induction on the number of blocks in c. If c is composed of exactly one block, the multiplicativity is trivial. Let us suppose that the property is true for all connected graphs composed of m blocks, and let us consider a connected graph c composed of $m+1$ blocks. Its vertices are labelled from 1 to N. We choose in c a leaf-block that we denote by b, and call S the set of its vertices. Without loss of generality, we can assume that the cutpoint linking b to c is labelled N, and that $S = \{1, 2, \dots, k, N\}$, $|S| = k+1$. Let us name $b_1, b_2, \dots, b_m$ the other blocks of c. We then have, using (16),

$$\begin{aligned}
w(c) &= \int_{\mathbb{R}^{d(N-1)}} \prod_{\{i,j\}\in c;\ \vec{x}_N=\vec{0}} f_{ij}\, d\vec{x}_1 d\vec{x}_2 \dots d\vec{x}_{N-1} \\
&= \int_{\mathbb{R}^{d(N-1)}} \prod_{\{i,j\}\in b;\ \vec{x}_N=\vec{0}} f_{ij} \prod_{\{i,j\}\in c\setminus b;\ \vec{x}_N=\vec{0}} f_{ij}\, d\vec{x}_1 d\vec{x}_2 \dots d\vec{x}_{N-1} \\
&= \int_{\mathbb{R}^{dk}} \prod_{\{i,j\}\in b;\ \vec{x}_N=\vec{0}} f_{ij}\, d\vec{x}_1 d\vec{x}_2 \dots d\vec{x}_k \\
&\qquad \times \int_{\mathbb{R}^{d(N-k-1)}} \prod_{\{i,j\}\in c\setminus b;\ \vec{x}_N=\vec{0}} f_{ij}\, d\vec{x}_{k+1} d\vec{x}_{k+2} \dots d\vec{x}_{N-1} \\
&= w(b)\cdot w(c\setminus b).
\end{aligned} \tag{20}$$

Since $c \setminus b$ is made of m blocks, the induction hypothesis can be used and we have

$$w(c) = w(b)\cdot w(b_1)\cdot w(b_2)\cdot \ldots \cdot w(b_m), \tag{21}$$

proving that the $w(c)$ function is block-multiplicative. □

1.3.1. *The virial expansion.* In order to better explain the thermodynamic behaviour of non ideal gases, Kamerlingh Onnes proposed, in 1901, a series expansion of the form

$$\frac{P}{kT} = \rho + \gamma_2(T)\rho^2 + \gamma_3(T)\rho^3 + \cdots, \tag{22}$$

where $\rho = \frac{N}{V}$ is the density, called the *virial expansion*. A benefit of Mayer's theory is a formal derivation of this expansion and an interpretation of the virial coefficients $\gamma_n(T)$, $n \geq 2$, as the total weight of all (labelled) 2-connected graphs with n vertices (see [31] and [16]):

$$\gamma_n(T) = -\frac{(n-1)}{n!}|\mathcal{B}[n]|_w. \tag{23}$$

1.4. **Functional equation for weighted connected graphs.** The block-multiplicativity of $w(c)$ allows us to establish a functional relation between the species $\mathcal{C}_w$ of weighted connected graphs, and $\mathcal{B}_w$, of weighted 2-connected graphs. But first, let us recall a few notions in the theory of species.

For any species F, the *derivative* F' of F is the species whose structures on any set U are F-structures on the set $U \cup \{*\}$, where $*$ is a non-labelled external element. See [1] for more information about species. The rooting operation $F \mapsto F^\bullet$ consists of choosing a "root" (i.e., any element) in the set on which the F-structure is built. It can be defined as

$$F^\bullet = X \cdot F', \tag{24}$$

where X denotes the species of one-element sets. Notice that in terms of exponential generating series we have $F'(z) = \frac{d}{dz}F(z)$ and $F^\bullet(z) = zF'(z)$.

Theorem 6 below, and its weighted version theorem 7, can be found in various forms in the mathematical and physical literature (see [1], [9], [10], [11], [12], [15], [17], [21], [22], [25], [31]).

Theorem 6. *Let $\mathcal{B}$ be a particular class of 2-connected graphs and $\mathcal{C}_\mathcal{B}$ the species of connected graphs with all blocks in $\mathcal{B}$. Then we have the combinatorial functional equation*

$$\mathcal{C}_\mathcal{B}' = E(\mathcal{B}'(\mathcal{C}_\mathcal{B}^\bullet)), \tag{25}$$

where E denotes the species of sets, and, in terms of exponential generating functions,

$$\mathcal{C}_\mathcal{B}'(z) = \exp\left(\mathcal{B}'(\mathcal{C}_\mathcal{B}^\bullet(z))\right). \tag{26}$$

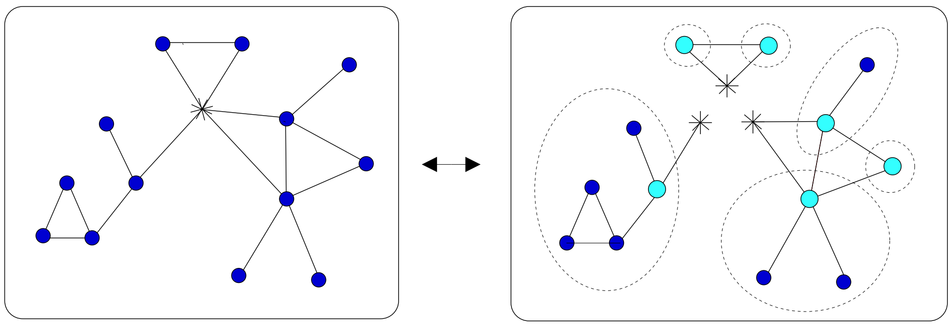

Figure 2. $\mathcal{C}_\mathcal{B}' = E(\mathcal{B}'(\mathcal{C}_\mathcal{B}^\bullet))$

Proof. Figure 2 will help us understand the isomorphism between the two classes of structures. On the left hand side of the figure we have a $\mathcal{C}'_{\mathcal{B}}$-structure, which is a $\mathcal{C}_{\mathcal{B}}$-structure over a set $U \cup \{*\}$. On the right hand side, the $*$ vertex has been split between the blocks to which it belongs. The vertices in pale colour are different from the others because they are forming $\mathcal{B}'$-structures with the $*$ vertices. Also each of these pale colour vertices is the root of a $\mathcal{C}^{\bullet}_{\mathcal{B}}$-structure, circled by a dotted line. We thus see that the $\mathcal{B}'$-structures are built on a set of $\mathcal{C}^{\bullet}_{\mathcal{B}}$-structures.

Therefore we have a bijection between the $\mathcal{C}'_{\mathcal{B}}$-structures on U and sets of $\mathcal{B}'(\mathcal{C}^{\bullet}_{\mathcal{B}})$-structures on U. Since this construction commutes with any relabelling, we have a species isomorphism which is expressed by (25). Formula (26) then follows automatically. □

Multiplying (25) by X gives

$$\mathcal{C}^{\bullet}_{\mathcal{B}} = X \cdot E(\mathcal{B}'(\mathcal{C}^{\bullet}_{\mathcal{B}})). \tag{27}$$

In terms of the exponential generating functions, we obtain the perhaps more familiar expression

$$\mathcal{C}^{\bullet}_{\mathcal{B}}(z) = z \cdot \exp\left(\mathcal{B}'(\mathcal{C}^{\bullet}_{\mathcal{B}}(z))\right). \tag{28}$$

Theorem 7. *Let w be a block-multiplicative weight function on connected graphs with all blocks in a particular species $\mathcal{B}$. Then we have*

$$\mathcal{C}^{\bullet}_{\mathcal{B},w} = X \cdot E\left(\mathcal{B}'_w(\mathcal{C}^{\bullet}_{\mathcal{B},w})\right). \tag{29}$$

Proof. The same proof as in Theorem 6 can be used. One has only to check that the weight of a global $\mathcal{C}'_{\mathcal{B}}$-structure is equal to the product of the weights of the individual structures that appear in the corresponding $E\left(\mathcal{B}'_w(\mathcal{C}^{\bullet}_{\mathcal{B},w})\right)$-structure. This is ensured by the block-multiplicativity of w. □

When we take the generating functions, formula (29) becomes

$$\mathcal{C}^{\bullet}_{\mathcal{B},w}(z) = z \exp\left(\mathcal{B}'_w(\mathcal{C}^{\bullet}_{\mathcal{B},w}(z))\right). \tag{30}$$

Note that the series $y = \mathcal{C}^{\bullet}_{\mathcal{B},w}(z)$ is a solution of the functional equation

$$y = z \exp\left(\mathcal{B}'_w(y)\right), \tag{31}$$

and the Lagrange inversion formula can be used to express the coefficients of y in terms of those of $R(t) := \exp(\mathcal{B}'_w(t))$.

2. The Gaussian model

2.1. **Relation between the Gaussian model and graph complexity.** Let

$$f_{ij} = -\exp(-\alpha\|\vec{x}_i - \vec{x}_j\|^2),\ \alpha > 0, \tag{32}$$

which corresponds to a soft repulsive potential at constant temperature. In this case, the function

$$f(r) = -\exp(-\alpha r^2) \tag{33}$$

satisfies the hypothesis of Proposition 3 and the second Mayer weight $w(c)$ can be explicitly computed for any connected graph c (see [31]). Here we give a complete proof and an extension of this result.

The graph complexity $\gamma(c)$ of a connected graph c is equal to the number of spanning subtrees of c. Let $e(c)$ denote the number of edges of c.

Theorem 8. *In dimension d, the second Mayer weight*

$$w(c) = \lim_{V\to\infty} \frac{1}{V} \int_{V^n} \prod_{\{i,j\}\in c} -\exp(-\alpha\|\vec{x}_i - \vec{x}_j\|^2) d\vec{x}_1 \ldots d\vec{x}_n \tag{34}$$

of a connected graph c with n vertices, has value

$$w(c) = (-1)^{e(c)} \left(\frac{\pi}{\alpha}\right)^{\frac{d(n-1)}{2}} \gamma(c)^{-\frac{d}{2}}. \tag{35}$$

Proof.

$$\begin{aligned} w(c) &= \lim_{V\to\infty} \frac{1}{V} \int_{V^n} \prod_{\{i,j\}\in c} -\exp(-\alpha\|\vec{x}_i - \vec{x}_j\|^2) d\vec{x}_1 \ldots d\vec{x}_n \\ &= (-1)^{e(c)} \lim_{V\to\infty} \frac{1}{V} \int_{V^n} \prod_{\{i,j\}\in c} \exp(-\alpha\|\vec{x}_i - \vec{x}_j\|^2) d\vec{x}_1 \ldots d\vec{x}_n. \end{aligned}$$

Without loss of generality, we can consider the vessel V as a d-dimensional hypercube centered at the origin, of the form of the Cartesian product $[-D, D]^d$ of intervals. Let us set $\vec{x}_i = (x_{i1}, x_{i2}, \ldots, x_{id})$. We have

$$\begin{aligned} w(c) &= (-1)^{e(c)} \lim_{D\to\infty} \frac{1}{(2D)^d} \int_{[-D,D]^{dn}} \prod_{\{i,j\}\in c} \exp\Big(- \alpha[(x_{i1} - x_{j1})^2 \\ &\quad + (x_{i2} - x_{j2})^2 + \ldots + (x_{id} - x_{jd})^2]\Big) dx_{11}dx_{12} \ldots dx_{1d}dx_{21} \ldots dx_{nd} \\ &= (-1)^{e(c)} \Bigg[\lim_{D\to\infty} \frac{1}{2D} \int_{[-D,D]^n} \\ &\qquad \prod_{\{i,j\}\in c} \exp(-\alpha(x_{i1} - x_{j1})^2) dx_{11}dx_{21} \ldots dx_{n1} \Bigg]^d, \end{aligned} \tag{36}$$

since the components of the vectors $\vec{x}_i$ are independent of each other and appear symmetrically in the Gaussian weight (32). Applying (16) and removing the unnecessary second lower index 1, we find

$$\begin{aligned} w(c) &= (-1)^{e(c)} \left[\int_{\mathbb{R}^{n-1}} \prod_{\{i,j\}\in c;\ x_n=0} \exp\left(-\alpha(x_i - x_j)^2\right) dx_1 \cdots dx_{n-1} \right]^d \\ &= (-1)^{e(c)} \left[\int_{\mathbb{R}^{n-1}} \exp\left(-\alpha \sum_{\{i,j\}\in c;\ x_n=0} (x_i - x_j)^2 \right) dx_1 \cdots dx_{n-1} \right]^d. \end{aligned}$$

Let $K = K(c)$ be the *Kirchhoff matrix*, of size $(n-1) \times (n-1)$, of the connected graph c, defined by

$$k_{ij} = \begin{cases} -1 & \text{if } \{i,j\} \in c, \\ d(i) & \text{if } \ i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the degree $d(i)$ takes into account edges attached to the "missing" vertex n. It is easily verified that

$$\sum_{\{i,j\} \in c;\ x_n = 0} (x_i - x_j)^2 = XKX^T, \tag{37}$$

for $X = (x_1, x_2, \ldots, x_{n-1})$, so that K is positive definite and we obtain, for the weight function,

$$w(c) = (-1)^{e(c)} \left[\int_{\mathbb{R}^{n-1}} \exp(-\alpha XKX^T) dx_1 \ldots dx_{n-1} \right]^d. \tag{38}$$

We then can use the following classical multidimensional Gaussian integral:

Proposition 9. *Let P be a $p \times p$ positive definite symmetric matrix. Then we have*

$$\int_{\mathbb{R}^p} \exp(-XPX^T) dX = \pi^{\frac{p}{2}} \det(P)^{-\frac{1}{2}}.$$

□

With $P = \alpha K$and $p = n-1$ we obtain

$$\begin{aligned} w(c) &= (-1)^{e(c)} \left[(\pi)^{\frac{n-1}{2}} \det(\alpha K)^{-1/2} \right]^d \\ &= (-1)^{e(c)} \left[\left(\frac{\pi}{\alpha}\right)^{\frac{n-1}{2}} \det(K)^{-1/2} \right]^d. \end{aligned} \tag{39}$$

We now use the spanning tree theorem of Kirchhoff (see [20]).

Theorem 10 (Kirchhoff)**.** *Let K be the Kirchhoff matrix of the connected graph c. Then, the number $\gamma(c)$ of spanning trees of c satisfies $\gamma(c) = \det(K)$.*

□

We finally obtain

$$w(c) = (-1)^{e(c)} \left(\frac{\pi}{\alpha}\right)^{\frac{d(n-1)}{2}} \gamma(c)^{-\frac{d}{2}}, \tag{40}$$

concluding the proof of Theorem 8. □

2.2. **An extension of the Gaussian potential.** Following a suggestion of participants (G. Andrews and H. Wilf) at the 54th Séminaire Lotharingien de Combinatoire in Lucelle, France, in April 2005, we can generalize this result to include the degree distribution of the vertices of the spanning subtrees. Let $T(c)$ be the set of all spanning subtrees of the connected graph c, and let $d_t(i)$ be the degree of the vertex i in the spanning subtree t.

Theorem 11. *With the potential function* f_{ij} *defined as*

$$f_{ij} = -\exp\left(-\alpha y_i y_j \|\vec{x}_i - \vec{x}_j\|^2\right), \tag{41}$$

where the y_i*'s are positive variables, the second Mayer weight function in* d *dimensions*

$$w(c) = \lim_{V\to\infty} \frac{1}{V} \int_{V^n} \prod_{\{i,j\}\in c} -\exp(-\alpha y_i y_j \|\vec{x}_i - \vec{x}_j\|^2) d\vec{x}_1 \dots d\vec{x}_n \tag{42}$$

of a connected graph c *with vertices* $\{1, 2, \dots, n\}$, *has value*

$$w(c) = (-1)^{e(c)} \left(\frac{\pi}{\alpha}\right)^{\frac{d(n-1)}{2}} \left(\sum_{t\in T(c)} y_1^{d_t(1)} y_2^{d_t(2)} \dots y_n^{d_t(n)}\right)^{-\frac{d}{2}}. \tag{43}$$

Proof. It suffices to define the matrix K by

$$k_{ij} = \begin{cases} -y_i y_j & \text{if } \{i,j\} \in c, \\ y_i \sum_{\{i,h\}\in c} y_h & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

As before, we have

$$\begin{aligned} w(c) &= \lim_{V\to\infty} \frac{1}{V} \int_{V^n} \prod_{\{i,j\}\in c} -\exp(-\alpha y_i y_j \|\vec{x}_i - \vec{x}_j\|^2) d\vec{x}_1 \dots d\vec{x}_n \\ &= (-1)^{e(c)} \left[\int_{\mathbb{R}^{n-1}} \exp(-\alpha X K X^T) dx_1 \cdots dx_{n-1}\right]^d \\ &= (-1)^{e(c)} \left[\left(\frac{\pi}{\alpha}\right)^{\frac{n-1}{2}} \det(K)^{-1/2}\right]^d. \end{aligned} \tag{44}$$

But Lovász shows (see [18], problem 4.10) that for such a matrix K,

$$\det(K) = \sum_{t\in T(c)} y_1^{d_t(1)} y_2^{d_t(2)} \dots y_n^{d_t(n)}; \tag{45}$$

therefore, (43) is established. □

Remark. It is possible to further extend this result by setting

$$f_{ij} = -\exp\left(-w_{i,j} \|\vec{x}_i - \vec{x}_j\|^2\right), \tag{46}$$

where the $w_{i,j}$'s are general (positive) weights associated to edges. In this case the weight of a connected graph c gives the edge enumerator of spanning subtrees of c. Indeed, invoking a more general form of Kirchhoff's formula (see, for example Sylvester [29], Borchardt [3] and Chaiken and Kleitman [5], we find that

$$w(c) = (-1)^{e(c)} (\pi)^{\frac{d(n-1)}{2}} \left(\sum_{t\in T(c)} \prod_{\{i,j\}\in t} w_{i,j}\right)^{-\frac{d}{2}}. \tag{47}$$

3. The hard-core continuum gas in one dimension

Consider N hard particles of diameter 1 on a line segment, of the form $[-D, D]$. The *hard-core* constraint translates into the interaction potential $\chi(|x_i - x_j| \geq 1)$ (with $\varphi(r) = \infty$, if $r < 1$, and $\varphi(r) = 0$, if $r \geq 1$) and the Mayer function f_{ij} is defined by

$$\begin{aligned} 1 + f_{ij} &= \chi(|x_i - x_j| \geq 1) \\ \Leftrightarrow f_{ij} &= -\chi(|x_i - x_j| < 1). \end{aligned} \tag{48}$$

The hypothesis of Proposition 3 is verified for the function

$$f(r) = -\chi(r < 1), \tag{49}$$

allowing us to write the weight function $w(c)$ of a connected graph c as

$$w(c) = (-1)^{e(c)} \int_{\mathbb{R}^{N-1}} \prod_{\{i,j\}\in c\,;\, x_N=0} \chi(|x_i - x_j| < 1)\, dx_1 \ldots dx_{N-1}. \tag{50}$$

With this potential, the pressure can still be expressed in terms of the exponential generating function of the weighted species of connected graphs, as in (18):

$$\frac{P}{kT} = \mathcal{C}_w(z). \tag{51}$$

3.1. **Global formulas.** It is known (see [4]) that for the hard-core gas, the pressure of the system is given by

$$\frac{P}{kT} = L(z), \tag{52}$$

where $L(z)$ denotes the *Lambert function*, defined by the functional equation

$$L(z) \exp(L(z)) = z. \tag{53}$$

Here, we give a combinatorial proof of this result.

Proposition 12. *In the thermodynamic limit $D \to \infty$, the pressure $\frac{P}{kT}$ of the continuous unidimensional hard-core gas model is given by (52).*

Proof. Note that the Lambert function satisfies $L(z) = -T(-z)$, where $T(z)$ is the exponential generating function of labelled rooted trees. Let us consider the particles on a segment of the form $[0, 2D]$. Then, since the $N!$ possible relative positions of the x_i give rise to integrals of equal value, the grand-canonical partition function can be written as

$$\begin{aligned} Z_{gr}(D, z) &= \sum_{N\geq 0} \frac{z^N}{N!} \int_{[0,\,2D]^N} \prod_{i<j} \chi(|x_i - x_j| \geq 1) dx_1 dx_2 \cdots dx_N \\ &= \sum_{N\geq 0} z^N \int_0^{2D} dx_1 \int_{x_1+1}^{2D} dx_2 \cdots \int_{x_{N-1}+1}^{2D} dx_N. \end{aligned} \tag{54}$$

Now the integral in (54) is the volume of the simplex

$$0 \leq x_1 \leq x_2 - 1 \leq x_3 - 2 \leq \cdots \leq x_N - N + 1 \leq 2D - N + 1,$$

and has value $(2D-N+1)^N/N!$. See for example Section 4.2 of [30] on the one-dimensional Tonks gas. It follows that

$$Z_{gr}(-D,-z) = \sum_{N\geq 0}(N+2D-1)^N z^N/N!\,, \tag{55}$$

which is the exponential generating function of structures which consist of functions of the form $f: U \to U+V$, where U is a finite set (of varying size N), V is a fixed set of size $|V| = 2D-1$ and $+$ denotes the disjoint union. By considering the saggital graph of such functions, obtained by taking the oriented edges $(u, f(u))$, $u \in U$, we see that the connected components consist of either connected endofunctions on subsets U' of U or rooted trees on U' pointing to an element of V. Thus there are $2D-1$ kinds of connected components of the sort "rooted trees". Let f_N be the number of connected endofunctions on an N-element set and let $F(z) = \sum_{N\geq 0} f_n z^N/n!$ be their generating function. Then we have

$$Z_{gr}(-D,-z) = \exp\left(F(z) + (2D-1)T(z)\right) \tag{56}$$

and, reverting back to D and z and taking the thermodynamic limit,

$$\begin{aligned}\frac{P}{kT} &= \lim_{D\to\infty} \frac{1}{2D}\log Z_{gr}(D,z)\\ &= \lim_{D\to\infty} \frac{1}{2D}(F(-z) - (2D+1)T(-z))\\ &= -T(-z) \;=\; L(z),\end{aligned}$$

which concludes the proof. □

Corollary 13. *Let N be an integer ≥ 1; the total weight $|\mathcal{C}[N]|_w$ of the set of all connected graphs over the set $[N] = \{1, 2, \ldots, N\}$ of vertices is given by*

$$\sum_{c\in\mathcal{C}[N]} w(c) = (-N)^{N-1}. \tag{57}$$

Proof. This follows immediately from the fact that

$$\mathcal{C}_w(z) = \frac{P}{kT} = -T(-z)$$

by extracting coefficients. □

We now invoke the functional equation (29) in the case where $\mathcal{B}$ is the species of all 2-connected graphs and hence $\mathcal{C}_{\mathcal{B}} = \mathcal{C}$, the species of all connected graphs, with the weight function w given by (50):

$$\mathcal{C}_w^\bullet(z) = z\,\exp\left(\mathcal{B}'_w(\mathcal{C}_w^\bullet(z))\right). \tag{58}$$

Proposition 14. *For the total weight of 2-connected graphs, we have*

$$\mathcal{B}_w(z) = z\log(1-z). \tag{59}$$

Conversely, formula (59) implies that $\mathcal{C}_w(z) = L(z)$.

Proof. It is clear from (58) that any one of the functions $\mathcal{C}_w(z)$ and $\mathcal{B}_w(z)$ determines uniquely the other, since $\mathcal{C}_w(z)$ is of the form $z+\dots$ and $\mathcal{B}_w'(z) = -2z+\dots$. Hence it suffices to prove that

$$L^\bullet(z) = z\exp\left(B'(L^\bullet(z))\right), \tag{60}$$

where the function $B(z)$ is defined by

$$B(z) = z\log(1-z), \tag{61}$$

in order to establish the Proposition. It is easily seen that

$$B'(z) = \log(1-z) - \frac{z}{1-z},$$

and that $L^\bullet(z) = zL'(z)$ satisfies

$$L^\bullet(z) = \frac{L(z)}{1+L(z)},$$

upon differentiation of (53). We then have

$$\begin{aligned}
z\exp(B'(L^\bullet(z))) &= z\exp\left(\log(1-L^\bullet(z)) - \frac{L^\bullet(z)}{1-L^\bullet(z)}\right)\\
&= z(1-L^\bullet(z))\exp\left(\frac{-L^\bullet(z)}{1-L^\bullet(z)}\right)\\
&= z\left(1-\frac{L(z)}{1+L(z)}\right)\exp\left(\frac{-\frac{L(z)}{1+L(z)}}{1-\frac{L(z)}{1+L(z)}}\right)\\
&= z\frac{1}{1+L(z)}\exp(-L(z))\\
&= \frac{L(z)}{1+L(z)}\\
&= L^\bullet(z),
\end{aligned}$$

which proves (60). □

Proposition 15. *Let N be an integer ≥ 2; the total weight $|\mathcal{B}[N]|$ of the set of all 2-connected graphs with N vertices is given by*

$$\sum_{c\in\mathcal{B}[N]} w(c) = -N(N-2)! \tag{62}$$

Proof. This follows immediately from (59) by extracting coefficients. □

Corollary 16. *The virial expansion for the hard core one-dimensional gas is given by*

$$\frac{P}{kT} = \frac{\rho}{1-\rho}. \tag{63}$$

Proof. Using equations (23) and (62), we see that $\gamma_n(T) = 1$, for all $n \geq 2$ and the result follows. □

Question 1: Are there direct combinatorial proofs of (57) and (62), independently of Proposition 12 ?

Question 2: Can we compute the individual weights $w(c)$ of given connected graphs c and interpret them in terms of other graph invariants?

3.2. **The Ehrhart polynomial.** While trying to answer these questions, we have made the following observation. Except for the sign, the weight

$$w(c) = (-1)^{e(c)} \int_{\mathbb{R}^{N-1}} \prod_{\{i,j\}\in c\,;\, x_N=0} \chi(|x_i - x_j| < 1)\, dx_1 \dots dx_{N-1} \tag{64}$$

can be seen as the volume of a convex polytope $\mathcal{P}(c)$ in $\mathbb{R}^N$ bounded by the constraints $|x_i - x_j| \leq 1$, for $\{i,j\} \in c$, with $x_N = 0$. We can compute this volume using Ehrhart polynomials (see [28]).

Theorem 17 (Ehrhart). *Let $\mathcal{P}$ be a convex polytope of dimension d in $\mathbb{R}^m$, with vertices having integer coordinates. Let $n\mathcal{P} = \{n\alpha : \alpha \in \mathcal{P}\}$ denote the n-fold expansion of $\mathcal{P}$, and $I(\mathcal{P}, n)$, the number of points with integer coordinates which lie inside $n\mathcal{P}$. Then $I(\mathcal{P}, n)$ is a polynomial function of n of degree d whose leading coefficient is the volume* Vol$(\mathcal{P})$ *of $\mathcal{P}$.*

□

In order to apply Ehrhart's theorem, we have to prove the following:

Proposition 18. *Let c be a connected graph with its N vertices labelled* $\{1, 2, \dots, N\}$, *and define the convex polytope $\mathcal{P}(c) \subset \mathbb{R}^N$ by*

$$\mathcal{P}(c) = \left\{ X \in \mathbb{R}^N \mid x_N = 0 \text{ and } |x_i - x_j| \leq 1 \ \ \forall \{i,j\} \in c \right\}, \tag{65}$$

where $X = (x_1, \dots, x_N)$. Then the vertices of $\mathcal{P}(c)$ have integer coordinates.

Proof. Notice that $\mathcal{P}(c)$ is of dimension $N-1$. Every vertex of $\mathcal{P}(c)$ is at the intersection of $N-1$ faces of the polytope. Each equation of the form $|x_i - x_j| = 1$, $\{i,j\} \in c$, generates two faces: one that satisfies the equation $x_i - x_j = 1$, and the other which satisfies $x_j - x_i = 1$, corresponding to the two possible orientations of the edge $\{i,j\}$. We can express all these equations in the matrix form

$$X\,\Omega = (1, 1, \dots, 1), \tag{66}$$

where Ω is the incidence matrix of the oriented graph obtained from c by replacing each edge by the two corresponding opposite oriented edges. The result then follows from the fact that this matrix is totally unimodular, i.e., all its subdeterminants have values 0, ± 1. See [26]. □

It follows that the volume of $\mathcal{P}(c)$ and the weight

$$w(c) = (-1)^{e(c)}\, \mathrm{Vol}(\mathcal{P}(c)) \tag{67}$$

can be obtained by computing the Ehrhart polynomial $I(\mathcal{P}(c), n)$. Notice that the n-fold expansion of $\mathcal{P}(c)$ is given by

$$n\mathcal{P}(c) = \left\{ X \in \mathbb{R}^N \mid x_N = 0 \text{ and } |x_i - x_j| \leq n \ \ \forall \{i,j\} \in c \right\}. \tag{68}$$

For all 2-connected graphs c having N vertices, with $N \leq 6$, we have carried out the computation of the Ehrhart polynomial $I(\mathcal{P}(c), n)$ and hence the volume $\mathrm{Vol}(\mathcal{P}(c))$, by counting the number of points with integer coordinates in the n-fold expansion $n\,\mathcal{P}(c)$, for $n = 2, \ldots, 6$. See Appendix B for the results. The weight of any connected graph c all of whose blocks have size at most 6 can then be deduced by block multiplicativity. The individual weights $w(c)$ for 2-connected graphs c of size up to 5 were already known. See Riddell and Uhlenbeck [23], p. 2063, for example.

3.3. **Particular weight values.** The numerical results have led us to conjecture and then prove a formula for the weight $w(K_N)$ of the complete graph with N vertices ($N \geq 2$). In this section, we also give two formulas for the weight $w(C_N)$ of the N-cycle ($N \geq 3$). The weight $w(K_N \backslash e)$ for the complete graph minus an edge ($N \geq 3$) is given in the following section.

3.3.1. *The complete graph K_N.*

Proposition 19. *For the complete graph K_N, we have*

$$w(K_N) = (-1)^{\binom{N}{2}} N. \tag{69}$$

Proof. Starting with formula (50), we have

$$\begin{aligned} w(K_N) &= (-1)^{\binom{N}{2}} \int_{\mathbb{R}^{N-1}} \prod_{\{i,j\} \in K_N;\, x_N = 0} \chi(|x_i - x_j| < 1)\, dx_1 \cdots dx_{N-1} \\ &= (-1)^{\binom{N}{2}} \int_{-1}^{1} \cdots \int_{-1}^{1} \prod_{1 \leq i < j \leq N-1} \chi(|x_i - x_j| < 1)\, dx_1 \cdots dx_{N-1}. \end{aligned} \tag{70}$$

Replacing each x_i by $x_i + 1$, for $i = 1, \ldots N-1$, yields

$$\begin{aligned} |w(K_N)| &= \int_{0}^{2} \cdots \int_{0}^{2} \prod_{1 \leq i < j \leq N-1} \chi(|x_i - x_j| < 1)\, dx_1 \cdots dx_{N-1} \\ &= (N-1)! \int_A \prod_{1 \leq i < j \leq N-1} \chi(x_i - x_j < 1)\, dx_1 \cdots dx_{N-1}, \end{aligned} \tag{71}$$

where A denotes the region $0 \leq x_1 \leq \cdots \leq x_{N-1} \leq 2$, by symmetry. Let us make the change of variables

$$\begin{aligned} y_i &= x_{i+1} - x_i, \text{ for } 1 \leq i \leq N-2, \\ y_{N-1} &= 2 - x_{N-1}, \end{aligned} \tag{72}$$

which is equivalent to

$$\begin{aligned} x_i &= 2 - y_i - \cdots - y_{N-1}, \text{ for } 1 \leq i \leq N-2, \\ x_{N-1} &= 2 - y_{N-1}. \end{aligned} \tag{73}$$

The domain of integration A is transformed into the region B defined by $y_i \geq 0$, for $i = 1, \ldots, N-1$, and

$$y_1 + \cdots + y_{N-1} \leq 2.$$

Moreover, in this region, the integrand of (71) is not zero, and has value 1, if and only if

$$y_1 + y_2 + \cdots + y_{N-2} < 1.$$

We can decompose the region B into the subregions B_1, where $y_1 + \cdots + y_{N-1} \le 1$, and B_2, where $1 \le y_1 + \cdots + y_{N-1} \le 2$ and (71) becomes

$$\begin{aligned}
|w(K_N)| &= (N-1)! \int_B \chi(y_1 + \cdots + y_{N-2} < 1)\, dy_1 \cdots dy_{N-1} \\
&= (N-1)! \int_{B_1 \uplus B_2} \chi(y_1 + \cdots + y_{N-2} < 1)\, dy_1 \cdots dy_{N-1} \\
&= (N-1)! \Bigg(\int_{\mathbb{R}_+^{N-1}} \chi(y_1 + \cdots + y_{N-1} < 1)\, dy_1 \cdots dy_{N-1} \\
&\quad + \int_{\mathbb{R}_+^{N-2}} \left(\int_{1-y_1-\cdots-y_{N-2}}^{2-y_1-\cdots-y_{N-2}} dy_{N-1} \right) \chi(y_1 + \cdots + y_{N-2} < 1)\, dy_1 \cdots dy_{N-2} \Bigg) \\
&= (N-1)! \left(\frac{1}{(N-1)!} + \frac{1}{(N-2)!} \right) \\
&= 1 + N - 1 = N.
\end{aligned}$$

□

3.3.2. *The Cycle graph C_N.*

Proposition 20. *For the (unoriented) cycle C_N with N vertices, we have*

$$w(C_N) = (-1)^N \frac{2^N}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin t}{t} \right)^N dt. \tag{74}$$

Proof. Starting with (50), let us make the change of variables

$$t_i = x_i - x_{i+1}, \text{ for } 1 \le i \le N-1,$$

of Jacobian 1. The conditions $|x_i - x_{i+1}| \le 1$ corresponding to the edges $\{i, i+1\}$ of C_N give $|t_i| \le 1$ and the condition $|x_1 - x_N| \le 1$ corresponding to the edge $\{1, N\}$ yields $|t_1 + \cdots + t_{N-1}| \le 1$. Then, writing $\chi(x) := \chi(|x| \le 1)$ for simplicity, we have

$$\begin{aligned}
w(C_N) &= (-1)^N \int_{\mathbb{R}^{N-1}} \prod_{\{i,j\} \in C_N\,;\, x_N = 0} \chi(x_i - x_j)\, dx_1 \ldots dx_{N-1} \\
&= (-1)^N \int_{\mathbb{R}^{N-1}} \chi(t_1 + t_2 + \cdots + t_{N-1}) \chi(t_1) \cdots \chi(t_{N-1})\, dt_1 \cdots dt_{N-1} \\
&= (-1)^N \operatorname{Vol}\{T \in \mathbb{R}^{N-1} \mid -\mathbf{1} \le T \le \mathbf{1},\ -1 \le \langle T, \mathbf{1} \rangle \le 1\}, \qquad (75)
\end{aligned}$$

where $T = (t_1, \ldots, t_{N-1})$, $\mathbf{1} = (1, \ldots, 1)$ and $\langle X, Y \rangle$ denotes the scalar product.

Let $U_1, U_2, \ldots, U_{N-1}$ be independent identically distributed uniform random variables on the interval $[-1, 1]$, with common density function $u(x) = \frac{1}{2}\chi(x)$ and let $S = U_1 + \cdots + U_{N-1}$. Then, by (75),

$$w(C_N) = (-1)^N\, 2^{N-1} \operatorname{Prob}(-1 \le S \le 1). \tag{76}$$

The density function $s(x)$ of S is given by the $(N-1)$-fold convolution product

$$s = u^{*(N-1)} = u * \cdots * u \quad (N-1 \text{ factors}),$$

where

$$(f * g)(x) := \int_{-\infty}^{\infty} f(\xi) g(x-\xi)\, d\xi.$$

We deduce from (76) that

$$\begin{aligned} w(C_N) &= (-1)^N\, 2^{N-1} \int_{-1}^{1} s(\xi)\, d\xi \\ &= (-1)^N\, 2^{N-1} 2 \int_{-\infty}^{\infty} s(\xi) u(\xi)\, d\xi \\ &= (-2)^N \int_{-\infty}^{\infty} u^{*(N-1)}(\xi) u(0-\xi)\, d\xi \\ &= (-2)^N\, u^{*N}(0). \end{aligned} \tag{77}$$

Since the Fourier transform $\hat{u}(t)$ of $u(x)$ is given by

$$\hat{u}(t) = \int_{-\infty}^{\infty} u(x) e^{-itx} dx = \frac{1}{2} \int_{-1}^{1} e^{-itx} dx = \frac{\sin t}{t} \tag{78}$$

and $\widehat{f * g}(t) = \hat{f}(t) \cdot \hat{g}(t)$, we get

$$\widehat{u^{*N}}(t) = (\hat{u}(t))^N = \left(\frac{\sin t}{t}\right)^N. \tag{79}$$

Taking the inverse Fourier transform on both sides, we deduce that

$$\begin{aligned} u^{*N}(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{u^{*N}}(t) e^{itx} dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin t}{t}\right)^N e^{itx} dt \end{aligned} \tag{80}$$

and (74) follows via the substitution $x = 0$. □

Corollary 21. *We have the asymptotic estimate*

$$w(C_N) \sim (-2)^N \left(\frac{3}{2\pi N}\right)^{\frac{1}{2}} \left(1 - \frac{3}{20N} - \frac{13}{1120N^2} + \dots\right) \tag{81}$$

as $N \to \infty$.

Proof. Make the local substitution $\frac{\sin t}{t} = \exp(-u^2)$ around the origin and integrate term by term. □

Proposition 22. *For the cycle* C_N *with* N *vertices, we also have*

$$w(C_N) = \frac{(-1)^N}{(N-1)!} \sum_{i=0}^{\lfloor \frac{N-1}{2} \rfloor} (-1)^i \binom{N}{i} (N-2i)^{N-1}. \tag{82}$$

Proof. Note first that the density function $u(x) = \frac{1}{2}\chi(|x| \le 1)$ satisfies

$$u = \frac{1}{2}(\delta_{-1} - \delta_1) * H(x) \quad \text{(a.e.)}, \tag{83}$$

where δ_a and H are defined by

$$(\delta_a * f)(x) = f(x-a), \qquad H(x) = \begin{cases} 1, & \text{if } x > 0, \\ 1/2, & \text{if } x = 0, \\ 0, & \text{if } x < 0. \end{cases}$$

Using (77), (83), commutativity of the convolution product and the binomial theorem, we obtain

$$\begin{aligned} w(C_N) &= (-1)^N((\delta_{-1} - \delta_1)^{*N} * H^{*N})(0) \\ &= (-1)^N \left(\sum_{i=0}^{N} (-1)^i \binom{N}{i} \delta_{2i-N} * \frac{(x^+)^{N-1}}{(N-1)!} \right) \Bigg|_{x=0} \\ &= \frac{(-1)^N}{(N-1)!} \sum_{i=0}^{N} (-1)^i \binom{N}{i} ((x+N-2i)^+)^{N-1} \Bigg|_{x=0} \\ &= \frac{(-1)^N}{(N-1)!} \sum_{i=0}^{\lfloor \frac{N-1}{2} \rfloor} (-1)^i \binom{N}{i} (N-2i)^{N-1}, \end{aligned}$$

since $H^{*N}(x) = (x^+)^{N-1}/(N-1)!$, where $x^+ := H(x)x$. □

Combinatorial proof of Proposition 22 for N even. We can give a combinatorial proof of (82) when N is even, based on the geometric properties of the Eulerian numbers $A_{n,k}$ $(n \ge 1,\ 1 \le k \le n)$. They can be defined by the Worpitzky formula (see [8])

$$A_{n,k} = \sum_{0 \le i \le k-1} (-1)^i (k-i)^n \binom{n+1}{i}. \tag{84}$$

These numbers admit a geometric interpretation using the volume of a closed region of $\mathbb{R}^n$ (see [8] and the note [27] following it):

$$A_{n,k}/n! = \mathrm{Vol}\{\, Y \in \mathbb{R}^n \mid \mathbf{0} \le Y \le \mathbf{1},\ k-1 \le \langle Y, \mathbf{1} \rangle \le k \,\}, \tag{85}$$

where $\mathbf{0}$ and $\mathbf{1}$ denote the constant vectors with entries 0 and 1, respectively.

Starting with (75) let us make the transformation

$$z_i := t_i + 1, \text{ for } 1 \le i \le N-1$$

and set $Z = (z_1, \ldots, z_{N-1})$ and $\mathbf{2} = (2, \ldots, 2)$. Then we see that

$$w(C_N) = (-1)^N \mathrm{Vol}\{\, Z \in \mathbb{R}^{N-1} \mid \mathbf{0} \le Z \le \mathbf{2},\ N-2 \le \langle Z, \mathbf{1} \rangle \le N \,\}. \tag{86}$$

If N is even, we can simplify the expression (86) further by setting $y_i := z_i/2$. We then find, using (85) and (84),

$$\begin{aligned} w(C_N) &= (-1)^N\, 2^{N-1}\,\mathrm{Vol}\,\{\, Y \in \mathbb{R}^{N-1} \mid \mathbf{0} \le Y \le \mathbf{1},\ N/2-1 \le \langle Y, \mathbf{1}\rangle \le N/2 \,\} \\ &= (-1)^N\, 2^{N-1} \frac{1}{(N-1)!}\, A_{N-1,N/2} \\ &= (-1)^N \frac{2^{N-1}}{(N-1)!} \sum_{0 \le i \le N/2-1} (-1)^i (N/2-i)^{N-1} \binom{N}{i} \\ &= \frac{(-1)^N}{(N-1)!} \sum_{0 \le i \le N/2-1} (-1)^i (N-2i)^{N-1} \binom{N}{i}, \end{aligned} \tag{87}$$

which establishes (82). □

Remark. For N odd (or general), by analogy with the Eulerian numbers, it is tempting to define numbers $B_{n,k}$ by the formula

$$B_{n,k}/n! = \mathrm{Vol}\,\{\, Z \in \mathbb{R}^n \mid \mathbf{0} \le Z \le \mathbf{2},\ k-1 \le \langle Z, \mathbf{1}\rangle \le k \,\} \tag{88}$$

so that $(N-1)!\, w(C_N) = B_{N-1,N-1} + B_{N-1,N}$. There remains to interpret these numbers combinatorially and to use them to establish (82) in complete generality.

3.4. **Graph homomorphisms.** As observed by Bodo Lass [13], it is possible to evaluate the volume of the polytope $\mathcal{P}(c)$ by decomposing it into a certain number $\nu(c)$ of subpolytopes which are all simplices of volume $1/(N-1)!$. Each subpolytope is obtained by fixing the integral parts and the relative positions of the fractional parts of the coordinates $x_1, \ldots, x_N$ of points $X \in \mathcal{P}(c)$. The number of such configurations will then yield $\nu(c)$ and we will have $\mathrm{Vol}(\mathcal{P}(c)) = \nu(c)/(N-1)!$.

In order to make this correspondence more precise, we consider the following fractional representation of real numbers

$$\mathbb{R} \to ([0,1] \times \mathbb{Z}) \ :\ x \mapsto (\xi_x, h_x), \tag{89}$$

where $h_x = \lfloor x \rfloor$ is the integral part of x and $\xi_x = x - h_x$ is the (positive) fractional part of x, so that $x = \xi_x + h_x$. For example,

$$0.25 \mapsto (0.25, 0),\quad 3.75 \mapsto (0.75, 3) \quad\text{and}\quad -1.25 \mapsto (0.75, -2).$$

See Figure 3. Notice that $0 \le \xi_x < 1$. However, for $x = 0$, it will be convenient to use the special representation $0 \mapsto (1.0, -1)$, as if 0 was infinitesimally negative.

In this representation, the condition $|x - y| < 1$ for two real numbers x and y translates into "$\xi_x \ne \xi_y$ and assuming $\xi_x < \xi_y$, then $h_x = h_y$ or $h_x = h_y + 1$". This can be visualized as follows: the slope of the line segment between x and y should be either null or negative. See Figure 3 where the interval $(x-1, x+1)$ is represented by the thick segments.

Now consider a connected graph c with vertex set $V = [N] = \{1, 2, \ldots, N\}$, and let $X = (x_1, \ldots, x_N)$ be a point in the polytope $\mathcal{P}(c)$. Let us write $x_i \mapsto (\xi_i, h_i)$ for the fractional representation of the coordinate x_i of X, $i = 1, \ldots, N$. Recall that $x_N = 0$ so that $\xi_N = 1.0$ and $h_N = -1$, with our convention.

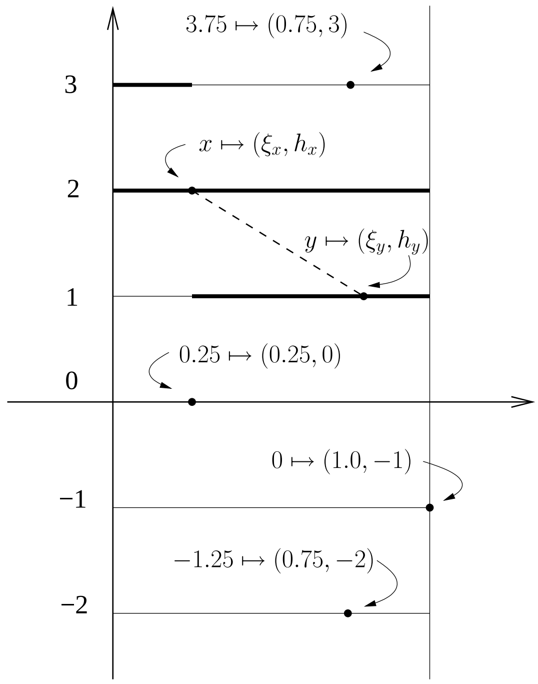

FIGURE 3. Fractional representation of real numbers

The volume of $\mathcal{P}(c)$ is not changed by removing all hyperplanes $\{x_i - x_j = k\}$, for $k \in \mathbb{Z}$. Hence, we can assume that all the fractional parts ξ_i are distinct. We form a subpolytope of $\mathcal{P}(c)$ by keeping the "heights" $h_1, h_2, \ldots, h_N$ fixed as well as the relative positions (total order) of the fractional parts $\xi_1, \xi_2, \ldots, \xi_N$. Let $h : V \to \mathbb{Z}$ denote the height function $i \mapsto h_i$ and $\beta : V \to [N]$ be the permutation of $[N]$ for which $\beta(i)$ gives the rank of ξ_i in this total order. Note that $\beta(N) = N$. For example, if $N = 5$ and $\xi_3 < \xi_4 < \xi_2 < \xi_1 < \xi_5$, then $\beta(1) = 4$, $\beta(2) = 3$, $\beta(3) = 1$, $\beta(4) = 2$ and $\beta(5) = 5$, i.e., $\beta = (4, 3, 1, 2, 5)$.

The corresponding subpolytope will be denoted by $\mathcal{P}(h, \beta)$. Let us choose a canonical point $X = X_{h,\beta}$ of $\mathcal{P}(h, \beta)$, say the *centroid*, obtained by setting $\xi_i = \beta(i)/N$, $i = 1, \ldots, N$. Using the fractional coordinates to represent this canonical point $X_{h,\beta}$ of $\mathcal{P}(h, \beta)$, and drawing a dotted line segment between x_i and x_j for each edge i, j of the graph c, we obtain a configuration in the plane which can be seen as an homomorphic image of c and which characterizes the subpolytope $\mathcal{P}(h, \beta)$. For example, with $N = 5$ and $c = C_5$, the 5-cycle, we can take $h = (0, 1, 1, 0, -1)$ and $\beta = (4, 3, 1, 2, 5)$ as above. This indeed defines a subpolytope $\mathcal{P}(h, \beta)$ of $\mathcal{P}(C_5)$, for which $X_{h,\beta} = (0.8, 1.6, 1.2, 0.4, 0)$. Figure 4 illustrates the corresponding configuration, where the homomorphic image of C_5 appears clearly.

Proposition 23. *Let c be a connected graph with vertex set $V = [N]$ and consider a function $h : V \to \mathbb{Z}$ and a bijection $\beta : V \to [N]$ satisfying $\beta(N) = N$. Then the pair (h, β) determines a valid subpolytope $\mathcal{P}(h, \beta)$ of $\mathcal{P}(c)$ if and only if the following*

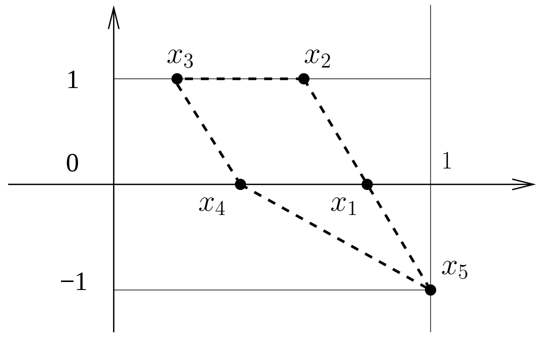

FIGURE 4. Fractional representation of a simplicial subpolytope of $\mathcal{P}(C_5)$

condition is satisfied:

$$\text{for any edge } \{i,j\} \text{ of } c, \quad \beta(i) < \beta(j) \ \text{ implies } \ h_i = h_j \text{ or } h_i = h_j + 1. \tag{90}$$

Proof. Let $X_{h,\beta} = (x_1, \dots, x_N)$ denote the canonical point associated with (h, β), i.e., where $x_i = h_i + \xi_i$, with $\xi_i := \beta(i)/N$, $i = 1, \dots, N$. Then the pair (h, β) determines a valid subpolytope $\mathcal{P}(h, \beta)$ of $\mathcal{P}(c)$ if and only if the point $X_{h,\beta}$ is in $\mathcal{P}(c)$. But the condition (90) expresses exactly that $|x_i - x_j| < 1$, whenever $\{i, j\}$ is an edge of c, that is, the defining condition of $\mathcal{P}(c)$. □

Proposition 24. *Let c be a connected graph and let (h, β) be such that $\beta(N) = N$ and condition (90) is satisfied. Then the volume of the associated subpolytope $\mathcal{P}(h, \beta)$ is equal to $1/(N-1)!$.*

Proof. The simplest case is when β is the identity, i.e., when $0 < \xi_1 < \xi_2 < \cdots < \xi_{N-1} < 1$, and all levels $h_1, h_2, \dots, h_{N-1}$ are 0. Since $x_N = 0$, this corresponds to the standard $(N-1)$-dimensional simplex whose volume is $1/(N-1)!$. But it is clear that the same is true for any β and that all the other subpolytopes $\mathcal{P}(h, \beta)$ are translates of these, hence also $(N-1)$-simplexes, of volume $1/(N-1)!$. □

Proposition 25. *Let c be a connected graph and let $\nu(c)$ be the number of pairs (h, β) such that the condition (90) is satisfied. Then the volume of the polytope $\mathcal{P}(c)$ defined by (65) is given by*

$$\mathrm{Vol}(\mathcal{P}(c)) = \nu(c)/(N-1)!. \tag{91}$$

Proof. It is clear that the polytope $\mathcal{P}(c)$ is the disjoint union of all its subpolytopes $\mathcal{P}(h, \beta)$ and the result follows from Proposition 24. □

Proposition 25 can be used to compute the weight of some infinite families of graphs, since $w(c) = (-1)^{e(c)}\mathrm{Vol}(\mathcal{P}(c))$. As a first example, we give an alternate proof of the formula (69): $w(K_N) = (-1)^{\binom{N}{2}} N$. Indeed, since all edges are present in the complete graph, any of the $(N-1)!$ permutations *beta* for which $\beta(N) = N$ can occur, by symmetry, and there are only N possible height sequences $h \circ \beta^{(-1)}$, of the form $(0,\ldots,0,-1,\ldots,-1)$, from $(0,\ldots,0,-1)$ to $(-1,\ldots,-1)$, which give rise to legal configurations (h,β). Hence $\nu(K_N) = N(N-1)!$ and the result follows.

Another example is the following, due to Bodo Lass:

Proposition 26. [13] *For $N \geq 3$, let $K_N\backslash e$ denote the complete graph on N vertices from which an arbitrary edge has been removed. Then we have*

$$w(K_N\backslash e) = (-1)^{\binom{N}{2}-1}\left(N + \frac{2}{(N-1)}\right). \tag{92}$$

Proof. We can assume that the missing edge is $e = \{1, N\}$. Note that all the configurations (h,β) of K_N are also valid here. In addition, there are two other possibilities for $h:$ a) set $h_1 = 1$, $h_N = -1$ and all other $h_i = 0$, so that $\beta(1)$ must be 1, and b) $h_1 = -2$ and all other $h_i = -1$, so that $\beta(1)$ must be $N-1$. In both cases β can be extended in $(N-2)!$ ways. $\square$

Finally, an elegant description of all the compatible configurations can be given for the cycle graph C_N on $[N]$, with edge set $\{\{i, i+1\} (\mathrm{mod}\ N) \mid i = 1,\ldots,N\}$. Indeed, following the cycle, the sequence of heights

$$h^* = (h_N = -1, h_1, h_2, \ldots, h_{N-1}, h_N = -1) \tag{93}$$

defines a path of Motzkin type, i.e., where the only permitted "moves" are rises (by 1), levels and descents (by 1), denoted by r, l and d, respectively. Starting and ending at height -1, the path h^* can thus be encoded by a generalized Motzkin word μ in the letters r, l, d. As an example, for the configuration of Figure 4, we have

$$h^* = (-1, 0, 1, 1, 0, -1) \quad \text{and} \quad \mu = rrldd\,.$$

Notice that the first step cannot be a descent and the last step, a rise. But the path is allowed to attain heights below -1. Thus the legal configurations of C_N can be classified according to the corresponding words μ and the compatible permutations β can be given a simple description in terms of μ. Details are left to the reader.

Remark. Olivier Bernardi [2] has partially answered Question 1 given at the end of Section 3.1. Using the subpolytope representations of this section, he has described an involution which establishes (57), namely that the sum of all weights $w(c)$ of connected graphs c over an N-element set is $(-N)^{N-1}$.

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Appendix A. Proof of Proposition 3

First we have to show that the function

$$F_{\vec{x}_N}(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_{N-1}) = \prod_{\{i,j\}\in c} f(|\vec{x}_i - \vec{x}_j|) \tag{94}$$

is integrable. In order to prove this, we need to show that $F_{\vec{x}_N}$ is measurable and that

$$I_{\vec{x}_N} := \int_{\mathbb{R}^{d(N-1)}} |F_{\vec{x}_N}(\vec{x}_1, \ldots, \vec{x}_{N-1})| d\vec{x}_1 \cdots d\vec{x}_{N-1} < \infty. \tag{95}$$

Since f is integrable, it is measurable and the function $f(|\vec{x}_i - \vec{x}_j|)$ is measurable for all $i, j \leq N$. This implies that the product $F_{\vec{x}_N}$ is measurable. In order to prove that $I_{\vec{x}_N} < \infty$, let us choose a spanning subtree a of the connected graph c. Since f is bounded, we can denote its bound by

$$A = \sup_{r\geq 0} |f(r)| < \infty. \tag{96}$$

We then have

$$\begin{aligned} I_{\vec{x}_N} &= \int_{\mathbb{R}^{d(N-1)}} \prod_{\{i,j\}\in a} |f(|\vec{x}_i - \vec{x}_j|)| \prod_{\{i,j\}\in c\setminus a} |f(|\vec{x}_i - \vec{x}_j|)| d\vec{x}_1 \cdots d\vec{x}_{N-1} \\ &\leq A^{|c\setminus a|} \int_{\mathbb{R}^{d(N-1)}} \prod_{\{i,j\}\in a} |f(|\vec{x}_i - \vec{x}_j|)| d\vec{x}_1 \cdots d\vec{x}_{N-1}. \end{aligned} \tag{97}$$

Under the change of variables

$$\vec{z}_i = \vec{x}_i - \vec{x}_N, \ i = 1, \ldots, N-1, \tag{98}$$

we have, for $i, j \neq N$,

$$\vec{x}_i - \vec{x}_j = \vec{x}_i - \vec{x}_N + \vec{x}_N - \vec{x}_j = \vec{z}_i - \vec{z}_j.$$

Since $\vec{x}_N$ is fixed, the Jacobian of this transformation is 1 and (97) becomes

$$I_{\vec{x}_N} \leq A^{|c\setminus a|} \int_{\mathbb{R}^{d(N-1)}} \prod_{\{i,j\}\in a;\, i,j<N} |f(|\vec{z}_i - \vec{z}_j|)| \prod_{\{i,N\}\in a} |f(|\vec{z}_i|)| d\vec{z}_1 \cdots d\vec{z}_{N-1}.$$

Let us give the edges of a an orientation towards the "root" N. Consider the function $\sigma : \{1, \ldots, N-1\} \to \{1, \ldots, N\}$ such that $\sigma(i)$ is the end of the arrow going out of vertex i. We can write

$$I_{\vec{x}_N} \le A^{|c\backslash a|} \int_{\mathbb{R}^{d(N-1)}} \prod_{\sigma(i)<N} |f(|\vec{z}_i - \vec{z}_{\sigma(i)}|)| \prod_{\{i,N\}\in a} |f(|\vec{z}_i|)| d\vec{z}_1 \dots d\vec{z}_{N-1}. \tag{99}$$

Let us consider the linear transformation

$$T : \begin{cases} \vec{y}_i = \vec{z}_i - \vec{z}_{\sigma(i)}, & \sigma(i) \neq N \\ \vec{y}_i = \vec{z}_i, & \sigma(i) = N \end{cases} \tag{100}$$

Let us also denote by T the corresponding $(N-1)\times(N-1)$ matrix. This means that for $Y = (\vec{y}_1, \dots, \vec{y}_{N-1})$ and $Z = (\vec{z}_1, \dots, \vec{z}_{N-1})$, we have $Y = TZ$ and if we denote by

$$Y_i = (y_{1i}, y_{2i}, \dots, y_{(N-1)\,i}),\ \ Z_i = (z_{1i}, z_{2i}, \dots, z_{(N-1)\,i}),\ \ i = 1, \dots, d,$$

the vectors made of the i-th components of every vector $\vec{y}_j$ and $\vec{z}_j$, for $j = 1, \dots, N-1$, we also have $Y_i = TZ_i$. For the Jacobian, we have

$$\left| \frac{\partial(\vec{z}_1, \dots, \vec{z}_{N-1})}{\partial(\vec{y}_1, \dots, \vec{y}_{N-1})} \right| = \frac{1}{|\det(T)|^d}.$$

But T is the incidence matrix of the tree a, implying that $|\det(T)| = 1$. Applying the transformation to (99), we obtain

$$\begin{aligned} I_{\vec{x}_N} &\le A^{|c\backslash a|} \int_{\mathbb{R}^{d\cdot(N-1)}} \prod_{i=1}^{N-1} |f(|\vec{y}_i|)| d\vec{y}_1 \dots d\vec{y}_{N-1} \\ &= A^{|c\backslash a|} \left(\int_{\mathbb{R}^d} |f(|\vec{y}|)| d\vec{y} \right)^{N-1}. \end{aligned}$$

Hence, it suffices to show that

$$\int_{\mathbb{R}^d} |f(|\vec{y}|)| d\vec{y} < \infty$$

to prove the integrability of $F_{\vec{x}_N}$. Going to spherical coordinates with radius $r = |\vec{y}|$, we have $d\vec{y} = r^{d-1} dr dS$ because the surface element dS on the sphere of radius 1, $S(0,1)$, is multiplied by r^{d-1} when the point $\vec{y}$ is at distance r from the origin. Since the surface of $S(0,1)$ is finite, we find

$$\begin{aligned} \int_{\mathbb{R}^d} |f(|\vec{y}|)| d\vec{y} &= \int_{S(0,1)} dS \int_0^\infty r^{d-1} |f(r)| dr \\ &= \int_{S(0,1)} dS \left(\int_0^1 r^{d-1} |f(r)| dr + \int_1^\infty r^{d-1} |f(r)| dr \right) \\ &\le \int_{S(0,1)} dS \left(\int_0^1 |f(r)| dr + \int_1^\infty r^{d-1} |f(r)| dr \right) \\ &\le \int_{S(0,1)} dS \left(A + \int_0^\infty r^{d-1} |f(r)| dr \right) \\ &< \infty, \end{aligned}$$

using (14). Therefore $F_{\vec{x}_N}$ is integrable. Moreover, the transformation (98) shows that its integral is independent of $\vec{x}_N$.

We now can prove the existence of the thermodynamic limit and show that

$$\begin{aligned} w(c) &:= \lim_{V\to\infty} \frac{1}{V} \int_{V^N} \prod_{\{i,j\}\in c} f_{ij}\, d\vec{x}_1 \cdots d\vec{x}_N \\ &= \int_{\mathbb{R}^{d(N-1)}} \prod_{\{i,j\}\in c;\ \vec{x}_N=\vec{0}} f_{ij} d\vec{x}_1 \cdots d\vec{x}_{N-1}, \end{aligned} \tag{101}$$

where $f_{ij} = f(|\vec{x}_i - \vec{x}_j|)$. Let us consider the vessel V like a ball with radius R centered at the origin, denoted by $B(0,R)$. More precisely,

$$V = B(0,R) = \{\vec{x} \in \mathbb{R}^d \mid |\vec{x}| \le R\}.$$

We denote its volume by $K(d)R^d$. The thermodynamic limit can then be written as

$$\lim_{R\to\infty} \frac{1}{K(d)R^d} \int_{B(0,R)^N} \prod_{\{i,j\}\in c} f_{ij}\, d\vec{x}_1 \dots d\vec{x}_N.$$

Let us study the function

$$\begin{aligned} \epsilon(R,\vec{x}_N) &:= \int_{\mathbb{R}^{d(N-1)}} \prod_{\{i,j\}\in c} f_{ij}\, d\vec{x}_1 \cdots d\vec{x}_{N-1} - \int_{B(0,R)^{N-1}} \prod_{\{i,j\}\in c} f_{ij}\, d\vec{x}_1 \cdots d\vec{x}_{N-1} \\ &= \int_{\mathbb{R}^{d(N-1)}\setminus B(0,R)^{N-1}} \prod_{\{i,j\}\in c} f_{ij}\, d\vec{x}_1 \cdots d\vec{x}_{N-1} \end{aligned}$$

under the condition that $|\vec{x}_N| < R$. If we show that

$$\lim_{R\to\infty} \frac{1}{K(d)R^d} \int_{B(0,R)} |\epsilon(R,\vec{x}_N)| d\vec{x}_N = 0, \tag{102}$$

the identity (101) will directly follow since then

$$\begin{aligned} &\lim_{R\to\infty} \frac{1}{K(d)R^d} \int_{B(0,R)^N} \prod_{\{i,j\}\in c} f_{ij}\, d\vec{x}_1 \cdots d\vec{x}_N \\ &\quad= \lim_{R\to\infty} \frac{1}{K(d)R^d} \int_{B(0,R)} \left[\int_{B(0,R)^{N-1}} \prod_{\{i,j\}\in c} f_{ij}\, d\vec{x}_1 \cdots d\vec{x}_{N-1} \right] d\vec{x}_N \\ &\quad= \lim_{R\to\infty} \frac{1}{K(d)R^d} \int_{B(0,R)} \left[\int_{\mathbb{R}^{d(N-1)}} \prod_{\{i,j\}\in c} f_{ij}\, d\vec{x}_1 \cdots d\vec{x}_{N-1} \ - \ \epsilon(R,\vec{x}_N) \right] d\vec{x}_N \\ &\quad= \lim_{R\to\infty} \frac{1}{K(d)R^d} \int_{B(0,R)} d\vec{x}_N \int_{\mathbb{R}^{d(N-1)}} \prod_{\{i,j\}\in c} f_{ij}\, d\vec{x}_1 \cdots d\vec{x}_{N-1} \ - \ 0 \\ &\quad= \int_{\mathbb{R}^{d(N-1)}} \prod_{\{i,j\}\in c;\vec{x}_N=\vec{0}} f_{ij}\, d\vec{x}_1 \cdots d\vec{x}_{N-1}, \end{aligned}$$

applying the translation $\vec{x}_i := \vec{x}_i - \vec{x}_N$, for $i = 1, \dots, N-1$.

Now let A be defined by (96) and let again a be a spanning subtree of c, rooted at the vertex N. We have

$$
\begin{aligned}
|\epsilon(R,\vec{x}_N)| &= \left| \int_{\mathbb{R}^{d(N-1)}\setminus B(0,R)^{N-1}} \prod_{\{i,j\}\in c} f_{ij}\, d\vec{x}_1 \cdots d\vec{x}_{N-1} \right| \\
&\le \int_{\mathbb{R}^{d(N-1)}\setminus B(0,R)^{N-1}} \prod_{\{i,j\}\in c} |f_{ij}|\, d\vec{x}_1 \cdots d\vec{x}_{N-1} \\
&= \int_{\mathbb{R}^{d(N-1)}\setminus B(0,R)^{N-1}} \prod_{\{i,j\}\in a} |f_{ij}| \prod_{\{i,j\}\in c\setminus a} |f_{ij}|\, d\vec{x}_1 \cdots d\vec{x}_{N-1} \\
&\le A^{|c\setminus a|} \int_{\mathbb{R}^{d(N-1)}\setminus B(0,R)^{N-1}} \prod_{\{i,j\}\in a} |f_{ij}|\, d\vec{x}_1 \cdots d\vec{x}_{N-1} \\
&= A^{|c\setminus a|} \int_{\mathbb{R}^{d(N-1)}\setminus B(0,R)^{N-1}} \prod_{\substack{\{i,j\}\in a \\ i,j\neq N}} |f_{ij}| \prod_{\{i,N\}\in a} |f_{iN}|\, d\vec{x}_1 \cdots d\vec{x}_{N-1}.
\end{aligned}
$$

We change the variables as in (98), considering $\vec{x}_N$ as constant, and we define

$$
D(R,\vec{x}_N) := \mathbb{R}^{d(N-1)} \setminus (B(0,R)^{N-1} - \vec{x}_N^{N-1}), \tag{103}
$$

where $B(0,R)^{N-1} - \vec{x}_N^{N-1}$ denotes the part $B(0,R)^{N-1}$ translated by $-\vec{x}_N^{N-1} = -(\vec{x}_N, \vec{x}_N, \ldots, \vec{x}_N) \in \mathbb{R}^{d(N-1)}$. Then we can write

$$
|\epsilon(R,\vec{x}_N)| \le A^{|c\setminus a|} \int_{D(R,\vec{x}_N)} \prod_{\{i,j\}\in a;\, i,j\neq N} |f(|\vec{z}_i - \vec{z}_j|)| \prod_{\{i,N\}\in a} |f(|\vec{z}_i|)|\, d\vec{z}_1 \ldots d\vec{z}_{N-1}.
$$

As before, we orient the edges of a towards the root N and define a function σ such that $\sigma(i)$ denotes the destination of the arrow that comes out of vertex i. We have

$$
|\epsilon(R,\vec{x}_N)| \le A^{|c\setminus a|} \int_{D(R,\vec{x}_N)} \prod_{\sigma(i)\neq N} |f(|\vec{z}_i - \vec{z}_{\sigma(i)}|)| \prod_{\sigma(i)=N} |f(|\vec{z}_i|)|\, d\vec{z}_1 \ldots d\vec{z}_{N-1}. \tag{104}
$$

Applying the linear transformation T of (100) to (104), we find

$$
|\epsilon(R,\vec{x}_N)| \le A^{|c\setminus a|} \int_{T(D(R,\vec{x}_N))} \prod_{i=1}^{N-1} |f(|\vec{y}_i|)|\, d\vec{y}_1 \ldots d\vec{y}_{N-1}, \tag{105}
$$

with

$$
\begin{aligned}
T\left(D(R,\vec{x}_N)\right) &= T\left(\mathbb{R}^{d(N-1)}\right) \setminus T\left(B(0,R)^{N-1} - \vec{x}_N^{N-1}\right) \\
&= \mathbb{R}^{d(N-1)} \setminus T\left(B(0,R)^{N-1} - \vec{x}_N^{N-1}\right).
\end{aligned}
$$

Since $|\vec{x}_N| < R$, the topological region $B(0,R)^{N-1} - \vec{x}_N^{N-1}$ contains the origin. Since T is continuous, this implies that $T(B(0,R)^{N-1} - \vec{x}_N^{N-1})$ contains a Cartesian product of balls $B(0, \gamma \cdot (R - |\vec{x}_N|))^{N-1}$, centered at the origin, where $\gamma > 0$ only depends on T. Thus we have

$$\begin{aligned}|\epsilon(R, \vec{x}_N)| &\leq A^{|c\backslash a|} \int_{\mathbb{R}^{d\cdot(N-1)}\backslash B(0,\gamma\cdot(R-|\vec{x}_N|))^{N-1}} \prod_{i=1}^{N-1} |f(|\vec{y}_i|)|\, d\vec{y}_1 \ldots d\vec{y}_{N-1} \\ &= A^{|c\backslash a|} \left[\left(\int_{\mathbb{R}^d} |f(|\vec{y}|)|\, d\vec{y} \right)^{N-1} - \left(\int_{B(0,\gamma\cdot(R-|\vec{x}_N|))} |f(|\vec{y}|)|\, d\vec{y} \right)^{N-1} \right].\end{aligned}$$

Since $r^{d-1} f(r)$ is integrable, we can set

$$\alpha = \int_{\mathbb{R}^d} |f(|\vec{y}|)| d\vec{y} \ < \ \infty \quad \text{and} \quad \beta = \int_{B(0,\gamma\cdot(R-|\vec{x}_N|))} |f(|\vec{y}|)| d\vec{y} \ < \ \infty.$$

Notice that $\alpha \geq \beta$. Since

$$\begin{aligned}\alpha^{N-1} - \beta^{N-1} &= (\alpha - \beta)(\alpha^{N-2} + \alpha^{N-3}\beta + \ldots + \alpha\beta^{N-3} + \beta^{N-2}) \\ &\leq (\alpha - \beta)(\alpha^{N-2} + \alpha^{N-2} + \ldots + \alpha^{N-2}) \\ &= (\alpha - \beta)(N-1)\alpha^{N-2},\end{aligned}$$

we have

$$|\epsilon(R, \vec{x}_N)| \leq A^{|c\backslash a|}(N-1)\alpha^{N-2} \left[\int_{\mathbb{R}^d} |f(|\vec{y}|)| d\vec{y} - \int_{B(0,\gamma\cdot(R-|\vec{x}_N|))} |f(|\vec{y}|)| d\vec{y} \right]. \quad (106)$$

Let us denote by k the constant $k = A^{|c\backslash a|}(N-1)\alpha^{N-2}$. Integrating both sides of (106) with respect to $\vec{x}_N$ and dividing by the volume of the ball $B(0, R)$ yields

$$\begin{aligned}&\frac{1}{K(d)R^d} \int_{B(0,R)} |\epsilon(R, \vec{x}_N)| d\vec{x}_N \\ &\qquad \leq k \cdot \frac{1}{K(d)R^d} \int_{B(0,R)} \left[\int_{\mathbb{R}^d} |f(|\vec{y}|)| d\vec{y} - \int_{B(0,\gamma(R-|\vec{x}_N|))} |f(|\vec{y}|)| d\vec{y} \right] d\vec{x}_N \\ &\qquad = k \frac{1}{K(d)R^d} \int_{B(0,R)} \int_{\mathbb{R}^d} |f(|\vec{y}|)| d\vec{y} d\vec{x}_N \\ &\qquad\qquad - \ k \frac{1}{K(d)R^d} \int_{B(0,R)} \int_{B(0,\gamma(R-|\vec{x}_N|))} |f(|\vec{y}|)| d\vec{y} d\vec{x}_N \\ &\qquad = k \int_{\mathbb{R}^d} |f(|\vec{y}|)| d\vec{y} - k \frac{1}{K(d)R^d} \int_{B(0,R)} \int_{B(0,\gamma(R-|\vec{x}_N|))} |f(|\vec{y}|)| d\vec{y} d\vec{x}_N\end{aligned}$$

Let us change the order of integration of the second integral. Its domain of integration is

$$\{(\vec{y}, \vec{x}_N) |\ 0 \leq |\vec{x}_N| < R \text{ and } 0 \leq |\vec{y}| < \gamma(R - |\vec{x}_N|)\}.$$

If we let $\vec{x}_N$ vary, we have, for a fixed $\vec{y}$,

$$\begin{aligned}0 \leq |\vec{y}| < \gamma \cdot (R - |\vec{x}_N|) &\Leftrightarrow \ 0 \leq \frac{|\vec{y}|}{\gamma} < R - |\vec{x}_N| \\ &\Leftrightarrow 0 \leq |\vec{x}_N| < R - \frac{|\vec{y}|}{\gamma}\end{aligned}$$

and we obtain

$$
\begin{aligned}
&\frac{1}{K(d)R^d}\int_{B(0,R)} |\epsilon(R,\vec{x}_N)|d\vec{x}_N \\
&\qquad \leq k\int_{\mathbb{R}^d} |f(|\vec{y}|)|d\vec{y} - \frac{k}{K(d)R^d}\int_{B(0,\gamma R)}\int_{B(0,R-\frac{|\vec{y}|}{\gamma})} d\vec{x}_N |f(|\vec{y}|)|d\vec{y} \\
&\qquad = k\int_{\mathbb{R}^d} |f(|\vec{y}|)|d\vec{y} - \frac{k}{K(d)R^d}\int_{B(0,\gamma R)} K(d)\left(R-\frac{|\vec{y}|}{\gamma}\right)^d |f(|\vec{y}|)|d\vec{y} \\
&\qquad = k\left(\int_{\mathbb{R}^d} |f(|\vec{y}|)|d\vec{y} - \int_{B(0,\gamma R)} \left(1-\frac{|\vec{y}|}{\gamma R}\right)^d |f(|\vec{y}|)|d\vec{y}\right) \\
&\qquad \to 0
\end{aligned}
$$

as $R \uparrow \infty$ by the Lebesgue dominated convergence theorem. This is easily seen by taking an increasing sequence of radii $R_n \uparrow \infty$ and considering the sequence of measurable functions

$$
\left(1-\frac{|\vec{y}|}{\gamma R_n}\right)^d |f(|\vec{y}|)|\,\chi(\vec{y}\in B(0,\gamma R_n), \quad n = 0,1,2,\ldots
$$

which are dominated by the integrable function $|f(|\vec{y}|)|$ and converges pointwise to $|f(|\vec{y}|)|$.

This establishes (102), and concludes the proof. □

Appendix B. Table for 2-connected graphs of size at most 6

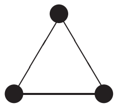
Key:

number	degree sequence of c		
graph c	Ehrhart Pol. in base n^i	nb of labellings	nb of spanning subtrees
	Ehrhart Pol. in base $\binom{n}{i}$	polytope's volume	volume $\times$ $(n-1)!$

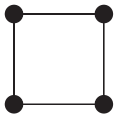
With 2 vertices:

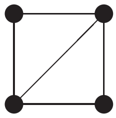
2	(1,1)		
	$2n+1$	1	1
	$1\binom{n}{0}+2\binom{n}{1}$	2	2

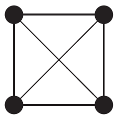
With 3 vertices:

<table>
<tr><td>3</td><td>(2,2,2)</td></tr>
<tr><td></td><td>$\begin{array}{l|c|c} 3n^2+3n+1 & 1 & 3 \\ \hline 1\binom{n}{0}+6\binom{n}{1}+6\binom{n}{2} & 3 & 6 \end{array}$</td></tr>
</table>

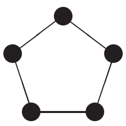
With 4 vertices:

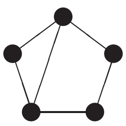
<table>
<tr><td>4.1</td><td>(2,2,2,2)</td></tr>
<tr><td></td><td>$\begin{array}{l|c|c} \frac{16}{3}\,n^3+8\,n^2+\frac{14}{3}\,n+1 & 3 & 4 \\ \hline 1\binom{n}{0}+18\binom{n}{1}+48\binom{n}{2}+32\binom{n}{3} & \frac{16}{3} & 32 \end{array}$</td></tr>
</table>

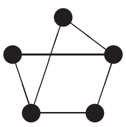
<table>
<tr><td>4.2</td><td>(3,3,2,2)</td></tr>
<tr><td></td><td>$\begin{array}{l|c|c} \frac{14}{3}\,n^3+7\,n^2+\frac{13}{3}\,n+1 & 6 & 8 \\ \hline 1\binom{n}{0}+16\binom{n}{1}+42\binom{n}{2}+28\binom{n}{3} & \frac{14}{3} & 28 \end{array}$</td></tr>
</table>

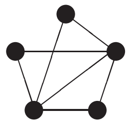
<table>
<tr><td>4.3</td><td>(3,3,3,3)</td></tr>
<tr><td></td><td>$\begin{array}{l|c|c} 4\,n^3+6\,n^2+4\,n+1 & 1 & 16 \\ \hline 1\binom{n}{0}+14\binom{n}{1}+36\binom{n}{2}+24\binom{n}{3} & 4 & 24 \end{array}$</td></tr>
</table>

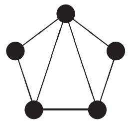
With 5 vertices:

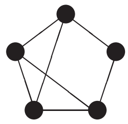
<table>
<tr><td>5.1</td><td>(2,2,2,2,2)</td></tr>
<tr><td></td><td>$\begin{array}{l|c|c} \frac{115}{12}\,n^4+\frac{115}{6}\,n^3+\frac{185}{12}\,n^2+\frac{35}{6}\,n+1 & 12 & 5 \\ \hline 1\binom{n}{0}+50\binom{n}{1}+280\binom{n}{2}+460\binom{n}{3}+230\binom{n}{4} & \frac{115}{12} & 230 \end{array}$</td></tr>
</table>

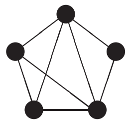
<table>
<tr><td>5.2</td><td>(3,3,2,2,2)</td></tr>
<tr><td></td><td>$\begin{array}{l|c|c} \frac{49}{6}\,n^4+\frac{49}{3}\,n^3+\frac{83}{6}\,n^2+\frac{17}{3}\,n+1 & 60 & 11 \\ \hline 1\binom{n}{0}+44\binom{n}{1}+240\binom{n}{2}+392\binom{n}{3}+196\binom{n}{4} & \frac{49}{6} & 196 \end{array}$</td></tr>
</table>

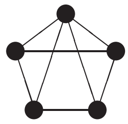
<table>
<tr><td>5.3</td><td>(3,3,2,2,2)</td></tr>
<tr><td></td><td>$\begin{array}{l|c|c} 8\,n^4+16\,n^3+14\,n^2+6\,n+1 & 10 & 12 \\ \hline 1\binom{n}{0}+44\binom{n}{1}+236\binom{n}{2}+384\binom{n}{3}+192\binom{n}{4} & 8 & 192 \end{array}$</td></tr>
</table>

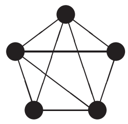
5.4	(4,4,2,2,2)
	$\begin{array}{l\|c\|c} \frac{15}{2}\,n^4+15\,n^3+\frac{27}{2}\,n^2+6\,n+1 & 10 & 20 \\ \hline 1\binom{n}{0}+42\binom{n}{1}+222\binom{n}{2}+360\binom{n}{3}+180\binom{n}{4} & \frac{15}{2} & 180 \end{array}$

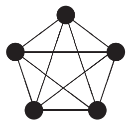
5.5	(4,3,3,2,2)
	$\begin{array}{l\|c\|c} \frac{29}{4}\,n^4+\frac{29}{2}\,n^3+\frac{51}{4}\,n^2+\frac{11}{2}\,n+1 & 60 & 21 \\ \hline 1\binom{n}{0}+40\binom{n}{1}+214\binom{n}{2}+348\binom{n}{3}+174\binom{n}{4} & \frac{29}{4} & 174 \end{array}$

5.6	(3,3,3,3,2)
	$\begin{array}{l\|c\|c} \frac{41}{6}\,n^4+\frac{41}{3}\,n^3+\frac{73}{6}\,n^2+\frac{16}{3}\,n+1 & 30 & 24 \\ \hline 1\binom{n}{0}+38\binom{n}{1}+202\binom{n}{2}+328\binom{n}{3}+164\binom{n}{4} & \frac{41}{6} & 164 \end{array}$

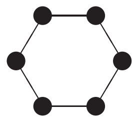
5.7	(4,4,3,3,2)
	$\begin{array}{l\|c\|c} \frac{19}{3}\,n^4+\frac{38}{3}\,n^3+\frac{35}{3}\,n^2+\frac{16}{3}\,n+1 & 30 & 40 \\ \hline 1\binom{n}{0}+36\binom{n}{1}+188\binom{n}{2}+304\binom{n}{3}+152\binom{n}{4} & \frac{19}{3} & 152 \end{array}$

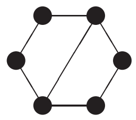
5.8	(4,3,3,3,3)
	$\begin{array}{l\|c\|c} 6\,n^4+12\,n^3+11\,n^2+5\,n+1 & 15 & 45 \\ \hline 1\binom{n}{0}+34\binom{n}{1}+178\binom{n}{2}+288\binom{n}{3}+144\binom{n}{4} & 6 & 144 \end{array}$

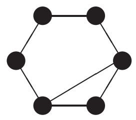
5.9	(4,4,4,3,3)
	$\begin{array}{l\|c\|c} \frac{11}{2}\,n^4+11\,n^3+\frac{21}{2}\,n^2+5\,n+1 & 10 & 75 \\ \hline 1\binom{n}{0}+32\binom{n}{1}+164\binom{n}{2}+264\binom{n}{3}+132\binom{n}{4} & \frac{11}{2} & 132 \end{array}$

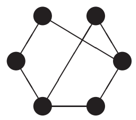
5.10	(4,4,4,4,4)
	$\begin{array}{l\|c\|c} 5\,n^4+10\,n^3+10\,n^2+5\,n+1 & 1 & 125 \\ \hline 1\binom{n}{0}+30\binom{n}{1}+150\binom{n}{2}+240\binom{n}{3}+120\binom{n}{4} & 5 & 120 \end{array}$

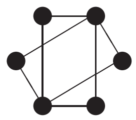
With 6 vertices, ordered according to the number of edges:

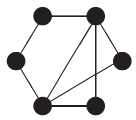
6.1	(2,2,2,2,2,2)		
	$\frac{88}{5}\,n^5+44\,n^4+46\,n^3+25\,n^2+\frac{37}{5}\,n+1$	60	6
	$1\binom{n}{0}+140\binom{n}{1}+1470\binom{n}{2}+4500\binom{n}{3}+5280\binom{n}{4}+2112\binom{n}{5}$	$\frac{88}{5}$	2112

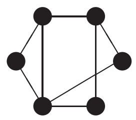
7.1	(3,3,2,2,2,2)		
	$\frac{72}{5}\,n^5+36\,n^4+\frac{118}{3}\,n^3+23\,n^2+\frac{109}{15}\,n+1$	180	15
	$1\binom{n}{0}+120\binom{n}{1}+1218\binom{n}{2}+3692\binom{n}{3}+4320\binom{n}{4}+1728\binom{n}{5}$	$\frac{72}{5}$	1728

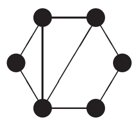
7.2	(3,3,2,2,2,2)		
	$\frac{439}{30}\,n^5+\frac{439}{12}\,n^4+\frac{118}{3}\,n^3+\frac{269}{12}\,n^2+\frac{211}{30}\,n+1$	360	14
	$1\binom{n}{0}+120\binom{n}{1}+1232\binom{n}{2}+3748\binom{n}{3}+4390\binom{n}{4}+1756\binom{n}{5}$	$\frac{439}{30}$	1756

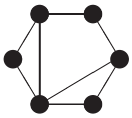
7.3	(3,3,2,2,2,2)		
	$\frac{419}{30}\,n^5+\frac{419}{12}\,n^4+38\,n^3+\frac{265}{12}\,n^2+\frac{211}{30}\,n+1$	180	16
	$1\binom{n}{0}+116\binom{n}{1}+1180\binom{n}{2}+3580\binom{n}{3}+4190\binom{n}{4}+1676\binom{n}{5}$	$\frac{419}{30}$	1676

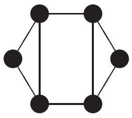
8.1	(4,4,2,2,2,2)		
	$\frac{64}{5}\,n^5+32\,n^4+\frac{112}{3}\,n^3+24\,n^2+\frac{118}{15}\,n+1$	15	32
	$1\binom{n}{0}+114\binom{n}{1}+1104\binom{n}{2}+3296\binom{n}{3}+3840\binom{n}{4}+1536\binom{n}{5}$	$\frac{64}{5}$	1536

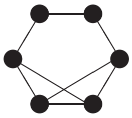
8.2	(4,4,2,2,2,2)		
	$\frac{194}{15}\,n^5+\frac{97}{3}\,n^4+36\,n^3+\frac{65}{3}\,n^2+\frac{106}{15}\,n+1$	180	28
	$1\binom{n}{0}+110\binom{n}{1}+1100\binom{n}{2}+3320\binom{n}{3}+3880\binom{n}{4}+1552\binom{n}{5}$	$\frac{194}{15}$	1552

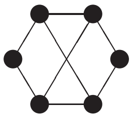
8.3	(4,3,3,2,2,2)		
	$\frac{37}{3}\,n^5+\frac{185}{6}\,n^4+\frac{104}{3}\,n^3+\frac{127}{6}\,n^2+7\,n+1$	360	32
	$1\binom{n}{0}+106\binom{n}{1}+1052\binom{n}{2}+3168\binom{n}{3}+3700\binom{n}{4}+1480\binom{n}{5}$	$\frac{37}{3}$	1480

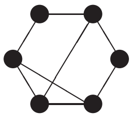
8.4	(4,3,3,2,2,2,2)		
	$\frac{127}{10}\,n^5+\frac{127}{4}\,n^4+\frac{106}{3}\,n^3+\frac{85}{4}\,n^2+\frac{209}{30}\,n+1$	720	29
	$1\binom{n}{0}+108\binom{n}{1}+1080\binom{n}{2}+3260\binom{n}{3}+3810\binom{n}{4}+1524\binom{n}{5}$	$\frac{127}{10}$	1524

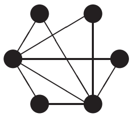
8.5	(4,3,3,2,2,2)
	$\begin{array}{c\|c\|c} \frac{188}{15}n^5+\frac{94}{3}n^4+\frac{104}{3}n^3+\frac{62}{3}n^2+\frac{34}{5}n+1 & 360 & 30 \\ \hline 1\binom{n}{0}+106\binom{n}{1}+1064\binom{n}{2}+3216\binom{n}{3}+3760\binom{n}{4}+1504\binom{n}{5} & \frac{188}{15} & 1504 \end{array}$

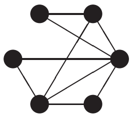
8.6	(3,3,3,3,2,2)
	$\begin{array}{c\|c\|c} \frac{188}{15}n^5+\frac{94}{3}n^4+\frac{104}{3}n^3+\frac{62}{3}n^2+\frac{34}{5}n+1 & 180 & 30 \\ \hline 1\binom{n}{0}+106\binom{n}{1}+1064\binom{n}{2}+3216\binom{n}{3}+3760\binom{n}{4}+1504\binom{n}{5} & \frac{188}{15} & 1504 \end{array}$

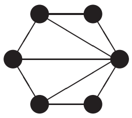
8.7	(3,3,3,3,2,2)
	$\begin{array}{c\|c\|c} \frac{361}{30}n^5+\frac{361}{12}n^4+\frac{100}{3}n^3+\frac{239}{12}n^2+\frac{199}{30}n+1 & 180 & 32 \\ \hline 1\binom{n}{0}+102\binom{n}{1}+1022\binom{n}{2}+3088\binom{n}{3}+3610\binom{n}{4}+1444\binom{n}{5} & \frac{361}{30} & 1444 \end{array}$

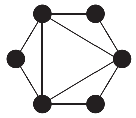
8.8	(3,3,3,3,2,2)
	$\begin{array}{c\|c\|c} \frac{176}{15}n^5+\frac{88}{3}n^4+\frac{100}{3}n^3+\frac{62}{3}n^2+\frac{104}{15}n+1 & 90 & 36 \\ \hline 1\binom{n}{0}+102\binom{n}{1}+1004\binom{n}{2}+3016\binom{n}{3}+3520\binom{n}{4}+1408\binom{n}{5} & \frac{176}{15} & 1408 \end{array}$

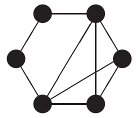
8.9	(3,3,3,3,2,2)
	$\begin{array}{c\|c\|c} \frac{117}{10}n^5+\frac{117}{4}n^4+\frac{98}{3}n^3+\frac{79}{4}n^2+\frac{199}{30}n+1 & 360 & 35 \\ \hline 1\binom{n}{0}+100\binom{n}{1}+996\binom{n}{2}+3004\binom{n}{3}+3510\binom{n}{4}+1404\binom{n}{5} & \frac{117}{10} & 1404 \end{array}$

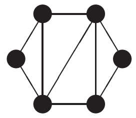
9.1	(5,5,2,2,2,2)
	$\begin{array}{c\|c\|c} \frac{62}{5}n^5+31n^4+\frac{110}{3}n^3+24n^2+\frac{119}{15}n+1 & 15 & 48 \\ \hline 1\binom{n}{0}+112\binom{n}{1}+1074\binom{n}{2}+3196\binom{n}{3}+3720\binom{n}{4}+1488\binom{n}{5} & \frac{62}{5} & 1488 \end{array}$

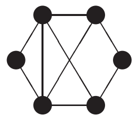
9.2	(5,4,3,2,2,2)
	$\begin{array}{c\|c\|c} \frac{349}{30}n^5+\frac{349}{12}n^4+\frac{100}{3}n^3+\frac{251}{12}n^2+\frac{211}{30}n+1 & 360 & 52 \\ \hline 1\binom{n}{0}+102\binom{n}{1}+998\binom{n}{2}+2992\binom{n}{3}+3490\binom{n}{4}+1396\binom{n}{5} & \frac{349}{30} & 1396 \end{array}$

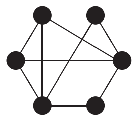
9.3	(5,3,3,3,2,2)
	$\begin{array}{c\|c\|c} \frac{169}{15}n^5+\frac{169}{6}n^4+32n^3+\frac{119}{6}n^2+\frac{101}{15}n+1 & 360 & 55 \\ \hline 1\binom{n}{0}+98\binom{n}{1}+964\binom{n}{2}+2896\binom{n}{3}+3380\binom{n}{4}+1352\binom{n}{5} & \frac{169}{15} & 1352 \end{array}$

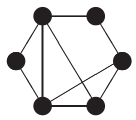
9.4	(4,4,4,2,2,2)
	$\begin{array}{c\|c\|c} \frac{113}{10}\,n^5+\frac{113}{4}\,n^4+32\,n^3+\frac{79}{4}\,n^2+\frac{67}{10}\,n+1 & 120 & 54 \\ \hline 1\binom{n}{0}+98\binom{n}{1}+966\binom{n}{2}+2904\binom{n}{3}+3390\binom{n}{4}+1356\binom{n}{5} & \frac{113}{10} & 1356 \end{array}$

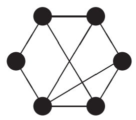
9.5	(4,4,3,3,2,2)
	$\begin{array}{c\|c\|c} 11\,n^5+\frac{55}{2}\,n^4+\frac{94}{3}\,n^3+\frac{39}{2}\,n^2+\frac{20}{3}\,n+1 & 180 & 56 \\ \hline 1\binom{n}{0}+96\binom{n}{1}+942\binom{n}{2}+2828\binom{n}{3}+3300\binom{n}{4}+1320\binom{n}{5} & 11 & 1320 \end{array}$

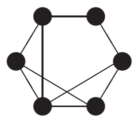
9.6	(4,4,3,3,2,2)
	$\begin{array}{c\|c\|c} \frac{169}{15}\,n^5+\frac{169}{6}\,n^4+32\,n^3+\frac{119}{6}\,n^2+\frac{101}{15}\,n+1 & 360 & 55 \\ \hline 1\binom{n}{0}+98\binom{n}{1}+964\binom{n}{2}+2896\binom{n}{3}+3380\binom{n}{4}+1352\binom{n}{5} & \frac{169}{15} & 1352 \end{array}$

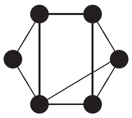
9.7	(4,4,3,3,2,2)
	$\begin{array}{c\|c\|c} \frac{163}{15}\,n^5+\frac{163}{6}\,n^4+\frac{94}{3}\,n^3+\frac{119}{6}\,n^2+\frac{34}{5}\,n+1 & 180 & 60 \\ \hline 1\binom{n}{0}+96\binom{n}{1}+934\binom{n}{2}+2796\binom{n}{3}+3260\binom{n}{4}+1304\binom{n}{5} & \frac{163}{15} & 1304 \end{array}$

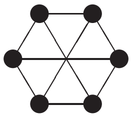
9.8	(4,4,3,3,2,2)
	$\begin{array}{c\|c\|c} \frac{161}{15}\,n^5+\frac{161}{6}\,n^4+\frac{94}{3}\,n^3+\frac{121}{6}\,n^2+\frac{104}{15}\,n+1 & 90 & 64 \\ \hline 1\binom{n}{0}+96\binom{n}{1}+926\binom{n}{2}+2764\binom{n}{3}+3220\binom{n}{4}+1288\binom{n}{5} & \frac{161}{15} & 1288 \end{array}$

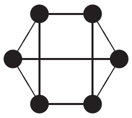
9.9	(4,4,3,3,2,2)
	$\begin{array}{c\|c\|c} \frac{161}{15}\,n^5+\frac{161}{6}\,n^4+\frac{92}{3}\,n^3+\frac{115}{6}\,n^2+\frac{33}{5}\,n+1 & 720 & 61 \\ \hline 1\binom{n}{0}+94\binom{n}{1}+920\binom{n}{2}+2760\binom{n}{3}+3220\binom{n}{4}+1288\binom{n}{5} & \frac{161}{15} & 1288 \end{array}$

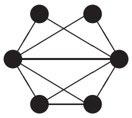
9.10	(4,3,3,3,3,2)
	$\begin{array}{c\|c\|c} \frac{51}{5}\,n^5+\frac{51}{2}\,n^4+\frac{88}{3}\,n^3+\frac{37}{2}\,n^2+\frac{97}{15}\,n+1 & 360 & 69 \\ \hline 1\binom{n}{0}+90\binom{n}{1}+876\binom{n}{2}+2624\binom{n}{3}+3060\binom{n}{4}+1224\binom{n}{5} & \frac{51}{5} & 1224 \end{array}$

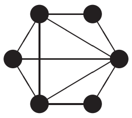
9.11	(4,3,3,3,3,2)
	$\begin{array}{c\|c\|c} \frac{103}{10}\,n^5+\frac{103}{4}\,n^4+\frac{88}{3}\,n^3+\frac{73}{4}\,n^2+\frac{191}{30}\,n+1 & 360 & 66 \\ \hline 1\binom{n}{0}+90\binom{n}{1}+882\binom{n}{2}+2648\binom{n}{3}+3090\binom{n}{4}+1236\binom{n}{5} & \frac{103}{10} & 1236 \end{array}$

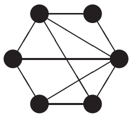
9.12	(4,3,3,3,3,2)
	$\begin{array}{c\|c\|c} \frac{158}{15}n^5+\frac{79}{3}n^4+30n^3+\frac{56}{3}n^2+\frac{97}{15}n+1 & 360 & 64 \\ \hline 1\binom{n}{0}+92\binom{n}{1}+902\binom{n}{2}+2708\binom{n}{3}+3160\binom{n}{4}+1264\binom{n}{5} & \frac{158}{15} & 1264 \end{array}$

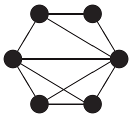
9.13	(3,3,3,3,3,3)
	$\begin{array}{c\|c\|c} \frac{48}{5}n^5+24n^4+28n^3+18n^2+\frac{32}{5}n+1 & 10 & 81 \\ \hline 1\binom{n}{0}+86\binom{n}{1}+828\binom{n}{2}+2472\binom{n}{3}+2880\binom{n}{4}+1152\binom{n}{5} & \frac{48}{5} & 1152 \end{array}$

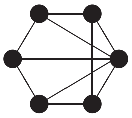
9.14	(3,3,3,3,3,3)
	$\begin{array}{c\|c\|c} \frac{49}{5}n^5+\frac{49}{2}n^4+28n^3+\frac{35}{2}n^2+\frac{31}{5}n+1 & 60 & 75 \\ \hline 1\binom{n}{0}+86\binom{n}{1}+840\binom{n}{2}+2520\binom{n}{3}+2940\binom{n}{4}+1176\binom{n}{5} & \frac{49}{5} & 1176 \end{array}$

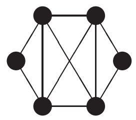
10.1	(5,5,3,3,2,2)
	$\begin{array}{c\|c\|c} \frac{31}{3}n^5+\frac{155}{6}n^4+\frac{92}{3}n^3+\frac{121}{6}n^2+7n+1 & 90 & 96 \\ \hline 1\binom{n}{0}+94\binom{n}{1}+896\binom{n}{2}+2664\binom{n}{3}+3100\binom{n}{4}+1240\binom{n}{5} & \frac{31}{3} & 1240 \end{array}$

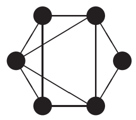
10.2	(5,4,4,3,2,2)
	$\begin{array}{c\|c\|c} \frac{301}{30}n^5+\frac{301}{12}n^4+\frac{88}{3}n^3+\frac{227}{12}n^2+\frac{199}{30}n+1 & 360 & 99 \\ \hline 1\binom{n}{0}+90\binom{n}{1}+866\binom{n}{2}+2584\binom{n}{3}+3010\binom{n}{4}+1204\binom{n}{5} & \frac{301}{30} & 1204 \end{array}$

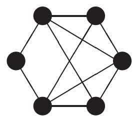
10.3	(5,4,3,3,3,2)
	$\begin{array}{c\|c\|c} \frac{287}{30}n^5+\frac{287}{12}n^4+28n^3+\frac{217}{12}n^2+\frac{193}{30}n+1 & 360 & 111 \\ \hline 1\binom{n}{0}+86\binom{n}{1}+826\binom{n}{2}+2464\binom{n}{3}+2870\binom{n}{4}+1148\binom{n}{5} & \frac{287}{30} & 1148 \end{array}$

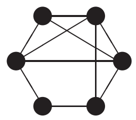
10.4	(5,4,3,3,3,2)
	$\begin{array}{c\|c\|c} \frac{59}{6}n^5+\frac{295}{12}n^4+\frac{86}{3}n^3+\frac{221}{12}n^2+13/2\,n+1 & 360 & 104 \\ \hline 1\binom{n}{0}+88\binom{n}{1}+848\binom{n}{2}+2532\binom{n}{3}+2950\binom{n}{4}+1180\binom{n}{5} & \frac{59}{6} & 1180 \end{array}$

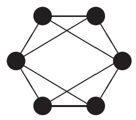
10.5	(5,3,3,3,3,3)
	$\begin{array}{c\|c\|c} \frac{55}{6}n^5+\frac{275}{12}n^4+\frac{80}{3}n^3+\frac{205}{12}n^2+\frac{37}{6}n+1 & 72 & 121 \\ \hline 1\binom{n}{0}+82\binom{n}{1}+790\binom{n}{2}+2360\binom{n}{3}+2750\binom{n}{4}+1100\binom{n}{5} & \frac{55}{6} & 1100 \end{array}$

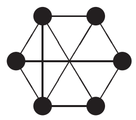
10.6	(4,4,4,4,2,2)
	$\begin{array}{l\|c\|c} 10\,n^5+25\,n^4+\frac{88}{3}\,n^3+19\,n^2+\frac{20}{3}\,n+1 & 90 & 100 \\ \hline 1\binom{n}{0}+90\binom{n}{1}+864\binom{n}{2}+2576\binom{n}{3}+3000\binom{n}{4}+1200\binom{n}{5} & 10 & 1200 \end{array}$

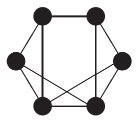
10.7	(4,4,4,3,3,2)
	$\begin{array}{l\|c\|c} \frac{47}{5}\,n^5+\frac{47}{2}\,n^4+\frac{82}{3}\,n^3+\frac{35}{2}\,n^2+\frac{94}{15}\,n+1 & 360 & 114 \\ \hline 1\binom{n}{0}+84\binom{n}{1}+810\binom{n}{2}+2420\binom{n}{3}+2820\binom{n}{4}+1128\binom{n}{5} & \frac{47}{5} & 1128 \end{array}$

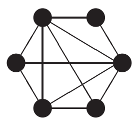
10.8	(4,4,4,3,3,2)
	$\begin{array}{l\|c\|c} \frac{139}{15}\,n^5+\frac{139}{6}\,n^4+\frac{82}{3}\,n^3+\frac{107}{6}\,n^2+\frac{32}{5}\,n+1 & 180 & 120 \\ \hline 1\binom{n}{0}+84\binom{n}{1}+802\binom{n}{2}+2388\binom{n}{3}+2780\binom{n}{4}+1112\binom{n}{5} & \frac{139}{15} & 1112 \end{array}$

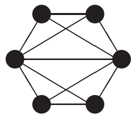
10.9	(4,4,4,3,3,2)
	$\begin{array}{l\|c\|c} \frac{28}{3}\,n^5+\frac{70}{3}\,n^4+\frac{82}{3}\,n^3+\frac{53}{3}\,n^2+\frac{19}{3}\,n+1 & 360 & 115 \\ \hline 1\binom{n}{0}+84\binom{n}{1}+806\binom{n}{2}+2404\binom{n}{3}+2800\binom{n}{4}+1120\binom{n}{5} & \frac{28}{3} & 1120 \end{array}$

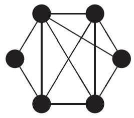
10.10	(4,4,3,3,3,3)
	$\begin{array}{l\|c\|c} \frac{136}{15}\,n^5+\frac{68}{3}\,n^4+\frac{80}{3}\,n^3+\frac{52}{3}\,n^2+\frac{94}{15}\,n+1 & 45 & 128 \\ \hline 1\binom{n}{0}+82\binom{n}{1}+784\binom{n}{2}+2336\binom{n}{3}+2720\binom{n}{4}+1088\binom{n}{5} & \frac{136}{15} & 1088 \end{array}$

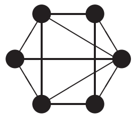
10.11	(4,4,3,3,3,3)
	$\begin{array}{l\|c\|c} \frac{44}{5}\,n^5+22\,n^4+26\,n^3+17\,n^2+\frac{31}{5}\,n+1 & 60 & 135 \\ \hline 1\binom{n}{0}+80\binom{n}{1}+762\binom{n}{2}+2268\binom{n}{3}+2640\binom{n}{4}+1056\binom{n}{5} & \frac{44}{5} & 1056 \end{array}$

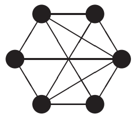
10.12	(4,4,3,3,3,3)
	$\begin{array}{l\|c\|c} \frac{89}{10}\,n^5+\frac{89}{4}\,n^4+26\,n^3+\frac{67}{4}\,n^2+\frac{61}{10}\,n+1 & 360 & 130 \\ \hline 1\binom{n}{0}+80\binom{n}{1}+768\binom{n}{2}+2292\binom{n}{3}+2670\binom{n}{4}+1068\binom{n}{5} & \frac{89}{10} & 1068 \end{array}$

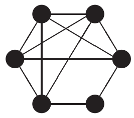
11.1	(5,5,4,3,3,2)
	$\begin{array}{l\|c\|c} \frac{133}{15}\,n^5+\frac{133}{6}\,n^4+\frac{80}{3}\,n^3+\frac{107}{6}\,n^2+\frac{97}{15}\,n+1 & 180 & 180 \\ \hline 1\binom{n}{0}+82\binom{n}{1}+772\binom{n}{2}+2288\binom{n}{3}+2660\binom{n}{4}+1064\binom{n}{5} & \frac{133}{15} & 1064 \end{array}$

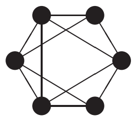
11.2	(5,5,3,3,3,3)
	$\begin{array}{lcc} \frac{26}{3}\,n^5+\frac{65}{3}\,n^4+26\,n^3+\frac{52}{3}\,n^2+\frac{19}{3}\,n+1 & 45 & 192 \\ \hline 1\binom{n}{0}+80\binom{n}{1}+754\binom{n}{2}+2236\binom{n}{3}+2600\binom{n}{4}+1040\binom{n}{5} & \frac{26}{3} & 1040 \end{array}$

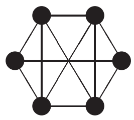
11.3	(5,4,4,4,3,2)
	$\begin{array}{lcc} \frac{87}{10}\,n^5+\frac{87}{4}\,n^4+26\,n^3+\frac{69}{4}\,n^2+\frac{63}{10}\,n+1 & 360 & 185 \\ \hline 1\binom{n}{0}+80\binom{n}{1}+756\binom{n}{2}+2244\binom{n}{3}+2610\binom{n}{4}+1044\binom{n}{5} & \frac{87}{10} & 1044 \end{array}$

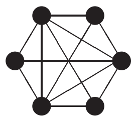
11.4	(5,4,4,3,3,3)
	$\begin{array}{lcc} \frac{124}{15}\,n^5+\frac{62}{3}\,n^4+\frac{74}{3}\,n^3+\frac{49}{3}\,n^2+\frac{91}{15}\,n+1 & 360 & 209 \\ \hline 1\binom{n}{0}+76\binom{n}{1}+718\binom{n}{2}+2132\binom{n}{3}+2480\binom{n}{4}+992\binom{n}{5} & \frac{124}{15} & 992 \end{array}$

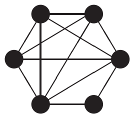
11.5	(5,4,4,3,3,3)
	$\begin{array}{lcc} \frac{41}{5}\,n^5+\frac{41}{2}\,n^4+\frac{74}{3}\,n^3+\frac{33}{2}\,n^2+\frac{92}{15}\,n+1 & 60 & 216 \\ \hline 1\binom{n}{0}+76\binom{n}{1}+714\binom{n}{2}+2116\binom{n}{3}+2460\binom{n}{4}+984\binom{n}{5} & \frac{41}{5} & 984 \end{array}$

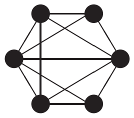
11.6	(4,4,4,4,4,2)
	$\begin{array}{lcc} \frac{42}{5}\,n^5+21\,n^4+\frac{76}{3}\,n^3+17\,n^2+\frac{94}{15}\,n+1 & 60 & 200 \\ \hline 1\binom{n}{0}+78\binom{n}{1}+732\binom{n}{2}+2168\binom{n}{3}+2520\binom{n}{4}+1008\binom{n}{5} & \frac{42}{5} & 1008 \end{array}$

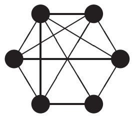
11.7	(4,4,4,4,3,3)
	$\begin{array}{lcc} \frac{241}{30}\,n^5+\frac{241}{12}\,n^4+24\,n^3+\frac{191}{12}\,n^2+\frac{179}{30}\,n+1 & 180 & 224 \\ \hline 1\binom{n}{0}+74\binom{n}{1}+698\binom{n}{2}+2072\binom{n}{3}+2410\binom{n}{4}+964\binom{n}{5} & \frac{241}{30} & 964 \end{array}$

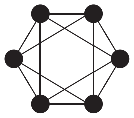
11.8	(4,4,4,4,3,3)
	$\begin{array}{lcc} 8\,n^5+20\,n^4+24\,n^3+16\,n^2+6\,n+1 & 90 & 225 \\ \hline 1\binom{n}{0}+74\binom{n}{1}+696\binom{n}{2}+2064\binom{n}{3}+2400\binom{n}{4}+960\binom{n}{5} & 8 & 960 \end{array}$

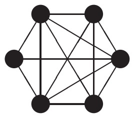
12.1	(5,5,5,3,3,3)
	$\begin{array}{lcc} \frac{39}{5}\,n^5+\frac{39}{2}\,n^4+24\,n^3+\frac{33}{2}\,n^2+\frac{31}{5}\,n+1 & 20 & 324 \\ \hline 1\binom{n}{0}+74\binom{n}{1}+684\binom{n}{2}+2016\binom{n}{3}+2340\binom{n}{4}+936\binom{n}{5} & \frac{39}{5} & 936 \end{array}$

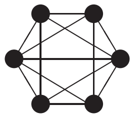
12.2	(5,5,4,4,4,2)		
	$8\,n^5+20\,n^4+\frac{74}{3}\,n^3+17\,n^2+\frac{19}{3}\,n+1$	60	300
	$1\binom{n}{0}+76\binom{n}{1}+702\binom{n}{2}+2068\binom{n}{3}+2400\binom{n}{4}+960\binom{n}{5}$	8	960

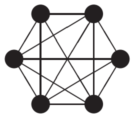
12.3	(5,5,4,4,3,3)		
	$\frac{229}{30}\,n^5+\frac{229}{12}\,n^4+\frac{70}{3}\,n^3+\frac{191}{12}\,n^2+\frac{181}{30}\,n+1$	180	336
	$1\binom{n}{0}+72\binom{n}{1}+668\binom{n}{2}+1972\binom{n}{3}+2290\binom{n}{4}+916\binom{n}{5}$	$\frac{229}{30}$	916

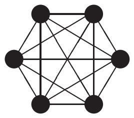
12.4	(5,4,4,4,4,3)		
	$\frac{37}{5}\,n^5+\frac{37}{2}\,n^4+\frac{68}{3}\,n^3+\frac{31}{2}\,n^2+\frac{89}{15}\,n+1$	180	360
	$1\binom{n}{0}+70\binom{n}{1}+648\binom{n}{2}+1912\binom{n}{3}+2220\binom{n}{4}+888\binom{n}{5}$	$\frac{37}{5}$	888

12.5	(4,4,4,4,4,4)		
	$\frac{36}{5}\,n^5+18\,n^4+22\,n^3+15\,n^2+\frac{29}{5}\,n+1$	15	384
	$1\binom{n}{0}+68\binom{n}{1}+630\binom{n}{2}+1860\binom{n}{3}+2160\binom{n}{4}+864\binom{n}{5}$	$\frac{36}{5}$	864

13.1	(5,5,5,4,4,3)		
	$7\,n^5+\frac{35}{2}\,n^4+22\,n^3+\frac{31}{2}\,n^2+6\,n+1$	60	540
	$1\binom{n}{0}+68\binom{n}{1}+618\binom{n}{2}+1812\binom{n}{3}+2100\binom{n}{4}+840\binom{n}{5}$	7	840

13.2	(5,5,4,4,4,4)		
	$\frac{34}{5}\,n^5+17\,n^4+\frac{64}{3}\,n^3+15\,n^2+\frac{88}{15}\,n+1$	45	576
	$1\binom{n}{0}+66\binom{n}{1}+600\binom{n}{2}+1760\binom{n}{3}+2040\binom{n}{4}+816\binom{n}{5}$	$\frac{34}{5}$	816

14.1	(5,5,5,5,4,4)		
	$\frac{32}{5}\,n^5+16\,n^4+\frac{63}{3}\,n^3+15\,n^2+\frac{89}{15}\,n+1$	15	864
	$1\binom{n}{0}+64\binom{n}{1}+570\binom{n}{2}+1660\binom{n}{3}+1920\binom{n}{4}+768\binom{n}{5}$	$\frac{32}{5}$	768

15.1	(5,5,5,5,5,5)		
	$6\,n^5+15\,n^4+20\,n^3+15\,n^2+6\,n+1$	1	1296
	$1\binom{n}{0}+62\binom{n}{1}+540\binom{n}{2}+1560\binom{n}{3}+1800\binom{n}{4}+720\binom{n}{5}$	6	720