COMPUTER-FREE EVALUATION OF AN INFINITE DOUBLE SUM VIA EULER SUMS

ALOIS PANHOLZER AND HELMUT PRODINGER

ABSTRACT. A short and computer-free proof (using Euler sums and multiple zeta functions) is provided for a double sum that was recently computed by Pemantle and Schneider using the computer software Sigma.

1. INTRODUCTION

The evaluation of the double sum

$$S := \sum_{j,k=1}^{\infty} \frac{H_j(H_{k+1} - 1)}{jk(k+1)(j+k)} = -\zeta(2) - 2\zeta(3) + 4\zeta(2)\zeta(3) + 2\zeta(5), \qquad (1)$$

(where $H_n := \sum_{k=1}^{n} \frac{1}{k}$ denote harmonic numbers) appears in [4] and was obtained using Carsten Schneider's software Sigma. Here, we will give a short proof that is completely computer-free. It uses Euler sums and multiple zeta functions.

The second order harmonic numbers which appear in the sequel are denoted by $H_n^{(2)} := \sum_{k=1}^{n} \frac{1}{k^2}$.

We split the sum S and apply partial fraction decomposition:

$$S = \sum_{k \geq 1} \frac{H_{k+1} - 1}{k(k+1)} \sum_{j \geq 1} \frac{H_j}{j(j+k)} = \sum_{k \geq 1} \frac{H_{k+1} - 1}{k^2(k+1)} \sum_{j \geq 1} H_j \left(\frac{1}{j} - \frac{1}{j+k} \right).$$

The inner sum is simplified as follows:

$$\sum_{j \geq 1} H_j \left(\frac{1}{j} - \frac{1}{j+k} \right) = \sum_{j \geq 1} \left(\frac{1}{j} - \frac{1}{j+k} \right) \sum_{l=1}^{j} \frac{1}{l} = \sum_{l \geq 1} \frac{1}{l} \sum_{j \geq l} \left(\frac{1}{j} - \frac{1}{j+k} \right).$$

The second sum here telescopes, and only k summands remain:

$$\sum_{l \geq 1} \frac{1}{l} \sum_{j \geq l} \left(\frac{1}{j} - \frac{1}{j+k} \right) = \sum_{l \geq 1} \frac{1}{l} \sum_{j=0}^{k-1} \frac{1}{l+j} = \sum_{l \geq 1} \frac{1}{l^2} + \sum_{j=1}^{k-1} \sum_{l \geq 1} \frac{1}{l(l+j)}$$

$$= \zeta(2) + \sum_{j=1}^{k-1} \frac{1}{j} \sum_{l \geq 1} \left(\frac{1}{l} - \frac{1}{l+j} \right) \qquad \text{[partial fraction decomposition]}$$

$$= \zeta(2) + \sum_{j=1}^{k-1} \frac{H_j}{j} \qquad\qquad \text{[again by telescoping]}$$

$$= \zeta(2) + \frac{H_k^2 + H_k^{(2)}}{2} - \frac{H_k}{k}.$$

Thus the task reduces to evaluate the following single sum:

$$S = \sum_{k=1}^{\infty} \frac{H_{k+1} - 1}{k^2(k+1)} \left(\zeta(2) + \frac{H_k^2 + H_k^{(2)}}{2} - \frac{H_k}{k} \right). \tag{2}$$

After partial fraction expansion (and shifting the index if necessary), sum S can be written as follows:

$$S = -2\zeta(2) + \frac{1}{2} \sum_{k \geq 1} \frac{H_k H_k^{(2)}}{k^2} + \frac{1}{2} \sum_{k \geq 1} \frac{H_k^3}{k^2} - \sum_{k \geq 1} \frac{H_k^2}{k^3} + (\zeta(2) - 1) \sum_{k \geq 1} \frac{H_k}{k^2} - 2 \sum_{k \geq 1} \frac{1}{k^2}.$$

For the final computation of S we require the following evaluations of Euler sums via values of the Riemann zeta function:

$$\sum_{k \geq 1} \frac{H_k H_k^{(2)}}{k^2} = \zeta(5) + \zeta(2)\zeta(3), \tag{3a}$$

$$\sum_{k \geq 1} \frac{H_k^3}{k^2} = 10\zeta(5) + \zeta(2)\zeta(3), \tag{3b}$$

$$\sum_{k \geq 1} \frac{H_k^2}{k^3} = \frac{7}{2}\zeta(5) - \zeta(2)\zeta(3), \tag{3c}$$

$$\sum_{k \geq 1} \frac{H_k}{k^2} = 2\zeta(3), \tag{3d}$$

from which equation (1) then follows.

Equations (3c) and (3d) can be found explicitly in [3]. In [3] one also finds the identities

$$\sum_{k \geq 1} \frac{H_k^3}{(k+1)^2} = \frac{15}{2}\zeta(5) + \zeta(2)\zeta(3), \tag{4a}$$

$$\sum_{k \geq 1} \frac{H_k}{k^4} = 3\zeta(5) - \zeta(2)\zeta(3), \tag{4b}$$

and due to

$$\sum_{k \geq 1} \frac{H_k^3}{k^2} = \sum_{k \geq 1} \frac{H_k^3}{(k+1)^2} + 3 \sum_{k \geq 1} \frac{H_k^2}{k^3} - 3 \sum_{k \geq 1} \frac{H_k}{k^4} + \sum_{k \geq 1} \frac{1}{k^5},$$

we obtain equation (3b) as well, using (4a), (4b) and (3c).

To show (3a) we will apply Theorem 2 of [2], which gives

$$\sum_{k \geq 1} \frac{H_{k-1} H_{k-1}^{(2)}}{k^2} = \zeta(2,1,2) + \zeta(2,2,1) + \zeta(2,3),$$

where the *multiple zeta functions* are defined by

$$\zeta(a_1, a_2, \ldots, a_m) = \sum_{k_1 > k_2 > \cdots > k_m \geq 1} \frac{1}{k_1^{a_1} k_2^{a_2} \cdots k_m^{a_m}}.$$

Since

$$\sum_{k\geq 1}\frac{H_k H_k^{(2)}}{k^2} = \sum_{k\geq 1}\frac{H_{k-1}H_{k-1}^{(2)}}{k^2} + \sum_{k\geq 1}\frac{H_{k-1}^{(2)}}{k^3} + \sum_{k\geq 1}\frac{H_{k-1}}{k^4} + \sum_{k\geq 1}\frac{1}{k^5}$$

$$= \sum_{k\geq 1}\frac{H_{k-1}H_{k-1}^{(2)}}{k^2} + \sum_{k>l\geq 1}\frac{1}{k^3 l^2} + \sum_{k>l\geq 1}\frac{1}{k^4 l} + \sum_{k\geq 1}\frac{1}{k^5},$$

we obtain

$$\sum_{k\geq 1}\frac{H_k H_k^{(2)}}{k^2} = \zeta(2,1,2) + \zeta(2,2,1) + \zeta(2,3) + \zeta(3,2) + \zeta(4,1) + \zeta(5).$$

Using the following evaluations of the multiple zeta function given in [1] resp. [2]:

$$\zeta(2,1,2) = \zeta(2,3) = \frac{9}{2}\zeta(5) - 2\zeta(2)\zeta(3),$$

$$\zeta(2,2,1) = \zeta(3,2) = -\frac{11}{2} + 3\zeta(2)\zeta(3),$$

$$\zeta(4,1) = 2\zeta(5) - \zeta(2)\zeta(3),$$

equation (3a) follows.

References

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INSTITUT FÜR DISKRETE MATHEMATIK UND GEOMETRIE, TECHNISCHE UNIVERSITÄT WIEN, WIEDNER HAUPTSTRASSE 8–10/104, A-1040 WIEN, AUSTRIA.
E-mail address: Alois.Panholzer@tuwien.ac.at

HELMUT PRODINGER, MATHEMATICS DEPARTMENT, STELLENBOSCH UNIVERSITY, 7602 STELLENBOSCH, SOUTH AFRICA.
E-mail address: hproding@sun.ac.za