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DECREASES AND DESCENTS IN WORDS

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ABSTRACT. The generating function for words by a multivariable statistic involving decrease, increase, descent and rise values is explicitly calculated by using the MacMahon Master Theorem and the properties of the first fundamental transformation on words. Applications to statistical study of the symmetric group are also given.

1. Introduction

In our recent papers [2, 3, 4, 5, 6, 7, 8, 9] that involve the calculations of factorial generating functions for the symmetric and hyperoctahedral groups, we have obtained several results on statistical distributions on *words*. In this paper we take up again the study of those word statistics in a more general context. They do not necessarily have counterparts on permutations, but are essential in this word calculus. The MacMahon Master Theorem [14, p. 97–98] and the first fundamental transformation on words [13, Chap. 10] will be our basic tools.

The *number of decreases* is a crucial statistic; as such it is at the origin of our word studies. The definition of *decrease* slightly differs from the definition of the classical *descent*. Let $w = x_1x_2\cdots x_n$ be an *arbitrary* word, whose letters are nonnegative integers. Recall that a positive integer i is said to be a *descent* (or *descent place*) of w if $1 \le i \le n-1$ and $x_i > x_{i+1}$. We say that i is a *decrease* of w if $1 \le i \le n-1$ and $x_i = x_{i+1} = \cdots = x_j > x_{j+1}$ for some j such that $i \le j \le n-1$. The letter x_i is said to be a *decrease value* (respectively *descent value*) of w. The set of all decreases (respectively descents) is denoted by $\mathrm{DEC}(w)$ (respectively $\mathrm{DES}(w)$). Each descent is a decrease, but not conversely. This means that $\mathrm{DES}(w) \subset \mathrm{DEC}(w)$. However, $\mathrm{DES}(w) = \mathrm{DEC}(w)$ when w is a word *without repetitions*.

In the present paper our intention is to go back to the study of the number of decreases, this time associated with several other word statistics, and derive the Ur-result that should have been at the origin of several of our statistical distribution studies. This Ur-result is stated in Theorem 1.1, but has two equivalent forms, as written in Theorems 1.2 and 1.3.

In parallel with the notion of decrease, we say that that a positive integer i is an *increase* (respectively a *rise*) of w if $1 \le i \le n$ and

$x_i = x_{i+1} = \cdots = x_j < x_{j+1}$ for some j such that $i \le j \le n$ (respectively if $1 \le i \le n$ and $x_i < x_{i+1}$). By convention, $x_{n+1} = +\infty$. The letter x_i is said to be *an increase value* (respectively *a rise value*) of w. Thus, the rightmost letter x_n is always a rise value. Again, the set of all increases (respectively rises) is denoted by $\mathrm{INC}(w)$ (respectively $\mathrm{RISE}(w)$). Each rise is an increase, but not conversely. This means that $\mathrm{RISE}(w) \subset \mathrm{INC}(w)$.

Furthermore, a position i $(1 \le i \le n)$ is said to be a *record* if $x_j \le x_i$ for all j such that $1 \le j \le i-1$. The letter x_i is said to be a *record value*. The set of all records of w is denoted by $\mathrm{REC}(w)$.

Introduce six sequences of commuting variables (X_i), (Y_i), (Z_i), (T_i), (Y_i'), (T_i') $(i = 0, 1, 2, \dots)$, and for each word $w = x_1 x_2 \dots x_n$ from $[0, r]^*$ define the *weight* $\psi(w)$ of $w = x_1 x_2 \dots x_n$ to be

$$(1.1)\quad \psi(w) := \prod_{i \in \mathrm{DES}} X_{x_i} \prod_{i \in \mathrm{RISE} \setminus \mathrm{REC}} Y_{x_i} \prod_{i \in \mathrm{DEC} \setminus \mathrm{DES}} Z_{x_i} \times \prod_{i \in (\mathrm{INC} \setminus \mathrm{RISE}) \setminus \mathrm{REC}} T_{x_i} \prod_{i \in \mathrm{RISE} \cap \mathrm{REC}} Y'_{x_i} \prod_{i \in (\mathrm{INC} \setminus \mathrm{RISE}) \cap \mathrm{REC}} T'_{x_i},$$

where the argument "(w)" has not been written for typographic reasons. For example, $i \in \mathrm{RISE} \setminus \mathrm{REC}$ stands for $i \in \mathrm{RISE}(w) \setminus \mathrm{REC}(w)$.

Example. For the word $w = 3\,2\,4\,4\,5\,5\,5\,3\,1\,1\,1\,4\,1\,3\,5$ the sets DES, DEC, INC, RISE, REC of w are indicated by bullets.

w	=	3	2	4	4	5	5	5	3	1	1	1	4	1	3	5
DES	=	•						•	•				•			
DEC	=	•				•	•	•	•				•			
RISE	=		•		•							•		•	•	•
INC	=		•	•	•					•	•	•		•	•	•
REC	=	•		•	•	•	•	•								•

We have $\psi(w) = X_3 Y_2 T_4' Y_4' Z_5 Z_5 X_5 X_3 T_1 T_1 Y_1 X_4 Y_1 Y_3 Y_5'$.

Now let C be the $(r+1) \times (r+1)$ matrix

$$(1.2)\quad C = \begin{pmatrix} 0 & \dfrac{X_1}{1-Z_1} & \dfrac{X_2}{1-Z_2} & \cdots & \dfrac{X_{r-1}}{1-Z_{r-1}} & \dfrac{X_r}{1-Z_r} \\ \dfrac{Y_0}{1-T_0} & 0 & \dfrac{X_2}{1-Z_2} & \cdots & \dfrac{X_{r-1}}{1-Z_{r-1}} & \dfrac{X_r}{1-Z_r} \\ \dfrac{Y_0}{1-T_0} & \dfrac{Y_1}{1-T_1} & 0 & \cdots & \dfrac{X_{r-1}}{1-Z_{r-1}} & \dfrac{X_r}{1-Z_r} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \dfrac{Y_0}{1-T_0} & \dfrac{Y_1}{1-T_1} & \dfrac{Y_2}{1-T_2} & \cdots & 0 & \dfrac{X_r}{1-Z_r} \\ \dfrac{Y_0}{1-T_0} & \dfrac{Y_1}{1-T_1} & \dfrac{Y_2}{1-T_2} & \cdots & \dfrac{Y_{r-1}}{1-T_{r-1}} & 0 \end{pmatrix}.$$

Theorem 1.1. *The generating function for the set $[0,r]^*$ by the weight ψ is given by*

$$\sum_{w\in[0,r]^*}\psi(w)=\frac{\prod\limits_{0\le j\le r}\Bigl(1+\dfrac{Y'_j}{1-T'_j}\Bigr)}{\det(I-C)},\tag{1.3}$$

where I is the identity matrix of order $(r+1)$.

Of course, the expression $1/\det(I-C)$ is too redolent of the MacMahon Master Theorem [14, p. 97–98] for not having it play a crucial role in the proof of (1.3). It does indeed. However we further need the properties of the first fundamental transformation, as it is developed in Cartier-Foata [1] and also in Lothaire [13, Chap. 10]. As mentioned earlier, Theorem 1.1 must be regarded as our Ur-result. Its proof is given in Section 2. Its two equivalent forms come next.

Theorem 1.2. *We also have:*

$$\sum_{w\in[0,r]^*}\psi(w)=\frac{\dfrac{\prod\limits_{0\le j\le r}\dfrac{1-Z_j}{1-Z_j+X_j}}{\prod\limits_{0\le j\le r}\dfrac{1-T'_j}{1-T'_j+Y'_j}}}{1-\sum\limits_{0\le l\le r}\dfrac{\prod\limits_{0\le j\le l-1}\dfrac{1-Z_j}{1-Z_j+X_j}}{\prod\limits_{0\le j\le l-1}\dfrac{1-T_j}{1-T_j+Y_j}}\dfrac{X_l}{1-Z_l+X_l}}.\tag{1.4}$$

By definition each letter equal to 0 cannot be a decrease value. Consequently, the weight $\psi(w)$ of each word w must not contain the variables X_0, Z_0. There is then another expression for the right-hand side of (1.4) which does not involve the variables X_0, Z_0. To obtain it we factor out $(1-Z_0)/(1-Z_0+X_0)$ from both numerator and denominator of the right-hand side, as done in the next theorem.

Theorem 1.3. *We also have:*

$$\sum_{w\in[0,r]^*}\psi(w)=\frac{\dfrac{\prod\limits_{1\le j\le r}\dfrac{1-Z_j}{1-Z_j+X_j}}{\prod\limits_{0\le j\le r}\dfrac{1-T'_j}{1-T'_j+Y'_j}}}{1-\sum\limits_{1\le l\le r}\dfrac{\prod\limits_{1\le j\le l-1}\dfrac{1-Z_j}{1-Z_j+X_j}}{\prod\limits_{0\le j\le l-1}\dfrac{1-T_j}{1-T_j+Y_j}}\dfrac{X_l}{1-Z_l+X_l}}.\tag{1.5}$$

Theorem 1.2 (and therefore Theorem 1.3) is proved in Section 3 by using a determinantal manipulation and summing the weights $\psi(w)$ according to their so-called *keys*. An alternate proof of Theorem 1.2, which is not reproduced in this paper, is based on the word-analog of the Kim–Zeng transformation [11] and follows the pattern developed in our previous paper [9]. Specializations of those two theorems for deriving generating functions *on words* only appear in Section 6. There is however a specialization of (1.5) that deserves a special development and is now presented.

Let γ be the homomorphism defined by the following substitutions of variables:

$$\gamma := \{X_j \leftarrow s\,Y_{j-1},\quad Z_j \leftarrow s\,Y_{j-1},\quad T_j \leftarrow Y_j,\quad T'_j \leftarrow Y'_j\}.$$

For each word $w = x_1x_2\cdots x_n \in [0,r]^*$ we then have:

$$\gamma\,\psi(w) = \prod_{x_i\in \mathrm{INC}\cap\mathrm{REC}} Y'_{x_i} \times \prod_{x_i\in\mathrm{DEC}} sY_{x_i-1} \times \prod_{x_i\in\mathrm{INC}\setminus\mathrm{REC}} Y_{x_i}. \tag{1.6}$$

Applying γ to (1.5) we get:

$$\sum_{w\in[0,r]^*}\gamma\,\psi(w) = \frac{\dfrac{\displaystyle\prod_{1\le j\le r}(1-sY_{j-1})}{\displaystyle\prod_{0\le j\le r}(1-Y'_j)}}{1-\displaystyle\sum_{1\le l\le r}\frac{\displaystyle\prod_{1\le j\le l-1}(1-sY_{j-1})}{\displaystyle\prod_{0\le j\le l-1}(1-Y_j)}sY_{l-1}}. \tag{1.7}$$

The above right-hand side can be further simplified as stated in the following theorem, whose proof is given in Section 4.

Theorem 1.4. *We have:*

$$\begin{aligned}\sum_{w\in[0,r]^*}&\gamma\,\psi(w)\\ &= \frac{(1-s)\displaystyle\prod_{0\le j\le r-1}(1-Y_j)\prod_{0\le j\le r-1}(1-s\,Y_j)}{\displaystyle\prod_{0\le j\le r}(1-Y'_j)\Bigl(\prod_{0\le j\le r-1}(1-Y_j)-s\prod_{0\le j\le r-1}(1-s\,Y_j)\Bigr)}.\end{aligned} \tag{1.8}$$

For each word $w = x_1x_2\cdots x_n$ let $\mathrm{inrec}\,w$ denote the *number* of letters of w, which are increase *and* record values. Also let $\mathrm{dec}\,w$ be the number of decreases in w and $\operatorname{tot} w = x_1 + x_2 + \cdots + x_n$ be the *sum* of the letters of w. Also make the substitution $Y'_j \leftarrow RY_j$, where R is a new variable and let $\psi_R := \gamma\,\psi\,|_{Y'_j \leftarrow R\,Y_j}$, so that (1.6) becomes:

$$\psi_R(w) = R^{\operatorname{inrec} w} s^{\operatorname{dec} w} \prod_{x_i \in \mathrm{DEC}} Y_{x_i - 1} \times \prod_{x_i \in \mathrm{INC}} Y_{x_i}. \tag{1.9}$$

On the other hand, let

$$H(Y) := \prod_{i \ge 0} (1 - Y_i)^{-1}; \tag{1.10}$$

$$H_r(Y) := \prod_{0 \le i \le r-1} (1 - Y_i)^{-1} \quad (r \ge 0). \tag{1.11}$$

Using the homomorphism ψ_R, identity (1.8) may be rewritten as:

$$\sum_{w \in [0,r]^*} \psi_R(w) = \frac{(1-s) H_{r+1}(RY)}{H_r(sY) - s H_r(Y)}. \tag{1.12}$$

The left-hand side of (1.12) can be further expressed as a series over the symmetric groups $\mathfrak{S}_n$ $(n = 0, 1, \ldots)$. If $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ is a permutation of $12\cdots n$, let $z = z_1 z_2 \cdots z_n$ be the word defined by:

$$z_i := \#\{j : i \le j \le n-1,\ \sigma(j) > \sigma(j+1)\} \quad (1 \le i \le n), \tag{1.13}$$

so that z_1 is the *number of descents*, $\operatorname{des}\sigma$, and $\operatorname{tot} z = z_1 + z_2 + \cdots + z_n$ is the *major index*, $\operatorname{maj}\sigma$, of σ. Also, let $\operatorname{exc}\sigma := \{i : 1 \le i \le n-1, \sigma(i) > i\}$ be the *number of excedances* and $\operatorname{fix}\sigma$ be the *number of fixed points* of σ. Finally, let NIW_n (respectively $\mathrm{NIW}_n(k)$) denote the set of words $c = c_1 c_2 \cdots c_n$, of length n, whose letters are integers satisfying $c_1 \ge c_1 \ge \cdots \ge c_n \ge 0$ (respectively $k \ge c_1 \ge c_1 \ge \cdots \ge c_n \ge 0$). With each pair $(\sigma, c) \in \mathfrak{S}_n \times \mathrm{NIW}_n$ we associate the monomial

$$Y_{(\sigma,c)} := \prod_{j < \sigma(j)} Y_{c_j + z_j - 1} \times \prod_{j \ge \sigma(j)} Y_{c_j + z_j}. \tag{1.14}$$

Theorem 1.5. *We have:*

$$\sum_{n \ge 0} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \operatorname{des}\sigma \le r}} R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} \sum_{c \in \mathrm{NIW}_n(r - \operatorname{des}\sigma)} Y_{(\sigma,c)} = \frac{(1-s) H_{r+1}(RY)}{H_r(sY) - s H_r(Y)} \quad (r \ge 0). \tag{1.15}$$

When r tends to infinity in (1.15), we get the identity

$$\sum_{n\ge 0}\sum_{\sigma\in\mathfrak{S}_n} R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} \sum_{c\in \mathrm{NIW}_n} Y_{(\sigma,c)} = \frac{(1-s)H(RY)}{H(sY)-sH(Y)}, \tag{1.16}$$

derived by Shareshian and Wachs [15, Theorem 2.1] using a quasi-symmetric function approach. However, starting from (1.15), we can obtain a *graded form* of (1.16) as follows. Let

$$Y(\sigma;t) := \sum_{k\ge 0} t^k \sum_{c\in \mathrm{NIW}_n(k)} Y_{(\sigma,c)}. \tag{1.17}$$

Theorem 1.6. *The graded form of* (1.16) *reads:*

$$\sum_{n\ge 0}\sum_{\sigma\in\mathfrak{S}_n} R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} t^{\operatorname{des}\sigma} Y(\sigma;t) = \sum_{r\ge 0} t^r \frac{(1-s)H_{r+1}(RY)}{H_r(sY)-sH_r(Y)}. \tag{1.18}$$

Section 5 starts with redescribing the Gessel–Reutenauer standardization [10] and showing how it is used to prove Theorem 1.5. The graded form (1.18) is deduced from Theorem 1.5 by a standard series manipulation. We end the paper by giving some specializations of our Ur-theorem and also by showing that the distribution of (fix, exc, des, maj) over the symmetric groups that was found earlier in [9] using the word-analog of the Kim–Zeng transformation [11], can also be deduced from our Ur-result in two different manners.

2. Proof of Theorem 1.1

A word $w = y_1y_2\cdots y_n \in [0,r]^*$ having no equal letters in succession is called an *h-derangement* (*horizontal derangement*). The set of all h-derangement words in $[0,r]^*$ is denoted by $[0,r]^*_h$. Let α be the substitution of variables defined by

$$\alpha := \{Z_i \leftarrow 0,\quad T_i \leftarrow 0,\quad T'_i \leftarrow 0\}.$$

Then

$$\alpha\,\psi(w) = \psi(w) = \prod_{i\in \mathrm{DES}} X_{x_i} \prod_{i\in \mathrm{RISE}\setminus \mathrm{REC}} Y_{x_i} \prod_{i\in \mathrm{RISE}\cap \mathrm{REC}} Y'_{x_i},$$

if w is an h-derangement and $\alpha\,\psi(w) = 0$ otherwise. The following specialization of Theorem 1.1 is obtained by taking the image of identity (1.3) under α.

Theorem 2.1. *We have*

$$\sum_{w\in[0,r]^*_h} \psi(w) = \frac{\prod_{0\le j\le r}\Bigl(1+Y'_j\Bigr)}{\det(I-\alpha\, C)}. \tag{2.1}$$

Even though Theorem 2.1 is a special case of Theorem 1.1, it can still be used as a lemma to prove Theorem 1.1. We proceed as follows.

Proof of Theorem 1.1. Let w be a word from $[0,r]^*$. The *key* of w is defined to be the h-derangement k derived from w by erasing all letters x_i such that $x_i = x_{i+1}$. For instance, the key of $w = 324455531114135$ is the h-derangement $k = 3245314135$.

Let β be the substitution of variables defined by

$$\beta := \{X_i \leftarrow X_i/(1-Z_i),\quad Y_i \leftarrow Y_i/(1-T_i),\quad Y'_i \leftarrow Y'_i/(1-T'_i)\}.$$

Then, the generating function for the set of all w whose key is k by the weight ψ is given by

$$\sum_{w,\,\mathrm{key}(w)=k} \psi(w) = \beta\,\psi(k).$$

Since $\beta\,\alpha\, C = C$ we have

$$\begin{aligned}
\sum_{w\in[0,r]^*} \psi(w) &= \sum_{k\in[0,r]^*_h}\ \sum_{\mathrm{key}(w)=k} \psi(w) \ = \sum_{k\in[0,r]^*_h} \beta\,\psi(k)\\
&= \beta\Bigl(\sum_{k\in[0,r]^*_h} \psi(k)\Bigr) \ = \ \beta\,\frac{\prod_{0\le j\le r}\Bigl(1+Y'_j\Bigr)}{\det(I-\alpha\, C)} \qquad \text{[by (2.1)]}\\
&= \frac{\prod_{0\le j\le r}\Bigl(1+\beta\, Y'_j\Bigr)}{\det(I-\beta\,\alpha\, C)} = \frac{\prod_{0\le j\le r}\Bigl(1+\dfrac{Y'_j}{1-T'_j}\Bigr)}{\det(I-C)}. \qquad \square
\end{aligned}$$

Proof of Theorem 2.1. A letter x_i which is a record and also a rise value is called a *riserec* value. For each h-derangement $k = x_1x_2\cdots x_n$ let w_0 be the nondecreasing word composed of all the riserec values of k and let k_0 be the word obtained from k by erasing all the riserec values of k. Since the letters of w_0 can be uniquely inserted into the word k_0 for reconstructing k, the map

$$k \mapsto (w_0, k_0) \tag{2.2}$$

is a bijection of the set of all h-derangements onto the the set of all pairs (w_0, k_0) such that w_0 is a nondecreasing word and k_0 is an h-derangement without any riserec value. Moreover, if a letter x_j of k is a record value and therefore becomes a letter, say, $x_{0,i}$ of w_0, then $x_{0,i}$ is a rise of w_0 if and only if x_j is a rise of k. Therefore

$$\psi(k) = \psi(w_0)\psi(k_0). \tag{2.3}$$

Next apply the first fundamental transformation to k_0 (see [13, § 10.5]), say, $u = \mathbf{F}_1(k_0)$. Let us recall how $\mathbf{F}_1$ is defined by means of an example. Start with the word $k_0 = 5\,3\,6\,5\,3\,2\,4\,6\,1\,2\,4\,3\,1$ and cut it before each record value to get $k_0 = 5\,3 \mid 6\,5\,3\,2\,4 \mid 6\,1\,2\,4\,3\,1 =: w_1 \mid w_2 \mid w_3$. In each compartment move the leftmost letter to the end to obtain the cyclic shifts $\delta w_1 = 3\,5$, $\delta w_2 = 5\,3\,2\,4\,6$, $\delta w_3 = 1\,2\,4\,3\,1\,6$ and form the two-row matrix $\begin{pmatrix}\delta w_1\,\delta w_2\,\delta w_3\\ w_1\ \ w_2\ \ w_3\end{pmatrix} = \begin{pmatrix}3\,5\,5\,3\,2\,4\,6\,1\,2\,4\,3\,1\,6\\ 5\,3\,6\,5\,3\,2\,4\,6\,1\,2\,4\,3\,1\end{pmatrix}$. Finally, rearrange the vertical biletters of that two-row matrix in such a way that the entries on the top row are in nondecreasing order, assuming that two biletters $\binom{a}{a'}$, $\binom{b}{b'}$ can commute only when $a \neq b$. We then obtain the two-row matrix $\mathbf{F}_1(k_0) := \begin{pmatrix}\overline{u}\\ u\end{pmatrix} = \begin{pmatrix}1\,1\,2\,2\,3\,3\,3\,4\,4\,5\,5\,6\,6\\ 6\,3\,3\,1\,5\,5\,4\,2\,2\,3\,6\,4\,1\end{pmatrix}$.

We can characterize the word $u = y_1y_2\cdots y_m$ when we start with an h-derangement k_0. Let $\overline{u} = z_1z_2\cdots z_m$ be the nondecreasing rearrangement of u (and of k_0). By construction $y_i \neq z_i$ for $1 \le i \le m$. Such a word u is called a *v-derangement* (vertical derangement). Denote the set of all v-derangements in $[0,r]^*$ by $[0,r]^*_v$. Then $\mathbf{F}_1$ provides a bijection of the set of all h-derangements onto the set of all pairs (w_0, u) such that w_0 is a nondecreasing word and u is a v-derangement:

$$k \mapsto (w_0, k_0) \mapsto (w_0, u) \quad \text{where} \quad \mathbf{F}_1(k_0) = \begin{pmatrix}\overline{u}\\ u\end{pmatrix}.$$

Example. $k = \mathbf{2}53\mathbf{5}653246124316\mathbf{6}$...

$$k = \mathbf{2}53\mathbf{5}65324612431\mathbf{6} \mapsto (w_0 = \mathbf{256}, k_0 = 5365324612431)$$
$$\mapsto (w_0 = \mathbf{256}, u = 6331554223641),$$

since
$$\mathbf{F}_1(k_0) = \begin{pmatrix}\overline{u}\\ u\end{pmatrix} = \begin{pmatrix}1\,1\,2\,2\,3\,3\,3\,4\,4\,5\,5\,6\,6\\ 6\,3\,3\,1\,5\,5\,4\,2\,2\,3\,6\,4\,1\end{pmatrix}.$$

Let Φ be the homomorphism generated by

$$\Phi\begin{pmatrix}i\\ j\end{pmatrix} := \begin{cases} X_j, & \text{if } j > i;\\ Y_j, & \text{if } j < i.\end{cases}$$

By the property [13, Chap. 10] of the first fundamental transformation, we have

$$\psi(k) = \psi(w_0)\psi(k_0) = \psi(w_0)\Phi\binom{\overline{u}}{u}. \tag{2.4}$$

Let $\mathrm{ND}(r)$ be the set of all *non-decreasing* words from $[0,r]^*$. It follows from the properties of the above bijections that

$$\sum_{w\in[0,r]_h^*} \psi(w) = \sum_{w_0\in \mathrm{ND}(r)} \psi(w_0) \sum_{u\in[0,r]_v^*} \Phi\binom{\overline{u}}{u};$$
$$\sum_{w_0\in \mathrm{ND}(r)} \psi(w_0) = \prod_{0\le j\le r} (1+Y'_j).$$

There remains to prove the identity:

$$\sum_{u\in[0,r]_v^*} \Phi\binom{\overline{u}}{u} = \frac{1}{\det(I-\alpha\, C)}. \tag{2.5}$$

The proof is based on the celebrated MacMahon Master Theorem, using the noncommutative version developed in [1, Chap. 4]. Also see [13, Chap. 10]. Consider the matrix

$$C'' = \begin{pmatrix} 0 & \binom{0}{1} & \binom{0}{2} & \cdots & \binom{0}{r-1} & \binom{0}{r} \\ \binom{1}{0} & 0 & \binom{1}{2} & \cdots & \binom{1}{r-1} & \binom{1}{r} \\ \binom{2}{0} & \binom{2}{1} & 0 & \cdots & \binom{2}{r-1} & \binom{2}{r} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{r-1}{0} & \binom{r-1}{1} & \binom{r-1}{2} & \cdots & 0 & \binom{r-1}{r} \\ \binom{r}{0} & \binom{r}{1} & \binom{r}{2} & \cdots & \binom{r}{r-1} & 0 \end{pmatrix}.$$

As shown in the references just mentioned, there holds the identity

$$\frac{1}{\det(I-C'')} = \sum_{u\in[0,r]_v^*} \binom{\overline{u}}{u}. \tag{2.6}$$

Applying Φ to both sides of (2.6) yields (2.5). ∎

3. Proof of Theorem 1.2

The proof of Theorem 1.2 is similar to the proof of Theorem 1.1 given in the previous section. First we prove the following specialization of Theorem 1.2.

Theorem 3.1. *We have:*

$$\sum_{w\in[0,r]^*_h} \psi(w) = \frac{\displaystyle\prod_{0\le j\le r} \frac{1+Y'_j}{1+X_j}}{1-\displaystyle\sum_{0\le l\le r}\prod_{0\le j\le l-1}\frac{1+Y_j}{1+X_j}\,\frac{X_l}{1+X_l}}. \tag{3.1}$$

Proof. Denote the left-hand side of (3.1) by LHS. From Theorem 2.1 we have

$$\text{LHS} = \frac{\displaystyle\prod_{0\le j\le r}(1+Y'_j)}{D},$$

where

$$D = \begin{vmatrix} 1 & -X_1 & -X_2 & \cdots & -X_{r-1} & -X_r \\ -Y_0 & 1 & -X_2 & \cdots & -X_{r-1} & -X_r \\ -Y_0 & -Y_1 & 1 & \cdots & -X_{r-1} & -X_r \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -Y_0 & -Y_1 & -Y_2 & \cdots & 1 & -X_r \\ -Y_0 & -Y_1 & -Y_2 & \cdots & -Y_{r-1} & 1 \end{vmatrix}.$$

In the above determinant subtract the r-th row from the $(r+1)$-st one; then the $(r-1)$-st from the r-th row; ..., the first row from the second. We obtain:

$$D = \begin{vmatrix} 1 & -X_1 & -X_2 & \cdots & -X_{r-1} & -X_r \\ -1-Y_0 & 1+X_1 & 0 & \cdots & 0 & 0 \\ 0 & -1-Y_1 & 1+X_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+X_{r-1} & 0 \\ 0 & 0 & 0 & \cdots & -1-Y_{r-1} & 1+X_r \end{vmatrix}.$$

Now expand the determinant by the cofactors of the *first row*. We get:

$$D = \prod_{1\le j\le r}(1+X_j) - \sum_{1\le l\le r}\Bigl(\prod_{0\le j\le l-1}(1+Y_j)\prod_{l+1\le j\le r}(1+X_j)\Bigr)X_l.$$

We further have:

$$\begin{aligned} D = \prod_{1\le j\le r}(1+X_j) &+ X_0\prod_{1\le j\le r}(1+X_j) \\ &- \sum_{0\le l\le r}\Bigl(\prod_{0\le j\le l-1}(1+Y_j)\prod_{l\le j\le r}(1+X_j)\Bigr)\frac{X_l}{1+X_l}. \end{aligned}$$

Hence,

$$\text{LHS} = \frac{\displaystyle\prod_{0\le j\le r} \frac{(1+Y_j')}{(1+X_j)}}{1 - \displaystyle\sum_{0\le l\le r} \prod_{0\le j\le l-1} \frac{(1+Y_j)}{(1+X_j)} \frac{X_l}{1+X_l}}. \qquad \square \tag{3.2}$$

Proof of Theorem 1.2. As in the proof of Theorem 1.1 we may write:

$$\sum_{w\in[0,r]^*} \psi(w) = \sum_{k\in[0,r]_h^*} \sum_{\mathrm{key}(w)=k} \psi(w) = \sum_{k\in[0,r]_h^*} \beta\,\psi(k) = \beta(\text{LHS}).$$

Using (3.2) it is immediate to verify that $\beta(\text{LHS})$ is equal to the right-hand side of (1.4). $\square$

4. Proof of Theorem 1.4

First, we may check that

$$\begin{aligned}
&\prod_{0\le j\le r} (1-sY_j) - \prod_{0\le j\le r} (1-Y_j)\\
=&\sum_{0\le l\le r} \prod_{0\le j\le l} (1-sY_j) \prod_{l+1\le j\le r} (1-Y_j) - \sum_{0\le l\le r} \prod_{0\le j\le l-1} (1-sY_j) \prod_{l\le j\le r} (1-Y_j)\\
=&(1-s) \sum_{0\le l\le r} Y_l \prod_{0\le j\le l-1} (1-sY_j) \prod_{l+1\le j\le r} (1-Y_j).
\end{aligned}$$

Proof of Theorem 1.4. Using (1.7) we have:

$$\begin{aligned}
&\frac{1}{1 - s \displaystyle\sum_{0\le l\le r-1} Y_l \prod_{0\le j\le l-1} (1-sY_j) / \prod_{0\le j\le l} (1-Y_j)}\\
&= \frac{\displaystyle\prod_{0\le j\le r-1} (1-Y_j)}{\displaystyle\prod_{0\le j\le r-1} (1-Y_j) - s \sum_{0\le l\le r-1} Y_l \prod_{0\le j\le l-1} (1-sY_j) \prod_{l+1\le j\le r-1} (1-Y_j)}\\
&= \frac{\displaystyle\prod_{0\le j\le r-1} (1-Y_j)}{\displaystyle\prod_{0\le j\le r-1} (1-Y_j) - s\Big(\prod_{0\le j\le r-1} (1-sY_j) - \prod_{0\le j\le r-1} (1-Y_j)\Big)/(1-s)}\\
&= \frac{(1-s) \displaystyle\prod_{0\le j\le r-1} (1-Y_j)}{\displaystyle\prod_{0\le j\le r-1} (1-Y_j) - s \prod_{0\le j\le r-1} (1-sY_j)}. \qquad \square
\end{aligned}$$

5. Proofs of Theorems 1.5 and 1.6

An updated version of the Gessel–Reutenauer standardization [10] is fully described in our previous paper [9], Section 5. The standardization consists of mapping each word w from $[0,r]^*$ of length n onto a pair (σ, c), where $\sigma \in \mathfrak{S}_n$ and $c = c_1 c_2 \cdots c_n$ is a word of length n, whose letters are nonnegative integers having the property: $r - \operatorname{des}\sigma \geq c_1 \geq c_2 \geq \cdots \geq c_n \geq 0$, the symbol $\operatorname{des}\sigma$ being the *number of descents* of σ. We recall the construction of the inverse $(\sigma, c) \mapsto w$ by means of an example.

$\mathrm{Id} =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$\rightarrow \sigma =$	**1**	$\underline{5}$	$\underline{6}$	$\underline{8}$	$\underline{13}$	$\underline{14}$	**7**	$\underline{17}$	4	**10**	$\underline{15}$	$\underline{18}$	$\underline{19}$	2	9	**16**	$\underline{20}$	$\underline{22}$	3	11	12	21
$z =$	4	4	4	4	4	4	3	3	2	2	2	2	2	1	1	1	1	1	0	0	0	0
$\rightarrow c =$	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\overline{c} =$	6	6	6	6	6	6	5	4	3	3	3	3	3	2	2	2	2	2	1	1	1	1
$\sigma =$	(**16**)	(12	18	22	21)	(**10**)	(**7**)	(4	8	17	20	11	15	9)	(2	5	13	19	3	6	14)	(**1**)
$\check{\sigma} =$	**16**	12	18	22	21	**10**	**7**	4	8	17	20	11	15	9	2	5	13	19	3	6	14	**1**
$\mapsto w =$	$2 \mid$	$\underline{3}$	$\underline{2}$	1	$1 \mid$	$3 \mid$	$5 \mid$	$\underline{6}$	$\underline{4}$	$\underline{2}$	1	$\underline{3}$	2	$3 \mid$	$\underline{6}$	$\underline{6}$	$\underline{3}$	1	$\underline{6}$	$\underline{6}$	$2 \mid$	6

In the example $n = 22$. The second row contains the values $\sigma(i)$ $(i = 1, 2, \ldots, n)$ of the *starting* permutation σ. The *excedances* $\sigma(i) > i$ are underlined, while the *fixed points* $\sigma(i) = i$ are written in boldface. The third row is the vector $z = z_1 z_2 \cdots z_n$ defined by (1.13) so that $z_1 = \operatorname{des}\sigma$ and $\operatorname{tot} z = z_1 + z_2 + \cdots + z_n$ is the *major index* of σ denoted by $\operatorname{maj}\sigma$, as already mentioned in the Introduction. The fourth row is the *starting* nonincreasing word $c = c_1 c_2 \cdots c_n$. The fifth row $\overline{c} = \overline{c}_1 \overline{c}_2 \cdots \overline{c}_n$ is the word defined by

$$\overline{c}_i := z_i + c_i \quad (1 \leq i \leq n).$$

In the sixth row the permutation σ is represented as the product of its disjoint cycles. Each cycle starts with its *minimum* element and those minimum elements are in *decreasing* order when reading the whole word from left to right. When removing the parentheses in the sixth row we obtain the seventh row denoted by $\check{\sigma} = \check{\sigma}(1)\check{\sigma}(2) \cdots \check{\sigma}(n)$. The bottom row is the word $w = x_1 x_2 \cdots x_n$ corresponding to the pair (σ, c) defined by

$$x_i := \overline{c}_{\check{\sigma}(i)} \quad (1 \leq i \leq n).$$

For instance, $\check{\sigma}(9) = 8$ and $\overline{c}(8) = 4$. Hence $x_9 = 4$. The decrease values of w have been underlined.

It can be verified that all the above steps are reversible and that $w \mapsto (\sigma, c)$ is a bijection of the set of all words from $[0,r]^*$ of length n onto the set of pairs (σ, c) such that $\sigma \in \mathfrak{S}_n$, $\operatorname{des}\sigma \leq r$ and $c = c_1 c_2 \cdots c_n$

is a word satisfying $r - \operatorname{des}\sigma \geq c_1 \geq c_2 \geq \cdots \geq c_n \geq 0$. Furthermore, x_i is a decrease value of w if and only if $\check\sigma(i) < \check\sigma(i+1)$, if and only if $\check\sigma(i) < \sigma(\check\sigma(i))$. Also x_i is an increase and record value of w if and only if $\check\sigma(i)$ is a fixed point of σ. Hence,

$$\begin{aligned}\psi_R(w) &= R^{\operatorname{inrec} w} s^{\operatorname{dec} w} \prod_{x_i \in \mathrm{DEC}} Y_{x_i - 1} \prod_{x_i \in \mathrm{INC}} Y_{x_i} \\ &= R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} \prod_{\check\sigma(i) < \sigma(\check\sigma(i))} Y_{\overline{c}_{\check\sigma(i)} - 1} \prod_{\check\sigma(i) \geq \sigma(\check\sigma(i))} Y_{\overline{c}_{\check\sigma(i)}} \\ &= R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} \prod_{j < \sigma(j)} Y_{\overline{c}_j - 1} \prod_{j \geq \sigma(j)} Y_{\overline{c}_j} \\ &= R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} \prod_{j < \sigma(j)} Y_{c_j + z_j - 1} \prod_{j \geq \sigma(j)} Y_{c_j + z_j} \\ &= R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} Y_{(\sigma,c)}.\end{aligned}$$

Consequently,

$$\sum_{w \in [0,r]^*} \psi_R(w) = \sum_{n \geq 0} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \operatorname{des}\sigma \leq r}} R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} \sum_{c \in \mathrm{NIW}_n(r - \operatorname{des}\sigma)} Y_{(\sigma,c)}. \tag{5.1}$$

This achieves the proof of Theorem 1.5 by taking identity (1.12) into account. ∎

For the proof of Theorem 1.6 we multiply both sides of (1.15) by t^r and sum over $r \geq 0$. We obtain:

$$\begin{aligned}&\sum_{r \geq 0} t^r \frac{(1-s) H_{r+1}(RY)}{H_r(sY) - s H_r(Y)} \\ &\quad = \sum_{r \geq 0} t^r \sum_{n \geq 0} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \operatorname{des}\sigma \leq r}} R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} \sum_{c \in \mathrm{NIW}_n(r - \operatorname{des}\sigma)} Y_{(\sigma,c)} \\ &\quad = \sum_{n \geq 0} \sum_{r \geq 0} t^r \sum_{0 \leq j \leq r} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \operatorname{des}\sigma = r - j}} R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} \sum_{c \in \mathrm{NIW}_n(j)} Y_{(\sigma,c)} \\ &\quad = \sum_{n \geq 0} \sum_{j \geq 0} t^j \sum_{r \geq j} t^{r-j} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \operatorname{des}\sigma = r - j}} R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} \sum_{c \in \mathrm{NIW}_n(j)} Y_{(\sigma,c)} \\ &\quad = \sum_{n \geq 0} \sum_{j \geq 0} t^j \sum_{k \geq 0} t^k \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \operatorname{des}\sigma = k}} R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} \sum_{c \in \mathrm{NIW}_n(j)} Y_{(\sigma,c)}\end{aligned}$$

$$= \sum_{n\ge 0}\sum_{\sigma\in\mathfrak{S}_n} R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} t^{\operatorname{des}\sigma} \sum_{j\ge 0} t^j \sum_{c\in \mathrm{NIW}_n(j)} Y_{(\sigma,c)}$$
$$= \sum_{n\ge 0}\sum_{\sigma\in\mathfrak{S}_n} R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} t^{\operatorname{des}\sigma} Y(\sigma;t). \quad \square$$

6. Specializations and q-Calculus

In Theorem 1.4, replace Y_j, Y'_j $(j\ge 0)$ by u. We get the identity

$$\sum_{w\in[0,r]^*} s^{\operatorname{dec} w} u^{\lambda w} = \frac{1-s}{(1-u)^{r+1}(1-us)^{-r} - s(1-u)}, \tag{6.1}$$

where λw denotes the length of w.

Now replace X_j $(j\ge 0)$ by us and the other variables by u in Theorem 1.2. We then recover the *classical* generating function for words by number of descents:

$$\sum_{w\in[0,r]^*} s^{\operatorname{des} w} u^{\lambda w} = \frac{1-s}{(1-u+us)^{r+1} - s}. \tag{6.2}$$

As was proved in our paper [9] (formula (1.15)), when multiplying (6.1) (and not (6.2)) by t^r and summing over $r\ge 0$ we get the generating function for the pair (exc, des) over the symmetric groups:

$$\sum_{r\ge 0} t^r \sum_{w\in[0,r]^*} s^{\operatorname{dec} w} u^{\lambda w} = \sum_{n\ge 0} \frac{u^n}{(1-t)^{n+1}} \sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{exc}\sigma} t^{\operatorname{des}\sigma}. \tag{6.3}$$

Now, recall the traditional notation of the q-ascending factorial

$$(a;q)_n = \begin{cases} 1, & \text{if } n=0;\\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n\ge 1.\end{cases}$$

Theorem 6.1. *The factorial generating function for the distributions of the vector* (fix, exc, des, maj) *over the symmetric groups* $\mathfrak{S}_n$ *is given by*

$$\sum_{n\ge 0} \frac{u^n}{(t;q)_{n+1}} \sum_{\sigma\in\mathfrak{S}_n} R^{\operatorname{fix}\sigma} s^{\operatorname{exc}\sigma} t^{\operatorname{des}\sigma} q^{\operatorname{maj}\sigma} = \sum_{r\ge 0} t^r \frac{1}{(uR;q)_{r+1}} \frac{(1-sq)\,(u;q)_r\,(usq;q)_r}{((u;q)_r - sq(usq;q)_r)}. \tag{6.4}$$

The theorem was proved in our previous paper [9] by means of the word-analog of the Kim–Zeng transformation [11] and the Gessel–Reutenauer standardization. Here, it is a simple consequence of Theorem 1.6 by applying the homomorphism ϕ generated by $\phi(Y_j) := uq^j$ $(j \geq 0)$ and $\phi(s) := sq$ to both sides of (1.18).

We proceed as follows. First, $\phi H_r(Y) = \prod_{0\leq j\leq r-1}(1-uq^j)^{-1} = 1/(u;q)_r$ and

$$\phi\frac{(1-s)H_{r+1}(RY)}{H_r(sY)-sH_r(Y)} = \frac{(1-sq)/(uR;q)_{r+1}}{1/(usq;q)_r - sq/(u;q)_r}$$
$$= \frac{(1-sq)(u;q)_r\,(usq;q)_r}{(uR;q)_{r+1}\,\big((u;q)_r - sq(usq;q)_r\big)}.$$

Then, for $(\sigma,c)\in\mathfrak{S}_n\times \mathrm{NIW}_n$

$$\phi Y_{(\sigma,c)} = u^n q^{\mathrm{tot}\,c+\mathrm{tot}\,z-\mathrm{exc}\,\sigma} = q^{\mathrm{maj}\,\sigma-\mathrm{exc}\,\sigma}u^n q^{\mathrm{tot}\,c};$$

$$\phi Y(\sigma;t) = q^{\mathrm{maj}\,\sigma-\mathrm{exc}\,\sigma}u^n\sum_{j\geq 0}t^j\sum_{c\in\mathrm{NIW}_n(j)}q^{\mathrm{tot}\,c} = q^{\mathrm{maj}\,\sigma-\mathrm{exc}\,\sigma}\frac{u^n}{(t;q)_{n+1}};$$

so that

$$\phi\big(R\,\mathrm{fix}^{\sigma}\, s^{\mathrm{exc}\,\sigma}t^{\mathrm{des}\,\sigma}Y(\sigma;t)\big) = R^{\mathrm{fix}\,\sigma}(sq)^{\mathrm{exc}\,\sigma}t^{\mathrm{des}\,\sigma}q^{\mathrm{maj}\,\sigma-\mathrm{exc}\,\sigma}\frac{u^n}{(t;q)_{n+1}}.$$

Hence, the image of identity (1.18) under ϕ gives back (6.4). ∎

There is still another proof, which we now describe. In identity (1.4) make the substitutions $X_j \leftarrow usq^j$, $Y_j \leftarrow uq^j$, $Z_j \leftarrow usq^j$, $T_j \leftarrow uq^j$, $Y'_j \leftarrow uRq^j$, $T'_j \leftarrow uRq^j$. The weight $\psi(w)$ becomes $s^{\mathrm{dec}\,w}R^{\mathrm{inrec}\,w}u^{\lambda w}q^{\mathrm{tot}\,w}$, and (1.4) yields the identity

$$\sum_{w\in[0,r]^*}s^{\mathrm{dec}\,w}R^{\mathrm{inrec}\,w}u^{\lambda w}q^{\mathrm{tot}\,w} = \frac{\dfrac{(us;q)_{r+1}}{(uR;q)_{r+1}}}{1-\displaystyle\sum_{0\leq l\leq r}\frac{(us;q)_l}{(u;q)_l}usq^l}. \tag{6.5}$$

Now, use the q-telescoping argument provided by Krattenthaler [12]:

$$\frac{(us;q)_l}{(u;q)_l}usq^l = \frac{sq}{1-sq}\Big(\frac{(us;q)_{l+1}}{(u;q)_l} - \frac{(us;q)_l}{(u;q)_{l-1}}\Big)\quad (1\leq l\leq r).$$

We obtain:

$$\sum_{w\in[0,r]^*}s^{\mathrm{dec}\,w}R^{\mathrm{inrec}\,w}u^{\lambda w}q^{\mathrm{tot}\,w} = \frac{1}{(uR;q)_{r+1}}\,\frac{(1-sq)\,(u;q)_r\,(usq;q)_r}{((u;q)_r - sq(usq;q)_r)}. \tag{6.6}$$

The summation can also be made over the symmetric groups by using the Gessel–Reutenauer standardization $w\mapsto(\sigma,c)$. This time only the following properties are needed:

$$\mathrm{dec}\,w = \mathrm{exc}\,\sigma;\quad \mathrm{tot}\,w = \mathrm{maj}\,\sigma + \mathrm{tot}\,c;\quad \mathrm{inrec}\,w = \mathrm{fix}\,\sigma.$$

Multiply (6.6) by t^r and sum over $r \geq 0$. We get:

$$\sum_{r\geq 0} t^r \sum_{n\geq 0} \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \operatorname{des}\sigma\leq r}} \sum_{c\in \mathrm{NIW}_n(r-\operatorname{des}\sigma)} s^{\operatorname{exc}\sigma} R^{\operatorname{fix}\sigma} u^n q^{\operatorname{maj}\sigma+\operatorname{tot} c}$$
$$= \sum_{r\geq 0} t^r \frac{1}{(uR;q)_{r+1}} \frac{(1-sq)\,(u;q)_r\,(usq;q)_r}{((u;q)_r - sq(usq;q)_r)}.$$

Following the same pattern as in the proof of Theorem 1.6 we again derive identity (6.4). ∎

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References

[1] Pierre Cartier, Dominique Foata. *Problèmes combinatoires de commutation et réarrangements*. Berlin, Springer-Verlag, 1969 (Lecture Notes in Math., 85), 88 pages. Also freely available on the *Sémin. Lothar. Combin.* website.

[2] Dominique Foata, Guo-Niu Han. Signed words and permutations, I: A fundamental transformation, *Proc. Amer. Math. Soc.*, vol. **135**, 2007, p. 31–40.

[3] ——, ——. Signed words and permutations, II; The Euler–Mahonian polynomials, *Electronic J. Combin.*, **11(2)**, #R22, 2005, 18 pages.

[4] ——, ——. Signed words and permutations, III; The MacMahon Verfahren, *Sémin. Lothar. Combin.*, **54**, [B25a], 2006, 20 pages.

[5] ——, ——. Signed words and permutations, IV; Fixed and pixed points, preprint 21 pages, *Israel J. Math.*, 2006, (to appear).

[6] ——, ——. Signed words and permutations, V; A sextuple distribution, preprint 24 pages, *Ramanujan J.*, 2007, (to appear).

[7] ——, ——. Fix-Mahonian Calculus, I: Two transformations, preprint 16 pages, 2006, *Europ. J. Combin.* (to appear).

[8] ——, ——. Fix-Mahonian Calculus, II: Further statistics, preprint 13 pages, 2006, *J. Combinatorial Theory, Ser. A* (to appear).

[9] ——, ——. Fix-Mahonian Calculus, III: A quadruple distribution, preprint 26 p., 2007, *Monatshefte für Math.* (to appear).

[10] Ira Gessel, Christophe Reutenauer. Counting permutations with given cycle structure and descent set, *J. Combin. Theory Ser. A*, vol. **64**, 1993, p. 189–215.

[11] Dongsu Kim, Jiang Zeng. A new decomposition of derangements, *J. Combin. Theory Ser. A*, vol. **96**, 2001, p. 192–198.

[12] Christian Krattenthaler. Personal communication, 2007.

[13] M. Lothaire. *Combinatorics on Words*. Addison-Wesley, London 1983 (Encyclopedia of Math. and its Appl., **17**).

[14] Percy Alexander MacMahon. *Combinatory Analysis*, vol. 1 and 2. Cambridge, Cambridge Univ. Press, 1915. (Reprinted by Chelsea, New York, 1955).

[15] John Shareshian, Michelle Wachs. q-Eulerian polynomials, excedance number and major index, *Electronic Research Announcements of the Amer. Math. Soc.*, vol. **13**, 2007, p. 33-45. See also the proceedings of the FPSAC 2007, Tianjin.

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