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# ON EVALUATIONS OF INFINITE DOUBLE SUMS AND TORNHEIM'S DOUBLE SERIES

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ABSTRACT. We consider generalizations of a sum, which was recently analyzed by Pemantle and Schneider using the computer software `Sigma`, and later also by Panholzer and Prodinger. Our generalizations include Tornheim's double series as a special case. We also consider alternating analogs of Tornheim's series. For Tornheim's double series and its alternating counterparts we provide short proofs for evaluation formulas, which recently appeared in the literature. We introduce finite Tornheim double sums and alternating analogs, and provide relations to finite multiple zeta functions, similarly to the infinite case. Besides, we discuss the evaluation of another double series, which also generalizes Tornheim's double series.

## 1. INTRODUCTION

Using Carsten Schneider's software `Sigma` the following result was obtained by Pemantle and Schneider [6]:

$$\sum_{j,k=1}^{\infty} \frac{H_j(H_{k+1}-1)}{jk(k+1)(j+k)} = -\zeta(2) - 2\zeta(3) + 4\zeta(2)\zeta(4) + 2\zeta(5), \tag{1}$$

where $\zeta(a) = \sum_{k\geq 1} 1/k^a$ denotes the Riemann zeta function, and $H_k = \sum_{l=1}^{k} 1/k$ the $k$-th harmonic number. Later a "computer-free" proof of this result was given by Panholzer and Prodinger [5]. This remarkable evaluation formula is our motivation to have a closer look at sums of the type

$$S := \sum_{j,k=1}^{\infty} \frac{H_k^{(u)} H_j^{(v)}}{j^r k^s (j+k)^t}, \tag{2}$$

where $S = S(r,s,t,u,v)$ and $H_k^{(u)} = \sum_{l=1}^{k} 1/l^u$ denotes the $k$-th harmonic number of order $u$, with $u \in \mathbb{N}$. For $u = 1$ we use the standard notation $H_k = H_k^{(1)}$. We want to mention that $H_k^{(0)} = k$. Note that the sum stated in (1) is not of type $S$ in the strict sense. However, one can treat modified sums of the forms

$$\sum_{j,k=1}^{\infty} \frac{H_{k+1}^{(u)} H_j^{(v)}}{j^r (k+1)^s (j+k)^t}$$

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in an analogous manner to $S$. We are interested in evaluating $S$ in terms of simpler Euler sums, as defined in (8). A further motivation for our study of series $S$ is Tornheim's double series $T = T(r,s,t)$ (sometimes also called Witten's zeta function), defined by

$$T := \sum_{j,k\geq 1} \frac{1}{j^r k^s (j+k)^t}, \tag{3}$$

which is just a special case of the $S$ series, namely $T(r,s,t) = S(r+1,s+1,t,0,0)$. Alternating series, analogous to Tornheim's double series $T(r,s,t)$ where introduced by Subbarao and Sitaramachandrarao in [7]; the series $A = A(r,s,t)$ and $R = R(r,s,t)$ are defined by

$$A := \sum_{j,k\geq 1} \frac{(-1)^{j+k}}{j^r k^s (j+k)^t}, \quad R := \sum_{j,k\geq 1} \frac{(-1)^{k}}{j^r k^s (j+k)^t}. \tag{4}$$

We will (re)address the question of evaluating Tornheim's double series $T$ and its alternating counterparts $A$ and $R$ in terms of zeta functions whenever $r+s+t$ is odd. This problem was considered by Huard, Williams and Zhang [3] for the series $T$ and recently by Tsumura [9], [10] for the alternating series $A$ and $R$. Concerning the series $R$ this extends the result of Tsumura [10].

We have to mention that quite recently, parallel and independently of this study, a slightly more general study of the sums $T$, $A$ and $R$, namely of the so-called $q$-analogs of $T$, $A$, $R$, have been carried out by Zhou, Cai and Bradley [12], using a similar approach relying on partial fraction decomposition. Hence, we will be very brief when discussing these sums and refer the interested reader to [12] for a more general study.

We will introduce three finite analogs of series $T$, $A$, $R$, namely the sums $T_N = T_N(r,s,t)$, $A_N = A_N(r,s,t)$ and $R_N = R_N(r,s,t)$, defined for $N \geq 1$ by

$$T_N := \sum_{k=1}^{N} \sum_{l=1}^{N-k} \frac{1}{k^r l^s (l+k)^t}, \quad A_N := \sum_{k=1}^{N} \sum_{l=1}^{N-k} \frac{(-1)^l}{k^r l^s (l+k)^t}, \tag{5}$$

and

$$R_N := \sum_{k=1}^{N} \sum_{l=1}^{N-k} \frac{(-1)^k}{k^r l^s (l+k)^t}. \tag{6}$$

We will prove that the sums $T_N$, $A_N$, $R_N$ satisfy finite versions of the relations for their infinite counterparts $T$, $A$, $R$. We also consider series $V = V(r,s,t,u)$, defined by

$$V = \sum_{j,k\geq 1} \frac{H_{j+k}^{(u)}}{j^r k^s (j+k)^t}, \tag{7}$$

which is also a generalization of Tornheim's double series $T$. To the best of our knowledge, neither the series $V$, $S$, nor the finite Tornheim series $T_N$, $A_N$ and $R_n$ have been previously treated in the literature.

It will turn out that we can express the sum $S$ in terms of (Euler) sums of the kind

$$\mathcal{T}(q;a_1,a_2) := \sum_{n\geq 1} \frac{H_{n-1}^{(a_1)} H_{n-1}^{(a_2)}}{n^q}, \quad \mathcal{M}(a,b,c;d) := \sum_{k=1}^{\infty} \frac{H_{k-1}^{(d)}}{k^a} \sum_{j=1}^{k-1} \frac{1}{j^b} \sum_{l=1}^{j-1} \frac{1}{l^c}, \tag{8}$$

and multiple zeta functions $\zeta(a_1, \ldots, a_l)$ defined by

$$\zeta(a_1, \ldots, a_l) := \sum_{n_1 > n_2 > \cdots > n_l \geq 1} \frac{1}{n_1^{a_1} n_2^{a_2} \ldots n_l^{a_l}}. \tag{9}$$

Furthermore we will state short (and independent) proofs for both the evaluation of Tornheim's double series $T$ and its alternating counterpart $A$, which appeared in [3], [9], and [12] by using earlier results concerning properties of multiple zeta functions of Borwein, Borwein and Girgensohn [1], and Flajolet and Salvy [4]. We show that a similar result is true for the series $R$, extending the result in [10].

The basic idea is to evaluate the series $T$, $A$ and $R$ in terms of (alternating) multiple zeta functions of at most two arguments, which is done by partial fraction decomposition. The results of [1] and [4] then provide the required evaluations of the arising multiple zeta functions of two arguments under certain conditions. For an $q$-series extension of this idea and the stated result we refer the reader to [12].

Concerning the series $V$, we will prove that if $r + s + t + u$ is even, series $V$ can be evaluated into multiple zeta functions of at most two arguments, using an earlier result of J. M. Borwein and Girgensohn [2].

It remains open to evaluate sums like $T$, $A$ and $R$ whenever $r + s + t$ is even for general $r, s, t$. For partial results in this direction we refer the reader to Tsumura [8].

Note that we mostly drop the dependence of the sums $R, S, T, A, V$ on $r, s, t, \ldots$, for the sake of simplicity.

In the first part of the next section we discuss briefly Tornheim's double series $T$ and related series $A$, $R$. The second part is devoted to the finite Tornheim sums $T_N$, $A_N$, $R_N$, and the sum $V$. Section 3 is then devoted to the study of the sum $S$ in generality.

## 2. TORNHEIM'S DOUBLE SERIES

Huard, Williams and Zhang [3] gave an explicit evaluation of $T(r, s, N - r - s)$ into rational linear combinations of the products $\zeta(2j)\zeta(N - 2j), 0 \leq j \leq (N - 3)/2$, when $N$ is odd, $N \geq 3$ and $r, s$ are nonnegative integers satisfying $1 \leq r + s \leq N$, $r \leq N - 1$ and $s \leq N - 2$. We show that their result is an immediate consequence of a result D. Borwein, J. M. Borwein and Girgensohn [1], which states that for odd weight $w = a_1 + a_2$ the multiple zeta function of two arguments $\zeta(a_1, a_2)$ can be evaluated into single valued zeta functions, and elementary observations.

We need the following well-known result.

**Lemma 1** (Partial fraction decomposition).

$$\frac{1}{m^r (m+n)^t} = \sum_{k=r}^{r+t-1} \binom{k-1}{r-1} \frac{(-1)^r}{n^k (m+n)^{r+t-k}} + \sum_{i=t}^{r+t-1} \binom{i-1}{t-1} \frac{(-1)^{i+t}}{n^i m^{r+t-i}}, \tag{10}$$

*for* $r, t \geq 1$.

The proof of the decomposition can easily be carried out by induction and is therefore omitted. Alternatively, one can use an *Ansatz* of the form

$$\frac{1}{m^r(m+n)^t} = \sum_{k=1}^{t} \frac{\alpha_k}{(n+m)^k} + \sum_{i=1}^{r} \frac{\beta_i}{m^i};$$

Multiplication with $(m+n)^t$ and subsequent differentiation(s) with respect to $m$ and evaluations at $m=-n$ then gives the numbers $\alpha_k$ and subsequently the $\beta_i$.

We will simplify $T = T(r,s,t)$ as follows. First we sum up $\frac{1}{j^r k^s (j+k)^t}$ over $i = j+k \geq 2$:

$$T = \sum_{i\geq 2} \frac{1}{i^t} \sum_{j+k=i} \frac{1}{j^r k^s} = \sum_{i\geq 1} \frac{1}{i^t} \sum_{k=1}^{i-1} \frac{(-1)^s}{(-k)^s (i-k)^r}. \tag{11}$$

Now we apply the partial fraction decomposition (10), change the order of summation and immediately obtain the following result:

$$T = \sum_{l=s}^{s+r-1} \binom{l-1}{s-1} \zeta(t+l, s+r-l) + \sum_{j=r}^{s+r-1} \binom{j-1}{r-1} \zeta(t+j, s+r-j). \tag{12}$$

Hence, we have obtained the following result.

**Theorem 1** (Huard, Williams, Zhang 1996)**.** *Whenever $w = r+s+t$ is odd, for $r,s,t \in \mathbb{N}$, Tornheim's double series $T(r,s,t)$ can be explicitly evaluated into single zeta functions.*

We can easily get explicit formulas simply by substituting the explicit expressions for $\zeta(a,b)$ found in [1] into (12) .

2.1. **Alternating series $A$.** Tsumura [9] gave in 2004 an evaluation formula of $A = A(r,s,t)$,

$$A = \sum_{m,n\geq 1} \frac{(-1)^{m+n}}{m^r n^s (m+n)^t}$$

into single valued zeta functions whenever $s+r+t$ is odd. We show how to derive such evaluations from a result of Flajolet and Salvy.

We introduce several alternating counterparts of the (multiple) zeta function, $\zeta(\bar{a}_1)$, $\zeta(\bar{a}_1, a_2)$, $\zeta(\bar{a}_1, a_2)$, and $\zeta(\bar{a}_1, \bar{a}_2)$

$$\zeta(\bar{a}_1, a_2) := \sum_{n_1 > n_2 \geq 1} \frac{(-1)^{n_1 - 1}}{n_1^{a_1} n_2^{a_2}}, \quad \zeta(\bar{a}_1) := \sum_{n\geq 1} \frac{(-1)^{n-1}}{n^{a_1}} = (1-2^{1-a_1})\zeta(a_1), \tag{13}$$

$$\zeta(\bar{a}_1, \bar{a}_2) := \sum_{n_1 > n_2 \geq 1} \frac{(-1)^{n_1 - 1}(-1)^{n_2 - 1}}{n_1^{a_1} n_2^{a_2}}, \quad \zeta(\bar{a}_1, \bar{a}_2) := \sum_{n_1 > n_2 \geq 1} \frac{(-1)^{n_2 - 1}}{n_1^{a_1} n_2^{a_2}}, \tag{14}$$

$$\tag{15}$$

and $\zeta(\bar{1}) = \log 2$.

Flajolet and Salvy have proved that for odd weight $w = a_1 + a_2$ the alternating multiple zeta function of two arguments $\zeta(\bar{a}_1, a_2)$ can be decomposed into (single valued) alternating zeta functions. As for the Tornheim's series, we will easily simplify $A = A(r, s, t)$ into a shape, where we the result of Flajolet and Salvy can be applied.

We sum up over $i = m + n$ to obtain

$$A = -\sum_{l=s}^{r+s-1} \binom{l-1}{s-1} \zeta(\overline{t+l}, s+r-l) - \sum_{j=r}^{s+r-1} \binom{j-1}{r-1} \zeta(\overline{t+j}, s+r-j). \qquad (16)$$

Hence, we have proven the following result.

**Theorem 2** (Tsumura 2004)**.** *Whenever $w = r + s + t$ is odd, for $r, s, t \in \mathbb{N}$, the double series $A(r, s, t)$ can be explicitly evaluated into single (alternating) zeta functions.*

Again, one can easily obtain explicit formulas applying the results of [4] to (16).

### 2.2. **Alternating series $R$.** Tsumura provided the following result in [10].

**Theorem 3** (Tsumura 2003)**.** *For $k \in \mathbb{N}$, the double series $R(2k+1, 2k+1, 2k+1)$ can be explicitly evaluated into single (alternating) zeta functions.*

We extend this result by deriving an evaluation formula for $R$

$$R = \sum_{j,k \geq 1} \frac{(-1)^k}{j^r k^s (j+k)^t}$$

into single valued (alternating) zeta functions, whenever $s + r + t$ is odd.

**Theorem 4.** *Whenever $w = r + s + t$ is odd, for $r, s, t \in \mathbb{N}$, the double series $R(r, s, t)$ can be explicitly evaluated into single (alternating) zeta functions.*

In order to do so we use again partial fraction decomposition, and a result of Flajolet and Salvy [4], which states that for odd weight $w = a_1 + a_2$ the alternating multiple zeta functions of two arguments $\hat{\zeta}(a_1, a_2)$ can be explicitly decomposed into (alternating) zeta functions. As before we easily obtain Summation of $\frac{(-1)^k}{j^r k^s (j+k)^t}$ over $i = k + j \geq 2$ gives

$$R = \sum_{i \geq 2} \frac{1}{i^t} \sum_{k+j=i} \frac{(-1)^k}{j^r k^s} = \sum_{i \geq 1} \frac{1}{i^t} \sum_{k=1}^{i-1} \frac{(-1)^{s+k}}{(-k)^s (i-k)^r}. \qquad (17)$$

Now we apply the partial fraction decomposition (10) and immediately get

$$R = \sum_{l=s}^{s+r-1} \binom{l-1}{s-1} \zeta(\overline{t+l}, \overline{s+r-l}) - \sum_{j=r}^{s+r-1} \binom{j-1}{r-1} \zeta(j+t, \overline{s+r-j}). \qquad (18)$$

By a combination of the results of [1] and [4] this proves the stated result.

2.3. **Finite (alternating) Tornheim sums.** It is our goal to provide formulas in the spirit of (12), (16) and (18) for finite (alternating) Tornheim sums $T_N$, $A_N$ and $R_N$.
Before we state our results, we have to introduce finite (alternating) zeta functions $\zeta_N(a)$, $\zeta_N(\bar{a})$, $\zeta_N(\bar{a}_1, a_2)$, $\zeta_N(\bar{a}_1, a_2)$, and $\zeta_N(\bar{a}_1, \bar{a}_2)$, defined as follows:

$$\zeta_N(a) := \sum_{k=1}^{N} \frac{1}{k^a} = H_N^{(a)}, \quad \zeta_N(\bar{a}) := \sum_{k=1}^{N} \frac{(-1)^{N-1}}{k^a},$$
$$\zeta_N(a_1, a_2) := \sum_{N \geq n_1 > n_2 \geq 1} \frac{1}{n_1^{a_1} n_2^{a_2}}, \quad \zeta_N(\bar{a}_1, a_2) := \sum_{N \geq n_1 > n_2 \geq 1} \frac{(-1)^{n_1 - 1}}{n_1^{a_1} n_2^{a_2}},$$
$$\zeta_N(\bar{a}_1, \bar{a}_2) := \sum_{N \geq n_1 > n_2 \geq 1} \frac{(-1)^{n_1 - 1}(-1)^{n_2 - 1}}{n_1^{a_1} n_2^{a_2}}, \quad \zeta_N(\bar{a}_1, \bar{a}_2) := \sum_{N \geq n_1 > n_2 \geq 1} \frac{(-1)^{n_2 - 1}}{n_1^{a_1} n_2^{a_2}};$$

note that $\zeta_N(a) = H_N^{(a)}$ denotes the $N$-th harmonic number of order $a$; and that, assuming $a, a_1 > 1$, the finite zeta functions converge to their infinite counterparts.

**Theorem 5.** *For $N, r, s \geq 1$ and $t \geq 0$ we have the following results:*

$$T_N = \sum_{l=s}^{s+r-1} \binom{l-1}{s-1} \zeta_N(t+l, s+r-l) + \sum_{j=r}^{s+r-1} \binom{j-1}{r-1} \zeta_N(t+j, s+r-j)$$
$$A_N = -\sum_{l=s}^{r+s-1} \binom{l-1}{s-1} \zeta_N(\overline{t+l}, s+r-l) - \sum_{j=r}^{s+r-1} \binom{j-1}{r-1} \zeta_N(\overline{t+j}, s+r-j)$$
$$R_N = \sum_{l=s}^{s+r-1} \binom{l-1}{s-1} \zeta_N(\overline{t+l}, \overline{s+r-l}) - \sum_{j=r}^{s+r-1} \binom{j-1}{r-1} \zeta_N(j+t, \overline{s+r-j}).$$

*Taking the limit with respect to $N$, assuming convergence, the finite Tornheim sums converge to their infinite counterparts $T$, $A$ and $R$.*

**Remark 1.** We expect that a $q$-analog of our result is also true. The author is currently investigating into this matter.

*Proof.* We start with the proof for the sum $T_N$. By taking differences with respect to $N$ we get for $N \geq 2$

$$T_N - T_{N-1} = \sum_{k=1}^{N-1} \frac{1}{(N-k)^s k^r N^t}.$$

Consequently, we obtain by iteration the result

$$T_N = \sum_{j=1}^{N} \sum_{k=1}^{j-1} \frac{1}{k^r (j-k)^s j^t}$$

The stated result follows now by applying partial fraction decomposition and interchanging of summation. The other sums can be treated along the same lines. $\square$

2.4. **A generalization.** The series $V = V(r,s,t,u)$, defined by

$$V = \sum_{j,k=1}^{\infty} \frac{H_{j+k}^{(u)}}{j^r k^s (j+k)^t}, \tag{19}$$

is an obvious generalization of Tornheim's double series, due to the relation $T(r,s,t) = V(r,s,t+1,0)$. It is a natural question to ask for evaluations of $V$ into (multiple) zeta functions. Of course, we do not expect that $V$ evaluates into single valued zeta functions for most values $r,s,t,u$. Indeed, we will prove the following result.

**Theorem 6.** *Whenever $w = r+s+t+u$ is even, for $r,s,t,w \in \mathbb{N}$, the series $V$ can be explicitly evaluated into zeta functions of at most two arguments.*

*Proof.* We proceed as for Tornheim's double series by summing up over $i = m+n \geq 2$:

$$V = \sum_{i\geq 1} \frac{H_i^{(u)}}{i^t} \sum_{k=1}^{i-1} \frac{(-1)^s}{(-k)^s (i-k)^r}. \tag{20}$$

After partial fraction decomposition (10) we obtain.

$$V = \sum_{i\geq 1} \frac{1}{i^t} \Bigg[ \sum_{l=1}^{r} \frac{H_i^{(u)} H_{i-1}^{(l)} \binom{s+r-l-1}{r-l}}{i^{s+r-l}} + \sum_{j=r}^{s+r-1} \frac{H_i^{(u)} H_{i-1}^{(s+r-j)} \binom{j-1}{r-1}}{i^j} \Bigg].$$

Next we express $V$ in terms of the sums $\mathcal{T}(q;a_1,a_2)$, as defined in (8), and obtain the following result, which generalizes the corresponding result for Tornheim's double series.

**Proposition 1.**

$$V = \sum_{l=1}^{r} \binom{s+r-l-1}{r-l} \big(\mathcal{T}(s+r+t-l;u,l) + \zeta(s+r+t+u-l,l)\big)$$
$$+ \sum_{j=r}^{s+r-1} \binom{j-1}{r-1} \big(\mathcal{T}(j+t;u,s+r-j) + \zeta(j+t+u,s+r-j)\big).$$

**Remark 2.** Similar relations for alternating analogs of the series $V$ are easily obtained. We leave the details to the interested reader.

We know by Lemma 3 that we can express the sums $\mathcal{T}(q;a_1,a_2)$ in terms of multiple zeta functions with at most 3 arguments. The following result will complete our proof.

**Lemma 2** (J. M. Borwein and Girgensohn, 1996)**.** *If $w = a_1+a_2+a_3$ is even or less than or equal to 10, then $\zeta(a_1,a_2,a_3)$ can be expressed as a rational linear combination of products of multiple zeta functions of at most two arguments.*

By application of this result we can decompose $\mathcal{T}(s+r+t-l;u,l)$ and $\mathcal{T}(j+t;u,s+r-j)$ into multiple zeta function of at most two arguments whenever $w = s+r+t-l+u+l = s+r+t+u$ is even, which finishes the proof. $\square$

## 3. Decomposition of $S$ into Euler sums and multiple zeta functions

In order to evaluate the sum $S$, we use the ideas employed in [5]. The main idea is to split the sum $S$ and apply partial fraction decomposition. We proceed by splitting $S$ and interchanging summation:

$$S = \sum_{j,k\geq 1} \frac{H_j^{(u)} H_k^{(v)}}{j^r k^s (j+k)^t} = \sum_{k\geq 1} \frac{H_k^{(v)}}{k^s} \sum_{j\geq 1} \frac{\sum_{l=1}^{j} \frac{1}{l^u}}{j^r (j+k)^t} = \sum_{k\geq 1} \frac{H_k^{(v)}}{k^s} \sum_{l\geq 1} \frac{1}{l^u} \sum_{j\geq l} \frac{1}{j^r (j+k)^t}. \tag{21}$$

Next we simplify the inner sum

$$\sum_{l\geq 1} \frac{1}{l^u} \sum_{j\geq l} \frac{1}{j^r (j+k)^t}$$

using partial fraction decomposition as given in Lemma 1.

$$\sum_{l\geq 1} \frac{1}{l^u} \sum_{j\geq l} \frac{1}{j^r (j+k)^t} = \sum_{l\geq 1} \frac{1}{l^u} \sum_{j\geq l} \left( \sum_{p=1}^{t} \frac{(-1)^r \binom{r+t-p-1}{t-p}}{k^{r+t-p} (j+k)^p} + \sum_{i=t}^{r+t-1} \frac{(-1)^{i+t} \binom{i-1}{t-1}}{k^i j^{r+t-i}} \right). \tag{22}$$

We isolate the terms $p = 1$ from the first sum and $i = r+t-1$ from the second sum arising due to the the partial fraction decomposition. These terms telescope and we obtain after summing up over $j \geq l$

$$\begin{aligned} \sum_{l\geq 1} \frac{1}{l^u} \sum_{j\geq l} \frac{1}{j^r (j+k)^t} = \sum_{l\geq 1} \frac{1}{l^u} \Bigg[ &\sum_{p=2}^{t} \frac{(-1)^r \binom{r+t-p-1}{t-p}}{k^{r+t-p}} \left( \zeta(p) - H_{l-1}^{(p)} - \sum_{j=0}^{k-1} \frac{1}{(l+j)^p} \right) \\ &+ \sum_{i=t}^{r+t-2} \frac{(-1)^{i+t} \binom{i-1}{t-1}}{k^i} \left( \zeta(r+t-i) - H_{l-1}^{(r+t-i)} \right) \\ &+ (-1)^{r+1} \frac{\binom{r+t-2}{t-1}}{k^{r+t-1}} \sum_{j=0}^{k-1} \frac{1}{l+j} \Bigg]. \end{aligned} \tag{23}$$

In order to carry out summation with respect to $l$ one has to be careful whether $u = 1$ or $u > 1$ (or due to symmetry $v = 1$ or $v > 1$). In the case $u > 1$ the interchange of summations is certainly justified, hence we restrict ourselves to the case $u > 1$ or $v > 1$. We use the partial fraction decomposition of

$$\frac{1}{l^u} \frac{1}{(l+j)^p} = \sum_{m=1}^{p} \frac{(-1)^u \binom{r+p-m-1}{p-m}}{j^{u+p-m} (l+j)^m} + \sum_{n=p}^{u+p-1} \frac{(-1)^{n+p} \binom{n-1}{p-1}}{j^n l^{u+p-n}}, \tag{24}$$

which leads to the following:

$$\sum_{l\geq 1} \frac{1}{l^u} \sum_{j=0}^{k-1} \frac{1}{(l+j)^p} =$$

$$
\begin{aligned}
&= \zeta(u+p) + \sum_{j=1}^{k-1} \sum_{l\geq 1} \left( \sum_{m=1}^{p} \frac{(-1)^u \binom{u+p-m-1}{p-m}}{j^{u+p-m}(l+j)^m} + \sum_{n=p}^{u+p-1} \frac{(-1)^{n+p}\binom{n-1}{p-1}}{j^n l^{u+p-n}} \right) \\
&= \zeta(u+p) + \sum_{j=1}^{k-1} \Bigg[ \sum_{m=2}^{p} \frac{(-1)^u \binom{u+p-m-1}{p-m}}{j^{u+p-m}} \Big( \zeta(m) - H_j^{(m)} \Big) \\
&\quad + \sum_{n=p}^{u+p-2} \frac{(-1)^{n+p}\binom{n-1}{p-1}}{j^n} \zeta(u+p-n) + \frac{(-1)^{u+1}\binom{u+p-2}{p-1} H_j}{j^{u+p-1}} \Bigg]. \qquad (25)
\end{aligned}
$$

By combining (22), (25) and carrying out the summation with respect to $k$ we obtain the result.

**Theorem 7.** *For $u > 1$ or $v > 1$, and $s, r, t \geq 1$ the sum $S := \sum_{j,k=1}^{\infty} \frac{H_k^{(u)} H_j^{(v)}}{j^r k^s (j+k)^t}$ can be expressed in terms of Euler sums and multiple zeta functions as follows:*

$$
\begin{aligned}
S &= \sum_{p=2}^{t} \binom{r+t-p-1}{t-p} (-1)^r \\
&\times \Bigg[ \Big( \zeta(v+s+r+t-p) + \zeta(s+r+t-p, v) \Big) \Big( \zeta(u)\zeta(p) - \zeta(u,p) - \zeta(u+p) \Big) \\
&+ \sum_{m=2}^{p} (-1)^u \binom{u+p-m-1}{p-m} \Big( -\zeta(s+v+r+t-p, u+p-m)\zeta(m) \\
&- \zeta(m)\mathcal{T}(s+r+t-p; v, u+p-m) + \mathcal{M}(s+r+t-p, u+p-m, m; v) \\
&+ \mathcal{T}(s+r+t-p; u+p, v) + \zeta(s+v+r+t-p, u+p-m, m) \\
&+ \zeta(s+v+r+t-p, u+p) \Big) \\
&- \sum_{n=p}^{u+p-2} \binom{n-1}{p-1} (-1)^{n+p} \Big( \zeta(s+v+r+t-p, n) + \mathcal{T}(s+r+t-p; v, n) \Big) \\
&\times \zeta(u+p-n) + (-1)^{u+1} \binom{u+p-2}{p-1} \Big( \zeta(s+v+r+t-p, u+p) \\
&+ \zeta(s+v+r+t-p, u+p-1, 1) + \mathcal{T}(s+r+t-p; v, u+p) \\
&+ \mathcal{M}(s+r+t-p, u+p-1, 1; v) \Big) \Bigg] \\
&+ \sum_{i=t}^{r+t-2} (-1)^{i+t} \binom{i-1}{t-1} \Big[ \zeta(s+v+i)\zeta(u)\zeta(r+t-i) \\
&+ \zeta(s+i, v)\zeta(u)\zeta(r+t-i) - \zeta(u, r+t-i) \Big]
\end{aligned}
$$

$$
\begin{aligned}
&+(-1)^{r+1}\binom{r+t-2}{t-1}\Bigg[\Big(\zeta(s+v+r+t-1)+\zeta(s+r+t-1,v)\Big)\zeta(u+1)\\
&+\sum_{n=1}^{u-1}(-1)^{n+1}\Big(\zeta(s+v+r+t-1,n)+\mathcal{T}(s+r+t-1;v,n)\Big)\zeta(u+1-n)\\
&+\ (-1)^{u+1}\Big(\zeta(s+v+r+t-1,u+1)+\zeta(s+v+r+t-1,u,1)\\
&+\ \mathcal{T}(s+r+t-1;v,u+1)+\mathcal{M}(s+r+t-1,u,1;v)\Big)\Bigg],
\end{aligned}
\tag{26}
$$

*with $\mathcal{T}(q;a_1,a_2)$, $\mathcal{M}(a,b,c;d)$ and $\zeta(a_1,\dots,a_l)$ as defined in* (8)*,* (9)*.*

Obviously this theorem is *very* involved, therefore we discuss a special case and relations between the sums $\mathcal{T}(q;a_1,a_2)$ and $\mathcal{M}(a,b,c;d)$.

3.1. **Case $t=1$.** When $t=1$ we can give a much simpler evaluation of the sum $S$. This is due to the simplifications in terms of the partial fractions. Nevertheless the derivation is completely similar to our earlier result.

**Proposition 2.** *The sum $S'=\sum_{j,k=1}^{\infty}\frac{H_j^{(u)}H_k^{(v)}}{j^r k^s (j+k)}$ can be expressed in terms of Euler sums and multiple zeta functions as follows:*

$$
\begin{aligned}
S'=&\sum_{i=1}^{r-1}(-1)^{i+1}\Big(\zeta(s+i+v)+\zeta(s+i,v)\Big)\Big(\zeta(u)\zeta(r+1-i)-\zeta(u,r+1-i)\Big)\\
&+(-1)^{r+1}\Big(\zeta(s+r+v)+\zeta(s+r,v)\Big)\zeta(u+1)\\
&+\sum_{i=1}^{u-1}(-1)^{i+r}\zeta(u+1-i)\Big(\zeta(s+v+r,i)+\mathcal{T}(s+r;v,i)\Big)\\
&+(-1)^{u+r}\Big(\zeta(s+v+r,u+1)+\zeta(s+v+r,u,1)\\
&+\ \mathcal{T}(s+r;v,u+1)+\mathcal{M}(s+r,u,1;v)\Big),
\end{aligned}
\tag{27}
$$

*which also holds if $0\le u,v\le 1$ with $u=1$ or $v=1$. The terms $\zeta(1)$ and $\zeta(1,l)$ should be interpreted as zero whenever occurring.*

3.2. **Some relations for sums $\mathcal{T}$ and $\mathcal{M}$.** We want to state some relations between the functions $\mathcal{T}(q;a_1,a_2)$, $\mathcal{M}(a,b,c;d)$ and multiple zeta functions. Such relations are convenient for simplifying the sum $S'$. Let

$$
\mathcal{T}(q;a_1,a_2):=\sum_{n=1}^{N}\frac{H_{n-1}^{(a_1)}H_{n-1}^{(a_2)}}{n^q},\quad \mathcal{M}_N(a,b,c;d):=\sum_{k=1}^{N}\frac{H_{k-1}^{(d)}}{k^a}\sum_{j=1}^{k-1}\frac{1}{j^b}\sum_{l=1}^{j-1}\frac{1}{l^c},
\tag{28}
$$

denote the finite counterparts of the sums $\mathcal{T}$ and $\mathcal{M}$, where the subscript $N$ signifies that the outermost sum stops at $N$. We define the same way the finite multiple zeta function

$$\zeta_N(a_1,\dots,a_l) := \sum_{N\geq n_1>n_2>\cdots>n_l\geq 1} \frac{1}{n_1^{a_1}n_2^{a_2}\dots n_l^{a_l}}. \tag{29}$$

The case of one or two arguments has already been used in the case of finite Tornheim series. At first we note that the following two lemmas relate $\mathcal{T}(q;a_1,a_2)$, $\mathcal{M}(a,b,c;d)$ with multiple zeta functions.

**Lemma 3.**

$$\mathcal{T}_N(q;a_1,a_2) = \zeta_N(q,a_1,a_2) + \zeta_N(q,a_2,a_1) + \zeta_N(q,a_1+a_2). \tag{30}$$

*Proof.*

$$\begin{aligned}\mathcal{T}_N(q;a_1,a_2) &= \sum_{n=1}^{N} \frac{H_{n-1}^{(a_1)}H_{n-1}^{(a_2)}}{n^q} = \sum_{n=1}^{N}\frac{1}{n^q}\sum_{k=1}^{n-1}\frac{1}{k^{a_1}}\Big(\sum_{l=1}^{k-1}\frac{1}{l^{a_2}} + \frac{1}{k^{a_2}} + \sum_{l=k+1}^{n-1}\frac{1}{l^{a_2}}\Big)\\ &= \zeta_N(q,a_1,a_2) + \zeta_N(q,a_1+a_2) + \sum_{n=1}^{N}\frac{1}{n^q}\sum_{l=2}^{n-1}\frac{1}{l^{a_2}}\sum_{k=1}^{l-1}\frac{1}{k^{a_1}}\\ &= \zeta_N(q,a_1,a_2) + \zeta_N(q,a_2,a_1) + \zeta_N(q,a_1+a_2).\end{aligned}$$

□

**Lemma 4.**

$$\mathcal{M}_N(a,b,c;d) = \zeta_N(d)\zeta_N(a,b,c) - \zeta_N(d,a,b,c) - \zeta_N(a+d,b,c). \tag{31}$$

*Proof.*

$$\begin{aligned}\mathcal{M}_N(a,b,c;d) &= \sum_{k=1}^{N}\frac{H_k^{(d)}}{k^a}\sum_{j=1}^{k-1}\frac{1}{j^b}\sum_{l=1}^{j-1}\frac{1}{l^c} - \zeta_N(a+d,b,c) = \zeta_N(d)\zeta_N(a,b,c)\\ &\quad - \sum_{k=1}^{N-1}\frac{1}{(k+1)^d}\sum_{i=1}^{k}\frac{1}{i^a}\sum_{j=1}^{i-1}\frac{1}{j^b}\sum_{l=1}^{j-1}\frac{1}{l^c} - \zeta_N(a+d,b,c)\\ &= \zeta_N(d)\zeta_N(a,b,c) - \zeta_N(d,a,b,c) - \zeta_N(a+d,b,c).\end{aligned} \tag{32}$$

□

Note that if $d=0$, then $\mathcal{M}(a,b,c;0) = \zeta(a-1,b,c) - \zeta(a,b,c)$.

The next Lemma provides some permutation formulæ similar to those appearing in [2]. These formulæ can be deduced by basic computation similar to the prove of Theorem 2 of [2].

**Lemma 5.**

$$\begin{aligned}(i)\ \mathcal{M}_N(a,b,c;d) &= \mathcal{T}_N(a;b,c,d) - \mathcal{T}_N(a;b+c,d) - \mathcal{M}_N(a,c,b;d),\\ (ii)\ \mathcal{M}_N(a,b,c;d) &= \zeta_N(b,c)\zeta_N(a,d) - \mathcal{T}_N(a+b;c,d) - \mathcal{M}_N(b,a,d;c),\\ (iii)\ \mathcal{M}_N(a,b,c;d) &= \zeta_N(b)\mathcal{T}_N(a;c,d) - \zeta_N(a,d)\zeta_N(c,b) + \mathcal{T}_N(a+c;b,d1)\end{aligned}$$

$$+\mathcal{M}_N(c,a,d;b)-\mathcal{T}_N(a;b+c,d)-\mathcal{M}_N(b,a,c;d)$$
$$-\mathcal{T}_N(a+b;c,d). \tag{33}$$

One can use Lemma 5 to obtain two identities for $\mathcal{M}(a,b,c;d)$. At first we assume $a=b$ and $c=d=1$ for $\mathcal{M}(a,b,c;d)$. By taking the limit $N\to\infty$ (ii) of Lemma 5 provides the formula

$$\mathcal{M}(a,a,1;1)=\frac{1}{2}\zeta^2(a,1)+\mathcal{T}(2a;1,1). \tag{34}$$

Evaluating $\mathcal{M}(a,b,1;a)$ can be done using $(ii)$ and $(i)$ of Lemma 5, which provides the formula

$$\mathcal{M}(a,b,1;a)=\zeta(b,1)\frac{\zeta^2(a)-\zeta(2(a))}{2}-\mathcal{T}(a+b;1,a)$$
$$-\frac{\mathcal{T}(b;a,a,1)-\mathcal{T}(b;2a,1)}{2}. \tag{35}$$

As mentioned before, such relations can be used to simplify the sum $S'$.

## 4. Further discussion

Let $W=W(r,s,t,u)$ denote the sum defined by

$$W=\sum_{j,k\geq 1}\frac{1}{j^r k^s (j+k)^t (j+2k)^u},$$

In [11] Tsumura showed that whenever $r+s+t+u$ is even, then $W$ can be evaluated into single valued zeta functions. We want to remark that a simple proof of Tsumura's result can be deduced exactly by the same methods used for the series $T$ and $A$ and $V$.

It would be interesting to study the sum defined by

$$\sum_{j,k=1}^{\infty}\frac{H_k^{(u)}H_j^{(v)}H_{j+k}^{(w)}}{j^r k^s (j+k)^t}, \tag{36}$$

which includes both sums $S$ and $V$ as special instances. However the study of these seems to be more difficult. Finally we offer some evaluations, which can be obtained as for sum (1),

$$\sum_{j,k=1}^{\infty}\frac{H_{j+k}}{jk(j+k)}=6\zeta(4),\quad \sum_{j,k=1}^{\infty}\frac{H_{j+k}H_j}{jk(j+k)}=11\zeta(5)+\zeta(2)\zeta(3), \tag{37}$$

$$\sum_{j,k=1}^{\infty}\frac{H_kH_j}{jk(j+k)}=2\zeta(5)+4\zeta(2)\zeta(3). \tag{38}$$

## 5. CONCLUSION

We have studied a generalization of a sum which recently appeared in literature. Furthermore we have presented simple proofs for the evaluation of Tornheim's double series and its alternating counterparts by reducing the problem to evaluation of (alternating) multiple zeta functions with two arguments, and also proved corresponding results for finite analogs of Tornheim's double series. We provided a criterion for the evaluation of the sum $V$, where $V$ generalizes Tornheim's double series.

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