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ACTIONS OF THE SYMMETRIC GROUP GENERATED BY COMPARABLE SETS OF INTEGERS AND SMITH INVARIANTS

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ABSTRACT. Lascoux and Schützenberger have shown that there exists a unique action of the symmetric group generated by the commutation of the column lengths of a two-column tableau and preserving the plactic class. We describe more general operators on pairs of comparable subsets of $\{1, \ldots, n\}$ which commute their cardinalities, and we prove that those operators define an action of the symmetric group by checking the braid relations on triples of sets of integers. The action of the symmetric group by Lascoux and Schützenberger appears in our construction as an extreme case as we only require the invariance of the shape and the weight of the insertion tableau. Instead of sets of positive integers one may take other equivalent objects as words in a two letter alphabet, and describe an action of the symmetric group on words congruent to key-tableaux defined by reflection crystal operators type based on non-standard pairing of parentheses. This construction arises naturally as a combinatorial description of the Smith invariants of certain sequences of products of matrices, over a local principal ideal domain, under a natural action of the symmetric group.

1. Introduction

In [20, 21] (see also [26]), A. Lascoux and M.-P. Schützenberger introduced the concept of key-tableau as a Young tableau whose columns are comparable under the inclusion order, and they have used this notion to study Demazure characters combinatorially, see [7]. For type A, Demazure characters are equivalent to key polynomials. Due to the action of the symmetric group on the set of tableaux, originally defined by Ehresmann in [8], the symmetric group acts on the set of key-tableaux. This action coincides with the one defined by the reflection crystal operators on the free algebra, based on the standard matching of parentheses, a particular parentheses matching on words in a two-letter alphabet. Reflection crystal operators are due to Lascoux and Schützenberger [19, 23], and they are equivalent to the ones coming from the theory of crystal graphs in the work of Kashiwara and Nakashima [24].

A column is a word in which every letter is strictly less than the previous one. Every word w can be uniquely written as a product of a minimal number of columns, and the sequence formed by their lengths is called the column-shape of w. A word w is frank if its shape is a permutation of the sequence formed by the column lengths of the only tableau in the Knuth class of w [21]. The action of the symmetric group on words congruent to key-tableaux defined by the reflection crystal operators preserve

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the Q-symbol, and has a translation on frank words within a Knuth class defined by *jeu de taquin* slides on two-column frank words [21]. In this paper, we drop the requirements of preserving respectively the Q-symbol and the Knuth class in those two constructions, and we define two families of actions of the symmetric group including, as particular cases, the actions of the symmetric group described above. The first one over frank words runs over tableaux of the same shape and weight, and the second over words congruent to key-tableaux. More specifically, we consider the ordered set $P[n]$ of sets of positive integers in $[n] = \{1, \dots, n\}$ and describe operators on pairs of comparable elements of $P[n]$ which generalize to non-congruent frank words the action of the symmetric group defined by *jeu de taquin* slides on two-column frank words within a Knuth class. This action of the symmetric group has a translation on words congruent to key-tableaux defined by reflection crystal operators type based on non-standard pairing of parentheses already considered in [6]. Such a translation is described by the dual Robinson–Schensted–Knuth correspondence [9, 17]. (See also [22].)

These two families of actions of the symmetric group arise naturally in the matrix context. Given an n by n non-singular matrix A, with entries in a local principal ideal domain with prime p, we write $A \sim \Delta_\alpha$ to mean that by Gaußian elimination one can reduce A to a diagonal matrix Δ_α with diagonal entries $p^{\alpha_1}, \dots, p^{\alpha_n}$, for unique non-negative integers $\alpha_1 \geq \dots \geq \alpha_n$, called the *Smith normal form* of A. The sequence $p^{\alpha_1}, \dots, p^{\alpha_n}$ defines the *invariant factors* of A, and $\alpha = (\alpha_1, \dots, \alpha_n)$ the *invariant partition* or the sequence of *Smith invariants* of A. It is known that partitions α, β, γ are respectively the Smith invariants of n by n non-singular matrices A, B, and C such that $AB = C$ if and only if there exists a Littlewood–Richardson tableau T of type (α, β, γ), that is, a tableau of shape γ/α which rectifies to the key-tableau of weight β (or Yamanouchi tableau of weight β) [10, 11]. In other words,

$$\text{there exists } U \sim I_n,\ \Delta_\alpha U \Delta_\beta \sim \Delta_\gamma \text{ if and only if } \quad c^{\gamma}_{\alpha,\beta} \neq 0, \tag{1.1}$$

where I_n denotes the n by n identity matrix, and $c^{\gamma}_{\alpha,\beta}$ is the Littlewood–Richardson coefficient of type (α, β, γ). The relationship between the Smith invariants of a product of matrices and the product of Schur functions was noticed earlier, with different approaches, by several authors, see P. Hall, J. A. Green, T. Klein, R. C. Thompson, *et al.* [1, 2, 3, 14, 16, 29]. (For an overview and other interrelations, see the survey by W. Fulton [11] as well as [10, 12, 13].)

One can show (1.1) by introducing the notion of a matrix realization of a pair (T, K), where T is a skew-tableau which rectifies to the key-tableau K [3, 4, 5]. (We warn the reader that French notation is used.) Such a matrix realization consists of a sequence of products of matrices

$$\Delta_\alpha, \Delta_\alpha U \Delta_{(1^{m_1})}, \Delta_\alpha U \Delta_{(1^{m_1})} \Delta_{(1^{m_2})}, \dots, \Delta_\alpha U \Delta_{(1^{m_1})} \Delta_{(1^{m_2})} \cdots \Delta_{(1^{m_t})}, \tag{1.2}$$

where U is an unimodular matrix, α is a partition of length $\leq n$, and $m = (m_1, \dots, m_t)$ is a non-negative integral vector with $m_i \leq n$ for all i. In particular, (1^{m_i}) denotes the partition whose Young diagram is a column of length m_i. Transposing the nested sequence of Smith invariants $\alpha^0 = \alpha \subseteq \alpha^1 \subseteq \dots \subseteq \alpha^t$, defined by that sequence of matrices, one gets a tableau T with skew-shape the transpose of α^t/α, and weight m. It is shown in [3, 6] that the column-reading word of T, from left to right, is congruent to a key-tableau, and the sequence of column words $J_t \cdots J_2\, J_1$, called the indexing-set word of T, with J_k the column-word of length m_k comprising the column-indices of the letter k in T, left to right, for $k \geq 1$, is a frank word.

The action of the simple transposition $s_i = (i, i+1)$ on $(m_1, \dots, m_t)$, by exchanging m_i and m_{i+1}, affords an action of s_i on the sequence of products of matrices (1.2), by exchanging $\Delta_{(1^{m_i})}$ and $\Delta_{(1^{m_{i+1}})}$. Therefore the operators $\{s_i\}$ generate an action of the symmetric group on the set of the semistandard Young tableaux defined by those sequences of products of matrices. This action is indeed two-fold: on the one hand, on the tableau column-reading words, one obtains an action of the symmetric group on words congruent to key-tableaux defined by reflection crystal operators based on non-standard matching of parenthesis; and on the other hand, on the tableau indexing-set words, it yields an action of the symmetric group on frank words. Checking the braid relations for $t = 3$, these actions turn out to be the ones generated by triples of sets of the ordered set $P[n]$.

The paper is organised as follows. In the next section we analyse the relation between reflection crystal operators on words over a two-letter alphabet and operators defined by *jeu de taquin slides* on two-column skew-tableaux. An important tool in this analysis is the dual RSK correspondence and its symmetry as well as its interpretation in terms of skew-tableaux. As an application of this duality, we study, in Subsection 2.3, the sequence of Smith invariants, equivalently the skew-tableaux, associated with the sequences (1.2) for $t = 2$. For this, we have to define two-column frank word *variants* of *jeu de taquin* and to show their relationship with non-standard pairing of parentheses on words congruent to two-letter key-tableaux. Moreover, variants of *jeu de taquin* and non-standard reflection crystal operators have a full interpretation in this matrix context, given in Theorem 2.4.

In Section 3, considering the sequence (1.2) for $t = 3$, we study the hexagon of skew-tableaux defined by

$$\begin{tikzcd}
 & \Delta_\alpha U \Delta_{(1^{m_2})}\Delta_{(1^{m_1})}\Delta_{(1^{m_3})} \arrow[r, dash, "s_2"] & \Delta_\alpha U \Delta_{(1^{m_2})}\Delta_{(1^{m_3})}\Delta_{(1^{m_1})} \arrow[dr, dash, "s_1"] & \\
\Delta_\alpha U \Delta_{(1^{m_1})}\Delta_{(1^{m_2})}\Delta_{(1^{m_3})} \arrow[ur, dash, "s_1"] \arrow[dr, dash, "s_2"'] & & & \Delta_\alpha U \Delta_{(1^{m_3})}\Delta_{(1^{m_2})}\Delta_{(1^{m_1})}. \\
 & \Delta_\alpha U \Delta_{(1^{m_1})}\Delta_{(1^{m_3})}\Delta_{(1^{m_2})} \arrow[r, dash, "s_1"'] & \Delta_\alpha U \Delta_{(1^{m_3})}\Delta_{(1^{m_1})}\Delta_{(1^{m_2})} \arrow[ur, dash, "s_2"'] &
\end{tikzcd}$$

We have two hexagons, one defined by the words of the skew-tableaux, and the other one defined by the indexing-set frank words. Since these hexagons are generated by Smith invariants, the matrix setting imposes conditions on their vertices, given in Lemma 3.2. Theorems 3.6 and 3.8 give a combinatorial interpretation of these conditions and characterize the variant *jeu de taquin* operators and the non-standard reflection crystal operators satisfying the braid relations, that is, closing these hexagons. Algorithm 3.11 shows that Lascoux–Schützenberger actions of the symmetric group on frank words and words congruent to key-tableaux are respectively obtained: from a particular shuffle decomposition of a three-column tableau; and from a shuffle decomposition of a three-letter Yamanouchi word. Again these actions of the symmetric group have a full interpretation in our matrix setting, given in Theorem 3.14. Finally, in Example 3.15, we exhibit two dual permutahedra in $\mathfrak{S}_4$, generated by *variants* of *jeu de taquin* operators and reflection crystal operators type based on *non-standard pairing.*

2. Variants on two-column *jeu de taquin*, non-standard pairing, and Smith invariants

2.1. **Tableaux and dual RSK.** A composition $m = (m_1, \dots, m_t, \dots)$ is a (finite or infinite) sequence of non-negative integers, almost all zero. A partition is a weakly decreasing composition, and we denote by $\beta(m)$ the partition obtained by rearranging the entries of the composition m. It is convenient to not distinguish between two compositions which only differ by a string of zeros at the end. The Young diagram of a partition $\gamma = (\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n \geq 0)$ is the set of ordered pairs of integers $Y(\gamma) = \{(i,j) : 1 \leq i \leq \gamma_j, 1 \leq j \leq n\}$. We identify a partition with its Young diagram. The conjugate partition $\gamma' = (\gamma'_1, \gamma'_2, \dots)$ of γ is the partition corresponding to the transpose of $Y(\gamma)$. A skew-diagram γ/α is the set difference $Y(\gamma) - Y(\alpha)$ of Young diagrams of partitions. A (semistandard) (skew) tableau T over the alphabet $[t] = \{1, \dots, t\}$ is a function (often called filling) $T : Y \longrightarrow [t]$ from a (skew) diagram Y to the positive integers in $[t]$ such that they are weakly increasing in each row and strictly increasing up each column. The shape $shape(T)$ of T is the domain of the tableau T. The *column-reading word* of the (skew) tableau T is $w = w^1 w^2 \dots$ where w^i is the column-word (strictly decreasing word) over $[t]$ comprising the ith column of T, left to right. Define the column-shape of T to be the composition $colshape(T) = (|w^1|, |w^2|, \dots)$, where $|w|$ denotes the length of w. Similarly, the row-reading word of T is $\cdots u^2 u^1$, where u^i is the row word (increasing word) over $[t]$ comprising the ith row of T, bottom to top, and the row-shape of T is the composition $(|u^1|, |u^2|, \dots)$. A *vertical strip* is a skew-tableau with rows of length at most one. The *weight* of T is the weight of the word w, that is, the sequence $(m_1, \dots, m_t)$ where m_i is the number of occurrences of the letter i in w. The column-indexing-set word of T is $J = J_t \cdots J_2 J_1$ where J_k is the column-word of length m_k defined by the set of column-indices of the letter k in T, left to right. An example of a Young tableau of skew-shape $(4,3,2,2)/(3,1)$ is

$$T = \begin{array}{cccc} 4 & 4 & & \\ 2 & 2 & & \\ \bullet & 1 & 3 & \\ \bullet & \bullet & \bullet & 2 \end{array} . \tag{2.1}$$

The column-reading word is 42 421 3 2, the column-indexing-set word is $J = 21\,3\,421\,2$ and the weight $(1,3,1,2)$.

A *standard tableau* is a tableau filled with the numbers $\{1, \dots, n\}$ where n is the number of cells or boxes in the (skew) Young diagram. The standardization $std(T)$ of a Young tableau T is the standard tableau obtained by simultaneously replacing the 1's in T from left to right by $1, 2, \dots, m_1$, the $2's$ by $m_1 + 1, \dots, m_1 + m_2$ etc., where $(m_1, \dots, m_t)$ is the weight of T. The transpose of a standard tableau T is still a standard tableau, of conjugate shape, and we denote it by T^τ.

Let $K(m)$ denote the key-tableau of weight m, that is, the tableau of weight m whose column-shape is $\beta'(m)$. There is an obvious bijection between compositions and key-tableaux since $K(m)$ is the tableau whose first m_j columns contain the letter j, for all j [26].

The overlap of a pair (u, v) of column-words u and v is the maximum number of rows of length two obtained with respect to shifting the column u down with respect to the column v. For instance, the overlap of the first two columns of the skew-tableau (2.1) is two, of the second and the third is one, and of the last two is zero. A skew-tableau is said to be in *compact form* if the number of rows of size two of any two consecutive columns is the overlap of the words comprising those columns. Thus a skew-tableau T in compact form is identified with its column-reading word seen as a sequence of column words. For instance the skew-tableau (2.1) is in compact form.

Let $[t]^*$ denote the free monoid over the alphabet $[t] = \{1, \dots, t\}$. The Knuth or plactic congruence $\equiv$ [19, 9, 17, 23] on the words over the alphabet $[t]$ is the congruence in $[t]^*$ defined by the transitive closure of the relations below, where x, y and z are letters in $[t]$, and a and b are words in $[t]^*$:

$$axzyb \equiv azxyb, \quad x \leq y < z,$$

$$ayzxb \equiv ayxzb, \quad x < y \leq z.$$

Given the word $w = x_1 \cdots x_N$, the row insertion of w produces a unique pair of tableaux $(P(w), Q(w))$ of the same shape. $P(w)$ is the unique tableau of partition shape whose column-reading word is Knuth equivalent to w, and $Q(w)$, often called the Q-symbol of w, is the standard tableau of the same shape as $P(w)$ such that $\mathit{shape}(Q(w)|_{[i]}) = \mathit{shape}(P(x_1 \cdots x_i))$, $1 \leq i \leq t$, where $Q(w)|_{[i]}$ denotes the restriction of the tableau $Q(w)$ to the letters in $[i]$. Two (skew) tableaux are said to be Knuth equivalent if their words are congruent, equivalently, one is obtained from the other by *jeu de taquin* slides [9, 23, 28].

Let $v = v^1 v^2 \dots v^n$, with each v^i a column-word (some of the v^i may be the empty word), be an n-column factorization of v. The factorization $v^1 v^2 \dots v^n$ may be identified, as a sequence of column words, with the skew-tableau in compact form where v^i is the word comprising the i-th column. The column-shape of $v^1 v^2 \dots v^n$, denoted by $\mathit{colshape}(v^1 v^2 \dots v^n)$, is the column-shape of that skew-tableau. A factorization is minimal if the number of factors is minimal. We write $\mathit{colshape}(v)$ (unique up to zeros) to mean the column-shape of the minimal factorization of v. Lascoux and Schützenberger called v *frank* word [21] if it has a factorization of column-shape (in fact minimal) a permutation of the non-zero parts of the conjugate shape of $P(v)$.

The dual RSK correspondence [17, 9] can be defined as a bijection from the set of finite sequences of column-words $(\cdots J_2 J_1)$ to pairs of tableaux (Q, P) of conjugate shapes, where $Q = P(\cdots J_2 J_1)$ and $\mathit{shape}(P|_{[i]}) = \mathit{shape}(P(J_i \cdots J_1))'$ for all i.

Let A and B be two totally ordered alphabets. We consider biwords of pairwise distinct biletters

$$\begin{pmatrix} u_1 & \cdots & u_k \\ v_1 & \cdots & v_k \end{pmatrix},$$

with $u_i \in A$ and $v_j \in B$. The biword is said to be in anti-lexicographic order if its biletters $\binom{u_i}{v_i}$ satisfy $u_i < u_{i+1}$ or $u_i = u_{i+1}$ and $v_i > v_{i+1}$.

Let $J = \{J_{t-i+1}\}_{1 \leq i \leq t}$, be a sequence of column-words over the alphabet $[n]$ in dual RSK correspondence with the tableau-pair (Q, P) of conjugate shapes. Let m_i be the length of J_i, $1 \leq i \leq t$, and consider the biword with no two identical biletters,

$$\Sigma' = \begin{pmatrix} J_t & \cdots & J_1 \\ t^{m_t} & \cdots & 1^{m_1} \end{pmatrix}. \tag{2.2}$$

By increasing rearrangement of Σ' with respect to the anti-lexicographic order, we get

$$\Sigma = \begin{pmatrix} 1^{f_1} & \cdots & n^{f_n} \\ w^1 & \cdots & w^n \end{pmatrix}, \tag{2.3}$$

with w^i a column word of length f_i. Clearly the word $J_t \cdots J_1$ can be seen as the column-indexing-set word of a skew-tableau with column reading word $w^1 \cdots w^n$, and conversely $w^n \cdots w^1$ as the column-indexing-set word of a skew-tableau with column-reading word $J_1 \cdots J_t$. Therefore, the dual RSK correspondence defines also a bijection between skew-tableaux in compact form and tableau-pairs of conjugate shapes. For instance, consider the skew-tableau (2.1). Construct Σ and Σ' accordingly. The bottom word of Σ is w and the top word of Σ' is J,

$$\Sigma = \begin{pmatrix} 11\,22\,3\,44 \\ 42\,21\,3\,32 \end{pmatrix} \quad \longleftrightarrow \quad \Sigma' = \begin{pmatrix} 1\,43\,421\,2 \\ 4\,33\,222\,1 \end{pmatrix}.$$

From Proposition 5, the symmetry theorem in [9, Appendix A.4.3], we have the following result.

Theorem 2.1. *Given the sequence $J = \{J_{t-i+1}\}_{1\le i\le t}$, of column-words over the alphabet $[n]$, there is one and only one word w over the alphabet $[t]$, with weight the reverse of the column shape of $\{J_{t-i+1}\}_{1\le i\le t}$, revcolshape$(\{J_{t-i+1}\}_{1\le i\le t})$, such that*
(a) $P(w) = P$.
(b) the Q-symbol or recording tableau of w satisfies $Q(w) = std(Q)^\tau$ with $Q = P(J)$.
Moreover J is a frank word of colshape$(\{J_{t-i+1}\}_{1\le i\le t})$ if and only if $P = K(revcolshape(w))$ [21, 27].

As frank words, in a congruence class, are completely determined by their Q-symbols, it follows that frank words, in a plactic class, are in bijection with the set of permutations of the non-zero parts of the conjugate shape of the tableau in that class [21] and we have an action of the symmetric group on frank words. Lascoux and Schützenberger have translated this action of the symmetric group on frank words in the language of *jeu de taquin* slides on two-column skew-tableaux either aligned in the bottom or in the top. This *jeu de taquin* operation can be extended to any two-column skew-tableaux in compact form to define an action of the symmetric group on skew-tableaux (see [22]).

2.2. **Key-tableaux and sequences of matrix products over a local principal ideal domain.** Let the symmetric group $\mathfrak{S}_t$ act on finite compositions $m = (m_1, \dots, m_t)$ via the left action $s_i m = (m_1, \dots, m_{i+1}, m_i, \dots, m_t)$, with s_i, $1 \le i \le t-1$, the simple transpositions of $\mathfrak{S}_t$. Then $\beta(m)$ is the unique partition in the orbit $\mathfrak{S}_t m$ and $\beta'(m)$ its conjugate. Recall that the column-shape of the key-tableau $K(m)$ is $\beta'(m)$. Considering the bijection between compositions and key-tableaux mentioned earlier, we identify $K(m)$ with the sequence of diagonal matrices $(\Delta_{(1^{m_1})}, \dots, \Delta_{(1^{m_t})})$ in the sense that the nested sequence of partitions $(1^{m_1}) \subseteq (1^{m_1}) + (1^{m_2}) \subseteq \dots \subseteq (1^{m_1}) + \dots + (1^{m_t}) = \beta'(m)$ defines the key-tableau $K(m)$ and, simultaneously, are the Smith invariants of the sequence of products of matrices $\Delta_{(1^{m_1})}$, $\Delta_{(1^{m_1})}\Delta_{(1^{m_2})}$, $\dots$,

$\Delta_{(1^{m_1})}\Delta_{(1^{m_2})}\cdots\Delta_{(1^{m_t})} = \Delta_{\beta'(m)}$. For instance,

$$K(10325) = \begin{array}{lllll} 5 & & & & \\ 4 & 5 & & & \\ 3 & 4 & 5 & & \\ 1 & 3 & 3 & 5 & 5 \end{array} \qquad \text{is identified with } (\Delta_{[1]}, \Delta_{\emptyset}, \Delta_{[3]}, \Delta_{[2]}, \Delta_{[5]}).$$

Let U be an n by n unimodular matrix, that is, $U \sim I_n$. Put $\Delta_\alpha UK(m)$ for the sequence

$$\Delta_\alpha, \Delta_\alpha U\Delta_{(1^{m_1})}, \Delta_\alpha U\Delta_{(1^{m_1})}\Delta_{(1^{m_2})}, \dots, \Delta_\alpha U\Delta_{(1^{m_1})}\Delta_{(1^{m_2})}\cdots\Delta_{(1^{m_t})} = \Delta_\alpha U\Delta_{\beta'(m)}.$$

The nested sequence of Smith invariants $\alpha^0 = \alpha \subseteq \alpha^1 \subseteq \dots \subseteq \alpha^t = \gamma$ defined by this sequence of products of matrices is such that α^{i+1}/α^i is a vertical strip of length m_{i+1}, for $i = 0, 1, \dots, t-1$. Thus $\Delta_\alpha UK(m)$ is identified with the tableau T of shape γ'/α' and weight m, with indexing-set words $J_{k+1} = \{i : \alpha_i^{k+1} = \alpha_i^k + 1\}$, $1 \le k \le t-1$. It is shown in [6] that the column-reading word w of T satisfies $P(w) = K(m)$ and thus, from Theorem 2.1, J is a frank word of shape the non-null parts of the reverse of m.

When we consider the action of the symmetric group on finite compositions m we are simultaneously defining an action of the symmetric group on sequences of matrices $\Delta_\alpha UK(m)$, where U is a fixed unimodular matrix and α a fixed partition, and therefore on tableaux of skew-shape whose rectifications are the key-tableaux $K(m)$. Thus, we obtain two families of actions of the symmetric group which are translations of each other: one over non-congruent frank words running over tableaux of column-shape $\beta(m)$ where the weight is a permutation of the entries of m; and the other one on words congruent to key-tableaux $K(m)$, not necessarily sharing the same Q-symbol. This duality is explained in the next section.

2.3. *Jeu de taquin on two-column words and reflection crystal operators.*

It is known that there is a duality between reflection crystal operators and *jeu de taquin* on two-column skew tableaux [18, 22]. The *jeu de taquin* slides exchanging the length of two consecutive columns i, $i+1$ of a t-column skew-tableau in compact form, counting right to left, is translated, by the dual RSK-correspondence, into the i-th reflection crystal operator on words over the alphabet $[t]$. In the particular case of frank words it is translated into the i-th reflection crystal operator on words congruent to key-tableaux. Define the operator Θ on a two-column skew-tableau $T = J_2J_1$ in compact form and row-shape $(1^s, 2^q, 1^r)$, for some $q, r, s \ge 0$, as follows. If $r > s$ ($r < s$), perform *jeu de taquin* slides in the first $|r-s|$ inside (outside) corners marked $\blacksquare$ of the skew-tableau T until they become outside (inside) corners in the second (first) column. In other words, we slide down (up) maximally up to $|r-s|$ positions the entries of the first (second) column; then we exchange the east (west) entries with the vacant neighbours $\blacksquare$. Then $\Theta T = T'$ is a two-column skew-tableau in compact form with row-shape $(1^r, 2^q, 1^s)$. Obviously $\Theta T' = T$. In particular, when $r = 0$ or $s = 0$, Θ is the *jeu de taquin* on frank words. For instance, the *jeu de taquin* slides, with respect to the corner $\blacksquare$ as below, define the operator Θ on T of row shape $(1,2^3,1^2)$, and on T' of row shape $(1^2,2^3,1)$

$$T=\begin{array}{cc} 7 & \\ 3 & 6\\ 2 & 5\\ 1 & 4\\ \blacksquare & 3\\ & 2\end{array}\quad\overset{\Theta}{\longleftrightarrow}\quad T'=\begin{array}{cc} 7 & \\ 6 & \blacksquare\\ 3 & 5\\ 2 & 4\\ 1 & 3\\ & 2\end{array}. \tag{2.4}$$

Let T be a t-column skew-tableau in compact form. Define the operator Θ_i on T as follows: apply Θ to the columns i and $i+1$ of T, counting right to left, and put the outcome t-column skew-tableau in compact form. As *jeu de taquin* preserves Knuth equivalence, we have $\Theta_i T\equiv T$.

Let $w=w_1w_2\dots w_g$, $w_i\in[t]$, be a word. An r-pairing of w is a set of indexed pairs (called r-pairs) (w_i,w_j) such that $1\le i<j\le k$, $w_i=r+1$, and $w_j=r$, and if (w_l,w_s) is another pair, then i,l,j,s are pairwise distinct. View each $r+1$ (respectively r) as a left (respectively right) parenthesis and ignore the other letters. The r-pairs of w are precisely the matched parentheses. Furthermore the subword of unpaired $r's$ and $(r+1)'s$ must be a subword of w of the form $r^k(r+1)^l$. In general, not every r-pairing gives the maximal number of r-pairs of w, and if $\tilde\theta_r$ is the operator which replaces the word $r^k(r+1)^l$ of unpaired $r's$ and $(r+1)'s$ in w (in the corresponding positions) by $r^l(r+1)^k$, unless certain conditions are imposed on the r-pairing, the maximal number of r-pairs of $\tilde\theta_r w$ and w may be different. For example, $w=12112$ has two 1-pairings with a maximal number of 1-pairs, and $\tilde\theta_1 w=1(21)22$ has just one 1-pairing. However, when either $k=0$ or $l=0$ they have always the same maximal number of r-pairs. In this case, as we shall see in the next subsection, the operator $\tilde\theta_r$ can be reduced to a variant of *jeu de taquin* on two-column frank words.

The standard r-pairing on w is the particular r-pairing obtained in the following way. Start with the subword $w'=x_1\cdots x_m$, the restriction of w to the alphabet $\{r,r+1\}$. Then, bracket every factor $r+1\,r$ of w'. The letters which are not bracket constitute a subword w'' of w'. Then, bracket every factor $r+1\,r$ of w''. Continue this procedure until it stops. It remains a word of the form $r^k(r+1)^l$. The reflection crystal operator, based on the standard r-pairing, denoted by θ_r, which replaces the word $r^k(r+1)^l$ of unpaired $r's$ and $(r+1)'s$ in w by $r^l(r+1)^k$, can be reduced to *jeu de taquin* slides on two-column skew tableaux.

Recalling the definition of the biwords (2.2) and (2.3), the operators θ_i on words over the alphabet $[t]$, and Θ_i on t-column skew-tableaux $J=\{J_{t-i+1}\}_{1\le i\le t}$, in compact form, are a translation of each other in the sense of the following commutative diagram

$$\begin{tikzcd} \Sigma=\begin{pmatrix} J\uparrow\\ w\end{pmatrix} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \Sigma'=\begin{pmatrix} \cdots J_{i+1}\ J_i\cdots\\ \cdots(i+1)^{q+k}\ i^{q+l}\cdots\end{pmatrix} \arrow[d, leftrightarrow]\\ \widetilde\Sigma=\begin{pmatrix} J\uparrow\\ \theta_i w\end{pmatrix} \arrow[r, leftrightarrow] & \widetilde\Sigma'=\begin{pmatrix} \cdots\Theta_i(J_{i+1}\ J_i)\cdots\\ \cdots(i+1)^{q+l}\ i^{q+k}\cdots\end{pmatrix}\end{tikzcd}, \tag{2.5}$$

where $J\uparrow$ indicates J by weakly increasing order.

As column words may be identified with their support sets, these operators can be formulated in $P[n]$ the set of all subsets of $[n]$. We consider on $P[n]$ two orders, one by letting $B \leq B'$ whenever there is an increasing injection $i : B \rightarrow B'$, that is, $x \leq i(x)$, and the other one by putting $B \triangleright B'$ whenever there exists a decreasing injection $B \leftarrow B' : j$, that is, $j(x) \leq x$.

Define the skew-tableau in compact form $\begin{matrix} A & \\ B & C \\ & D \end{matrix}$, where $A \cup B$, $C \cup D$ are columns such that $B \leq C$, $|B| = |C| = q$, $|A| = s$, and $|D| = r$. Suppose $r > s$. Slide down maximally the column B along $C \cup D$ such that the weakly increasing order in rows is preserved, and define the column $X \subseteq C \cup D$ whose entries have west adjacent neighbours in B. Then θ_i can be based in any pairing of parentheses defined by any increasing injection $j : B \rightarrow X$. We have the equivalence, $\Theta_i T \equiv T$ if and only if the operator θ_i preserves the Q-symbol, that is, $Q(w) = Q(\theta_i w)$. For instance, for the two-column tableau T (2.4), we have the following diagram

$$\begin{array}{ccc} \Sigma = \begin{pmatrix} 1\,2\,2\,3\,3\,4\,5\,6\,7 \\ 2\,2\,1\,2\,1\,1\,1\,1\,2 \end{pmatrix} & \longleftrightarrow & \Sigma' = \begin{pmatrix} T \\ 2^{3+1}\ 1^{3+2} \end{pmatrix} \\ \updownarrow & & \updownarrow \\ \widetilde{\Sigma} = \begin{pmatrix} 1\,2\,2\,3\,3\,4\,5\,6\,7 \\ 2\,2\,1\,2\,1\,1\,1\,2\,2 \end{pmatrix} & \longleftrightarrow & \widetilde{\Sigma}' = \begin{pmatrix} \Theta_1 T \\ 2^{3+2} 1^{3+1} \end{pmatrix} \end{array},$$

where $w = (2(21)(21)1)1^2 2 \rightarrow \theta_1 w = (2(21)(21)1)12^2$.

From the diagram (2.5), we have the following result.

Theorem 2.2. *The following statements are equivalent:*

(*a*) *The operators Θ_i, $1 \leq i \leq t-1$, define an action of the symmetric group $\mathfrak{S}_t$ on the set of t-column words, equivalently, on the t-column skew-tableaux in the compact form. Moreover, $\Theta_i T \equiv T$, $1 \leq i \leq t-1$.*

(*b*) [19, 23] *The operators θ_i, $1 \leq i \leq t-1$, define an action of the symmetric group on all words over the alphabet $[t]$, and preserve the Q-symbol.*

2.4. **Two-column frank word variants of *jeu de taquin*, pairing of parentheses, and Smith invariants.** In this section, we define *variants* of *jeu de taquin* on two-column frank words and show its relationship with a non-standard pairing of parentheses on words congruent to two-letter key-tableaux. As an application we describe the Smith invariants, equivalently, the skew-tableaux on a two-letter alphabet, associated with the sequences $\Delta_\alpha UK(m)$ and $\Delta_\alpha UK(s_1 m)$ with $m = (m_1, m_2)$ and s_1 the elementary transposition $(1\,2)$.

Restrict the *jeu de taquin* operation Θ to a two-column tableau or contre-tableau (a two-column skew-tableau such that the pair of columns is aligned at the top) $J_2\, J_1$ [9, 28], and denote by $\tilde{\Theta}$ a variant of Θ which runs as follows. If $J_2 J_1$ is a contre-tableau (tableau), slide vertically the entries of the column J_2 (J_1) along the column J_1 (J_2) such that the row weakly increasing order is preserved, and every common label to the two columns never has a vacant west (east) neighbour. Then exchange the vacant positions with the east (west) neighbours. In particular, when the first (second) column J_2 (J_1) is slid down (up) maximally such that the row weakly increasing order is preserved, we get the outcome of the *jeu de taquin* operation. For instance,

$$\Theta: \begin{array}{cc} 2 & 5 \\ 1 & 4 \\ \blacksquare & 3 \\ \blacksquare & 2 \end{array} \longleftrightarrow \begin{array}{cc} \blacksquare & 5 \\ \blacksquare & 4 \\ 2 & 3 \\ 1 & 2 \end{array} \longleftrightarrow \begin{array}{cc} 5 & \blacksquare \\ 4 & \blacksquare \\ 2 & 3 \\ 1 & 2 \end{array}, \tag{2.6}$$

$$\tilde{\Theta}: \begin{array}{cc} 2 & 5 \\ 1 & 4 \\ \blacksquare & 3 \\ \blacksquare & 2 \end{array} \longleftrightarrow \begin{array}{cc} \blacksquare & 5 \\ 2 & 4 \\ \blacksquare & 3 \\ 1 & 2 \end{array} \longleftrightarrow \begin{array}{cc} 5 & \blacksquare \\ 2 & 4 \\ 3 & \blacksquare \\ 1 & 2 \end{array} \longleftrightarrow \begin{array}{cc} 5 & \blacksquare \\ 3 & \blacksquare \\ 2 & 4 \\ 1 & 2 \end{array}. \tag{2.7}$$

Obviously, $\tilde{\Theta}(J_2J_1)$ and $\Theta(J_2J_1)$ are not congruent unless $\tilde{\Theta} = \Theta$, but $\tilde{\Theta}(J_2J_1)$ is a frank word with the same shape and weight as $\Theta(J_2J_1)$.

Suppose that w is congruent to the key-tableau of weight $(0^{r-1}, m_r, m_{r+1})$. Without loss of generality, assume $m_{r+1} \le m_r$. Let $J_{r+1}J_r$ be a frank word of shape $(m_{r+1}, m_r, 0^{r-1})$, such that sorting the biletters of the biword $\Sigma' = \begin{pmatrix} J_{r+1}J_r \\ (r+1)^{m_{r+1}} r^{m_r} \end{pmatrix}$, by weakly increasing rearrangement of the biletters for the anti-lexicographic order, we get $\Sigma = \begin{pmatrix} J_{r+1}J_r \uparrow \\ w \end{pmatrix}$. Consider an r-pairing in w defined by an increasing injection $i : J_{r+1} \longrightarrow J_r$ such that $J_r \cap J_{r+1} \subseteq i(J_{r+1})$. To perform $\tilde{\theta}_r w$ based on this r-pairing means to apply an operator $\tilde{\Theta}$ on $J_{r+1}J_r$ (denoted by $\tilde{\Theta}_r$) which exchanges the vacant entries of the first column with the corresponding east neighbours consisting of $J_r \setminus i(J_{r+1})$ in the second column J_r. Conversely, an operator $\tilde{\Theta}_r$ on $J_{r+1}J_r$ means an operator $\tilde{\theta}_r$ on w, where the r-pairing on w is defined by any increasing injection $i : J_{r+1} \longrightarrow J_r$ such that $\tilde{\Theta}J_{r+1}J_r = [J_{r+1} \cup (J_r \setminus B)]B$, where $J_r \cap J_{r+1} \subseteq i(J_{r+1}) = B$. When $\tilde{\Theta}_r = \Theta_r$ we get the standard pairing of parentheses on w and hence θ_r. Thus the operators $\tilde{\Theta}_r$, Θ_r and $\tilde{\theta}_r$, θ_r are respectively translated into each other, according the following commutative diagram,

$$\begin{array}{ccc} \Sigma = \begin{pmatrix} J_{r+1}J_r \uparrow \\ w \end{pmatrix} & \longleftrightarrow & \Sigma' = \begin{pmatrix} J_{r+1}J_r \\ (r+1)^{m_{r+1}} r^{m_r} \end{pmatrix} \\ \updownarrow & & \updownarrow \\ \widetilde{\Sigma} = \begin{pmatrix} J_{r+1}J_r \uparrow \\ \tilde{\theta}_r w \end{pmatrix} & \longleftrightarrow & \widetilde{\Sigma}' = \begin{pmatrix} \tilde{\Theta}(J_{r+1}J_r) \\ (r+1)^{m_r} r^{m_{r+1}} \end{pmatrix} \end{array}. \tag{2.8}$$

If the row insertion of w gives the pair of tableaux (P, Q) then the row insertion of $\tilde{\theta}_r w$ leads to the pair $(\theta_r P, Q')$, where Q and Q' are distinct tableaux unless $\tilde{\theta}_r = \theta_r$. As a Knuth class is not closed under the action of the operator $\tilde{\Theta}_r$, $\tilde{\theta}_r$ does not preserve the Q-symbol but we have $\theta_r w \equiv \tilde{\theta}_r w$. For instance, in (2.6), any increasing injection $\{1, 2\} \to \{2, 3\}$ defines a standard pairing of parentheses, giving rise to $\theta_1 : (2(21)1)1 \to (2(21)1)2$; and in (2.7), any increasing injection $\{1, 2\} \to \{2, 4\}$ defines a pairing of parentheses, giving rise to $\tilde{\theta}_1 : (2(21)11) \to (2(21)21)$.

We are now in the position to describe the Smith invariants, equivalently, the skew-tableaux on a two-letter alphabet associated with the sequences $\Delta_\alpha UK(m)$ and

$\Delta_\alpha UK(s_1 m)$. We let P_σ be the permutation matrix associated with the permutation $\sigma \in \mathfrak{S}_n$.

Lemma 2.3. [5] *(a) Let U be an n by n unimodular matrix. Then, there exists $\sigma \in \mathcal{S}_n$ such that $U = TP_\sigma QL$, where T is an n by n upper triangular matrix, with $1'$s along the main diagonal, Q is an n by n upper triangular matrix, with $1'$s along the main diagonal, and multiples of p above it, and L is an n by n lower triangular matrix, with units along the main diagonal.*

(b) By elementary operations on the left and on the right, $\Delta_\alpha UK(m)$ may be considered equal to $\Delta_\alpha P_\sigma QK(m)$, with $\sigma \in \mathfrak{S}_n$.

(c) The Smith normal form of $\Delta_\alpha P_\sigma QD_{[m_1]}$, with $\sigma \in \mathfrak{S}_n$, is the diagonal matrix Δ_{α^1} where $\alpha \subseteq \alpha^1$ is a vertical strip of length m_1.

Theorem 2.4. [5] *Let T and T' be respectively the tableaux defined by the sequences $\Delta_\alpha UK(m)$ and $\Delta_\alpha UK(s_1 m)$, with indexing-set words J_2J_1, $J_2'J_1'$ respectively, and words w, w' respectively. Then,*

(a) J_2J_1, $J_2'J_1'$ are frank words such that $\tilde{\Theta}_1 J_2J_1 = J_2'J_1'$, for some operator $\tilde{\Theta}_1$.

(b) $w \equiv K(m)$ and $w' = \tilde{\theta}_1 w \equiv K(s_1 m)$, for some operator $\tilde{\theta}_1$.

Conversely, if T and T' are respectively tableaux with indexing-set frank words J_2J_1 and $J_2'J_1'$ satisfying $J_2'J_1' = \tilde{\Theta}_1 J_2J_1$, then there exists an unimodular matrix U such that $\Delta_\alpha UK(m)$ and $\Delta_\alpha U'K(s_1 m)$ define the tableaux T and T' respectively.

Example 2.5. Let $U = P_{4321}T_{14}(p)$, where $T_{14}(p)$ is the elementary matrix obtained from the identity by placing the prime p in position $(1,4)$. With $\alpha = (2,1)$ the sequences $\Delta_\alpha UK(3,2)$ and $\Delta_\alpha UK(2,3)$ define, respectively, $T = \begin{array}{llll} 2 & & & \\ \bullet & 1 & 2 & \\ \bullet & \bullet & 1 & 1 \end{array}$ and $T' = \begin{array}{llll} 2 & & & \\ \bullet & 2 & 2 & \\ \bullet & \bullet & 1 & 1 \end{array}$. The words $w = 21211$ of T and $w' = 22211$ of T' satisfy $\tilde{\theta}_1 w = w' \equiv \theta_1 w$, where $\tilde{\theta}_1$ is the operator based on the parentheses matching $(21(21)1)$. However, if we choose $U' = P_{3241}T_{24}(p)$, the sequences $\Delta_\alpha U'\ K(3,2)$ and $\Delta_\alpha U'\ K(2,3)$ define, respectively, T and $T'' = \begin{array}{llll} 2 & & & \\ \bullet & 1 & 2 & \\ \bullet & \bullet & 1 & 2 \end{array}$. In this case, the word w'' of T'' satisfy $\theta_1 w = w''$. The corresponding operations on the indexing frank words are given as follows:

$$\Theta: \begin{array}{cc} \blacksquare & 4 \\ 3 & 3 \\ 1 & 2 \end{array} \longleftrightarrow \begin{array}{cc} 4 & \blacksquare \\ 3 & 3 \\ 1 & 2 \end{array} \qquad \tilde{\Theta}: \begin{array}{cc} 3 & 4 \\ 1 & 3 \\ \blacksquare & 2 \end{array} \longleftrightarrow \begin{array}{cc} 3 & 4 \\ 1 & 3 \\ 2 & \blacksquare \end{array} \longleftrightarrow \begin{array}{cc} 3 & \blacksquare \\ 2 & 4 \\ 1 & 3 \end{array}. \tag{2.9}$$

The operators Θ_r (θ_r) can be extended to frank words with more than two columns (words on a t-letter alphabet, $t \geq 2$) [19, 23]. Under certain conditions, operators $\tilde{\Theta}_r$ $(\tilde{\theta}_r)$ can be extended, as well, to frank words with more than two columns (words on a t-letter alphabet, $t \geq 2$). For this, we generalize a criterion, by Lascoux and Schützenberger in [21], to test whether the concatenation of a frank word with a column word is a frank word. Denote, respectively, by $L(J)$ and $R(J)$ the left and right columns of a frank word J.

Theorem 2.6. [21] *The concatenation JJ' of two frank words J and J' is frank if and only if $R(H)L(H')$ is frank for any pair of frank words H, H' such that $H \equiv J$ and $H' \equiv J'$.*

Notice that when J, J' are column-words, JJ' is frank if and only if JJ' is a tableau or a contre-tableau. Therefore, we deduce the following criterion for the concatenation of a column with a frank word.

Corollary 2.1. *Let $J = J_k \cdots J_1$ be a frank word and J_{k+1} a column. Then, $J_{k+1}J$ is frank if and only if $J_{k+1}J_k$ and $\overline{J}_k J_{k-1} \cdots J_1$ are frank words, where $\overline{J}_{k+1}\,\overline{J}_k = \Theta_k(J_{k+1}J_k)$.*

The criterion given by this corollary can be generalized to operators $\tilde{\Theta}$.

Corollary 2.2. *Let $J = J_k \cdots J_1$ be a frank word and J_{k+1} a column. Then, $J_{k+1}J$ is frank if and only if $J_{k+1}J_k$ and $\widetilde{J}_k J_{k-1} \cdots J_1$ are frank words, where $\widetilde{J}_{k+1}\,\widetilde{J}_k = \tilde{\Theta}_k(J_{k+1}J_k)$ for some operator $\tilde{\Theta}_k$.*

Proof. The necessary condition is a consequence of the previous corollary. Conversely, assume the existence of an operator $\tilde{\Theta}_k$ in the required conditions, and let $\overline{J}_{k+1}\,\overline{J}_k = \Theta_k(J_{k+1}J_k)$. Clearly, we have $\overline{J}_k \leq \widetilde{J}_k$, and also $\overline{J}_{k+1} \triangleright \widetilde{J}_{k+1}$, since $|\overline{J}_k| = |\widetilde{J}_k|$. By the hypotheses, the product $\widetilde{J}_k L(H)$ is frank, for any frank word $H \equiv J_{k-1} \cdots J_1$. This means that either $\widetilde{J}_k \leq L(H)$, or $\widetilde{J}_k \triangleright L(H)$. By transitivity, we find that either $\overline{J}_k \leq L(H)$, or $\overline{J}_k \triangleright L(H)$, *i.e.*, $\overline{J}_k L(H)$ is frank. Thus, by Theorem 2.6, the word $\overline{J}_k J_{k-1} \cdots J_1$ is frank, and therefore, by the previous corollary, $J_{k+1}J$ is frank. $\square$

The following theorem was proved in [6]. Here we give a different proof based on indexing-set words.

Theorem 2.7. *Let T be the tableau defined by $\Delta_\alpha U\ K(m)$, with word w and J the indexing set word. Then $P(w) = K(m)$ and J is a frank word of column-shape the reverse of m.*

Proof. Let $J = J_t \ldots J_1$ with column-shape the reverse of m. We will prove, by induction on $t \geq 1$, that $J_t \cdots J_1$ is a frank word. When $t = 1$ the result is trivial, and the case $t = 2$ is a consequence of Theorem 2.4 (see [5]). So, let $t > 2$ and let T be the tableau defined by $\Delta_\alpha UK(m_1, \ldots, m_t)$. By the inductive step, the word $J_{t-1} \cdots J_1$ is frank, since the sequence $\Delta_\alpha UK(m_1, \ldots, m_{t-1})$ defines the tableau T' with indexing-set word $J_{t-1} \ldots J_1$ and weight $(m_1, \ldots, m_{t-1})$.

By the Smith normal form theorem (see for instance [25]), there is a partition $\bar{\alpha}$ and an unimodular matrix U' such that by elementary row operations, $\Delta_{\bar{\alpha}} UD_{[m_1]} \cdots D_{[m_{t-2}]}$ can be reduced to $\Delta_{\bar{\alpha}} U'$. The sequence $\Delta_{\bar{\alpha}} U'K(m_{t-1}, m_t)$ defines the tableau $\overline{T}$ with indexing-set word J_{t-1}, J_t, and weight (m_{t-1}, m_t). By the case $t = 2$, the word J_tJ_{t-1} is frank. Moreover, by Theorem 2.4, we find that if $\overline{T}'$ is the tableau defined by the sequence $\Delta_{\bar{\alpha}} U, K(m_t, m_{t-1})$, the indexing sets $\overline{J}_{t-1}, \overline{J}_t$ of $\overline{T}'$ satisfy $\overline{J}_t\overline{J}_{t-1} = \tilde{\Theta}_{t-1}(J_tJ_{t-1})$ for some operator $\tilde{\Theta}_{t-1}$.

Finally, notice that $\Delta_\alpha UK(m_1, \ldots, m_{t-2}, m_t)$ defines the tableau $\widetilde{T}$ with indexing-set word $\overline{J}_{t-1}\ J_{t-2}\ \ldots\ J_1$, and weight $(m_1, \ldots, m_{t-2},\ m_t)$. By the inductive step, $\overline{J}_{t-1}J_{t-2} \cdots J_1$ is a frank word. Thus, by Corollary 2.2, the word $J_t \cdots J_1$ is frank, and therefore, $w \equiv K(m)$. $\square$

3. An action of the symmetric group on Young tableaux of skew-shape

Let s_i denote the elementary transposition $(i, i+1)$ of $\mathfrak{S}_t$, $1 \leq i \leq t$. Let U be an n by n unimodular matrix and $\beta = (\beta_1, \beta_2, \beta_3)$ a partition. We consider the following hexagon

(3.1)
$$\begin{tikzcd}[column sep=small]
 & \Delta_\alpha UK(\beta_2,\beta_1,\beta_3) \arrow[r, dash, "s_2"] & \Delta_\alpha UK(\beta_2,\beta_3,\beta_1) \arrow[dr, dash, "s_1"] & \\
\Delta_\alpha UK(\beta_1,\beta_2,\beta_3) \arrow[ur, dash, "s_1"] \arrow[dr, dash, "s_2"'] & & & \Delta_\alpha UK(\beta_3,\beta_2,\beta_1). \\
 & \Delta_\alpha UK(\beta_1,\beta_3,\beta_2) \arrow[r, dash, "s_1"'] & \Delta_\alpha UK(\beta_3,\beta_1,\beta_2) \arrow[ur, dash, "s_2"'] &
\end{tikzcd}$$

From the discussion in Section 2.2, we may look at (3.1) as a hexagon whose vertices are tableaux of skew-shape such that the words are congruent to a key-tableau $K(\beta_{i_1}, \beta_{i_2}, \beta_{i_3})$, and the indexing-set frank words have column-shape the reverse of $(\beta_{i_1}, \beta_{i_2}, \beta_{i_3})$ with $(\beta_{i_1}, \beta_{i_2}, \beta_{i_3})$ running over the orbit $\mathfrak{S}_3\beta$. Therefore, we have two hexagons, one defined by the words of the skew-tableaux and the other one defined by the indexing-set frank words. These hexagons are copies of each other since operations based on pairing of parentheses can be reduced to variations of the *jeu de taquin* on two-column frank words and *vice versa*. Taking into account Theorems 2.4 and 2.7, the next statement follows from the hexagon above. Given $\sigma \in \mathfrak{S}_t$, *rev* denotes the longest permutation of $\mathfrak{S}_t$. Let α_1, α_2 be two operators satisfying the Moore–Coxeter relations of the symmetric group $\mathfrak{S}_3$, we put $\langle \alpha_1, \alpha_2 \rangle := \{\alpha_i \alpha_j \dots \alpha_k : i\, j \dots k \in [2]^*\}$.

Theorem 3.1. *Let $\sigma \in \langle s_1, s_2 \rangle$, $\theta \in \langle \theta_1, \theta_2 \rangle$ and $\Theta \in \langle \Theta_1, \Theta_2 \rangle$ with the same reduced word. Let σT be the tableau defined by $\Delta_\alpha UK(\sigma\beta)$, with word σw and indexing-set frank word σJ of column-shape the reverse of $\sigma\beta$. Then $\{\sigma T : \sigma \in \langle s_1, s_2 \rangle\}$ are the vertices of a hexagon such that:*

(*a*) *there exist operators $\tilde\theta_1$ and $\tilde\theta_2$ satisfying the Moore–Coxeter relations of the symmetric group $\mathfrak{S}_3$, where $\tilde\theta \in \langle \tilde\theta_1, \tilde\theta_2 \rangle$, with the same reduced word as θ, verifies $\sigma w = \tilde\theta w \equiv \theta K(\beta) = K(\sigma\beta)$,*

(3.2)
$$\begin{tikzcd}[column sep=small]
 & \tilde\theta_1 w \arrow[r, dash] & \tilde\theta_2\tilde\theta_1 w \arrow[dr, dash] & \\
w \arrow[ur, dash] \arrow[dr, dash] & & & \tilde\theta_1\tilde\theta_2\tilde\theta_1 w. \\
 & \tilde\theta_2 w \arrow[r, dash] & \tilde\theta_1\tilde\theta_2 w \arrow[ur, dash] &
\end{tikzcd}$$

(*b*) *there exist operators $\tilde\Theta_1$ and $\tilde\Theta_2$ satisfying the Moore–Coxeter relations of the symmetric group $\mathfrak{S}_3$, where $\tilde\Theta \in \langle \tilde\Theta_1, \tilde\Theta_2 \rangle$, with the same reduced word as Θ, verifies $\sigma J = \tilde\Theta J$,*

(3.3)
$$\begin{tikzcd}[column sep=small]
 & \tilde\Theta_1 J = J_3G_2G_1 \arrow[r, dash] & \tilde\Theta_2\tilde\Theta_1 J = F_3F_2G_1 \arrow[dr, dash] & \\
J = J_3J_2J_1 \arrow[ur, dash] \arrow[dr, dash] & & & \tilde\Theta_1\tilde\Theta_2\tilde\Theta_1 J = F_3XH_1. \\
 & \tilde\Theta_2 J = L_3L_2J_1 \arrow[r, dash] & \tilde\Theta_1\tilde\Theta_2 J = L_3H_2H_1 \arrow[ur, dash] &
\end{tikzcd}$$

Our aim is to describe explicitly the operators $\tilde{\theta}_i$ and $\tilde{\Theta}_i$ closing respectively the hexagons (3.2) and (3.3), that is, those operators $\tilde{\theta}_i$ and $\tilde{\Theta}_i$ such that $\tilde{\theta}_1\tilde{\theta}_2\tilde{\theta}_1 w = \tilde{\theta}_2\tilde{\theta}_1\tilde{\theta}_2 w$ and $\tilde{\Theta}_1\tilde{\Theta}_2\tilde{\Theta}_1 J = \tilde{\Theta}_2\tilde{\Theta}_1\tilde{\Theta}_2 J$ respectively.

In fact the hexagon (3.1) and, hence, hexagon (3.3), obey the following conditions. (The translation of these conditions to hexagon (3.2) will be done later.)

Lemma 3.2. [5] *The operators closing hexagons* (3.1) *and* (3.3) *obey the following conditions:*

(a) *If* L_3L_2 *and* F_3F_2 *are, respectively, the indexing-set frank words of* $\Delta_\alpha UK(\beta_1, \beta_3)$ *and* $\Delta_\alpha UK(\beta_2, \beta_3)$*, then there holds the inequality* $F_2 \le L_2$*.*

(b) *If* L_3H_2 *and* J_3G_2 *are, respectively, the indexing-set frank words of* $\Delta_\alpha UK(\beta_3, \beta_1)$ *and* $\Delta_\alpha UK(\beta_2, \beta_1)$*, then there holds the inequality*$G_2 \le H_2$*.*

(c) *The operators* $\tilde{\Theta}_1$ *and* $\tilde{\Theta}_2$ *defining the hexagon (3.3) are such that* $\tilde{\Theta}_2[\tilde{\Theta}_1 J] = F_3F_2G_1$ *with* $F_2 \le L_2$*, and* $\tilde{\Theta}_1[\tilde{\Theta}_2 J] = L_3H_2H_1$ *with* $G_2 \le H_2$*.*

Remark 3.3. The condition (c), in the previous lemma, imposed on the operators of the hexagon (3.3) does not come from the braid relations of the operators $\tilde{\Theta}_i$. As can be seen in the example below, there are operators $\tilde{\Theta}_1$ and $\tilde{\Theta}_2$ which close the hexagon and do not satisfy the conditions in (c).

$$
\begin{tikzcd}
 & \begin{array}{ccc} 1 & 3 & \\ & 2 & 4 \end{array} \arrow[r, dash, "\tilde{\Theta}_2"] & \begin{array}{ccc} 2 & & \\ 1 & 3 & 4 \end{array} \arrow[dr, dash, "\tilde{\Theta}_1"] & \\
\begin{array}{ccc} 1 & 2 & 4 \\ & & 3 \end{array} \arrow[ur, dash, "\tilde{\Theta}_1"] \arrow[dr, dash, "\tilde{\Theta}_2"'] & & & \begin{array}{ccc} 2 & & \\ 1 & 3 & 4 \end{array} \\
 & \begin{array}{ccc} 1 & 2 & 4 \\ & & 3 \end{array} \arrow[r, dash, "\tilde{\Theta}_1"'] & \begin{array}{ccc} 1 & 3 & \\ & 2 & 4 \end{array} \arrow[ur, dash, "\tilde{\Theta}_2"'] &
\end{tikzcd}.
$$

We start to analyse the hexagon (3.3) under the conditions in (c) of the previous lemma. The Knuth class of a key-tableau over a three-letter alphabet as well as any frank word with three columns can be characterized in terms of the shuffling operation. This characterization gives a combinatorial explanation of our hexagons (3.1), (3.2) and (3.3). Indeed by Greene's theorem [15], the set of all shuffles of the columns of a key-tableau are contained in its Knuth class. However, under certain conditions we have equality.

Theorem 3.4. [6] *Let* K *be a key-tableau with first column* A*. Then, the Knuth class of* K *is equal to the set of all shuffles of its columns if and only if each of its column is either an interval of* A *or is obtained from an interval of* A *by removing a single letter.*

This criterion can be easily applied considering the planar representation of the weight of the key-tableau. For instance $K(2, 0, 1, 2, 4, 2, 3)$ is the shuffle of its columns, since each column in the planar representation of the weight $(2, 0, 1, 2, 4, 2, 3)$,

$$
\begin{array}{c|cccc}
7 & \bullet & \bullet & \bullet & \\
6 & \bullet & \bullet & & \\
5 & \bullet & \bullet & \bullet & \bullet \\
4 & \bullet & \bullet & & \\
3 & \bullet & & & \\
1 & \bullet & \bullet & & \\
\hline
 & 1 & 2 & 3 & 4
\end{array}
$$

has, at most, one gap of size 1. Each column is either an interval of $A = \{1, 3, 4, 5, 6, 7\}$ or is obtained from an interval of A removing one letter.

Corollary 3.1. *The following statements are equivalent:*

(a) *The Knuth class of a key-tableau over a three-letter alphabet is the set of all shuffles of its columns.*

(b) *J is a three-column frank word if and only if J has one of the following forms*

$$\text{(I)}\ \begin{array}{ccc} A_1^1 & & \\ A_1^2 & A_2^2 & \\ A_1^3 & A_2^3 & A_3^3 \end{array};\quad \text{(II)}\ \begin{array}{ccc} & A_2^1 & \\ A_1^2 & A_2^2 & \\ A_1^3 & A_2^3 & A_3^3 \end{array};\quad \text{(III)}\ \begin{array}{ccc} & & A_3^1 \\ A_1^2 & & A_3^2 \\ A_1^3 & A_2^3 & A_3^3 \end{array};$$

$$\text{(IV)}\ \begin{array}{ccc} & & A_3^1 \\ & A_2^2 & A_3^2 \\ A_1^3 & A_2^3 & A_3^3 \end{array};\quad \text{(V)}\ \begin{array}{ccc} A_1^1 & & \\ A_1^2 & & A_3^2 \\ A_1^3 & A_2^3 & A_3^3 \end{array};\quad \text{(VI)}\ \begin{array}{ccc} & A_2^1 & \\ & A_2^2 & A_3^2 \\ A_1^3 & A_2^3 & A_3^3 \end{array},$$

where $A_1^3 \leq A_2^3 \leq A_3^3$, with $|A_1^3| = |A_2^3| = |A_3^3|$; $A_i^r \cap A_i^s = \emptyset$, for $r \neq s$, $i = 1, 2, 3$, and $A_1^2 \leq A_2^2$, $A_1^2 \leq A_3^2$, $A_2^2 \leq A_3^2$, with $|A_1^2| = |A_2^2| = |A_3^2|$.

Proof. This follows from the previous theorem. $\square$

Every three-column frank word is a shuffle of row words of length ≤ 3. Clearly the operators $\tilde{\Theta}_i$ acting row words of length ≤ 3 close the hexagons, and satisfy trivially the conditions (c) of Lemma 3.2. In this case, we have indeed $\tilde{\Theta}_i = \Theta_i$. But if the column-shape of the frank word is any permutation of $(1, 1, 2)$, the action of the operator Θ_i does not always split into an action on row words of length 3 and on row words of length 1.

Let $J = \begin{array}{ccc} c^2 & b^4 & a^4 \\ & & a^2 \end{array}$ be a three-column contre-tableau. We have to distinguish three situations. Either we have $c^2 \leq a^2 < b^4 \leq a^4$ or $c^2 \leq b^4 \leq a^2 < a^4$ or $a^2 < c^2 \leq b^4 \leq a^4$.

We say that a hexagon closes in the Knuth class when all elements appearing in the hexagon belong to the same Knuth class.

If $c^2 \leq a^2 < b^4 \leq a^4$, there is only one hexagon closing in the Knuth class of J, namely the one defined by $\tilde{\Theta}_i = \Theta_i$. It satisfies conditions (c) of Lemma 3.2, but the operators do not split into operators acting on row words. The full collection of hexagons satisfying conditions (c) of Lemma 3.2, in this case, is described as follows. When $c^2 = a^2$, the following (closing in the Knuth class) is the only hexagon satisfying conditions (c) of Lemma 3.2,

$$\begin{tikzcd}[column sep=small]
& \begin{array}{ccc} & b^4 & a^4 \\ c^2 & a^2 & \end{array} \arrow[r, dash, "\Theta_2"] & \begin{array}{ccc} b^4 & & a^4 \\ c^2 & a^2 & \end{array} \arrow[dr, dash, "\Theta_1"] & \\
J = \begin{array}{ccc} & b^4 & a^4 \\ c^2 & & a^2 \end{array} \arrow[ur, dash, "\Theta_1"] \arrow[dr, dash, "\Theta_2"'] & & & \begin{array}{ccc} b^4 & & a^4 \\ c^2 & a^2 & \end{array} \\
& \begin{array}{ccc} & b^4 & a^4 \\ c^2 & & a^2 \end{array} \arrow[r, dash, "\Theta_1"'] & \begin{array}{ccc} & b^4 & a^4 \\ c^2 & a^2 & \end{array} \arrow[ur, dash, "\Theta_2"'] &
\end{tikzcd}.$$

When $c^2 < a^2$, besides the hexagon closing in the Knuth class, there is still another hexagon (obviously not closing in the Knuth class) in the conditions (c) of Lemma 3.2,

$$
\begin{tikzcd}[column sep=small]
 & \begin{array}{ccc} & b^4 & a^4 \\ c^2 & a^2 & \end{array} \arrow[r, dash, "\tilde{\Theta}_2"] & \begin{array}{ccc} c^2 & & a^4 \\ a^2 & b^4 & \end{array} \arrow[dr, dash, "\Theta_1"] & \\
J = \begin{array}{ccc} & b^4 & a^4 \\ c^2 & & a^2 \end{array} \arrow[ur, dash, "\Theta_1"] \arrow[dr, dash, "\Theta_2"'] & & & \begin{array}{ccc} c^2 & & a^4 \\ a^2 & b^4 & \end{array} \\
 & \begin{array}{ccc} & b^4 & a^4 \\ c^2 & & a^2 \end{array} \arrow[r, dash, "\Theta_1"'] & \begin{array}{ccc} & b^4 & a^4 \\ c^2 & a^2 & \end{array} \arrow[ur, dash, "\tilde{\Theta}_2"'] &
\end{tikzcd}
.
$$

If $c^2 \leq b^4 \leq a^2 < a^4$, we have two hexagons closing in the conditions (c) of Lemma 3.2: one closing in the Knuth class of J, where the operators $\tilde{\Theta}_i = \Theta_i$ can be split by their action on the row words $c^2 \leq b^4 \leq a^2$ and a^4 (for convenience J is written as a shuffle of those rows),

$$
\begin{tikzcd}[column sep=small]
 & \begin{array}{ccc} c^2 & b^4 & a^2 \\ & a^4 & \end{array} \arrow[r, dash, "\Theta_2"] & \begin{array}{ccc} c^2 & b^4 & a^2 \\ a^4 & & \end{array} \arrow[dr, dash, "\Theta_1"] & \\
J = \begin{array}{ccc} c^2 & b^4 & a^2 \\ & & a^4 \end{array} \arrow[ur, dash, "\Theta_1"] \arrow[dr, dash, "\Theta_2"'] & & & \begin{array}{ccc} c^2 & b^4 & a^2 \\ a^4 & & \end{array} \\
 & \begin{array}{ccc} c^2 & b^4 & a^2 \\ & & a^4 \end{array} \arrow[r, dash, "\Theta_1"'] & \begin{array}{ccc} c^2 & b^4 & a^2 \\ & a^4 & \end{array} \arrow[ur, dash, "\Theta_2"'] &
\end{tikzcd}
;
$$

and the other one, when $b^4 < a^2$, not closing in the Knuth class of J, where the operators $\tilde{\Theta}_i$ are split by their action on the row words $c^2 \leq b^4 < a^4$ and a^2,

$$
\begin{tikzcd}[column sep=small]
 & \begin{array}{ccc} c^2 & b^4 & a^4 \\ & a^2 & \end{array} \arrow[r, dash, "\Theta_2"] & \begin{array}{ccc} c^2 & b^4 & a^4 \\ a^2 & & \end{array} \arrow[dr, dash, "\Theta_1"] & \\
J = \begin{array}{ccc} c^2 & b^4 & a^4 \\ & & a^2 \end{array} \arrow[ur, dash, "\tilde{\Theta}_1"] \arrow[dr, dash, "\Theta_2"'] & & & \begin{array}{ccc} c^2 & b^4 & a^4 \\ a^2 & & \end{array} \\
 & \begin{array}{ccc} c^2 & b^4 & a^4 \\ & & a^2 \end{array} \arrow[r, dash, "\tilde{\Theta}_1"'] & \begin{array}{ccc} c^2 & b^4 & a^4 \\ & a^2 & \end{array} \arrow[ur, dash, "\Theta_2"'] &
\end{tikzcd}
.
$$

If $a^2 < c^2 \leq b^4 \leq a^4$, there is only one hexagon satisfying conditions (c) of Lemma 3.2, the one closing in the Knuth class of J, which splits over the rows $c^2 \leq b^4 \leq a^4$ and a^2.

Example 3.5. From the discussion above, there are only two hexagons in the conditions (c) of Lemma 3.2 having the contre-tableau $J = \begin{array}{ccc} 1 & 2 & 4 \\ & & 3 \end{array}$ as a vertex. The second one gives the frank words in the Knuth class.

$$
\begin{tikzcd}[column sep=small]
 & \begin{array}{ccc} 1 & 3 & \\ & 2 & 4 \end{array} \arrow[r, dash, "\Theta_2"] & \begin{array}{ccc} 3 & & \\ 1 & 2 & 4 \end{array} \arrow[dr, dash, "\Theta_1"] & \\
J = \begin{array}{ccc} 1 & 2 & 4 \\ & & 3 \end{array} \arrow[ur, dash, "\tilde{\Theta}_1"] \arrow[dr, dash, "\Theta_2"'] & & & \begin{array}{ccc} 3 & & \\ 1 & 2 & 4 \end{array} \\
 & \begin{array}{ccc} 1 & 2 & 4 \\ & & 3 \end{array} \arrow[r, dash, "\tilde{\Theta}_1"'] & \begin{array}{ccc} 1 & 3 & \\ & 2 & 4 \end{array} \arrow[ur, dash, "\Theta_2"'] &
\end{tikzcd}
,
$$

$$
\begin{tikzcd}
& \begin{smallmatrix}1&4&\\&2&3\end{smallmatrix} \arrow[r, dash, "\Theta_2"] & \begin{smallmatrix}4&&\\1&2&3\end{smallmatrix} \arrow[dr, dash, "\Theta_1"] & \\
J = \begin{smallmatrix}1&2&3\\&&4\end{smallmatrix} \arrow[ur, dash, "\Theta_1"] \arrow[dr, dash, "\Theta_2"'] & & & \begin{smallmatrix}4&&\\1&2&3\end{smallmatrix} \\
& \begin{smallmatrix}1&2&4\\&&3\end{smallmatrix} \arrow[r, dash, "\Theta_1"'] & \begin{smallmatrix}1&4&\\&2&3\end{smallmatrix} \arrow[ur, dash, "\Theta_2"'] &
\end{tikzcd}
.
$$

In the case of the contre-tableau $J' = \begin{smallmatrix}1&3&4\\&&2\end{smallmatrix}$ where $1 < 2 < 3 < 4$, we have also only two hexagons in the conditions (c) of Lemma 3.2. The first hexagon gives the frank words in the Knuth class:

$$
\begin{tikzcd}
& \begin{smallmatrix}1&3&\\&2&4\end{smallmatrix} \arrow[r, dash, "\Theta_2"] & \begin{smallmatrix}1&&\\3&2&4\end{smallmatrix} \arrow[dr, dash, "\Theta_1"] & \\
J' = \begin{smallmatrix}1&3&4\\&&2\end{smallmatrix} \arrow[ur, dash, "\Theta_1"] \arrow[dr, dash, "\Theta_2"'] & & & \begin{smallmatrix}1&&\\3&2&4\end{smallmatrix} \\
& \begin{smallmatrix}1&3&4\\&&2\end{smallmatrix} \arrow[r, dash, "\Theta_1"'] & \begin{smallmatrix}1&2&\\&3&4\end{smallmatrix} \arrow[ur, dash, "\Theta_2"'] &
\end{tikzcd}
,
$$

$$
\begin{tikzcd}
& \begin{smallmatrix}1&3&\\&2&4\end{smallmatrix} \arrow[r, dash, "\tilde\Theta_2"] & \begin{smallmatrix}1&&\\2&3&4\end{smallmatrix} \arrow[dr, dash, "\Theta_1"] & \\
J' = \begin{smallmatrix}1&3&4\\&&2\end{smallmatrix} \arrow[ur, dash, "\Theta_1"] \arrow[dr, dash, "\Theta_2"'] & & & \begin{smallmatrix}1&&\\2&3&4\end{smallmatrix} \\
& \begin{smallmatrix}1&3&4\\&&2\end{smallmatrix} \arrow[r, dash, "\Theta_1"'] & \begin{smallmatrix}1&2&\\&3&4\end{smallmatrix} \arrow[ur, dash, "\tilde\Theta_2"'] &
\end{tikzcd}
.
$$

The meaning of Remark 3.3 becomes now clear. As any three-column frank word is a shuffle of rows of length ≤ 3, and frank words with column-shape equals to a permutation of $(2,1,1)$, then, given a three-column frank word, it is always possible to construct a hexagon satisfying conditions (c) of Lemma 3.2 and having that frank word as a vertex. Define such a hexagon as a *shuffle* of those above. Our next main theorem makes this *shuffle* precise and moreover shows that any hexagon on three-column frank words satisfying conditions (c) of Lemma 3.2 is exactly a shuffle of those hexagons.

Theorem 3.6. *Let $J = J_3J_2J_1$ be a three-column contre-tableau. The following assertions are equivalent.*

(a) *There exist operators $\tilde\Theta_1$ and $\tilde\Theta_2$ defining the hexagon (3.3) such that $\tilde\Theta_2[\tilde\Theta_1 J] = F_3F_2G_1$ with $F_2 \leq L_2$, and $\tilde\Theta_1[\tilde\Theta_2 J] = L_3H_2H_1$ with $G_2 \leq H_2$.*

(*b*) *The contre-tableau J has a decomposition, as below, giving rise to the hexagon of frank words with the same weight, with T a tableau,*

(3.4)

$$
\begin{array}{ccccccc}
 & & \begin{array}{ccc} A_3^5 & A_2^5 & A_1^5 \\ & A_2^4 & A_1^4 \\ & A_2^3 & A_1^3 \\ A_3^2 & A_1^2 & \\ & A_1^1 & \end{array} & \xrightarrow{\tilde{\Theta}_2} & \begin{array}{ccc} A_3^5 & A_2^5 & A_1^5 \\ A_2^4 & & A_1^4 \\ A_2^3 & & A_1^3 \\ A_3^2 & A_1^2 & \\ A_1^1 & & \end{array} & & \\
 & \nearrow^{\tilde{\Theta}_1} & & & & \searrow^{\tilde{\Theta}_1} & \\
J= \begin{array}{ccc} A_3^5 & A_2^5 & A_1^5 \\ & A_2^4 & A_1^4 \\ & A_2^3 & A_1^3 \\ A_3^2 & & A_1^2 \\ & & A_1^1 \end{array} & & & & & & T= \begin{array}{ccc} A_3^5 & A_2^5 & A_1^5 \\ A_2^4 & & A_1^4 \\ A_2^3 & A_1^3 & \\ A_3^2 & A_1^2 & \\ A_1^1 & & \end{array}, \\
 & \searrow_{\tilde{\Theta}_2} & & & & \nearrow_{\tilde{\Theta}_2} & \\
 & & \begin{array}{ccc} A_3^5 & A_2^5 & A_1^5 \\ & A_2^4 & A_1^4 \\ A_2^3 & & A_1^3 \\ A_3^2 & & A_1^2 \\ & & A_1^1 \end{array} & \xrightarrow{\tilde{\Theta}_1} & \begin{array}{ccc} A_3^5 & A_2^5 & A_1^5 \\ & A_2^4 & A_1^4 \\ A_2^3 & A_1^3 & \\ A_3^2 & A_1^2 & \\ & A_1^1 & \end{array} & &
\end{array}
$$

such that the sets A_i^j are pairwise disjoint in each column J_i,

$$A_{i+1}^j \leq A_i^j, \quad \textit{with} \quad |A_{i+1}^j| = |A_i^j|;$$

$$A_3^2 \leq A_1^2 < A_2^4 \leq A_1^4, \quad A_1^2 \cap A_2^4 = \emptyset, \textit{with} \quad |A_3^2| = |A_1^2| = |A_2^4| = |A_1^4|;$$

and $J_1 \cap A_2^5 \subseteq A_1^5$; $(J_1 \setminus A_1^5) \cap A_2^4 \subseteq A_1^4$; $[J_1 \setminus (A_1^5 \cup A_1^4)] \cap A_2^3 \subseteq A_1^3$; $[J_2 \cup (A_1^2 \cup A_1^1)] \cap A_3^2 \subseteq A_1^2$; *and* $[J_2 \cup (A_1^2 \cup A_1^1)] \cap A_3^5 \subseteq A_2^5$.

Proof. $(b) \Rightarrow (a)$ By previous corollary, the vertices of the hexagon (3.4) are frank words with the same weight, and clearly satisfy (c) of Lemma 3.2.

$(a) \Rightarrow (b)$ The frank words $J_3J_2J_1$ and $J_3G_2G_1$ are, respectively, in the conditions (IV) and (II) of Corollary 3.1 and satisfy $\tilde{\Theta}_1 J_3J_2J_1 = J_3G_2G_1$. Then

$$\begin{aligned} &G_1 \subseteq J_1,\ |G_1| = |J_2|,\ J_2 \leq G_1,\ J_1 \cap J_2 \subseteq G_1 \text{ and} \\ &G_2 = J_2 \cup (J_1 \setminus G_1),\ J_3 \leq G_2. \end{aligned} \tag{3.5}$$

Since the frank word $\tilde{\Theta}_2(J_3J_2J_1) = L_3L_2J_1$ satisfy conditions (III) of Corollary 3.1 we have $L_2 \subseteq J_2$, $|L_2| = |J_3|$, $J_3 \leq L_2 \leq J_1$ $J_2 \cap J_3 \subseteq L_2$ and $L_3 = J_3 \cup (J_2 \setminus L_2)$. Again the frank word $F_3F_2G_1 = \tilde{\Theta}_2(J_3G_2G_1)$ satisfy (V) of Corollary 3.1. Then

$$\begin{aligned} &F_2 \subseteq G_2,\ |F_2| = |J_3|,\ J_3 \leq F_2 \leq G_1,\ G_2 \cap J_3 \subseteq F_2 \text{ and} \\ &F_3 = J_3 \cup (G_2 \setminus F_2). \end{aligned} \tag{3.6}$$

By (3.5) and (3.6), we have $F_2 \subseteq G_2 = J_2 \cup (J_1 \setminus G_1)$. Thus, we may write $F_2 = A_2^5 \cup A_1^2$, with $A_2^5 \subseteq J_2$ and $A_1^2 \subseteq J_1 \setminus G_1$. Moreover, since $J_3 \leq F_2$, we may also write $J_3 = A_3^5 \cup A_3^2$, where $A_3^5 \leq A_2^5$ e $A_3^2 \leq A_1^2$ satisfy $|A_3^5| = |A_2^5|$, $|A_3^2| = |A_1^2|$, $G_2 \cap A_3^5 \subseteq A_2^5$ and $G_2 \cap A_3^2 \subseteq A_1^2$. We define $A_1^1 = J_1 \setminus (G_1 \cup A_1^2)$, therefore $J_1 \setminus G_1 = A_1^1 \cup A_1^2$.

The frank word $F_3XH_1 = \tilde{\Theta}_1 F_3F_2G_1$ satisfy (I) of Corollary 3.1. Then

$$\begin{aligned} &H_1 \subseteq G_1,\ |H_1| = |F_2|,\ F_2 \leq H_1,\ F_2 \cap G_1 \subseteq H_1 \text{ and} \\ &F_3 \triangleright X = F_2 \cup (G_1 \setminus H_1) \triangleright H_1. \end{aligned} \tag{3.7}$$

Since $F_2 = A_2^5 \cup A_1^2 \leq H_1$, we can define $A_1^5 = \min\{Z \subseteq H_1 : |Z| = |A_2^5| \text{ and } A_2^5 \leq Z\}$, where the minimum is taken with respect to $\leq$, and $A_1^4 = H_1 \setminus A_1^5$. As $H_1 \subseteq G_1$, put $A_1^3 = G_1 \setminus H_1$. We have $H_1 = A_1^5 \cup A_1^4$ and $X = A_2^5 \cup A_1^2 \cup A_1^3$. From $F_2 \leq H_1$ and the definition of A_1^5, we get $A_3^5 \leq A_2^5 \leq A_1^5$ and $A_3^2 \leq A_1^2 < A_1^4$, where $A_1^2 \cap A_1^4 = \emptyset$. Note that from (3.5) and (3.7), we obtain $J_1 \cap A_2^5 \subseteq A_1^5$. By Lemma 3.2

$$F_2 \leq L_2. \tag{3.8}$$

Now we consider the bottom edges of our hexagon (3.3). Since the frank word $L_3H_2H_1 = \tilde{\Theta}_1(L_3L_2J_1)$ satisfy (II) of Corollary 3.1 we have

$$\begin{aligned} &H_1 \subseteq J_1,\ |H_1| = |L_2|,\ L_2 \leq H_1,\ L_2 \cap J_1 \subseteq H_1 \text{ and} \\ &L_3 \leq H_2 = L_2 \cup (J_1 \setminus H_1) \triangleright H_1. \end{aligned} \tag{3.9}$$

By Lemma 3.2, (c), we have

$$G_2 \leq H_2. \tag{3.10}$$

Finally, since $F_3XH_1 = \tilde{\Theta}_2(L_3H_2H_1)$, we have $X \subseteq H_2,\ |X| = |L_3|,\ L_3 \leq X,\ H_2 \cap L \subseteq X$ and $F_3 = L_3 \cup (H_2 \setminus X)$.

By (3.9) and $A_2^5 \cup A_1^2 \cup A_1^3 = X_2 \subseteq H_2 = L_2 \cup A_1^1 \cup A_1^2 \cup A_1^3$, we conclude that $A_2^5 \subseteq L_2 \cup A_1^1$. Since A_2^5 and A_1^1 are disjoint sets, it follows $A_2^5 \subseteq L_2$. Define $A_2^4 = L_2 \setminus A_2^5$ and $A_2^3 = J_2 \setminus L_2$. As $|L_2| = |H_1|$, we also have $|A_1^4| = |A_2^4|$, $|A_1^3| = |A_2^3|$, $(J_1 \setminus A_1^5) \cap A_2^4 \subseteq A_1^4$ and $(J_1 \setminus (A_1^5 \cup A_1^4)) \cap A_2^3 \subseteq A_1^3$. Moreover from the inequality $L_2 \leq H_1$, we get $A_2^4 \leq A_1^4$. By (3.8) and (3.5), we get $A_1^2 < A_2^4$ with $A_1^2 \cap A_1^4 = \emptyset$, and by (3.10), we have $A_2^3 \leq A_1^3$. $\square$

Considering all tableaux of a given shape and weight, this theorem defines an action of the symmetric group on frank words in the union of the Knuth classes of those tableaux. See Example 3.15.

A right (left) key $\widetilde{K}_+(T)$ ($\widetilde{K}_-(T)$) of the tableau T is the key-tableau of the same shape as T whose j-th column is the first (last) column of any skew-tableau in a hexagon (3.4) with the following property: its first (last) column has the same length as the j-th column of T (cf. [19]).

From the hexagon (3.4) we get, respectively, a right key of T,

$$\widetilde{K}_+(T) = \begin{matrix} A_1^5 & A_1^5 & A_1^5 \\ A_1^4 & A_1^4 & A_1^4 \\ A_1^3 & A_1^3 & \\ A_1^2 & & \\ A_1^1 & & \end{matrix},$$

and a left key of T,

$$\widetilde{K}_-(T) = \begin{matrix} A_3^5 & A_3^5 & A_3^5 \\ A_2^4 & & \\ A_2^3 & A_2^3 & \\ A_3^2 & A_3^2 & A_3^2 \\ A_1^1 & & \end{matrix},$$

with $\widetilde{K}_+(T) \geq \widetilde{K}_-(T)$.

Example 3.7. Below we give two decompositions of the tableau $T = \begin{matrix} 5 & & \\ 4 & & \\ 3 & 3 & \\ 2 & 2 & 5 \end{matrix}$ leading to different left and right keys. The second hexagon gives the frank words in the Knuth class of T:

(3.11)

$$
\begin{tikzcd}[column sep=small]
& \begin{matrix} & 3 & 3 \\ & 5 & 5 \\ 2 & 2 & \\ & 4 & \end{matrix} \arrow[rr, dash, "\Theta_2"] & & \begin{matrix} 3 & & 3 \\ 5 & & 5 \\ 2 & 2 & \\ 4 & & \end{matrix} \arrow[dr, dash, "\tilde{\Theta}_1"] & \\
\begin{matrix} & 3 & 3 \\ & 5 & 5 \\ 2 & & 2 \\ & & 4 \end{matrix} \arrow[ur, dash, "\Theta_1"] \arrow[dr, dash, "\tilde{\Theta}_2"'] & & & & \begin{matrix} 3 & 3 & \\ 5 & & 5 \\ 2 & 2 & \end{matrix} = T \\
& \begin{matrix} 3 & & 3 \\ & 5 & 5 \\ 2 & & 2 \\ & & 4 \end{matrix} \arrow[rr, dash, "\Theta_1"'] & & \begin{matrix} & 3 & \\ & 5 & 5 \\ 2 & 2 & \\ & 4 & \end{matrix} \arrow[ur, dash, "\Theta_2"'] &
\end{tikzcd}
\ , \ K_+(T) = \begin{matrix} 5 & & \\ 4 & & \\ 3 & 5 & \\ 2 & 3 & 5 \end{matrix} ,
$$

$$
\begin{tikzcd}[column sep=small]
& \begin{matrix} & 3 & 3 \\ & 5 & 5 \\ 2 & 2 & \\ & 4 & \end{matrix} \arrow[rr, dash, "\Theta_2"] & & \begin{matrix} 3 & & 3 \\ 5 & & 5 \\ 2 & 2 & \\ 4 & & \end{matrix} \arrow[dr, dash, "\Theta_1"] & \\
\begin{matrix} & 3 & 3 \\ & 5 & 5 \\ 2 & & 2 \\ & & 4 \end{matrix} \arrow[ur, dash, "\Theta_1"] \arrow[dr, dash, "\Theta_2"'] & & & & \begin{matrix} 3 & & 3 \\ 5 & 5 & \\ 2 & 2 & \\ 4 & & \end{matrix} = T \\
& \begin{matrix} & 3 & 3 \\ 5 & & 5 \\ 2 & & 2 \\ & & 4 \end{matrix} \arrow[rr, dash, "\Theta_1"'] & & \begin{matrix} & 3 & 3 \\ 5 & 5 & \\ 2 & 2 & \\ & 4 & \end{matrix} \arrow[ur, dash, "\Theta_2"'] &
\end{tikzcd}
\ , \ K_+(T) = \begin{matrix} 5 & & \\ 4 & & \\ 3 & 5 & \\ 2 & 3 & 3 \end{matrix} .
$$

We may now describe the hexagon (3.2). Without loss of generality, we may consider the hexagon (3.4) in simplified form in the sense that the sets A_i^j are singular,

$$J = \begin{tikzcd}[column sep=small]
 & \begin{array}{ccc} c^5 & b^5 & a^5 \\ & b^4 & a^4 \\ & b^3 & a^3 \\ c^2 & a^2 & \\ & a^1 & \end{array} \arrow[r, dash, "\tilde{\Theta}_2"] & \begin{array}{ccc} c^5 & b^5 & a^5 \\ b^4 & & a^4 \\ b^3 & & a^3 \\ c^2 & a^2 & \\ a^1 & & \end{array} \arrow[dr, dash, "\tilde{\Theta}_1"] & \\
\begin{array}{ccc} c^5 & b^5 & a^5 \\ & b^4 & a^4 \\ & b^3 & a^3 \\ c^2 & & a^2 \\ & & a^1 \end{array} \arrow[ur, dash, "\tilde{\Theta}_1"] \arrow[dr, dash, "\tilde{\Theta}_2"'] & & & \begin{array}{ccc} c^5 & b^5 & a^5 \\ b^4 & & a^4 \\ b^3 & a^3 & \\ c^2 & a^2 & \\ a^1 & & \end{array} \\
 & \begin{array}{ccc} c^5 & b^5 & a^5 \\ & b^4 & a^4 \\ b^3 & & a^3 \\ c^2 & & a^2 \\ & & a^1 \end{array} \arrow[r, dash, "\tilde{\Theta}_1"'] & \begin{array}{ccc} c^5 & b^5 & a^5 \\ & b^4 & a^4 \\ b^3 & a^3 & \\ c^2 & a^2 & \\ & a^1 & \end{array} \arrow[ur, dash, "\tilde{\Theta}_2"'] &
\end{tikzcd}, \tag{3.12}$$

with $c^5 \leq b^5 \leq a^5$, $b^3 \leq a^3$, and $c^2 \leq a^2 < b^4 \leq a^4$. The contre-tableau J is therefore split into the frank word $Y_1 = c^2b^4a^4a^2$ of shape $(1,1,2)$, and the row words $X_2 = c^5b^5a^5, X_3 = b^3a^3$, and $X_4 = a^1$. Let $X_1 = c^2a^2b^4a^4$. We consider the biwords with pairwise distinct biletters

$$\Sigma' = \begin{pmatrix} J_3 & J_2 & J_1 \\ 3^2 & 2^3 & 1^5 \end{pmatrix} \longleftrightarrow \Pi = \begin{pmatrix} c^2a^2b^4a^4 & c^5b^5a^5 & b^3a^3 & a^1 \\ 3\ 1\ 2\ 1 & 3\ 2\ 1 & 2\ 1 & 1 \end{pmatrix} \longleftrightarrow \Sigma = \begin{pmatrix} (J_3J_2J_1)\uparrow \\ w \end{pmatrix}, \tag{3.13}$$

where Π is obtained by sorting the biletters of Σ', and Σ is obtained by sorting the biletters of Π in weakly increasing rearrangement for the anti-lexicographic order. Since $(J_3J_2J_1)\uparrow$ is a shuffle of the words X_1, X_2, X_3 and X_4, then w is a shuffle of 3121, 321, 21 and 1 such that the biword Σ is a shuffle of $\begin{pmatrix} X_1 \\ 3121 \end{pmatrix}$, $\begin{pmatrix} X_2 \\ 321 \end{pmatrix}$, $\begin{pmatrix} X_3 \\ 21 \end{pmatrix}$ and $\begin{pmatrix} X_4 \\ 1 \end{pmatrix}$. Therefore the hexagon (3.2) is a "*shuffle*" of four elementary hexagons,

$$\begin{tikzcd}[column sep=small]
 & \begin{pmatrix} c^2a^2b^4a^4 \\ 3\ 2\ 2\ 1 \end{pmatrix} \arrow[r, dash, "\theta_2"] & \begin{pmatrix} c^2a^2b^4a^4 \\ 3\ 2\ 3\ 1 \end{pmatrix} \arrow[dr, dash, "\theta_1"] & \\
\begin{pmatrix} c^2a^2b^4a^4 \\ 3\ 1\ 2\ 1 \end{pmatrix} \arrow[ur, dash, "\theta_1"] \arrow[dr, dash, "\theta_2"'] & & & \begin{pmatrix} c^2a^2b^4a^4 \\ 3\ 2\ 3\ 1 \end{pmatrix} \\
 & \begin{pmatrix} c^2a^2b^4a^4 \\ 3\ 1\ 2\ 1 \end{pmatrix} \arrow[r, dash, "\theta_1"'] & \begin{pmatrix} c^2a^2b^4a^4 \\ 3\ 2\ 2\ 1 \end{pmatrix} \arrow[ur, dash, "\theta_2"'] &
\end{tikzcd} \tag{3.14}$$

$$\begin{tikzcd}[column sep=small]
 & \begin{pmatrix} c^5a^5b^5 \\ 3\ 2\ 1 \end{pmatrix} \arrow[r, dash, "\theta_2"] & \begin{pmatrix} c^5a^5b^5 \\ 3\ 2\ 1 \end{pmatrix} \arrow[dr, dash, "\theta_1"] & \\
\begin{pmatrix} c^5a^5b^5 \\ 3\ 2\ 1 \end{pmatrix} \arrow[ur, dash, "\theta_1"] \arrow[dr, dash, "\theta_2"'] & & & \begin{pmatrix} c^5a^5b^5 \\ 3\ 2\ 1 \end{pmatrix} \\
 & \begin{pmatrix} c^5a^5b^5 \\ 3\ 2\ 1 \end{pmatrix} \arrow[r, dash, "\theta_1"'] & \begin{pmatrix} c^5a^5b^5 \\ 3\ 2\ 1 \end{pmatrix} \arrow[ur, dash, "\theta_2"'] &
\end{tikzcd} \tag{3.15}$$

$$\begin{tikzcd}
& \begin{pmatrix} b^3a^3 \\ 2\,1 \end{pmatrix} \arrow[r, dash, "\theta_2"] & \begin{pmatrix} b^3a^3 \\ 3\,1 \end{pmatrix} \arrow[dr, dash, "\theta_1"] & \\
\begin{pmatrix} b^3a^3 \\ 2\,1 \end{pmatrix} \arrow[ur, dash, "\theta_1"] \arrow[dr, dash, "\theta_2"'] & & & \begin{pmatrix} b^3a^3 \\ 3\,2 \end{pmatrix} \\
& \begin{pmatrix} b^3a^3 \\ 3\,1 \end{pmatrix} \arrow[r, dash, "\theta_1"'] & \begin{pmatrix} b^3a^3 \\ 3\,2 \end{pmatrix} \arrow[ur, dash, "\theta_2"'] &
\end{tikzcd} \tag{3.16}$$

$$\begin{tikzcd}
& \begin{pmatrix} a^1 \\ 2 \end{pmatrix} \arrow[r, dash, "\theta_2"] & \begin{pmatrix} a^1 \\ 3 \end{pmatrix} \arrow[dr, dash, "\theta_1"] & \\
\begin{pmatrix} a^1 \\ 1 \end{pmatrix} \arrow[ur, dash, "\theta_1"] \arrow[dr, dash, "\theta_2"'] & & & \begin{pmatrix} a^1 \\ 3 \end{pmatrix} \\
& \begin{pmatrix} a^1 \\ 1 \end{pmatrix} \arrow[r, dash, "\theta_1"'] & \begin{pmatrix} a^1 \\ 2 \end{pmatrix} \arrow[ur, dash, "\theta_2"'] &
\end{tikzcd} \tag{3.17}$$

A Yamanouchi tableau is a key-tableau whose shape and weight coincide. A Yamanouchi word is a word congruent to a Yamanouchi tableau. By Corollary 3.1, every three-letter Yamanouchi word w is a shuffle of $k \geq 0$ words 3121, $l_1 \geq 0$ words 321, $l_2 \geq 0$ words 21 and $l_3 - k \geq 0$ words 1, that, by abuse of notation, we shall write $w = shuffle((3121)^k, (321)^{l_1}, (21)^{l_2}, 1^{l_3-k})$.

Theorem 3.8. *The hexagon* (3.2) *is a "shuffle" of the hexagons defined by the bottom rows of the four hexagons* (3.14), (3.15), (3.16), *and* (3.17) *with appropriate multiplicities. That is, there exist a shuffle of* $k \geq 0$ *words* 3121, $l_1 \geq 0$ *words* 321, $l_2 \geq 0$ *words* 21, *and* $l_3 - k \geq 0$ *words* 1, $w = shuffle((3121)^k, (321)^{l_1}, (21)^{l_2}, 1^{l_3-k})$, *such that*

(*a*) $\tilde{\theta}_i w = shuffle((\theta_i 3121)^k, (\theta_i 321)^{l_1}, (\theta_i 21)^{l_2}, (\theta_i 1)^{l_3-k})$, $i = 1, 2$;

(*b*) $\tilde{\theta}_i \tilde{\theta}_j w = shuffle((\theta_i\theta_j 3121)^k, (\theta_i\theta_j 321)^{l_1}, (\theta_i\theta_j 21)^{l_2}, (\theta_i\theta_j 1)^{l_3-k})$, $1 \leq i \neq j \leq 2$;

(*c*) $\tilde{\theta}_1\tilde{\theta}_2\tilde{\theta}_1 w = shuffle((\theta_1\theta_2\theta_1 3121)^k, (\theta_1\theta_2\theta_1 321)^{l_1}, (\theta_1\theta_2\theta_1 21)^{l_2}, (\theta_1\theta_2\theta_1 1)^{l_3-k})$.

Example 3.9. The hexagon (3.11) gives rise to the hexagon below, where the operators are based on non-standard pairing of parentheses

$$\begin{tikzcd}
& \underline{3}\,\underline{2}\,2\,1\,\bar{2}\,\underline{2}\,\underline{1} \arrow[r, dash, "\theta_2"] & \underline{3}\,\underline{2}\,3\,1\,\bar{3}\,\underline{3}\,\underline{1} \arrow[dr, dash, "\tilde{\theta}_1"] & \\
\underline{3}\,\underline{1}\,2\,1\,\bar{1}\,\underline{2}\,\underline{1} \arrow[ur, dash, "\theta_1"] \arrow[dr, dash, "\tilde{\theta}_2"'] & & & \underline{3}\,\underline{2}\,3\,2\,\bar{3}\,\underline{3}\,\underline{1} \\
& \underline{3}\,\underline{1}\,3\,1\,\bar{1}\,\underline{2}\,\underline{1} \arrow[r, dash, "\theta_1"'] & \underline{3}\,\underline{2}\,3\,2\,\bar{2}\,\underline{2}\,\underline{1} \arrow[ur, dash, "\theta_2"'] &
\end{tikzcd} \tag{3.18}$$

(the barred letters indicate the subwords 3121 and 1 in the shuffle).

Remark 3.10. The following example is the translation of Remark 3.3 to hexagon (3.2). The hexagon

$$\begin{tikzcd}
& 3221 \arrow[r, dash, "\tilde{\theta}_2"] & 3321 \arrow[dr, dash, "\tilde{\theta}_1"] & \\
3211 \arrow[ur, dash, "\tilde{\theta}_1"] \arrow[dr, dash, "\tilde{\theta}_2"'] & & & 3321 \\
& 3211 \arrow[r, dash, "\tilde{\theta}_1"'] & 3221 \arrow[ur, dash, "\tilde{\theta}_2"'] &
\end{tikzcd}$$

is not a shuffle of the two hexagons (3.15) and (3.17).

Next we show that the family of actions of $\mathfrak{S}_3$ defined by the operators $\tilde{\theta}_i$ ($\tilde{\Theta}_i$), $i = 1, 2$, based on shuffle decompositions of a three-letter Yamanouchi word w (three-column tableau) as shown in the previous theorems, includes the action defined by the operators θ_i (Θ_i), $i = 1, 2$. This is achieved in the following algorithm, where a special shuffle decomposition for a three-letter Yamanouchi word w is exhibited. Using (3.13), it follows that the hexagon (3.4) contains, in particular, the action defined by the operators Θ_i. That is, the Lascoux–Schützenberger action of the symmetric group: on frank words is obtained from a particular shuffle decomposition of a three-column tableau into rows of length ≤ 3, and tableaux of column-shape $(2, 1, 1)$; and on words congruent to key-tableaux is obtained from a shuffle decomposition of a three-letter Yamanouchi word into words 3121 and column-words 321, 21, 1.

Let $w = w_1 \cdots w_l$ be a word of length l. We denote by $w_{|A}$ the subword of w obtained by suppressing the letters not in A. If $X \subseteq [l]$, then $w|X$ is the subword of w defined by the letters of w in positions X. If $X, Y \subseteq [l]$ with $X \cap Y = \emptyset$, then $w|(X, Y)$ is the shuffle of the subwords $w|X$ and $w|Y$ defined by the letters of w in positions $X \cup Y$. By induction, we define $w|(X_1, \ldots, X_k)$, for any $k \geq 0$, putting the empty word for $k = 0$. Given the positive integers $i \leq j$, we put $[i, j]$ for the set $\{i, i + 1, \ldots, j\}$.

Algorithm 3.11. Let w be a Yamanouchi word over a three-letter alphabet. Our algorithm is consists of three steps.

Step 1. Consider the subword $w_{|\{2,1\}}$ and bracket every factor 21 of $w_{|\{2,1\}}$. The letters which are not bracketed constitute a subword of $w_{|\{2,1\}}$. Then bracket every factor 21 of this subword. Again, the letters which are not bracketed constitute a subword. Continue this procedure until it stops, that is, until we get a word consisting of l_1 non-bracketed letters $1's$ in w. This bracketing process enables us to decompose w as

$$w|(I_1, \ldots, I_{l_3+l_2}, J_1, \ldots, J_{l_3}, K_1, \ldots, K_{l_1}), \tag{3.19}$$

where $w|I_l = 21$, $l \in [l_3 + l_2]$, $w|J_l = 3$, $l \in [l_3]$, and $w|K_l = 1$, $l \in [l_1]$.

Step 2. Let w' be the subword of w obtained by removing all letters 1 belonging to the factors $w|I_l$, for all $l \in [l_3 + l_2]$. As in the previous step, we bracket all the successive factors 32 and 31 of w'. We get a new decomposition (3.19), by making the unions of k sets J_l with k sets K_l, for some integer $0 \leq q \leq \min\{l_3, l_1\}$, and making the unions of the remaining $l_3 - q$ sets J_l with $l_3 - q$ sets I_l:

$$w|(F_1, \ldots, F_q, G_1, \ldots, G_{l_3-q}, I_1, \ldots, I_{l_2+q}, K_1, \ldots, K_{l_1-q}),$$

where $w|F_l = 31$, $l \in [q]$, $w|G_l = 321$, $l \in [l_3 - q]$, $w|I_l = 21$, $l \in [l_2 + q]$, and $w|K_l = 1$, $l \in [l_1 - q]$ (reordering the sets I_i's, J_j's and K_l's in (3.19) if necessary).

Step 3. Finally, let w'' be the subword of w obtained by removing the subwords $w|G_l = 321$ and $w|K_l = 1$, for all $l \geq 1$. As before, we bracket all the successive factors 3121 of w''. This operator consists of the union of the q sets F_l with q sets I_l. The decomposition of w obtained in this way, is denoted by $w|(I_1^*, \ldots, I_{l_3+l_2+l_1-q}^*)$, where $w|I_l^* = 3121$, $l \in [q]$, $w|I_l^* = 321$, $l \in [q + 1, l_3]$, $w|I_l^* = 21$, $l \in [l_3 + 1, l_3 + l_2]$, and $w|I_l^* = 1$, $l \in [l_3 + l_2 + 1, l_3 + l_2 + l_1 - q]$.

The next example illustrates the application of the previous algorithm.

Example 3.12. Let $w = 33121121 \equiv K(4,2,2)$. Following the first step of Algorithm 3.11, we bracket all the successive factors 21 of $w_{|\{1,2\}}$, that is, 331(21)1(21), obtaining in this way the decomposition

$$w = w|(\{4,5\},\{7,8\},\{1\},\{2\},\{3\},\{6\}),$$

where $w|\{4,5\} = w|\{7,8\} = 21$, $w|\{1\} = w|\{2\} = 3$ and $w|\{3\} = w|\{6\} = 1$. Next, let $w' = 3312-12-$ (where $-$ indicates the place of the suppressed letters of w) be the subword of w obtained by removing the letters 1 belonging to $w|\{4,5\}$ and $w|\{7,8\}$, and bracket all the successive factors 31 and 32 of w'. Thus, we have $w' = 3(31)2-12-$, with the letters 3 and 1 belonging to $w|\{2\}$ and $w|\{3\}$, respectively; and then, we have $w'_1 = (3--2)-12-$, with the letters 3 and 2 of this factor belonging to $w|\{1\}$ and $w|\{4,5\}$, respectively. Then, we get the decomposition

$$w = w|(\{1,4,5\},\{7,8\},\{2,3\},\{6\}),$$

with $w|\{1,4,5\} = 321$, $w|\{7,8\} = 21$, $w|\{2,3\} = 31$ and $w|\{6\} = 1$. Finally, let $w'' = -31---21$ be the subword of w obtained by removing the subwords $w|(\{1,4,5\} = 321$ and $w|\{6\} = 1$. This word has only one factor 3121 and thus we get the decomposition

$$w = w|(\{2,3,7,8\}^*,\{1,4,5\}^*,\{6\}^*) = \underline{3}\,\overline{3\,1}\,\underline{2\,1}\,1\,\overline{2\,1},$$

where the underlined letters define 3121, the overlined letters define 321 and the remaining letter defines the shuffle component 1. It is easy to check that the parentheses matching operations induced by this decomposition are the standard ones:

$$\begin{tikzcd}
 & \underline{3}\,\overline{3\,2}\,\underline{2\,1}\,2\,\overline{2\,1} \arrow[r, dash, "\theta_2"] & \underline{3}\,\overline{3\,2}\,\underline{2\,1}\,3\,\overline{3\,1} \arrow[dr, dash, "\theta_1"] & \\
\underline{3}\,\overline{3\,1}\,\underline{2\,1}\,1\,\overline{2\,1} \arrow[ur, dash, "\theta_1"] \arrow[dr, dash, "\theta_2"'] & & & \underline{3}\,\overline{3\,2}\,\underline{2\,1}\,3\,\overline{3\,1} \\
 & \underline{3}\,\overline{3\,1}\,\underline{2\,1}\,1\,\overline{2\,1} \arrow[r, dash, "\theta_1"'] & \underline{3}\,\overline{3\,2}\,\underline{2\,1}\,2\,\overline{2\,1} \arrow[ur, dash, "\theta_2"'] &
\end{tikzcd} \tag{3.20}$$

Theorem 3.13. *Let w be a Yamanouchi word over a three-letter alphabet and consider the decomposition $w|(I_1^*,\dots,I_q^*)$ given by Algorithm 3.11. Then, for each $ij\cdots k \in [2]^*$, $\theta_i\theta_j\cdots\theta_k(w) = w^{ij\cdots k}$ satisfy $w^{ij\cdots k}|I_l = \theta_i\theta_j\cdots\theta_k(w|I_l)$, for all $l = 1,\dots,q$.*

Proof. By the construction of $w|(I_1^*,\dots,I_q^*)$, it is clear that $\theta_i\theta_j\cdots\theta_k(w) = w^{ij\cdots k}$, for $ij\cdots k \in \{1,21,121\}$. For the computation of $\theta_2(w)$, we must match all successive factors 32 of $w_{|\{2,3\}}$, until we get the subword 2^{l_2}, for some non-negative integer l_2. Each matched pair 32 belongs either to a component 321, or 3121, while the letters 2 of the subword 2^{l_2} belong to components 21. The word $\theta_2(w)$ is then obtained by replacing in w the subword 2^{l_2} by 3^{l_2}. Since $\theta_2(3121) = 3121$, $\theta_2(321) = 321$ and $\theta_2(21) = 31$, it follows that $\theta_2(w) = w^2$.

Finally, consider the subword $\theta_1\theta_2(w)$, obtained by matching the successive factors 21 of $\theta_2(w)_{|\{1,2\}}$. By the construction of $w|(I_1^*,\dots,I_q^*)$, each one of the matched pairs 21 belongs either to a component 321 or 3121. On the other hand, each letter of the subword $1^{l_2+l_1}$, obtained by removing the matched pairs, is itself a component or it represents the leftmost letter 1 of a component 3121. Since $\theta_1\theta_2(3121) = 3221$ and $\theta_1\theta_2(1) = 2$, we get $\theta_1\theta_2(w) = w^{12}$. $\square$

Finally, our construction in Theorem 3.6 has a matrix interpretation.

Theorem 3.14. [5] *To each hexagon* (3.4) *it corresponds a hexagon* (3.1). *That is, given a hexagon* (3.4), $\exists U \sim I$ *such that, for some partition* α, $\{\Delta_\alpha UK(\sigma\beta) : \sigma \in \mathfrak{S}_3\}$ *is a hexagon whose indexing-set frank words are those of* (3.4).

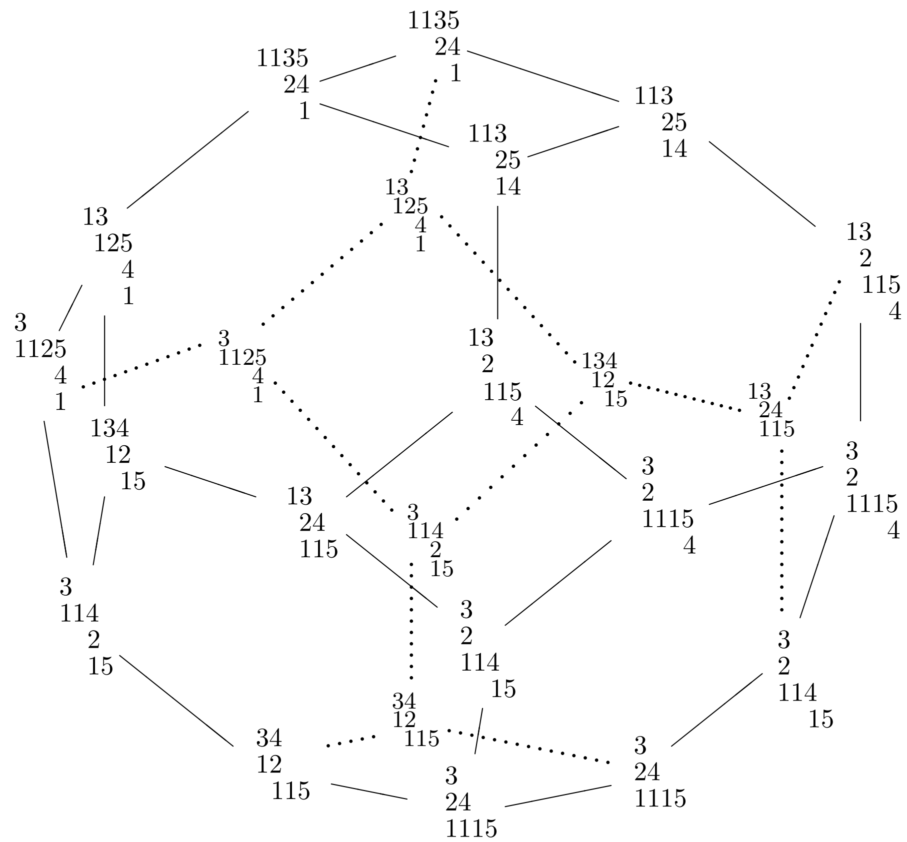
Next we give an example of our construction in Theorems 3.6 and 3.8 in the case of $\mathfrak{S}_4$.

Example 3.15. Dual permutahedra in $\mathfrak{S}_4$ generated by *variants of jeu de taquin* and *non-standard reflection crystal operators.*

Consider the Yamanouchi word $w = 4312211 \in [4]^*$ and the contre-tableau $J = 1\,1\,32\,451$. The biwords $\Sigma = \begin{pmatrix} 1112345 \\ 4312211 \end{pmatrix}$ and $\Sigma' = \begin{pmatrix} 1\,1\,32\,451 \\ 4\,3\,22\,111 \end{pmatrix}$ correspond by the dual RSK to the pair (K, P), with $K = 4321\,21\,1$ the Yamanouchi tableau of shape $(3, 2, 1, 1)$, and $P = 321\,51\,1\,4$.

The vertices of the following permutahedron in $\mathfrak{S}_4$ contains the contre-tableau $J = 1\,1\,32\,541 \equiv P = 321\,51\,1\,4$ and the tableau $T = 321\,41\,1\,5$ of the same shape and weight as P. The remaining ones are frank words either in the Knuth classes of P or T.

(3.21)

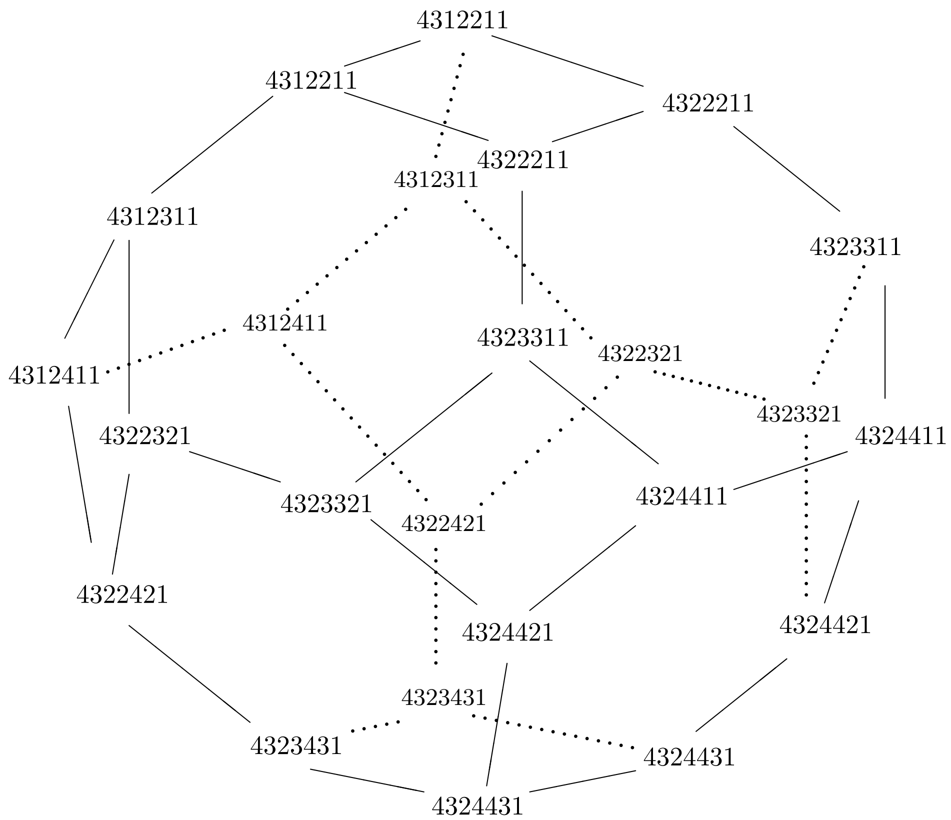


The vertices of the corresponding dual permutahedron in $\mathfrak{S}_4$ contain the Yamanouchi words $4312211 \equiv K(3,2,1,1)$ and $4324431 \equiv K(1,1,2,3)$ with Q-symbols

$$Q = \begin{array}{lll} 6 & & \\ 3 & & \\ 2 & 7 & \\ 1 & 4 & 5 \end{array} = (std\, P)^{\tau} \quad \text{and} \quad Q' = \begin{array}{lll} 7 & & \\ 3 & & \\ 2 & 6 & \\ 1 & 4 & 5 \end{array} = (std\, T)^{\tau}, \text{ respectively.}$$

The Q-symbols of the remain ones are either Q or Q'.

(3.22)



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