Littlewood-Richardson coefficients and the hive model

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Schur functions

- Let n be a fixed positive integer and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ a sequence of indeterminates.
- Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition of weight $|\lambda|$ and length $\ell(\lambda) \leq n$, so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$.
- Then the Schur function $s_\lambda(\mathbf{x})$ is defined by:

$$s_\lambda(\mathbf{x}) = \frac{\left| x_i^{n+\lambda_j-j} \right|_{1\leq i,j\leq n}}{\left| x_i^{n-j} \right|_{1\leq i,j\leq n}}.$$

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- The Schur functions form a $\mathbb{Z}$-basis of Λ_n, the ring of polynomial symmetric functions of $x_1, \ldots, x_n$.

LR-coefficients

- Any product of Schur functions can be expressed as a linear sum of Schur functions.

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$$s_\lambda(\mathbf{x})\ s_\mu(\mathbf{x}) = \sum_\nu c_{\lambda\mu}^\nu\ s_\nu(\mathbf{x})$$

- Each Littlewood-Richardson coefficient $c_{\lambda\mu}^\nu$ is a non-negative integer.
- They may be evaluated by means of the Littlewood-Richardson rule.

LR-rule

- Fill the boxes of the Young diagram F^λ with 0's. Then fill the boxes of the skew Young diagram $F^{\nu/\lambda}$ with μ_i entries i for $i = 1, 2, \ldots, n$.
- $c_{\lambda\mu}^{\nu}$ is the number of such diagrams with entries weakly increasing across rows, strictly increasing down columns, and satisfying the lattice permutation rule.

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- Ex. $n = 3$, $\lambda = (2, 1, 0)$, $\mu = (3, 2, 0)$, $\nu = (4, 3, 1)$

0	0	1	1
0	1	2	
2			

0	0	1	1
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1			

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- Hence $c^\nu_{\lambda\mu} = 2$.

Stretched LR coefficients

- Littlewood-Richardson coefficient $c^{\nu}_{\lambda\mu}$
- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ stretching parameter $t \in \mathbb{N}$
- Stretched partition $t\lambda = (t\lambda_1, t\lambda_2, \ldots, t\lambda_n)$
- Stretched Littlewood-Richardson coefficient $c^{t\nu}_{t\lambda, t\mu}$

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- Ex: $n = 3$, $\lambda = (2, 1, 0)$, $\mu = (3, 2, 0)$, $\nu = (4, 3, 1)$
  - $t = 1$: $c^{431}_{21,32} = 2$
  - $t = 2$: $c^{862}_{42,64} = 3$
  - $t = 3$: $c^{12\,93}_{63,94} = 4$
  - $\cdots$
  - suggests $c^{t\nu}_{t\lambda,t\mu} = t + 1$.

LR coefficients and polynomials

Ex: Let $c^{\nu}_{421,532} = c$ and $c^{t\nu}_{t(421),t(532)} = P(t)$.

$$c = 1 \quad \nu = (953) \qquad P(t) = 1$$

$$c = 2 \quad \nu = (9431) \qquad P(t) = (t+1)$$

$$c = 3 \quad \nu = (8441) \qquad P(t) = (t+1)(t+2)/2$$

$$c = 4 \quad \nu = (8531) \qquad P(t) = (t+1)(t+2)(t+3)/6$$

$$c = 4 \quad \nu = (7442) \qquad P(t) = (t+1)^2$$

$$c = 5 \quad \nu = (7541) \qquad P(t) = (t+1)(t+2)(2t+3)/6$$

$$c = 6 \quad \nu = (7532) \qquad P(t) = (t+1)^2(t+2)/2$$

$$c = 7 \quad \nu = (74321) \qquad P(t) = (t+1)(t+2)(t^2+3t+6)/6$$

Generating function for LR-polynomials

Ex: Let $F(z) = G(z)/(1-z)^d = \sum_{t=0}^{\infty} P(t)\, z^t$.

$$
\begin{array}{llll}
c=1 & \nu=(953) & d=1 & G(z)=1 \\
c=2 & \nu=(9431) & d=2 & G(z)=1 \\
c=3 & \nu=(8441) & d=3 & G(z)=1 \\
c=4 & \nu=(8531) & d=4 & G(z)=1 \\
c=4 & \nu=(7442) & d=3 & G(z)=1+z \\
c=5 & \nu=(7541) & d=4 & G(z)=1+z \\
c=6 & \nu=(7532) & d=4 & G(z)=1+2z \\
c=7 & \nu=(74321) & d=5 & G(z)=1+2z+z^2
\end{array}
$$

Further example

Ex: $n = 7,\ \lambda = (433210),\ \mu = (432210),\ \nu = (7444321)$.

- LR coefficient $c^{\nu}_{\lambda\mu} = 13$
- LR polynomial

$$c^{t\nu}_{t\lambda.t\mu} = 1/10080 \times (t+1)(t+2)(t+3)(t+4)(t+5) \times (5t+21)(t^2+2t+4)$$

- where $10080 = 5!\ 84$
- $d = 8$ and $G(z) = 1 + 4z + 12z^2 + 3z^3$

Polynomial behaviour

Theorem For all λ, μ, ν such that $c^{\nu}_{\lambda\mu} > 0$ there exists

- a polynomial $P^{\nu}_{\lambda\mu}(t)$ in t with $P^{\nu}_{\lambda\mu}(0) = 1$
- such that $P^{\nu}_{\lambda\mu}(t) = c^{t\nu}_{t\lambda,t\mu}$ for all positive integers t.

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Conjectures

- coefficients in $P^{\nu}_{\lambda\mu}(t)$ are all rational and non-negative.
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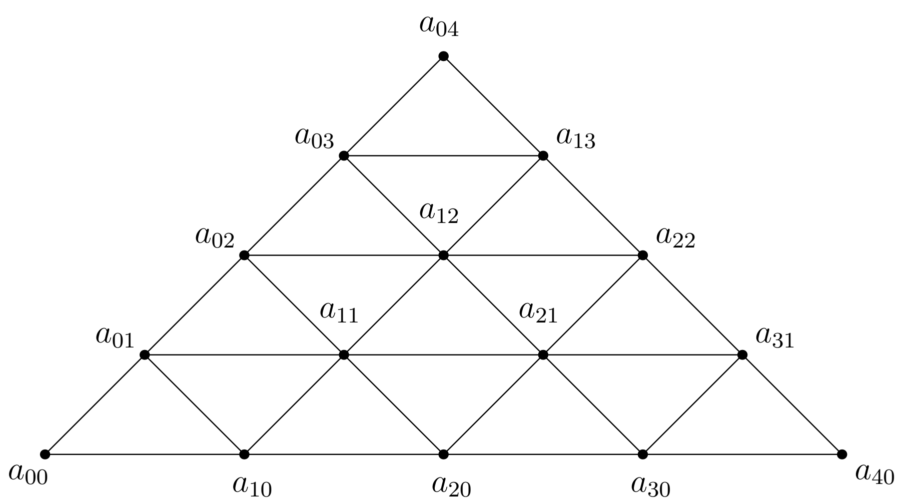
Problems

- predict degree of polynomial
- explain origin of factors of form $(t+1)(t+2)\cdots(t+m)$
- prove (if true) and account for positivity of coefficients

Integer hives

- n-hive with vertex labels $a_{ij} \in \mathbb{Z}$ for $0 \leq i, j, i + j \leq n$.

  Ex: $n = 4$



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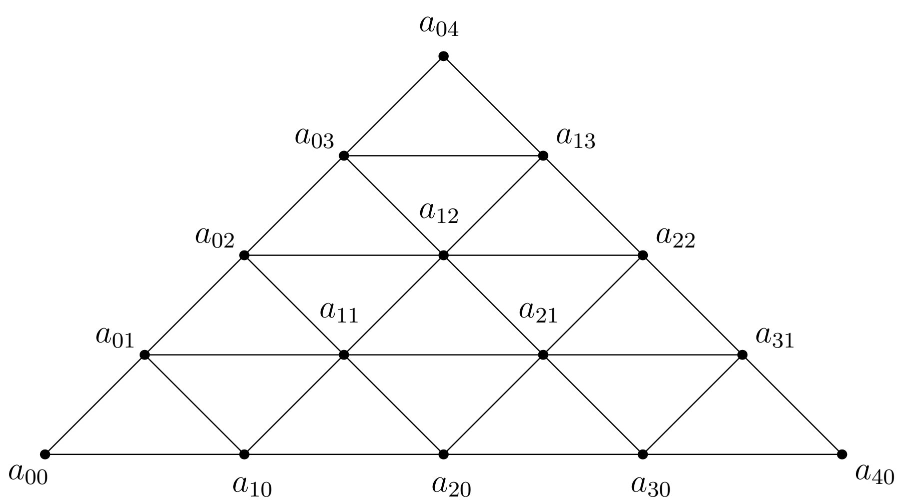
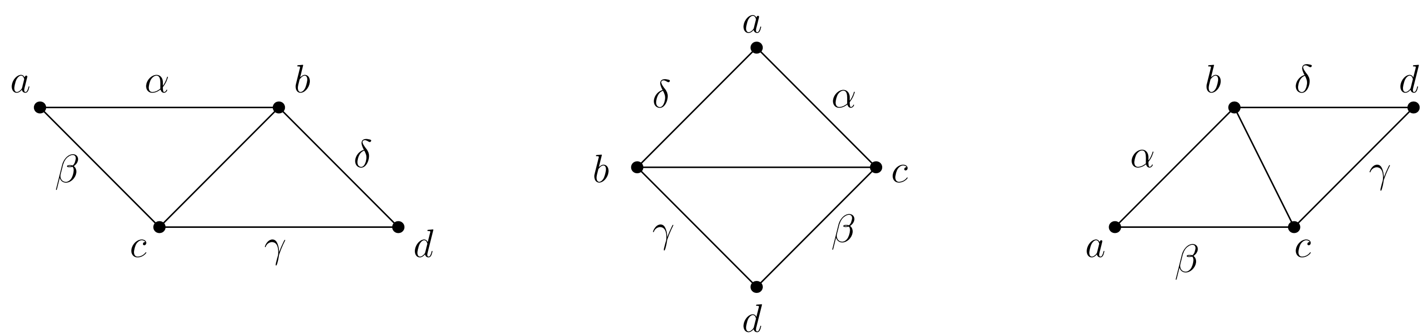


- Vertex labels increase from left to right
- Edge labels non-negative differences between neighbouring vertex labels

$$\alpha = a_{i,j+1} - a_{ij}, \quad \beta = a_{i+1,j-1} - a_{ij}, \quad \gamma = a_{i+1,j} - a_{ij}.$$

Hive conditions

- Distinct types of rhombi, with vertex and edge labels:



$R1$: $R2$: $R3$:

- Note: $\alpha, \beta, \gamma, \delta \geq 0$ and $\alpha + \delta = \beta + \gamma$.

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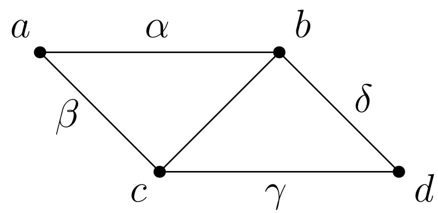
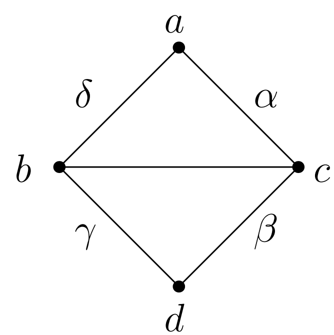
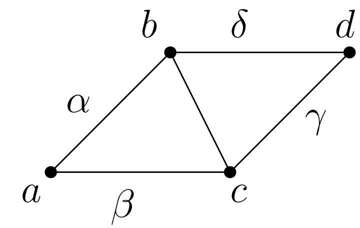


$R1$:



$R2$:



$R3$:

- Note: $\alpha, \beta, \gamma, \delta \geq 0$ and $\alpha + \delta = \beta + \gamma$.
- Hive conditions in terms of vertex labels:

$$b + c \geq a + d.$$

- Hive conditions in terms of edge labels:

$$\alpha \geq \gamma \quad \text{and} \quad \beta \geq \delta.$$

LR-hives vertex labels

Definition An LR-hive is an integer n-hive for which

- all rhombi of type R1, R2 and R3 satisfy the hive conditions;
- boundaries determined by partitions λ, μ, ν with $\ell(\lambda), \ell(\mu), \ell(\nu) \leq n$ and $|\lambda| + |\mu| = |\nu|$;

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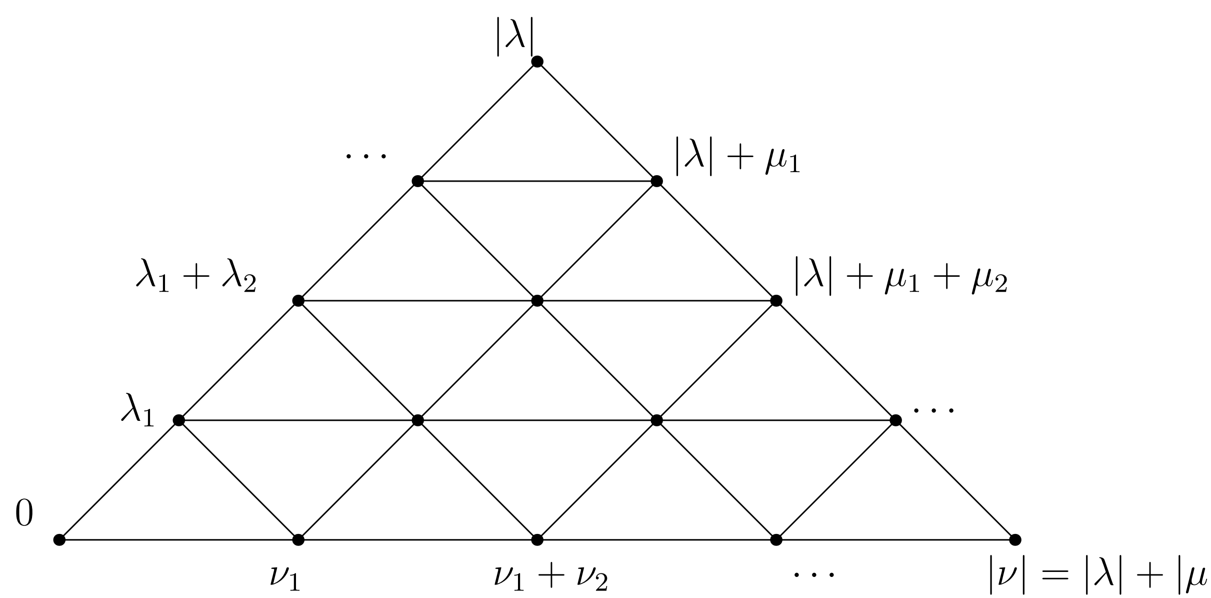
- boundary vertex labels



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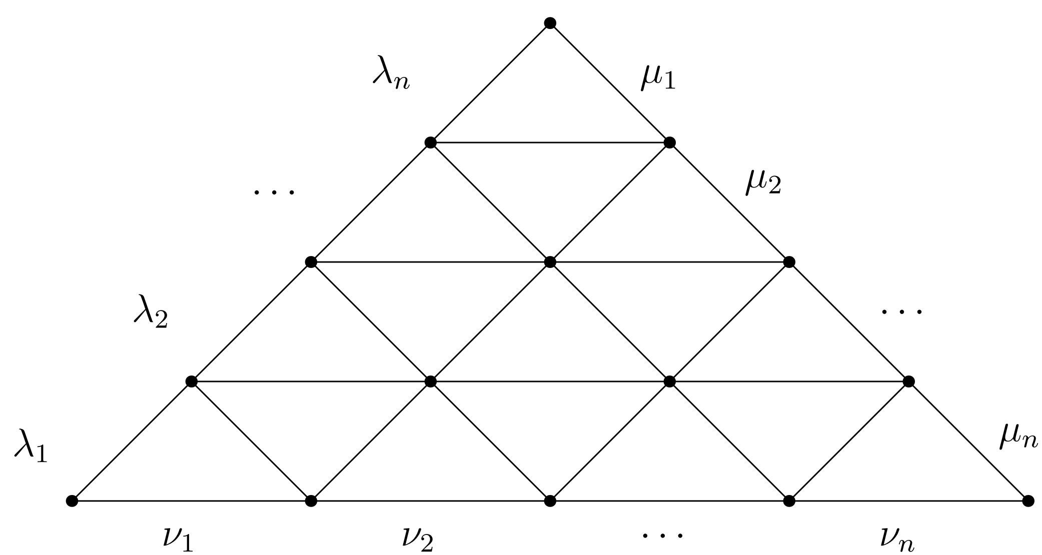

Bijection between LR-diagrams and LR-hives

Example: $n = 3$, $\lambda = (320)$, $\mu = (210)$ and $\nu = (431)$.

- $D =$ Littlewood-Richardson diagram;
- $G =$ Generalised Gelfand-Zetlin pattern;
- $Z =$ Zeros and cumulative row sums of G;
- $H =$ LR-hive $=$ reorientation of lower triangular part of Z.

$$D = \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 2 \\ \cline{1-3} 1 \\ \cline{1-1} \end{array} \quad \Longleftrightarrow \quad G = \begin{array}{ccccccc} 4 & & 3 & & 1 & & \\ & 4 & & 3 & & 1 & \\ & & 4 & & 2 & & 1 \\ & & & 3 & & 2 & & 0 \end{array}$$

$$\Longleftrightarrow \quad Z = \begin{array}{ccccccc} 0 & 4 & 7 & 8 & & & \\ & 0 & 4 & 7 & 8 & & \\ & & 0 & 4 & 6 & 7 & \\ & & & 0 & 3 & 5 & 5 \end{array} \quad \Longleftrightarrow \quad H = \begin{array}{ccccccc} & & & 5 & & & \\ & & 5 & & 7 & & \\ & 3 & & 6 & & 8 & \\ 0 & & 4 & & 7 & & 8 \end{array}$$

LR-hives showing that $c^{9964}_{753,742} = 6$

0
9 7
18 16 12
24 24 21 15
28 28 26 22 15

0
9 7
18 16 12
24 23 21 15
28 28 26 22 15

0
9 7
18 16 12
24 24 20 15
28 28 26 22 15

0
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24 23 20 15
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Theorems

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  The LR-polynomial $P^{\nu}_{\lambda\mu}(t)$ can be identified as the Ehrhart quasi-polynomial $i(\mathcal{P},t) = \#\{t\mathcal{P} \cap \mathbb{Z}^m\}$, of a rational convex polytope $\mathcal{P}$ defined by the LR-hive boundary conditions and the set of LR-hive inequalities: $a + d \leq b + c$ for each rhombus.

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- Note: Even though $\mathcal{P}$ may be rational but not integer the Ehrhart quasi-polynomial $i(\mathcal{P},t)$ is polynomial.

Construction of convex polytopes

- Let $m = (n-2)(n-1)/2 \; = \; \#$ interior points of an n-hive
- Let $v = (a_{11}, a_{12}, ...) \in \mathbb{R}^m$ be vector of interior labels
- Then polytope $\mathcal{P}$ is d-dimensional convex hull of these integer points in $\mathbb{R}^m$.

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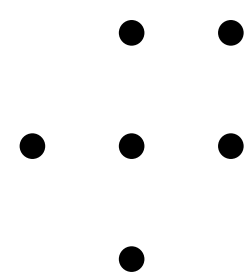
Ex: $\lambda = (753)$, $\mu = (742)$, $\nu = (9964)$, $n = 4$, $m = 3$,

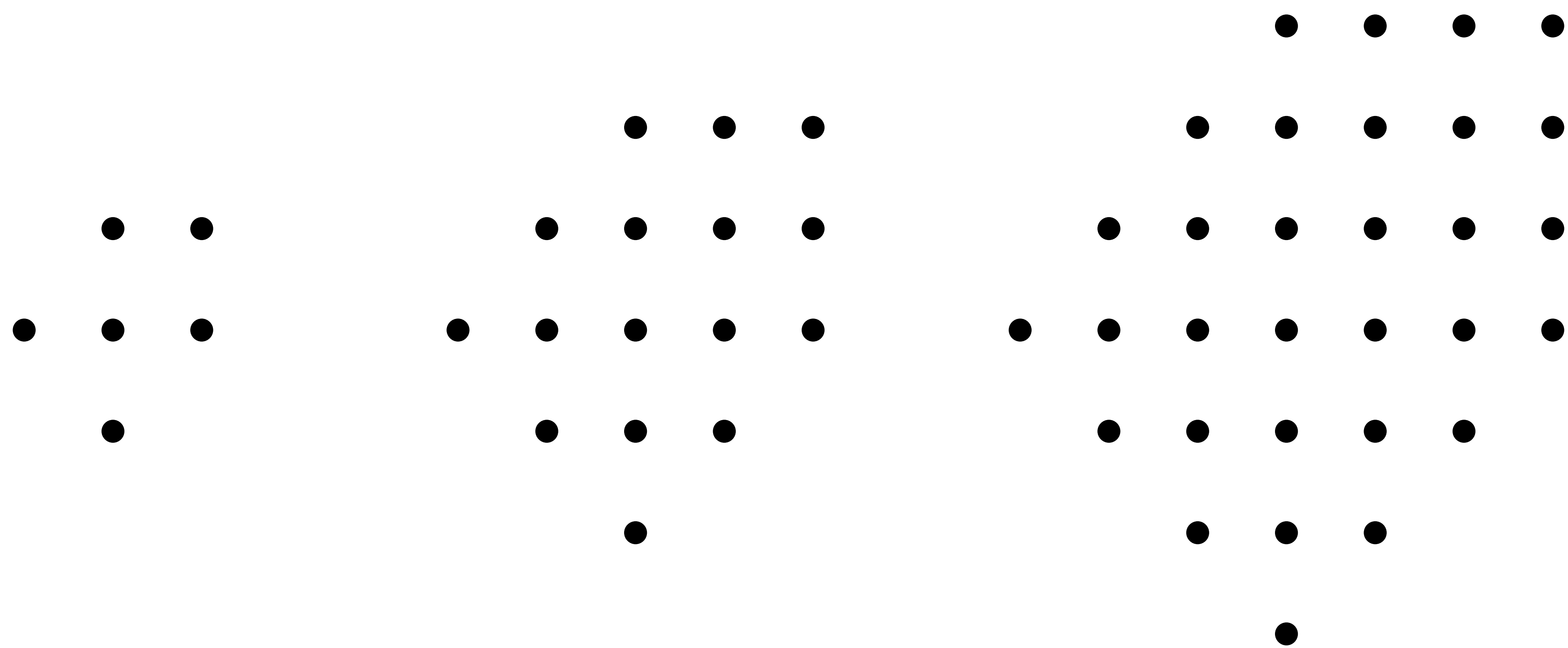
- Interior vertex labels $v = (a_{11}, a_{12}, a_{21})$

  $$(16, 21, 24), (16, 21, 23), (16, 20, 24),$$
  $$(16, 20, 23), (16, 20, 22), (16, 19, 23).$$

- Integer points of $\mathcal{P}$:  dimension $d = 2$.

Scaling convex polytope

- Expand $\mathcal{P}$ by scaling with t
- Identify and count all integer points to give $\mathcal{P}(t)$



- $P(1) = 6$, $P(2) = 16$, $P(3) = 31$, $\cdots$, $P(t) = \frac{1}{2}(5t^2 + 5t + 2)$.

Linear factors

Origin of some linear factors in LR-polynomials.

- Let $\mathcal{P}$ be an LR hive polytope, and $\overline{\mathcal{P}}$ its interior.
- For $t \in \mathbb{N}$: $P^{\nu}_{\lambda\mu}(t) = i(\mathcal{P}, t) = \#\{t\mathcal{P} \cap \mathbb{Z}^d\}$.
- Ehrhart reciprocity: $i(\mathcal{P}, -t) = (-1)^d \#\{t\overline{\mathcal{P}} \cap \mathbb{Z}^d\}$.
- For $m \in \mathbb{N}$: $P^{\nu}_{\lambda\mu}(-m) = i(\mathcal{P}, -m) = (-1)^d \#\{m\overline{\mathcal{P}} \cap \mathbb{Z}^d\}$.
- Hence $P^{\nu}_{\lambda\mu}(-m) = 0$ and $P^{\nu}_{\lambda\mu}(t)$ contains a factor $(t + m)$ if and only if $m\mathcal{P}$ contains no interior integer points.

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- Hence $P^{\nu}_{\lambda\mu}(-m) = 0$ and $P^{\nu}_{\lambda\mu}(t)$ contains a factor $(t+m)$ if and only if $m\mathcal{P}$ contains no interior integer points.

Corollary $P^{\nu}_{\lambda\mu}(t)$ contains $(t+1)(t+2)\cdots(t+m)$ as a factor if $m\mathcal{P}$ contains no interior integer points.

Problem: predict maximum value of m.

Construction of convex polytopes

Ex: $\lambda = (210)$, $\mu = (320)$, $\nu = (431)$, $n = 3$, $d = 1$

$$\begin{array}{ccccccc} & & & 5 & & & \\ & & 5 & & 7 & & \\ & 3 & & {\color{blue}a} & & 8 & \\ 0 & & 4 & & 7 & & 8 \end{array}$$

with $a = 6, 7$

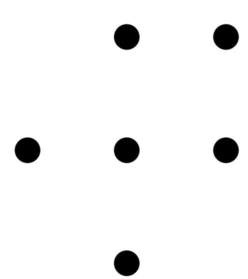
- $\mathcal{P} \cap \mathbb{Z} = \quad \bullet \quad \bullet$ no interior points
- $2\mathcal{P} \cap \mathbb{Z} = \quad \bullet \quad \bullet \quad \bullet$ one interior point
- implies $P(t)$ contains a factor $(t+1)$ but no factor $(t+2)$. In fact $P(t) = (t+1)$.

Construction of convex polytopes

Ex: $\lambda = (753)$, $\mu = (742)$, $\nu = (9964)$, $n = 4$, $d = 2$

$$\begin{array}{ccccccccc} & & & & 15 & & & & \\ & & & 15 & & 22 & & & \\ & & 12 & & {\color{blue}b} & & 26 & & \\ & 7 & & {\color{blue}16} & & {\color{blue}c} & & 28 & \\ 0 & & 9 & & 18 & & 24 & & 28 \end{array}$$

with $(b, c) = \begin{cases} (21, 24) & (21, 23) \\ (20, 24) & (20, 23) \\ (20, 22) & (19, 23) \end{cases}$

- $\mathcal{P} \cap \mathbb{Z}^2$:  one interior point
- implies no factor $(t + m)$. In fact $P(t) = \frac{1}{2}(5t^2 + 5t + 2)$.

Degrees of LR-polynomials

- For $c^\nu_{\lambda\mu} > 0$ the LR-rule implies $\ell(\lambda), \ell(\mu) \leq \ell(\nu)$.
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- For $t \in \mathbb{N}$, $P^{\nu}_{\lambda\mu}(t)$ is the number of scaled LR n-hives with boundary labels scaled by t and interior vertex labels subject to the same scaled linear inequalities.
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Degree bound $\deg P^{\nu}_{\lambda\mu}(t) \leq (n-1)(n-2)/2$ with $n = \ell(\nu)$.

First example

Ex: $n = 5$, degree bound $(n-1)(n-2)/2 = 6$.

- $\lambda = (9, 7, 6, 2, 0)$, $\mu = (13, 5, 3, 1, 0)$, $\nu = (14, 12, 11, 5, 4)$.
- $P^{\nu}_{\lambda\mu}(t) = (t+1)$ so that $\deg P^{\nu}_{\lambda\mu}(t) = 1$.

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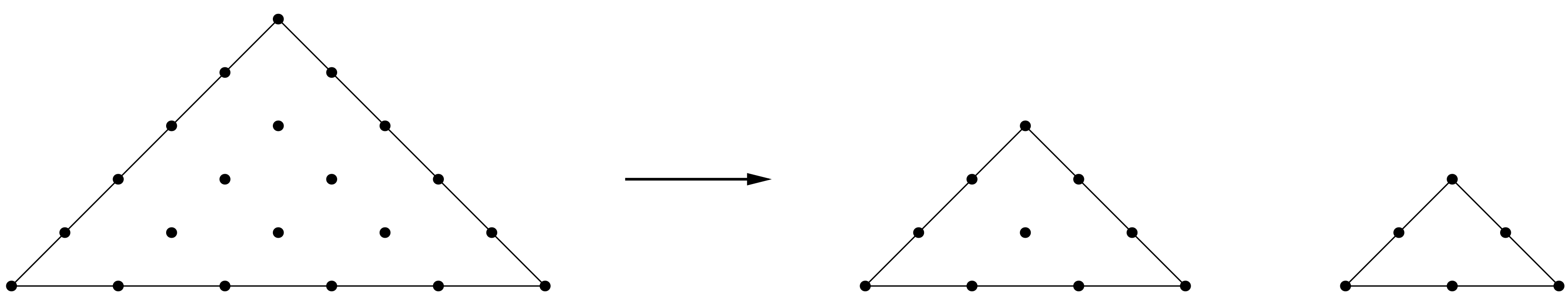
Origin of mismatch - factorisation

- $P^{\nu}_{\lambda\mu}(t) = P^{\nu_K}_{\lambda_I\,\mu_J}(t)\; P^{\nu_{\overline{K}}}_{\lambda_{\overline{I}}\,\mu_{\overline{J}}}(t)$.
- LR-hives for $n = 5$ are fixed by two smaller subhives of sizes $r = 3$ and $n - r = 2$.



LR factorisation example

Ex: $n=5$, $r=3$, $n-r=2$:

- $\lambda=(9,7,6,2,0)$, $\mu=(13,5,3,1,0)$, $\nu=(14,12,11,5,4)$.
- $I=\{1,2,4\}$, $J=\{2,3,4\}$, $K=\{2,3,5\}$.
- $\lambda_I=(9,7,2)$, $\mu_J=(5,3,1)$, $\nu_K=(12,11,4)$
- $\lambda_{\overline{I}}=(6,0)$, $\mu_{\overline{J}}=(13,0)$, $\nu_{\overline{K}}=(14,5)$

LR-coefficient:

$$c^{(14,12,11,5,4)}_{(9,7,6,2,0),(13,5,3,1,0)} = c^{(12,11,4)}_{(9,7,2),(5,3,1)}\; c^{(14,5)}_{(6,0),(13,0)} = 2\cdot 1 = 2.$$

LR-polynomial:

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$$= (t+1)\cdot 1 = (t+1).$$

LR factorisation example

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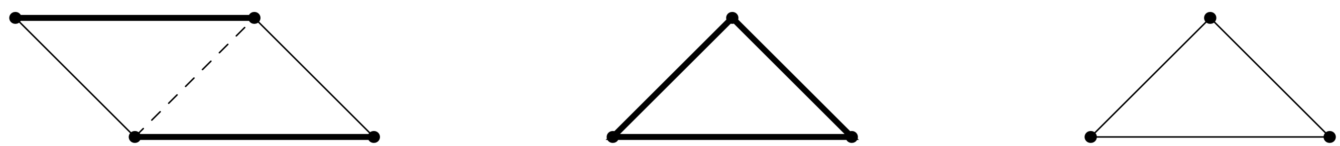
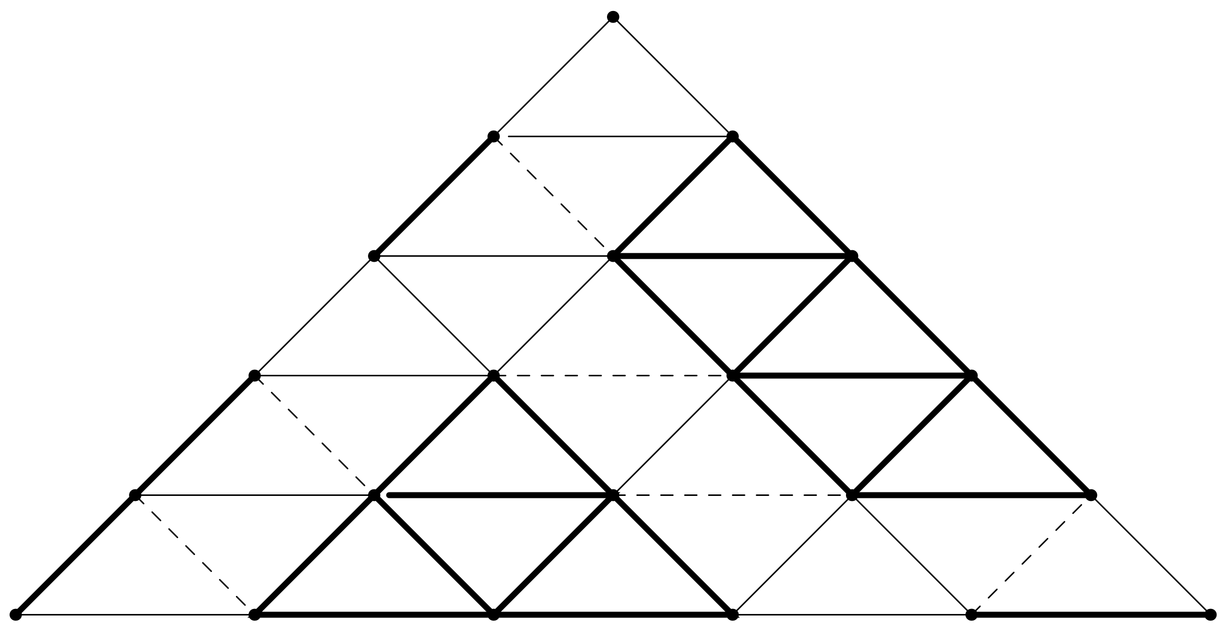
Puzzles

Definition A puzzle is a diagram on a triangular lattice in which edges are distinguished so that it is composed of copies of the following pieces oriented in any way so as to fit:





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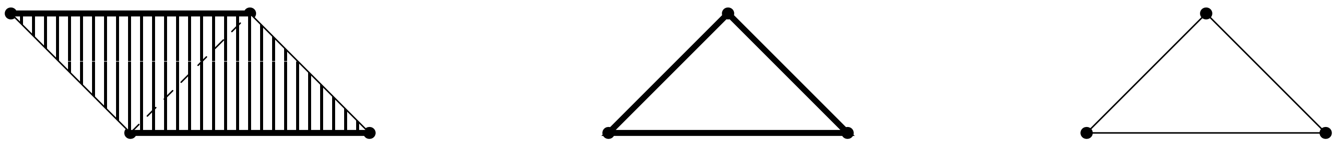

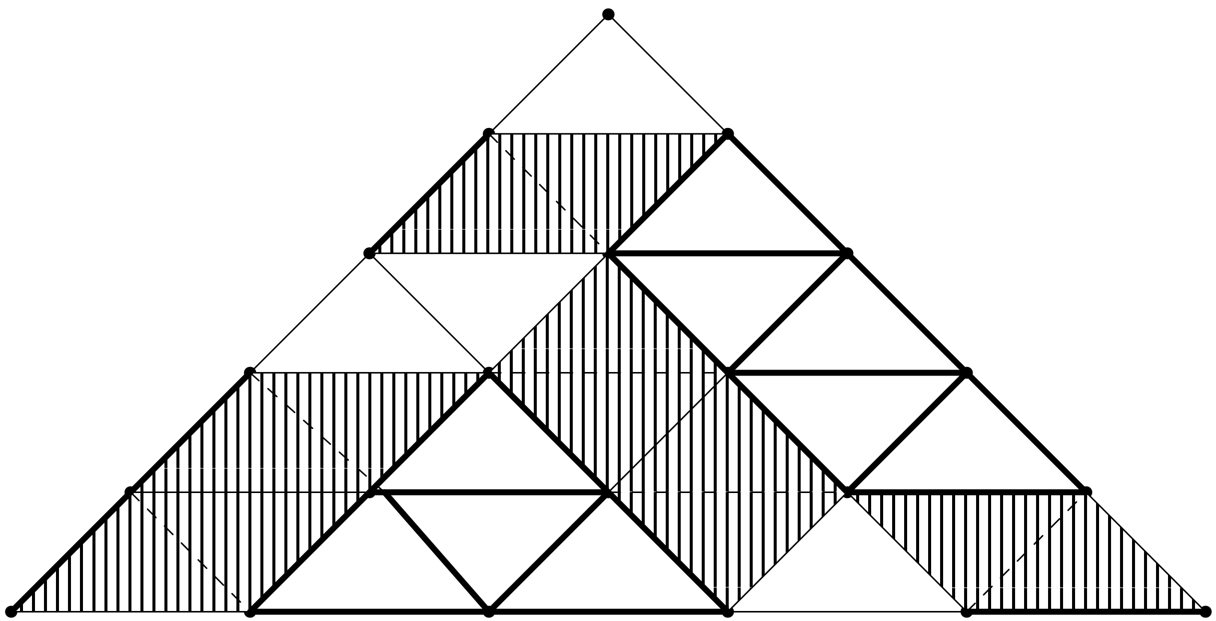

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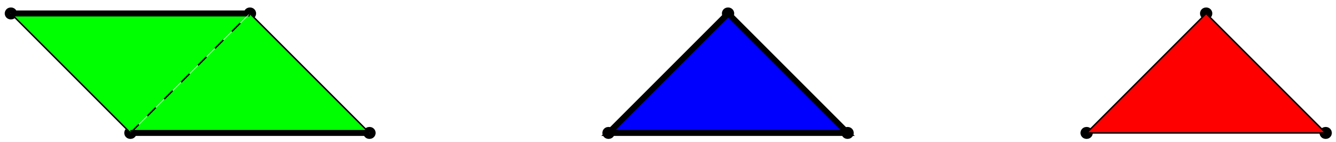
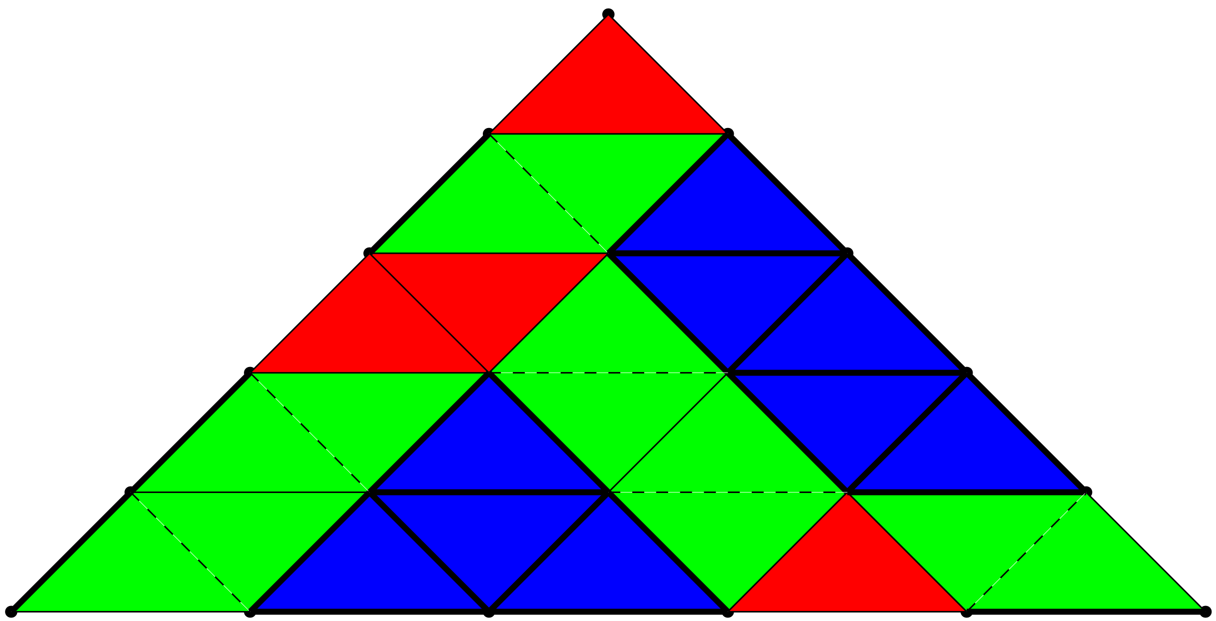
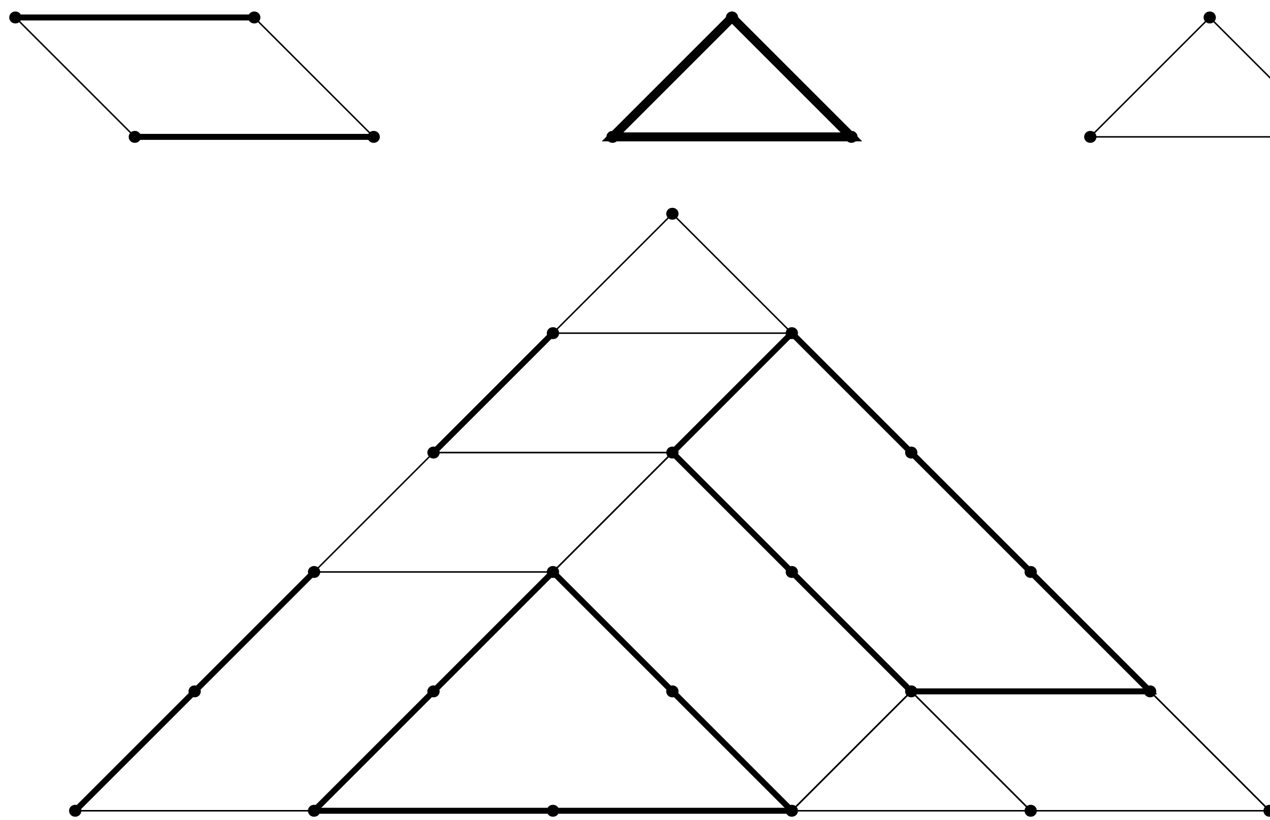




Hive plan

Definition A hive plan is made up of corridors, dark rooms and light rooms obtained by deleting interior edges of a puzzle:



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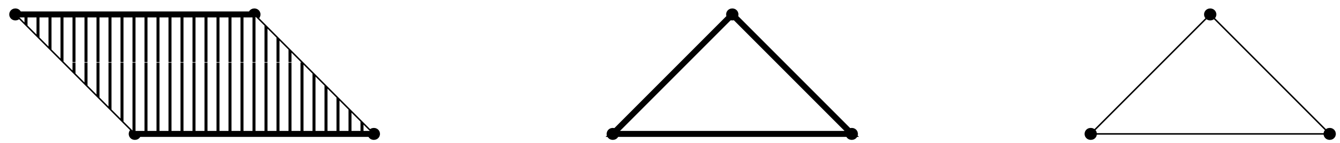
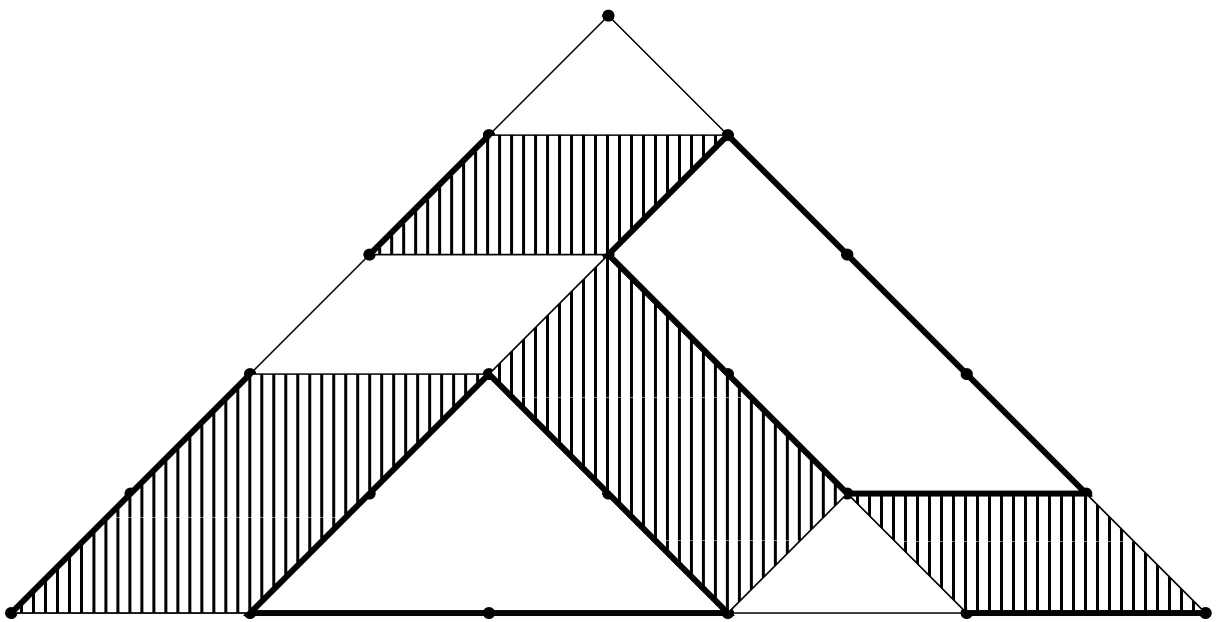
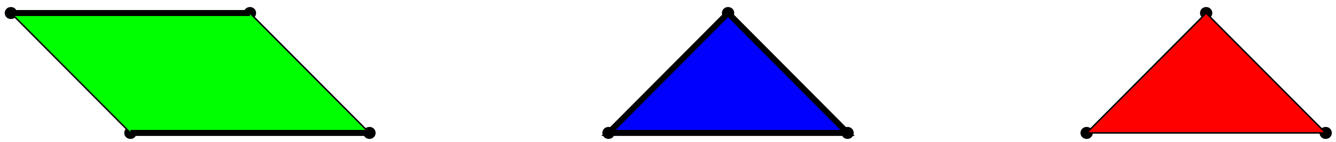
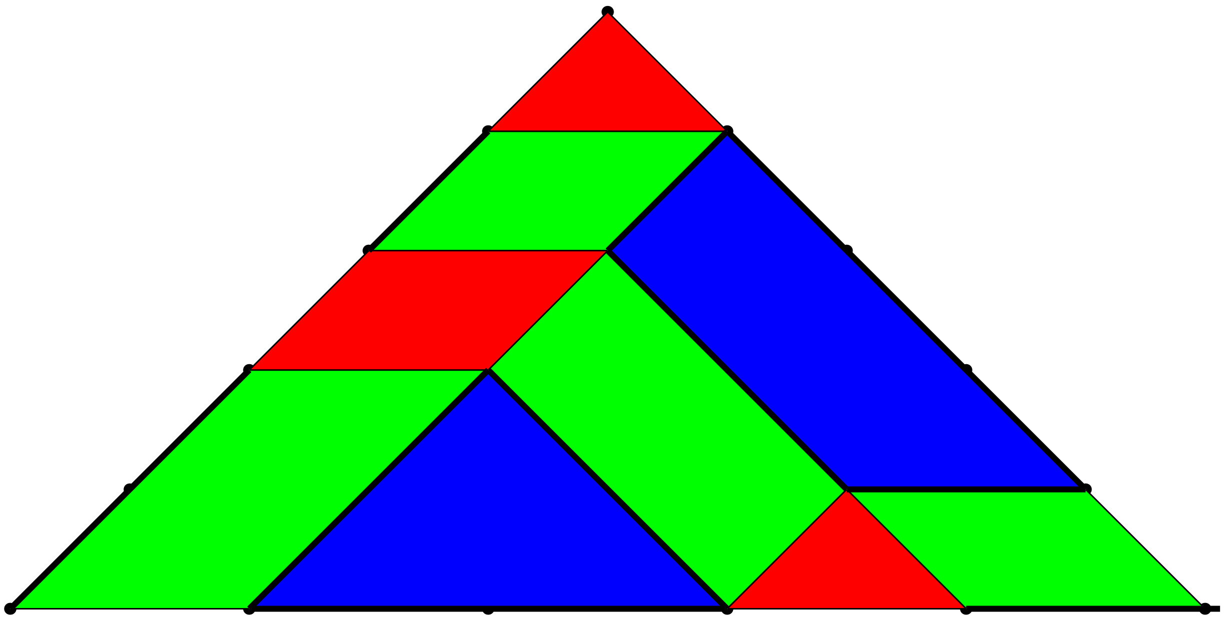




Hive plan

Definition A hive plan is made up of corridors, blue rooms and red rooms obtained by deleting interior edges of a puzzle:





Link between puzzles and Horn triples

- (I, J, K) is Horn triple if it specifies the positions of the thick edges on the boundary of any puzzle. It is essential if the puzzle with these boundary thick edges is unique.

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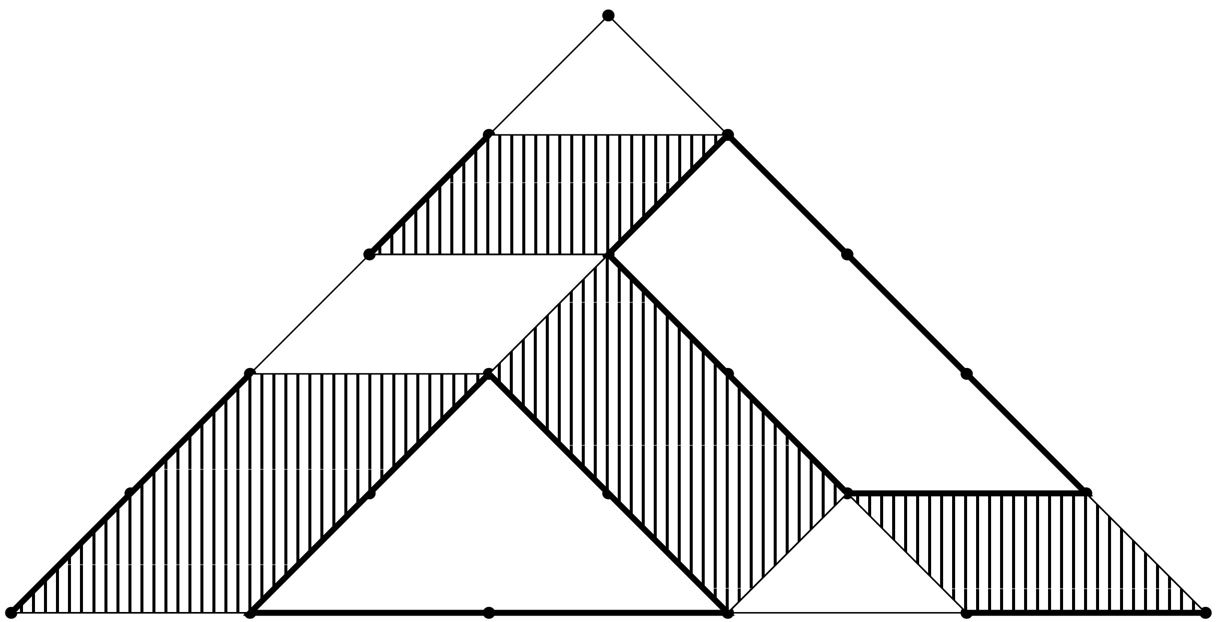
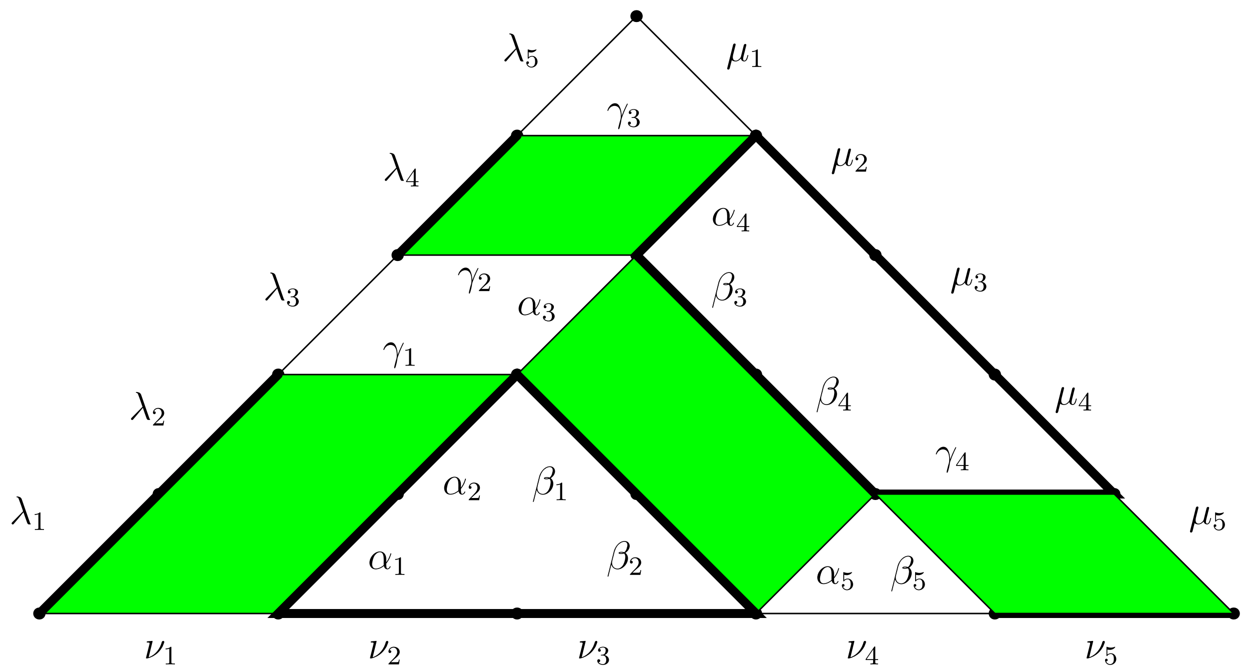
- For $I = (1, 2, 4)$, $J = (2, 3, 4)$ and $K = (2, 3, 5)$ we have:



Each Horn triple defines an inequality



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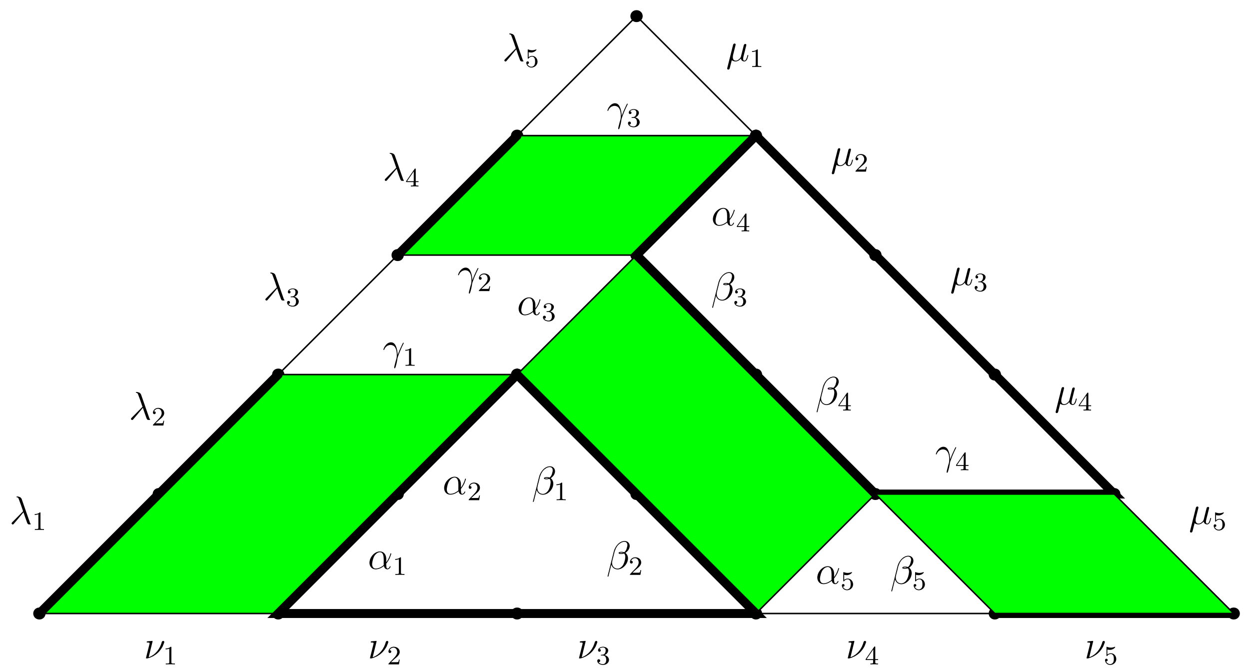

$$\nu_2 + \nu_3 + \nu_5 \leq (\nu_2 + \nu_3) + \gamma_4 = (\alpha_1 + \alpha_2 + \beta_1 + \beta_2) + \gamma_4$$
$$\leq \lambda_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_4 \leq \lambda_1 + \lambda_2 + \beta_1 + \beta_2 + \gamma_4$$
$$\leq \lambda_1 + \lambda_2 + \beta_3 + \beta_2 + \gamma_4 \leq \lambda_1 + \lambda_2 + (\beta_3 + \beta_4 + \gamma_4)$$
$$= \lambda_1 + \lambda_2 + (\alpha_4 + \mu_2 + \mu_3 + \mu_4) \leq \lambda_1 + \lambda_2 + \lambda_4 + \mu_2 + \mu_3 + \mu_4$$

That is $|\nu_K| \leq |\lambda_I| + |\mu_J|$.

Horn inequalities and non-zero conditions

Theorem The LR-coefficient $c^{\nu}_{\lambda\mu} > 0$ if and only if

- $|\nu| = |\lambda| + |\mu|$
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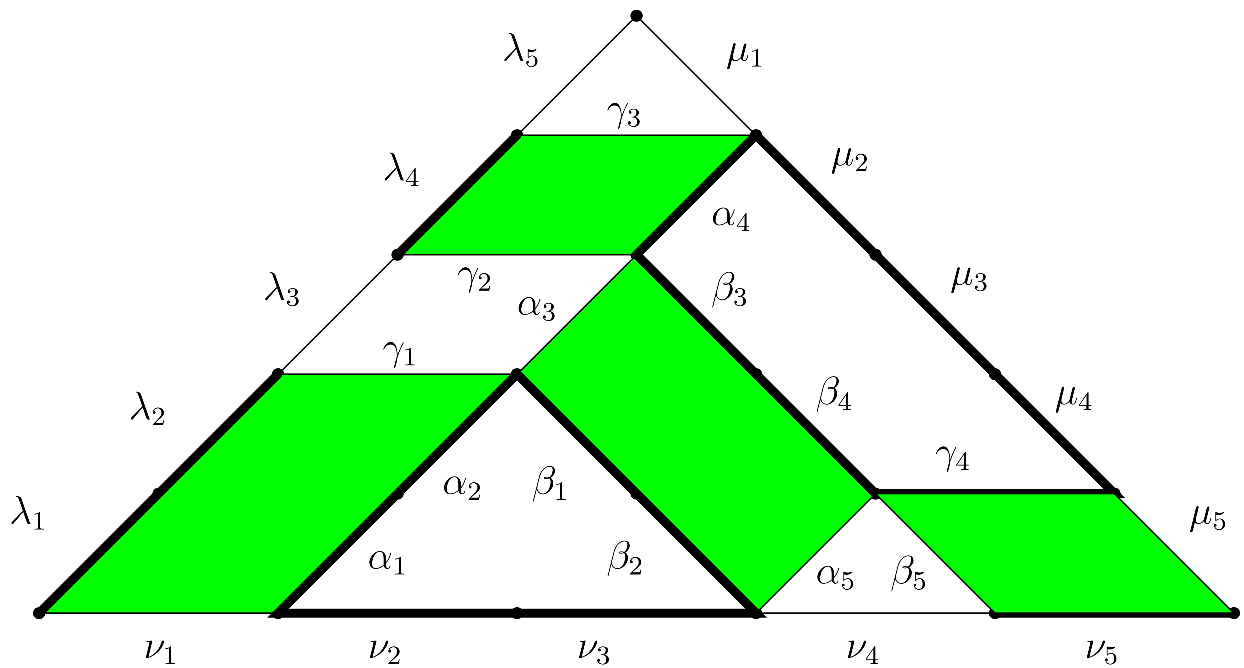
Corollary $c^{t\nu}_{t\lambda,t\mu} > 0$ if and only if $c^{\nu}_{\lambda\mu} > 0$

Proof

- $|t\nu| - |t\lambda| - |t\mu| = t(|\nu| - |\lambda| - |\mu|)$
- $|t\nu_K| - |t\lambda_I| - |t\mu_J| = t(|\nu_K| - |\lambda_I| - |\mu_J|$

Consequences of any Horn equality



- All sequences of inequalities become equalities.

$$\nu_2+\nu_3+\nu_5=(\nu_2+\nu_3)+\gamma_4=(\alpha_1+\alpha_2+\beta_1+\beta_2)+\gamma_4$$
$$=\lambda_1+\alpha_2+\beta_1+\beta_2+\gamma_4=\lambda_1+\lambda_2+\beta_1+\beta_2+\gamma_4$$
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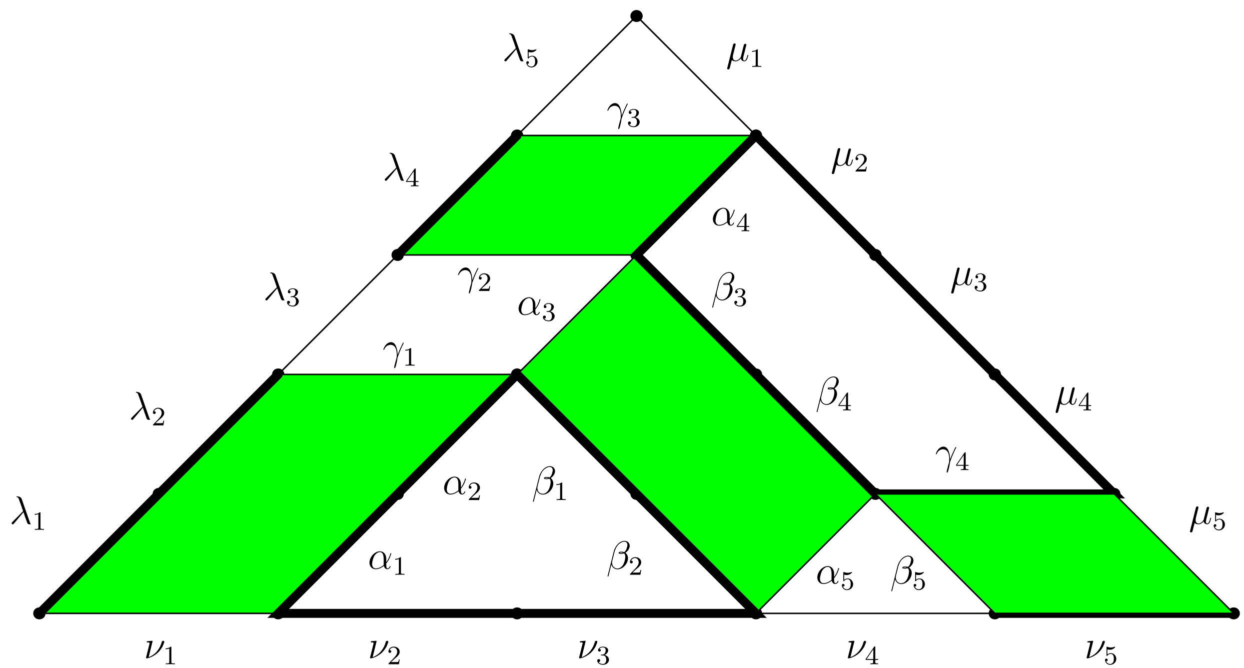
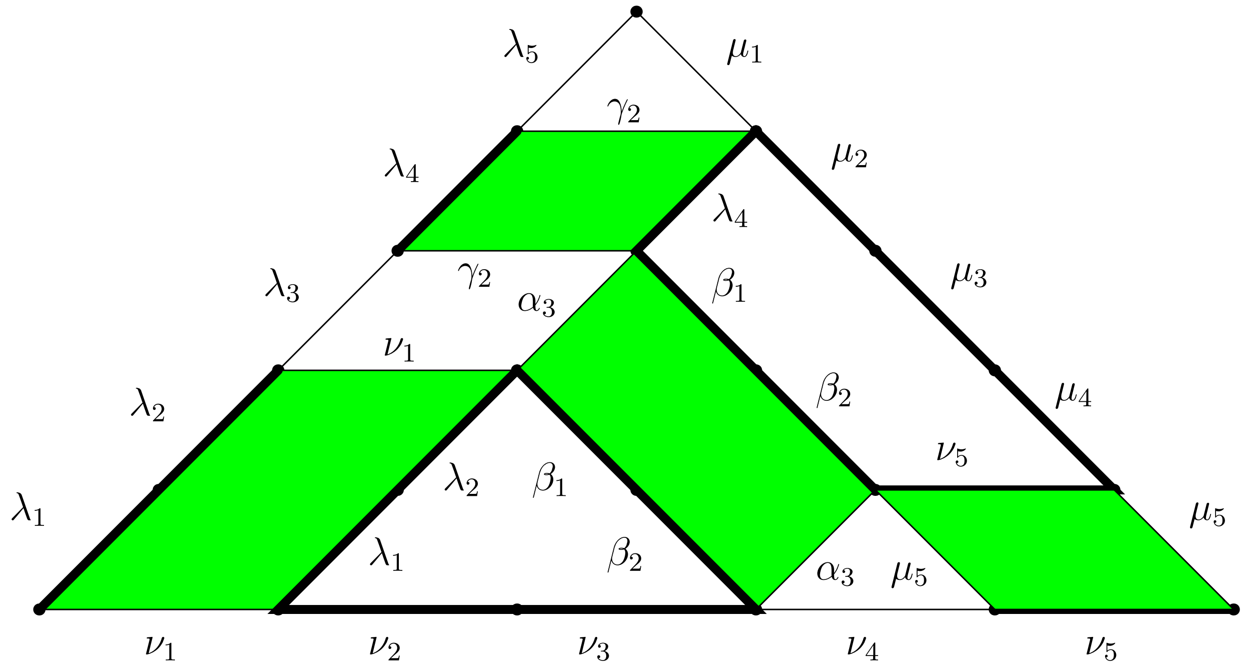
Edge label equalities



- Equalities imply: $\nu_5 = \gamma_4$, $\alpha_1 = \lambda_1$, $\alpha_2 = \lambda_2$, $\beta_1 = \beta_3$, $\beta_2 = \beta_4$, $\alpha_4 = \lambda_4$.
- In addition: $\beta_5 + \nu_5 = \gamma_4 + \mu_5$, $\nu_1 + \alpha_1 + \alpha_2 = \lambda_1 + \lambda_2 + \gamma_1$, $\gamma_2 + \alpha_4 = \lambda_4 + \gamma_3$.
- These then imply: $\beta_5 = \mu_5$, $\nu_1 = \gamma_1$, $\gamma_2 = \gamma_3$.

Factorisation of LR-hives



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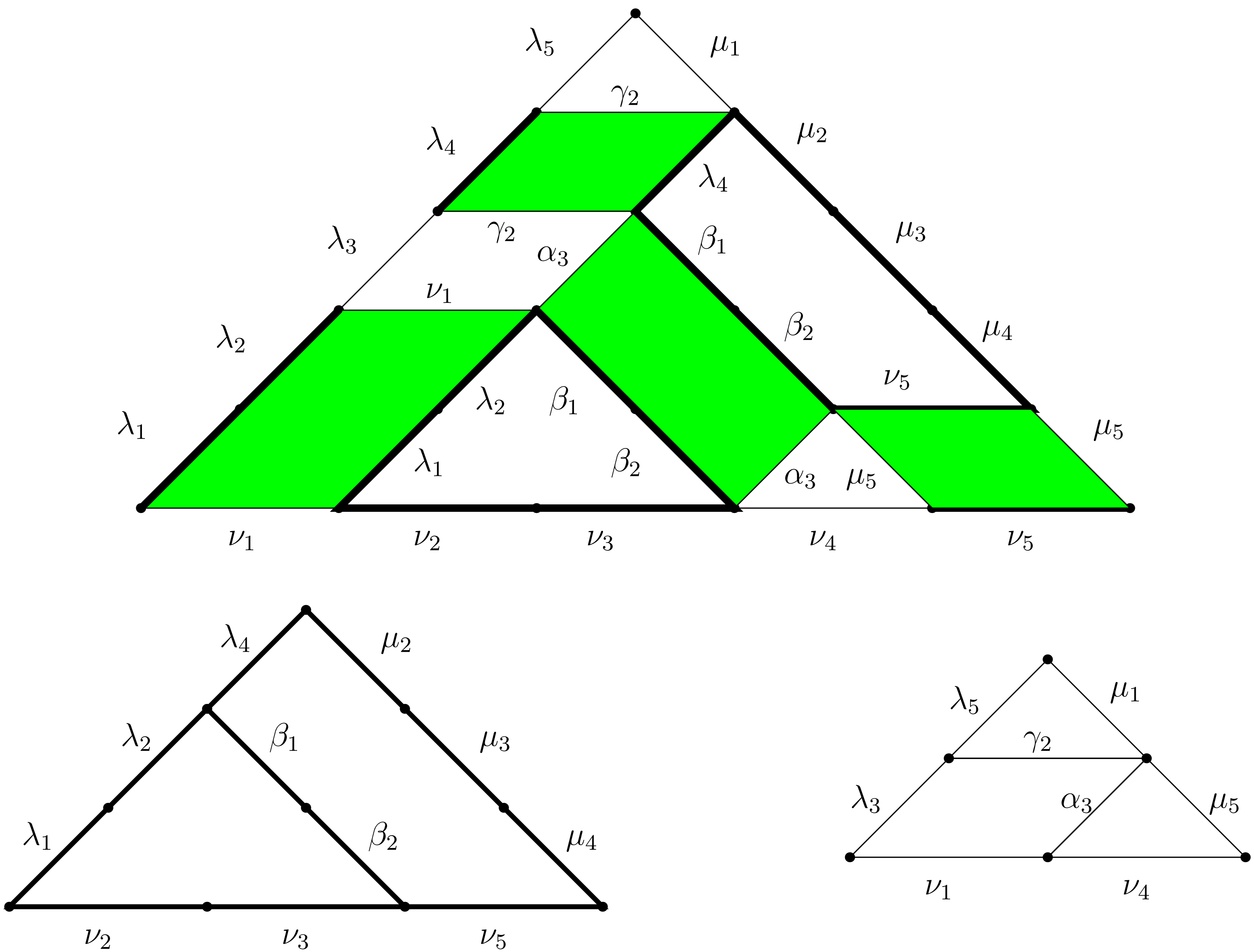
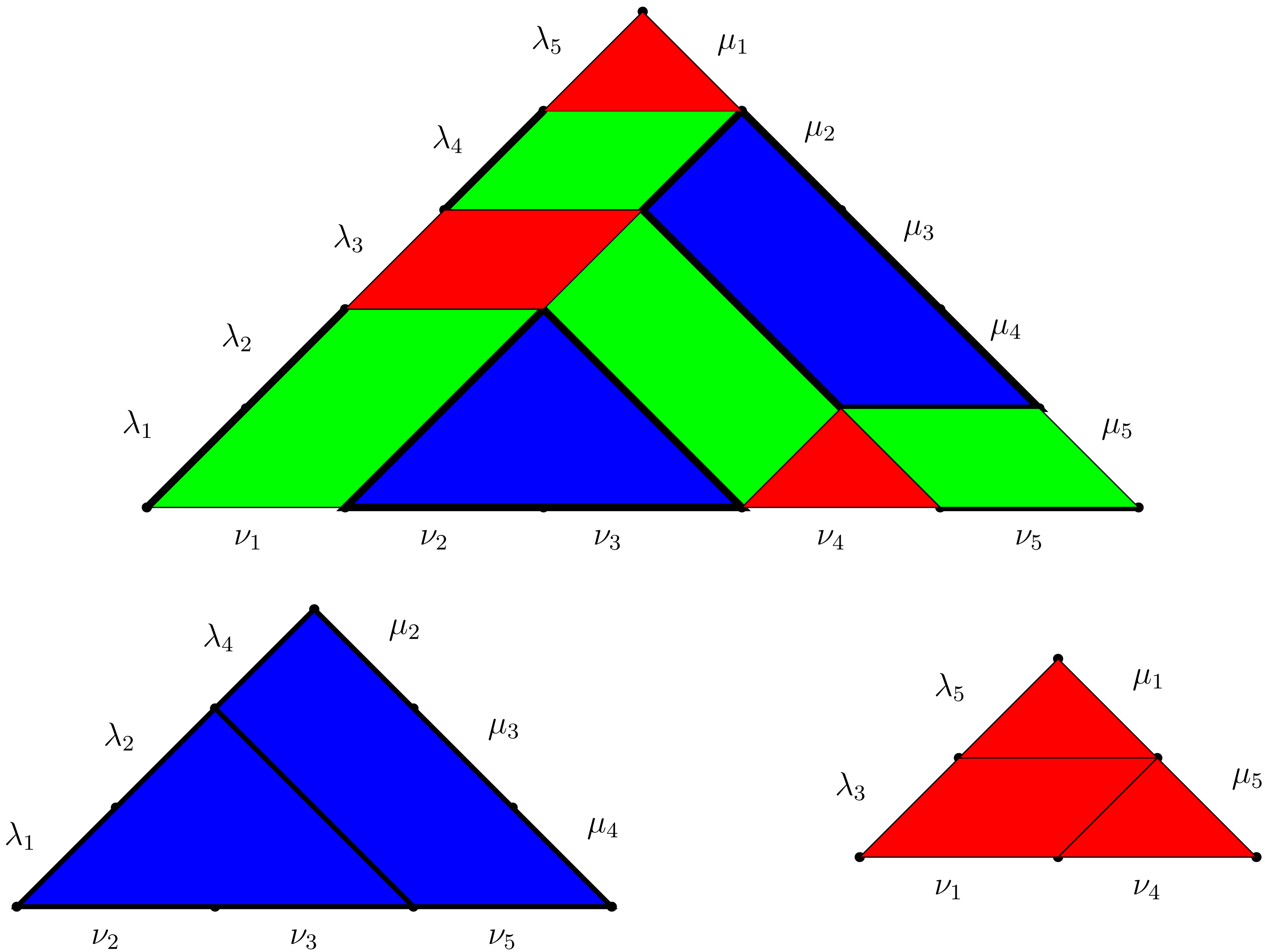


Illustration of H_n and subhives H_r,H_{n-r}



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- Definition If all essential Horn inequalities are strict then both $c^{\nu}_{\lambda\mu}$ and $P^{\nu}_{\lambda\mu}(t)$ are said to be primitive.

LR factorisation example

Ex: $n = 5$, $r = 3$, $n - r = 2$:

- $\lambda = (9, 7, 6, 2, 0)$, $\mu = (13, 5, 3, 1, 0)$, $\nu = (14, 12, 11, 5, 4)$.
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Degree bound for a primitive example

Ex: $n = 6$, degree bound $(n-1)(n-2)/2 = 10$.

- $\lambda = (4,3,3,1,0,0)$, $\mu = (4,2,1,1,1,0)$, $\nu = (6,5,4,2,2,1)$.
- $P^{\nu}_{\lambda\mu}(t) = (t+1)(t+2)(t+3)(t+4)/24$.
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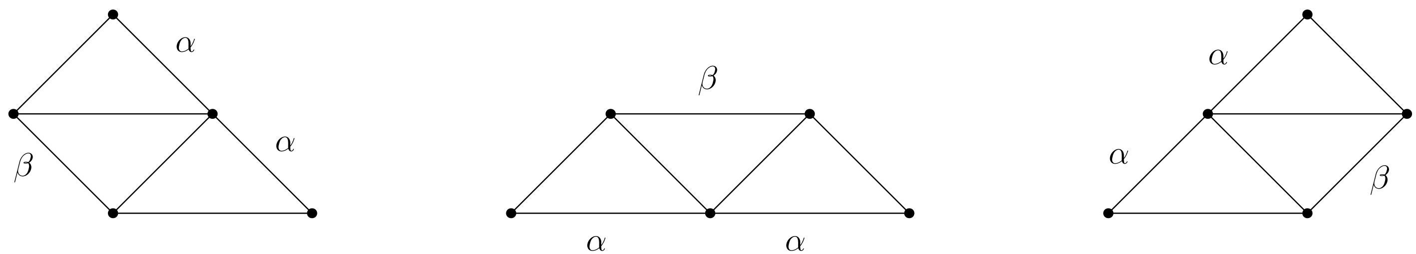
Origin of mismatch - partitions have equal parts.

Five-vertex equal edge constraints

- Equal edge constraints on 5-vertex subdiagrams
- In each case $\alpha \geq \beta \geq \alpha$ so that $\beta = \alpha$.



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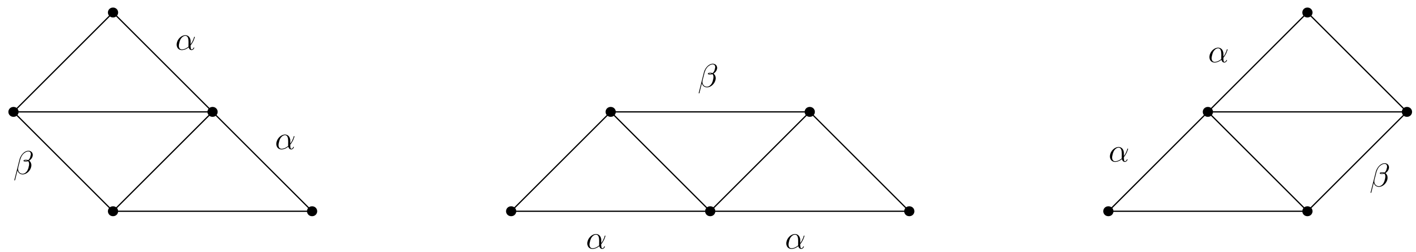
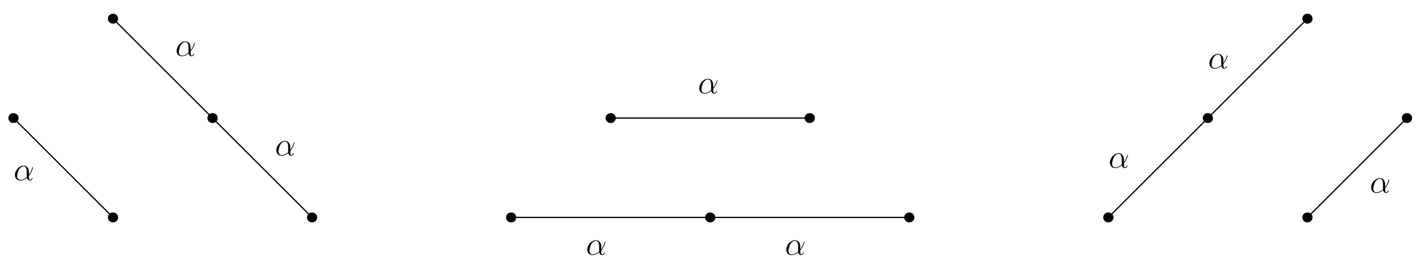
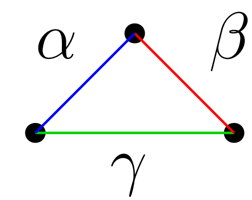


- Consecutive equal edges force neighbouring equal edge.
- Retain skeleton consisting of only equal edges.



Skeleton of an LR-hive and degree bounds

- Apply 5-vertex equal edge procedure to LR n-hive.
- Work inwards from boundaries specified by λ, μ, ν.
- Invoke triangular hive condition $\alpha + \beta = \gamma$:



- Result is skeletal graph $G_{n;\lambda\mu\nu}$ of hive.
- Let $d(G_{n;\lambda\mu\nu})$ be number of components of $G_{n;\lambda\mu\nu}$ not connected to the boundary.

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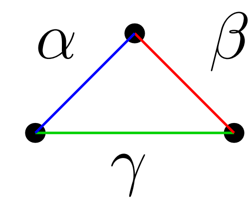

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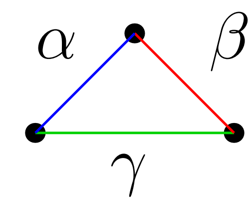

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Conjecture If $P^{\nu}_{\lambda\mu}(t)$ is primitive then $\deg P^{\nu}_{\lambda\mu}(t) = d(G_{n;\lambda\mu\nu})$.

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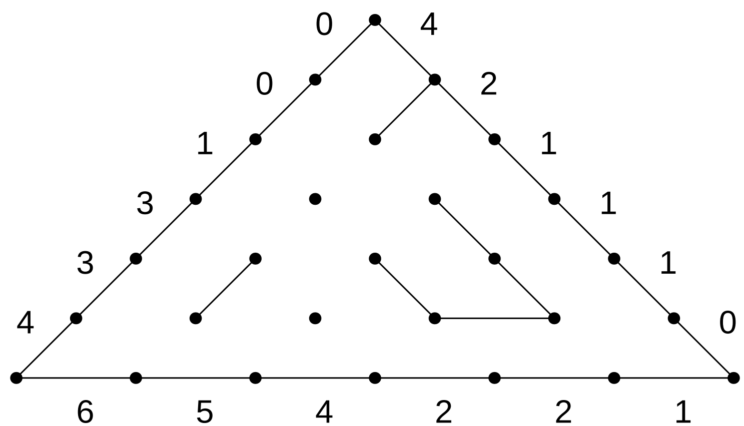

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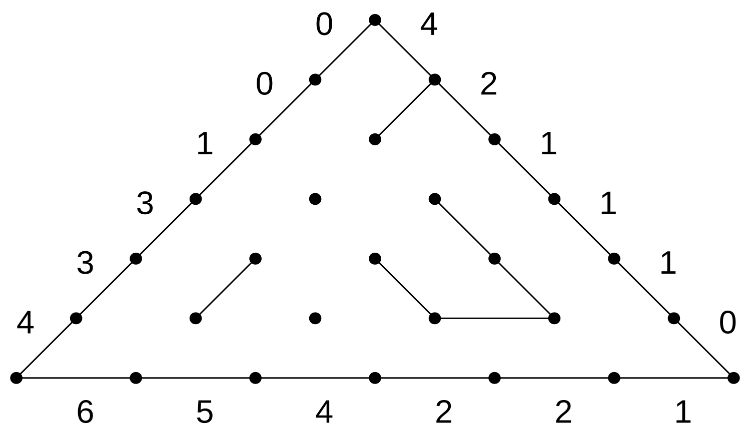

- $P^{\nu}_{\lambda\mu}(t) = (t+1)(t+2)(t+3)(t+4)/24$.
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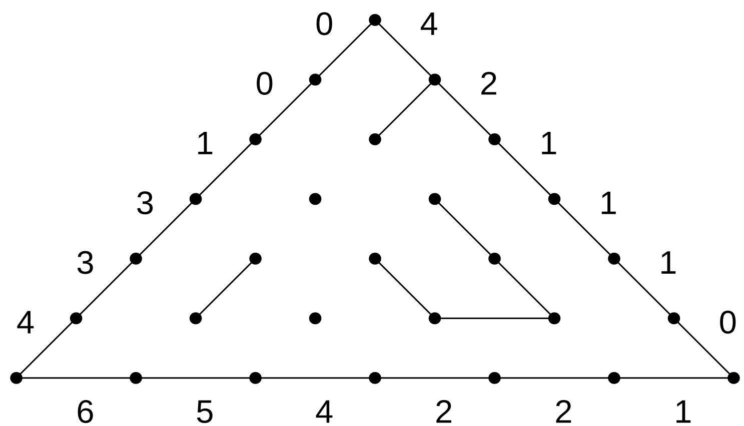

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Skeleton graph degree bound is saturated.

Degree of LR-polynomial

Ex: $n = 7$, $\lambda = (4332100)$, $\mu = (4322100)$, $\nu = (7444321)$.

- $$P^{\nu}_{\lambda\mu}(t) = (t+1)(t+2)(t+3)(t+4)(t+5) \times (5t+21)(t^2+2t+4)/10080.$$

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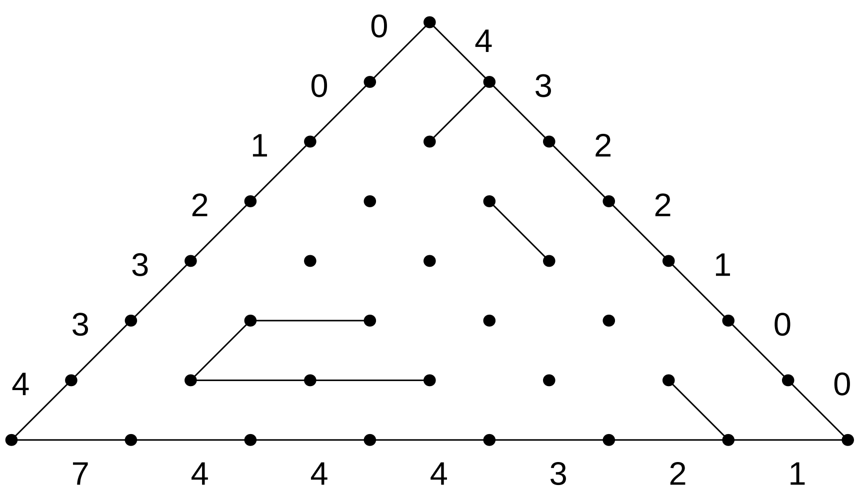

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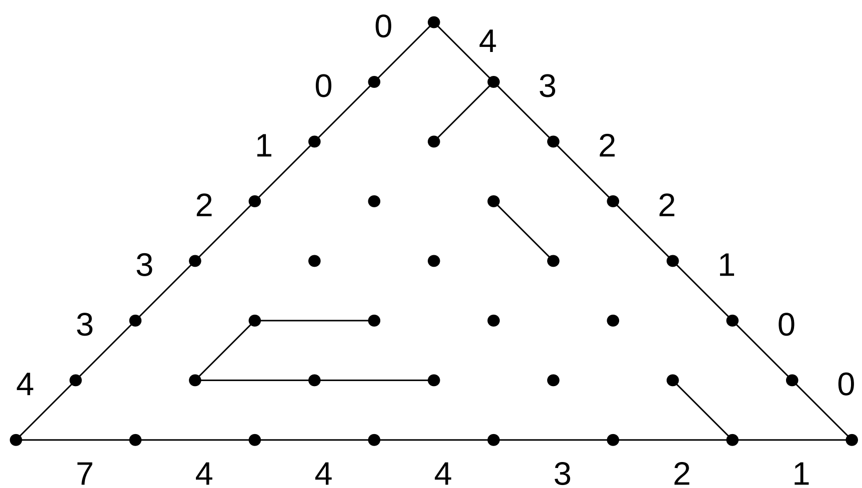

- $\deg P^{\nu}_{\lambda\mu}(t) = 8 = d(G_{n;\lambda\mu\nu})$.

Degree of primitive LR-polynomial

Ex: $n = 6$, $\lambda = (221100)$, $\mu = (221100)$, $\nu = (332211)$.

- $P^{\nu}_{\lambda\mu}(t) = \frac{1}{2}(t+1)(t+2)$.
- Now construct skeleton - multi-stage process

Degree of primitive LR-polynomial

Ex: $n = 6$, $\lambda = (221100)$, $\mu = (221100)$, $\nu = (332211)$.

- $P^{\nu}_{\lambda\mu}(t) = \frac{1}{2}(t+1)(t+2)$.
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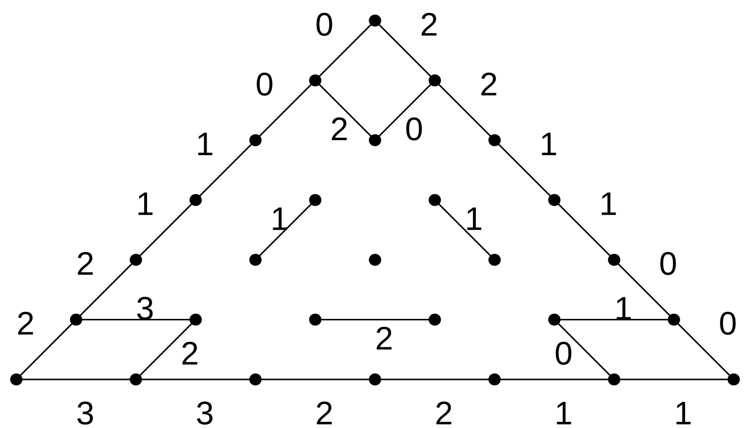

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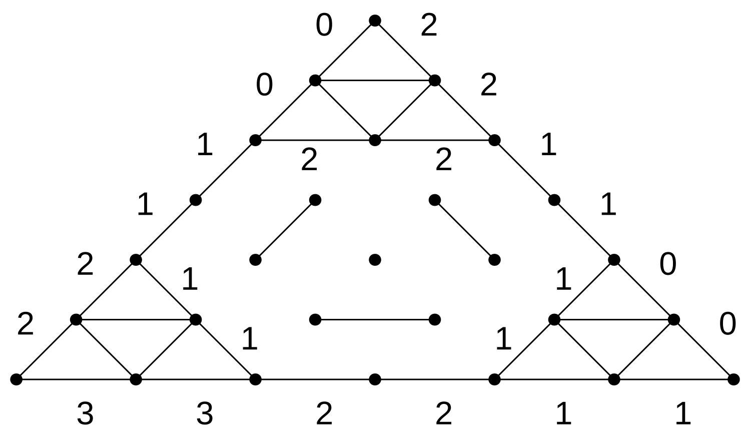

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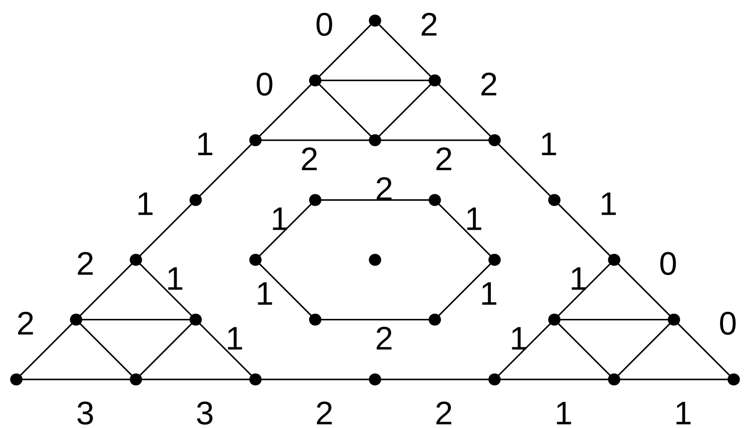

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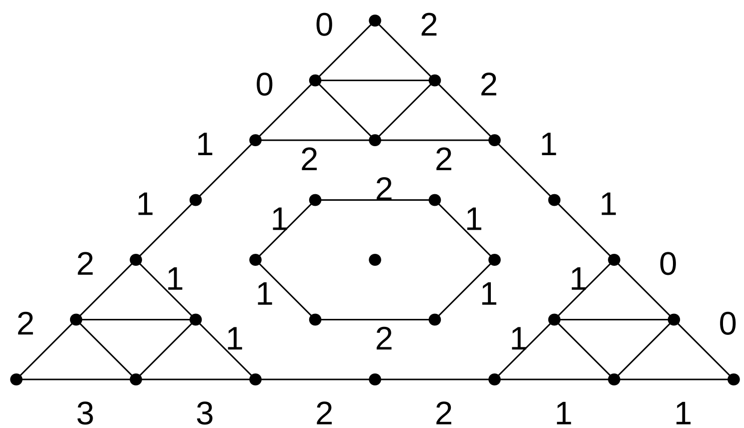
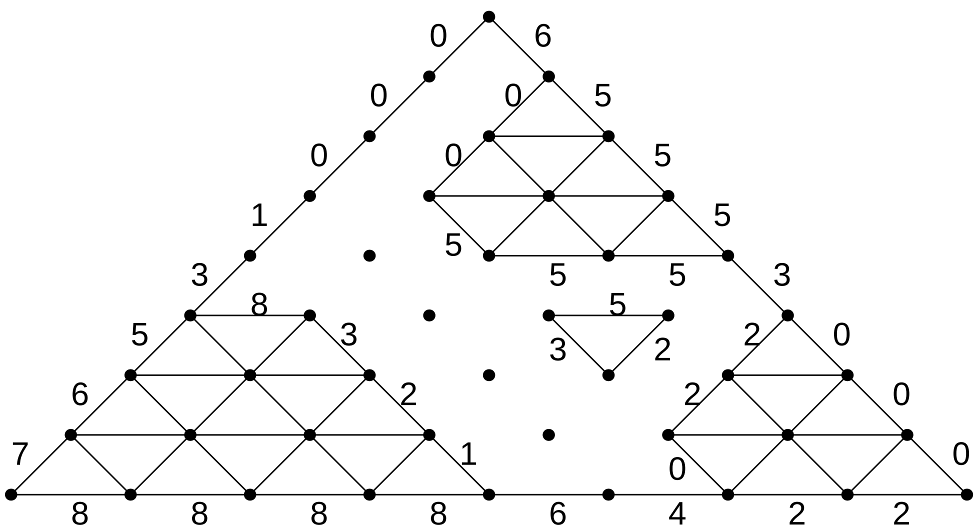


- $\deg P^{\nu}_{\lambda\mu}(t) = 2 = d(G_{n;\lambda\mu\nu})$.

Counterexample to skeleton degree bound

Ex: $n = 8$, $\lambda = (76531000)$, $\mu = (65553000)$, $\nu = (88886422)$.

- $P^{\nu}_{\lambda\mu}(2) = (t+1)(t+2)(t+3)(t+4)/24$
- Now construct skeleton



- $d(G_{n;\lambda\mu\nu}) = 5$.

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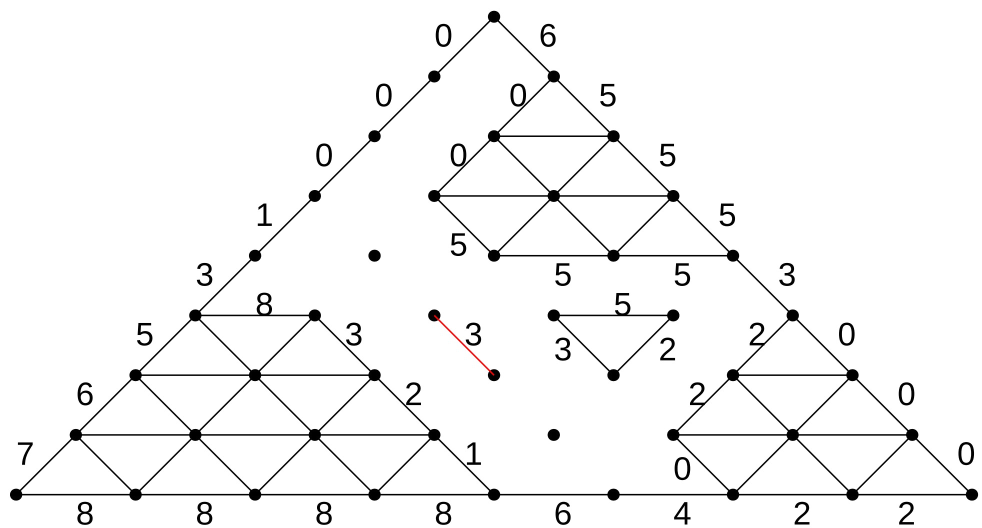

- $\deg P^{\nu}_{\lambda\mu}(t) = 4 < 5 = d(G_{n;\lambda\mu\nu})$.

Linear factors

Origin of linear factors in LR-polynomials.

- Let $\mathcal{P}$ be an LR hive polytope, and $\overline{\mathcal{P}}$ its interior.
- For $t \in \mathbb{N}$: $P^{\nu}_{\lambda\mu}(t) = i(\mathcal{P}, t) = \#\{t\mathcal{P} \cap \mathbb{Z}^d\}$.
- Ehrhart reciprocity: $i(\mathcal{P}, -t) = (-1)^d \#\{t\overline{\mathcal{P}} \cap \mathbb{Z}^d\}$.
- For $m \in \mathbb{N}$: $P^{\nu}_{\lambda\mu}(-m) = i(\mathcal{P}, -m) = (-1)^d \#\{m\overline{\mathcal{P}} \cap \mathbb{Z}^d\}$.
- Hence $P^{\nu}_{\lambda\mu}(-m) = 0$ and $P^{\nu}_{\lambda\mu}(t)$ must contain a linear factor $(t + m)$ if $m\mathcal{P}$ contains no interior integer points.

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Anticipate: $P^{\nu}_{\lambda\mu}(t)$ may contain $(t+1)(t+2)\cdots(t+M)$.

Problem: Determine M.

Possible continuation in t

- For $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ let $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ with $\overline{x}_i = x_i^{-1}$ for $i = 1, 2, \ldots, n$.
- For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ let $\tilde{\lambda} = (\lambda_n, \ldots, \lambda_2, \lambda_1)$.
- $$s_{t\lambda}(\mathbf{x}) = \frac{\left|x_i^{t\lambda_j+n-j}\right|}{\left|x_i^{n-j}\right|} \implies s_{-m\lambda}(\mathbf{x}) = \frac{\left|x_i^{-m\lambda_j+n-j}\right|}{\left|x_i^{n-j}\right|}.$$
- This gives $s_{-m\lambda}(\mathbf{x}) = \dfrac{\left|\overline{x}_i^{m\lambda_{n-k+1}+n-k}\right|}{\left|\overline{x}_i^{n-k}\right|} = s_{m\tilde{\lambda}}(\overline{\mathbf{x}})$.

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Definition For $c_{\lambda\mu}^{\nu} > 0$ and any positive integer m, let

$$c_{-m\lambda,-m\mu}^{-m\nu} = c_{m\tilde{\lambda},m\tilde{\mu}}^{m\tilde{\nu}}.$$

LR polynomials for negative t

Conjecture: Let $c^{\nu}_{\lambda\mu} > 0$ be simple, all Horn inequalities strict,

then $$P^{\nu}_{\lambda\mu}(-m) = c^{m\tilde{\nu}}_{m\tilde{\lambda},m\tilde{\mu}},$$

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Standardization:

- $s_{m\tilde{\lambda}}(\mathbf{x}) = 0$ or $\pm s_{\rho}(\mathbf{x})$ for some partition ρ.
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Two types of zero:

- $s_{m\tilde{\lambda}}(\overline{x}) = 0$, $s_{m\tilde{\mu}}(\overline{x}) = 0$, $s_{m\tilde{\nu}}(\overline{x}) = 0$.
- $c_{\rho\sigma}^{\tau} = 0$.

Simple example

Ex: $n = 7$, $\lambda = (433210)$, $\mu = (432210)$, $\nu = (7444321)$.

- $P^{\nu}_{\lambda\mu}(t) = (t+1)(t+2)(t+3)(t+4)(t+5)$
  $\cdot\, (5t+21)(t^2+2t+4)/10080.$

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Type one zeros for $m = 1, 2, 3$ since:

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No more zeros for $m > 5$ since for $m = 6$: $c^{\tau}_{\rho\sigma} = 3$.

Simple and non-simple examples

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- $P^{\nu}_{\lambda\mu}(t) = (t+1)(t+2)(t+3)(t+4)(t+5)$
  $\cdot\ (5t+21)(t^2+2t+4)/10080$.
- $c^{m\tilde{\nu}}_{m\tilde{\lambda},m\tilde{\mu}} = 0, 0, 0, 0, 0, 3, 39, 247$ for $m = 1, 2, 3, 4, 5, 6, 7, 8$.
- $P^{\nu}_{\lambda\mu}(-m) = 0, 0, 0, 0, 0, 3, 39, 247$ for $m = 1, 2, 3, 4, 5, 6, 7, 8$.

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Non-simple: $n = 6$, $\lambda = (221100)$, $\mu = (221100)$, $\nu = (332211)$.

- $P^{\nu}_{\lambda\mu}(t) = (t+1)(t+2)/2$.
- $c^{m\tilde{\nu}}_{m\tilde{\lambda},m\tilde{\mu}} = 0, 0, 0, 3, 6,$ for $m = 1, 2, 3, 4, 5$.
- $P^{\nu}_{\lambda\mu}(-m) = 0, 0, 3, 6, 10$ for $m = 1, 2, 3, 4, 5$.

Non-primitive example

Non-primitive: $n = 5$, $\lambda = (9, 7, 6, 2, 0)$, $\mu = (13, 5, 3, 1, 0)$, $\nu = (14, 12, 11, 5, 4)$, $P^{\nu}_{\lambda\mu}(t) = (t + 1)$.

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Primitive factors

- $n = 3$, $\lambda_I = (9,7,2)$, $\mu_J = (5,3,1)$, $\nu_K = (12,11,4)$.
- $n = 2$, $\lambda_{\overline{I}} = (6,0)$, $\mu_{\overline{J}} = (13,0)$, $\nu_{\overline{K}} = (14,5)$.

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Conjecture: $c^{\nu}_{\lambda\mu} > 0$ is primitive, all essential Horn inequalities strict, if and only if $c^{m\tilde{\nu}}_{m\tilde{\lambda},m\tilde{\mu}} \neq 0$ for some positive integer m.