PDSAWs in a Wedge and Nestings of Matchings and Permutations

Martin Rubey

March 31, 2008

Self Avoiding Walks

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- a SAW is a nearest neighbour walk on a lattice, that does not intersect itself (we consider only 2-dimensional walks)
- used as model for linear polymers in a solution
- physicists want to know the asymptotic number of SAWs with n steps:

$$c_n = \#\text{SAWs with } n \text{ steps} \sim A\mu^n n^{\gamma-1}(1+\dots)$$

μ is the growth constant – it depends on the lattice:

- $\mu_\triangle \approx 4.15$ [Jensen]
- $\mu_\square \approx 2.64$ [Guttmann & Jensen]
- $\mu_{\hexagon} = \sqrt{2+\sqrt{2}}$ [Nienhuis]

$\gamma = 43/32$ is the critical exponent – it is universal! [Nienhuis]

(all numbers and existence of γ only conjectured)

Partially Directed SAWs

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- a PDSAW is a SAW in the square lattice without west steps,
- its generating function is

$$\sum_{n\geq 0} c_n t^n = \frac{1+t}{1-2t-t^2} = 1 + 3t + 7t^2 + 17t^3 + \ldots$$

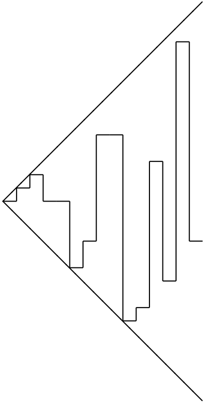
and

$$c_n = \frac{1}{2}\left((1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right).$$

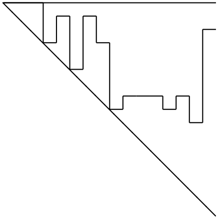
  - growth constant $\mu = 1 + \sqrt{2}$
  - critical exponent $\gamma = 1$

PDSAWs in a Wedge

- confine PDSAWs to a wedge:



or



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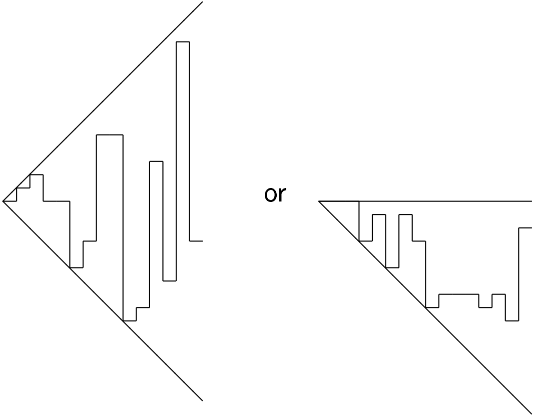

- the growth constant $\mu = 1 + \sqrt{2}$ is independent of the angle of the wedge! [Rensburg & Prellberg & Rechnitzer]

PDSAWs in a Wedge: [Rensburg & Prellberg & Rechnitzer]

- for the symmetric wedge defined by $y = \pm x$, with

$$P = \sqrt{(1-t^2)(1-5t^2)} \text{ and } Q = (1-3t^2-P)/2t$$

they find that

$$\sum_{n\geq 0} c_n t^n = \frac{1}{1-2t-t^2}\left(1-(1-t^2-P)\sum_{n\geq 0}(-1)^n t^{n^2} Q^n\right)$$
$$= 1+t+t^2+3t^3+5t^4+13t^5+\ldots$$

and critical exponent $\gamma = 1$.

PDSAWs in a Wedge: [Rensburg & Prellberg & Rechnitzer]

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  and critical exponent $\gamma = 1$.
- for the asymmetric wedge between $y = 0$ and $y = -x$ the generating function is slightly more complicated, with critical exponent $\gamma = \frac{1}{2}$.

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- generating function according to north steps:

$$\frac{1}{(1-q)^n} \sum_{i \geq 0} (-1)^i \left(\binom{2n}{n-i} - \binom{2n}{n-i-1} \right) q^{\binom{i+1}{2}}$$

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  \# PDSAWs in the symmetric wedge with n east steps and N north steps
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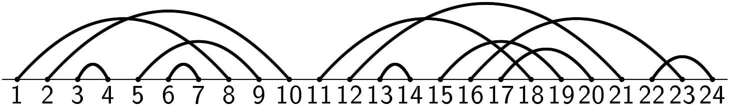
  # PDSAWs in the symmetric wedge with n east steps and N north steps
  $= \#$ matchings of $[2n]$ with N crossings
  $= \#$ matchings of $[2n]$ with N nestings

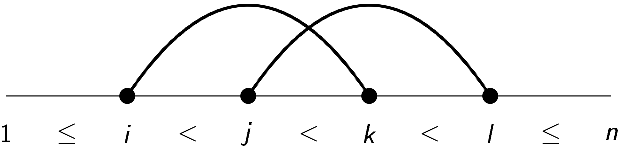
Matchings

a matching of [24]:

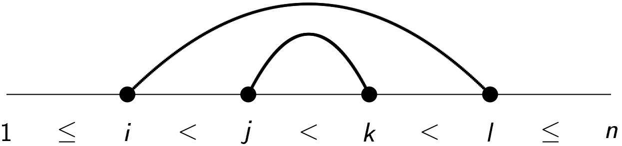


Nestings and Crossings in Matchings

A crossing:



A nesting:



\# matchings with N nestings and C crossings
$=$ # matchings with N crossings and C nestings [Kasraoui & Zeng]

Andrew Rechnitzer's Challenge

Find a *bijective* proof!

Matchings $\leftrightarrow$ PDSAWs.

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Find a *bijective* proof!

Matchings $\leftrightarrow$ weighted Dyck paths $\leftrightarrow$ PDSAWs.

Matchings according to Nestings and «histoires d'Hermite»

Theorem (Françon & Viennot, Flajolet)

Matchings of the set $[2n]$ *with*

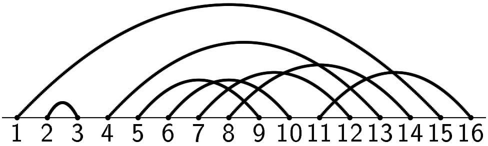
- N *nestings and*
- 1 *being matched with* M

are in bijection with weighted Dyck paths with n *down steps,*

- *total weight* N *and*
- M *being the position of the first down step with weight* 0.

The down steps of the Dyck path are weighted with a nonnegative integer smaller than the initial height of the step.
(The position of the first step in a Dyck path is one, the second step has position two, etc.)

Matchings according to Nestings and «histoires d'Hermite»



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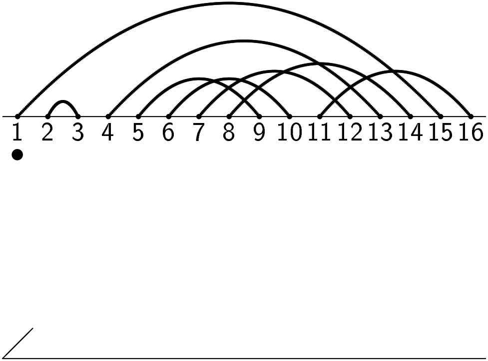

- translate openers to up-steps without weight

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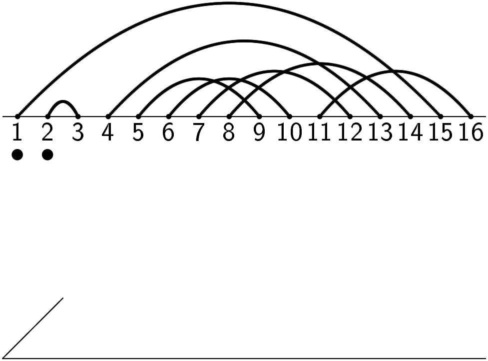

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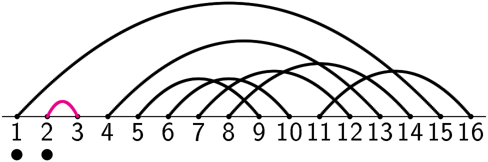

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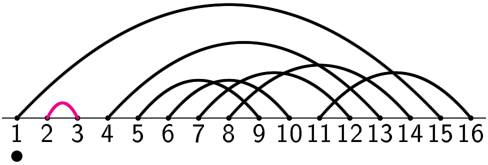

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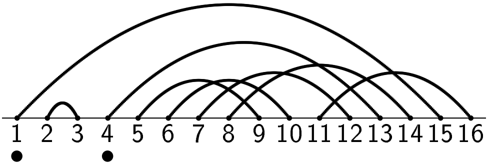

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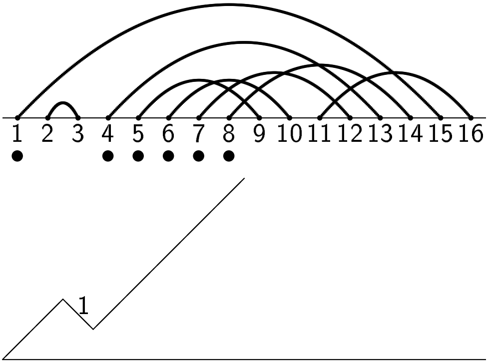

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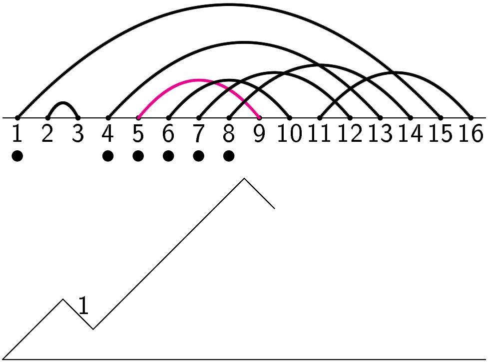

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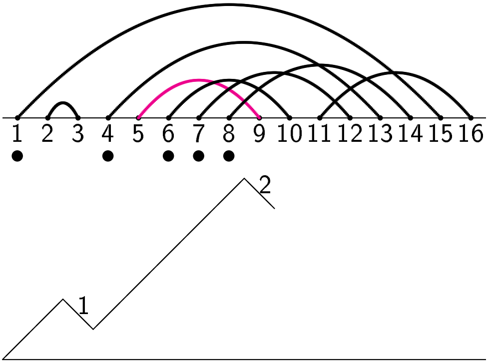

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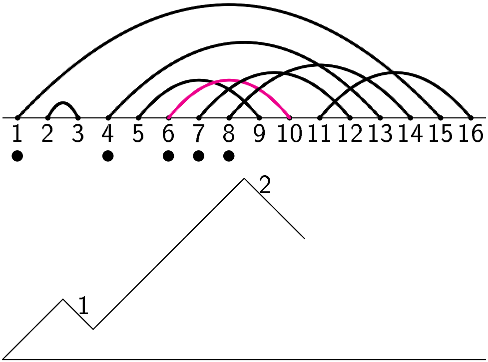

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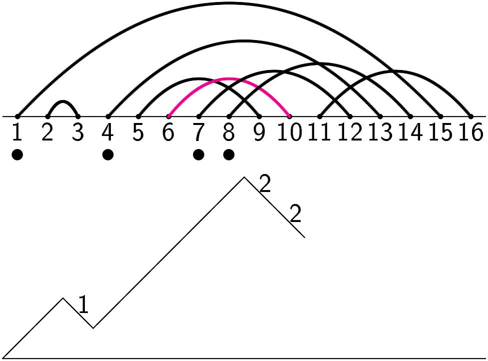

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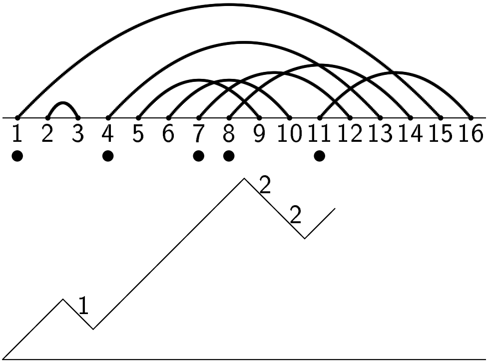

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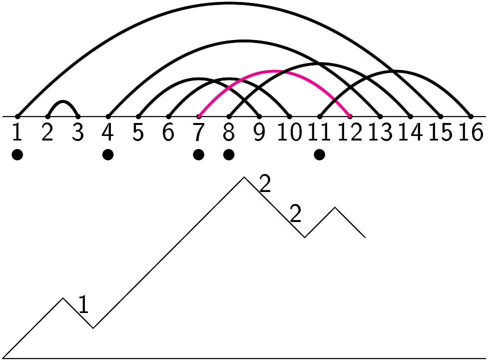

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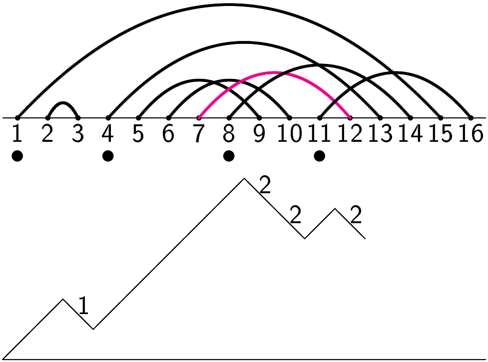

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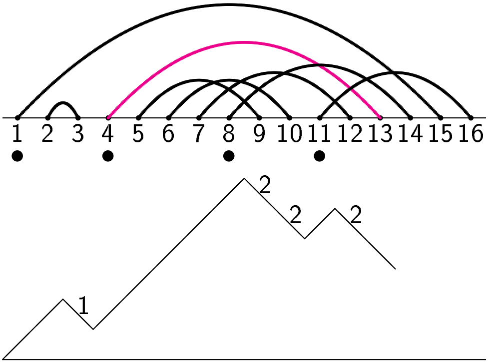

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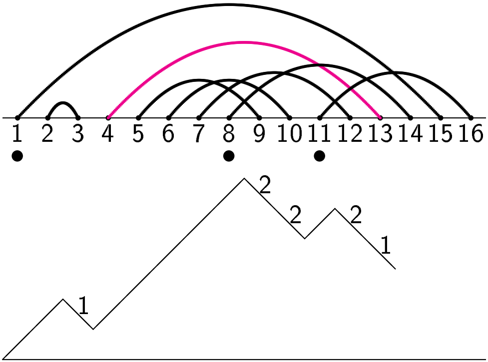

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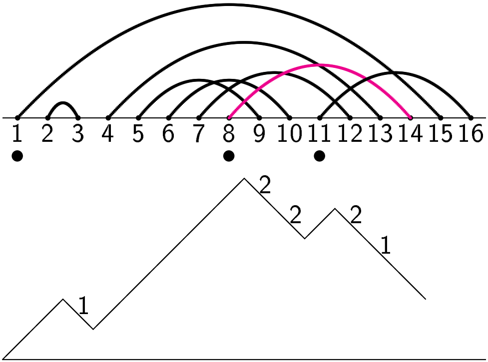

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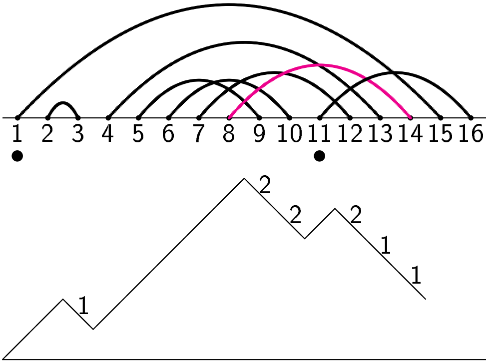

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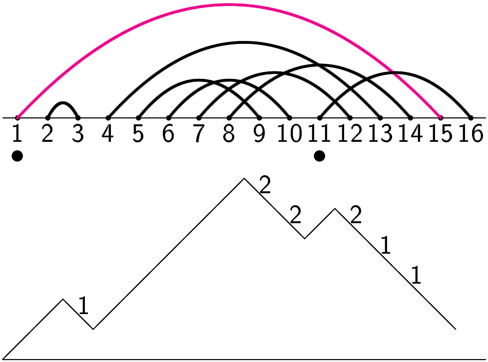

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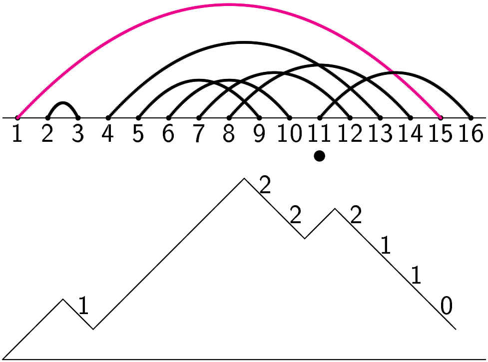

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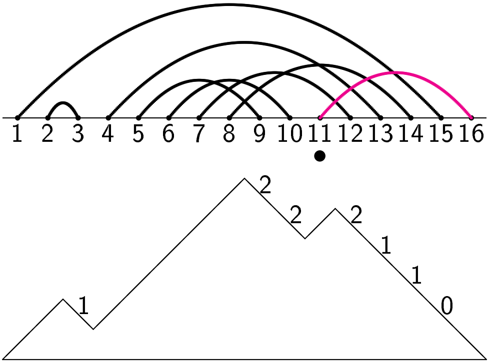

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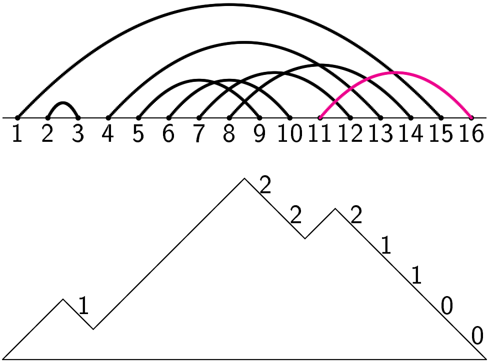

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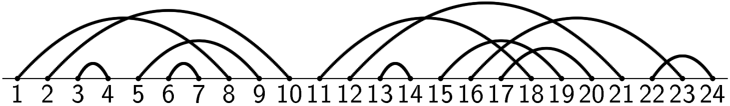
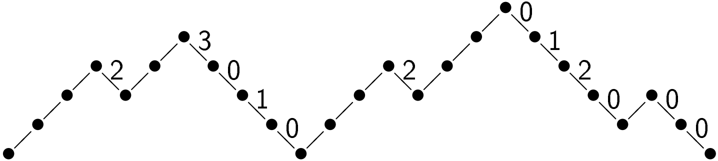
Prime factors of Matchings and Dyck Paths

This bijection preserves prime factors!

- a matching with 2 prime factors:



- the corresponding weighted Dyck path:



«histoires d'Hermite» and PDSAWs in the symmetric wedge

Theorem (R.)

Weighted Dyck paths with n down steps,

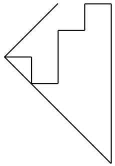
- *total weight N and*
- *M being the position of the first down step with weight 0,*

are in bijection with PDSAWs in the symmetric wedge with n east steps,

- *N north steps and*
- *length of the last descent equal to $M - 1$.*

(The length of the last descent is the number of south steps after the last east step of the PDSAW.)

«histoires d'Hermite» and PDSAWs in the symmetric wedge



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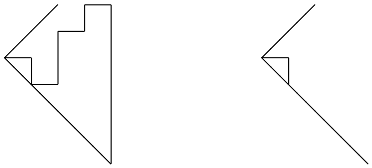

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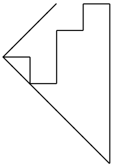

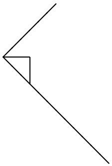

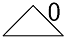

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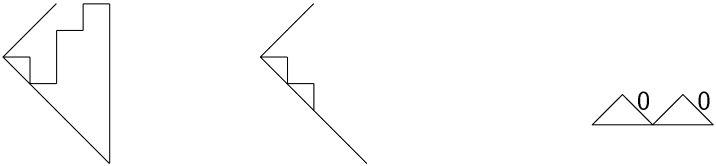

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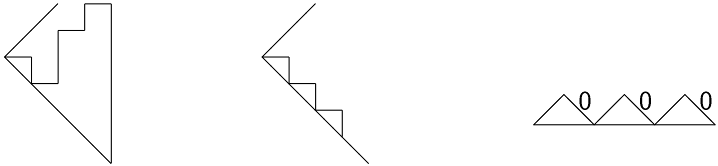

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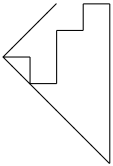
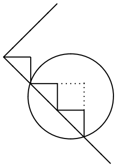
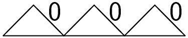
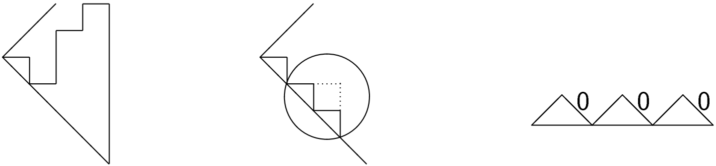
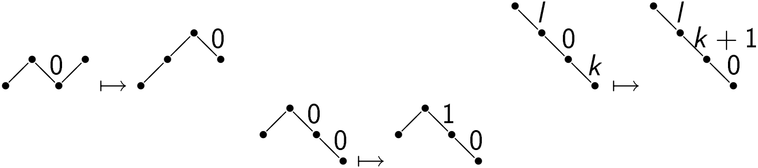






«histoires d'Hermite» and PDSAWs in the symmetric wedge





for $k > 0$:

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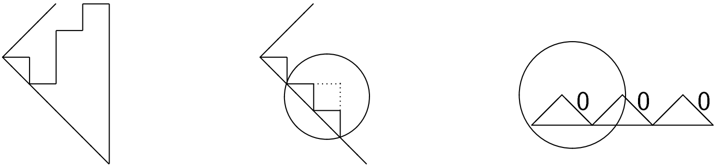

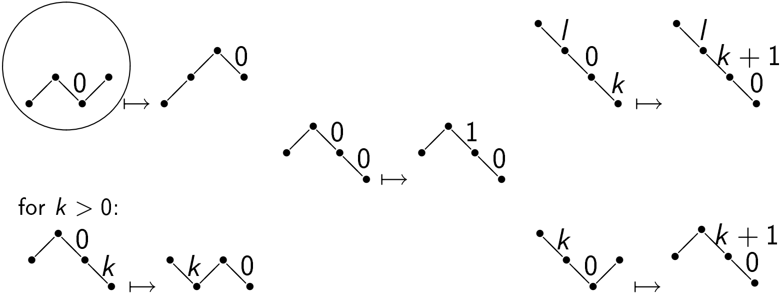

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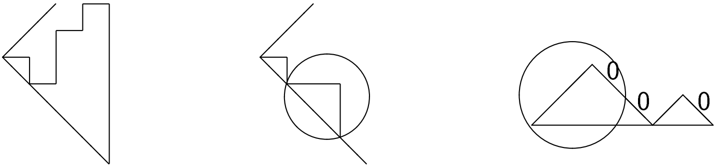

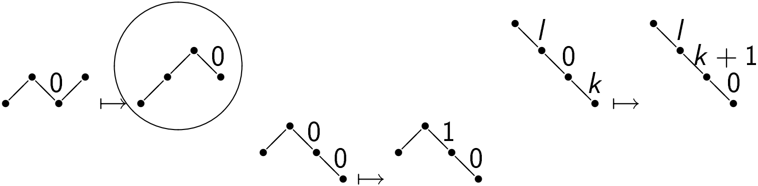

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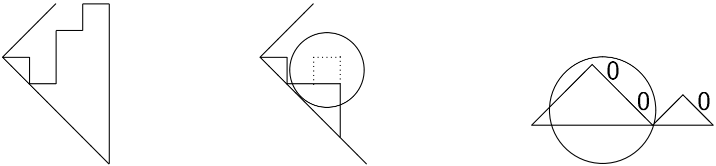

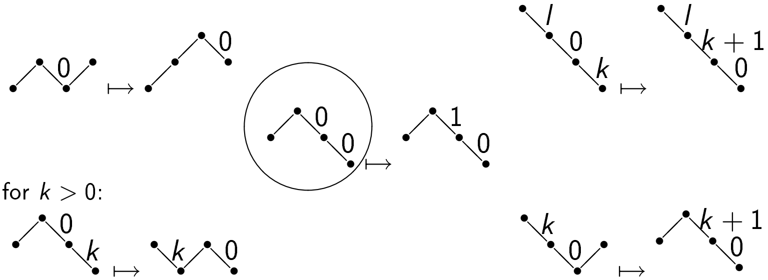

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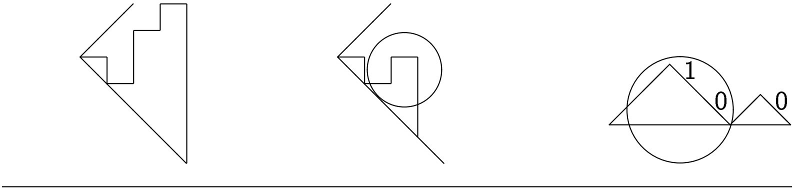

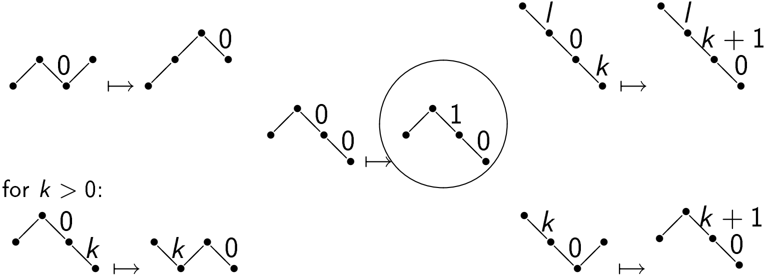

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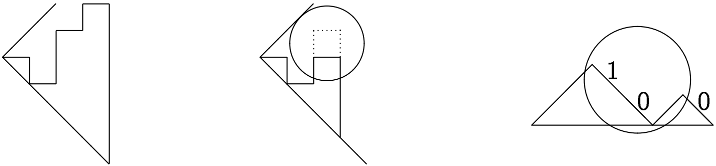

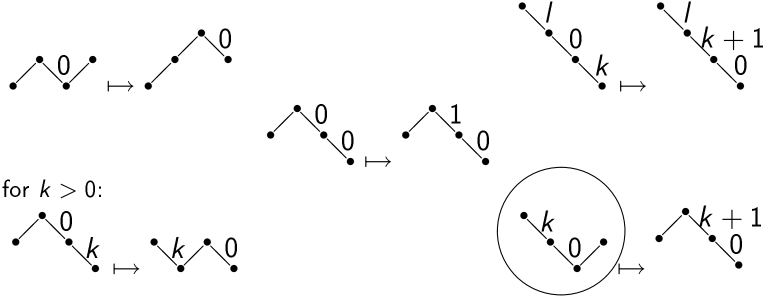

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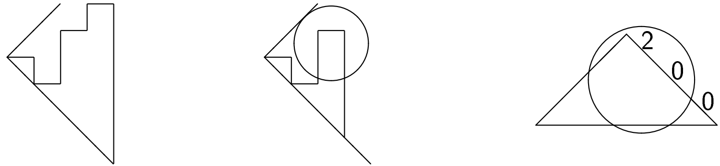

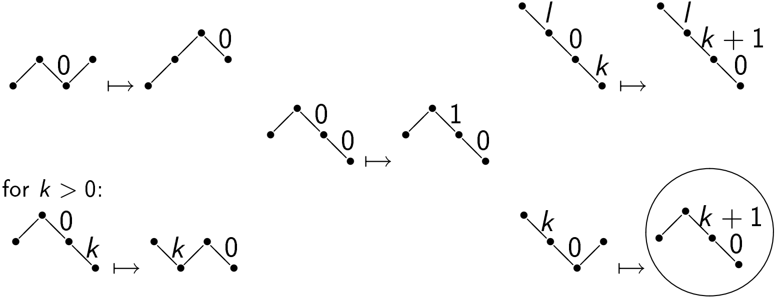

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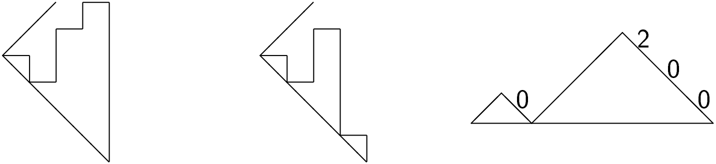

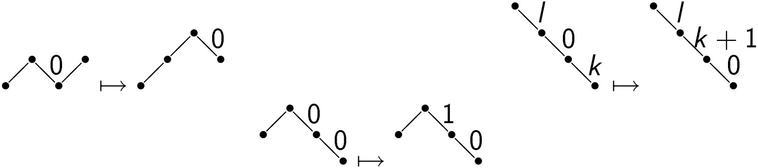

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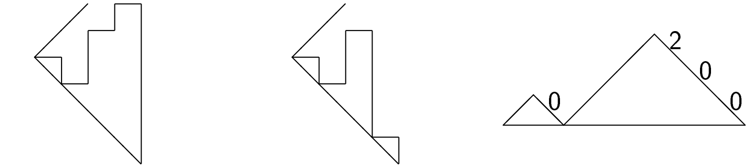
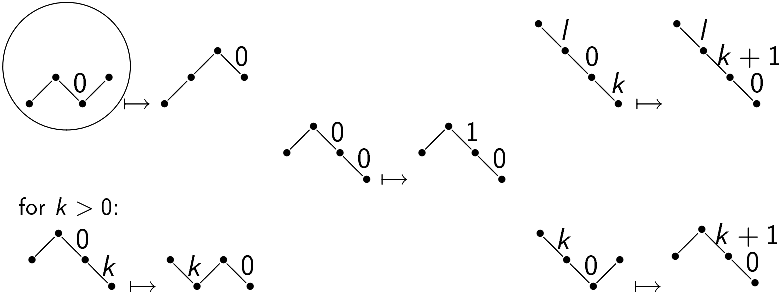


last east step

$<$ next-to-last	$=$ next-to-last	$>$ next-to last



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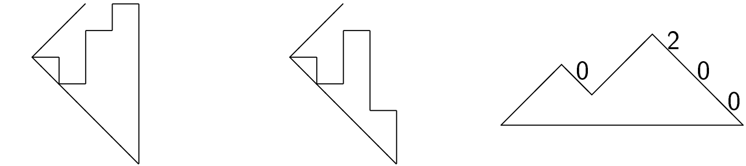

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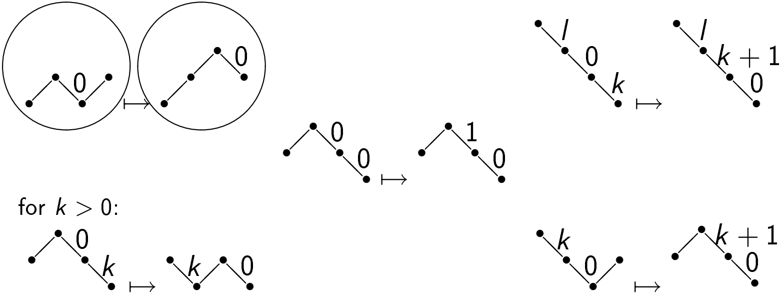

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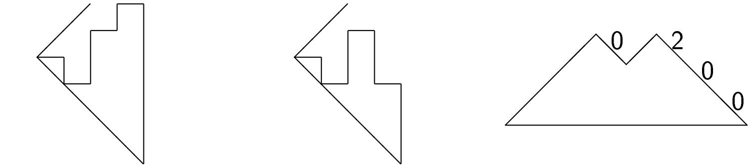

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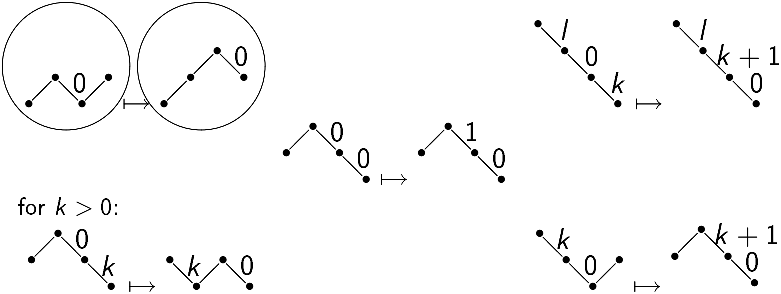

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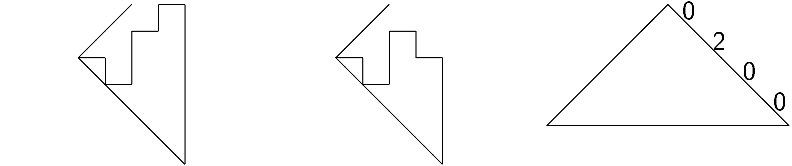

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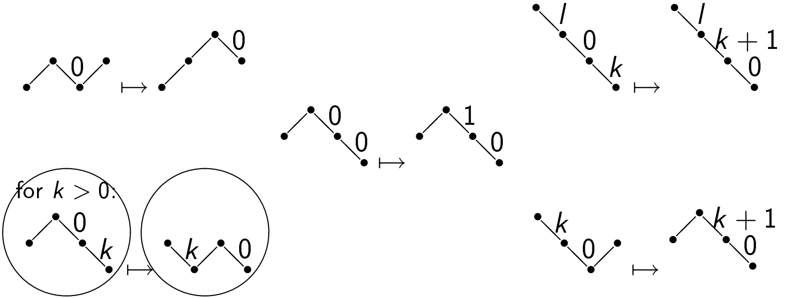

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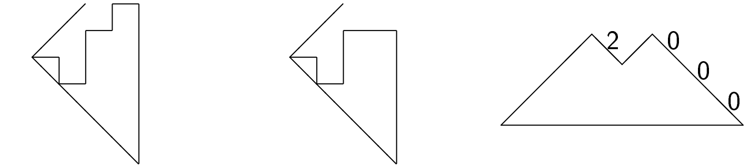

last east step

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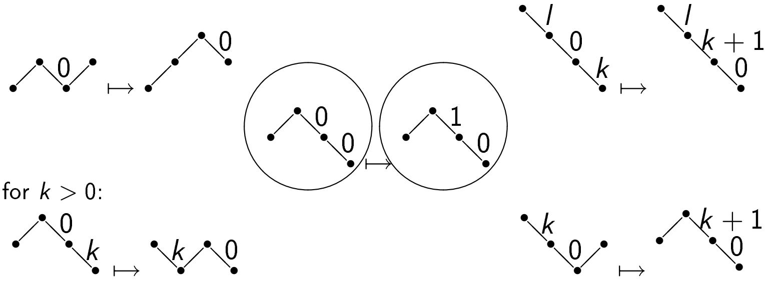

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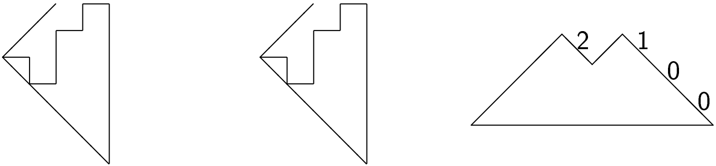

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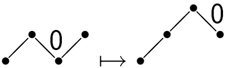

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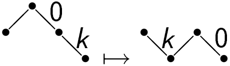
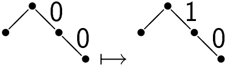
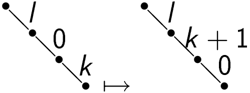
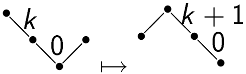


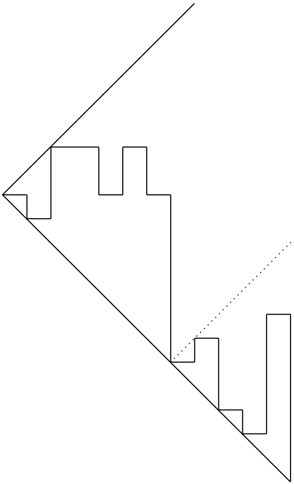




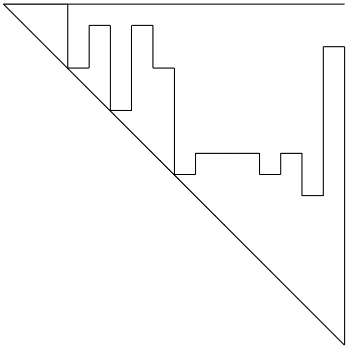


Prime factors of PDSAWs in the symmetric wedge

Again, prime factors are preserved, the first factor of the matching being mapped to the last prime factor of the PDSAW. The PDSAW with 2 prime factors corresponding to the matching before:



What about PDSAWs in the asymmetric wedge?



What about PDSAWs in the asymmetric wedge?

- total number with n east steps
  $c_n = (2n-1)!! = \#\text{permutations of } [n]$
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  $\#$ PDSAWs in the asymmetric wedge with n east steps and N north steps
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  (consider possible heights of the east steps)
- generating function according to north steps: [Williams]

$$\sum_{k=1}^{n} q^{-k^2} \sum_{i=0}^{k-1} (-1)^i [k-i]_q^n q^{ki} \left(\binom{n}{i} q^{k-i} + \binom{n}{i-1} \right)$$

or [Corteel & R.]

$$\sum_{k=0}^{n} (-1)^k \left(\binom{2n}{n-k} - \binom{2n}{n-k-1} \right) \left(1 + \sum_{j=1}^{k} q^{j(k-j)} \left(q^j - 1\right) \right)$$

- Online Encyclopedia of Integer Sequences:
  \# PDSAWs in the asymmetric wedge with n east steps and N north steps
  $=$ \# permutations of $[n]$ with N nestings!

Nestings and Crossings in Permutations

permutations with N nestings and C crossings
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PDSAWs in the asymmetric wedge

Theorem (Foata & Zeilberger, R.)

There are bijections between permutations of the set $[n]$ *with*

- N *nestings and*
- 1 *being mapped to* M,

weighted bicoloured Motzkin paths with n *steps,*

- *total weight* N *and*
- M *being the position of the first east or south-east step with weight* 0, *and*

PDSAWs between $y = -x$ *and the* x*-axis, with* n *east steps,*

- N *north steps and*
- *length of the last descent equal to* M.

PDSAWs in the asymmetric wedge

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Moreover, prime factors are preserved, the first factor of the permutation being mapped to the last prime factor of the PDSAW.

Future Work

- Find a bijective proof analogous to Penaud's for the generating function for permutations according to nestings.

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- Can we obtain the generating function for PDSAWs ending anywhere according to the total number of steps using our bijections?
- What about set partitions? They share many of the properties of matchings and permutations concerning nestings and crossings, but is there a corresponding familiy of PDSAWs?