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HYPEROCTAHEDRAL SPECIES

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J'ai descendu avec Pierre plusieurs rapides. Parfois c'étais sur l'eau, parfois c'étais en concepts.
Merci, Nantel

Abstract. We introduce hyperoctahedral species ($\mathcal{H}$-species) or species of type B, which are analogous to the classical tensor species, but on which we consider the action of the groups of signed permutations. We give a bistrong monoidal functor, a functor which preserves Hopf monoids, between the monoidal categories of species and $\mathcal{H}$-species. We also define bilax monoidal functors (functors which preserve the structure of bimonoids) between the category of $\mathcal{H}$-species and the category of graded vector spaces. Using these functors, the combinatorial Hopf algebra DQSym is shown to arise from the cofree comonoid on the exponential species.

1. Introduction

The theory of species of Joyal [9] has open a lot of interesting problems in combinatorics. In particular, Bergeron [6] used species to find a combinatorial explanation of plethystic substitution and Mendez and Nava [16] studied colored species. Others were interested in generalizing species for modules of other groups. Hetyei *et al.* [13] were motivated by various enumerative questions on the n-dimensional cube and defined cubical species, modules for the hyperoctahedral groups. In [12], Henderson generalized Joyal's species, called B_r-modules, to study characters of the representations of $W(r, n)$, the wreath product of the cyclic group of order r and the symmetric group $\mathcal{S}_n$, on cohomology of De Concini–Procesi compactifications. Others have been interested in finding relationships between species and well-known algebras such as the descent algebra [17] or combinatorial Hopf algebras [2].

In [2], Aguiar and Mahajan have introduced bilax monoidal functors (functors preserving bimonoids) from the category of species (**Sp**) to the category of graded vector spaces (**gVec**). In particular, they showed that many different graded Hopf algebras can arise from one Hopf monoid in species. In this paper, we make an analogous study with hyperoctahedral species as done by Aguiar and Mahajan [2] in the case of species. Hyperoctahedral species, or $\mathcal{H}$-species, which are given in terms of $\mathcal{H}$-sets, are modules of the hyperoctahedral groups. Therefore, the role played by the symmetric groups in species is played by the groups of signed permutations in this setting. We denote by $\mathbf{Sp}^{\mathcal{H}}$ the category of $\mathcal{H}$-species. Endowed with a symmetric tensor product, it turns into a monoidal category and therefore Hopf monoids can be

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defined. We construct $\mathcal{H}$-species from usual species by way of a bistrong monoidal functor $\mathcal{S}$ sending the regular representation of $!\mathcal{S}_n$ to the regular representation of $\mathcal{B}_n$. We also introduce bilax monoidal functors $K^{\mathcal{H}}$ and $\widetilde{K^{\mathcal{H}}}$ to generate graded vector spaces from $\mathcal{H}$-species. Composing these functors ($K^{\mathcal{H}}\mathcal{S}$, $\widetilde{K^{\mathcal{H}}\mathcal{S}}$), we get new graded Hopf algebras from usual species. We also give a natural transformation which links our work and the work of Aguiar and Mahajan [2]. We finally discuss an interesting example: the graded Hopf algebra DQSym of diagonally symmetric functions, see [3], arises from the cofree comonoid on the exponential species.

Many constructions we use are very similar to the ones of Aguiar and Mahajan [2]. In particular, the tensor product on $\mathcal{H}$-species and the functors from $\mathcal{H}$-species to graded vector spaces parallel the same notions for species. The advantage of a similar setting and similar constructions is that as many as four (based on the functors defined in this paper) graded Hopf algebras arise from a a single species. Therefore, natural transformations between our functors and Aguiar and Mahajan's functors, for example

$$\begin{tikzcd} & \mathbf{Sp}^{\mathcal{H}} \arrow[dr, "\widetilde{K^{\mathcal{H}}}"] \arrow[d, phantom, "\Downarrow \widetilde{\alpha^{\mathcal{S}}}"] & \\ \mathbf{Sp} \arrow[ur, "\mathcal{S}"] \arrow[rr, "\overline{K}"'] & {} & \mathbf{gVec}, \end{tikzcd} \tag{1.1}$$

given here by $\widetilde{\alpha^{\mathcal{S}}}$, give morphisms of graded Hopf algebras in the same family. For example, Aguiar and Mahajan proved that the Hopf algebra of set compositions and the Hopf algebra QSym of quasi-symmetric functions arise from a particular Hopf monoid. Using the functors defined in this paper, we prove that DQSym is in the same family (see Theorem 35) as is the Hopf algebra of $\mathcal{H}_{\mathcal{S}}$-set compositions (see Example 29). Furthermore, the natural transformation $\widetilde{\alpha^{\mathcal{S}}}$ yields the morphism of Hopf algebras DQSym $\to$ QSym sending a bicomposition to the composition obtained by adding each component of each bipart (see Example 36). This paper is only the starting point of the study of $\mathcal{H}$-species. Further exploration is done in [8], but a lot of interesting problems remain to be done.

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2. Preliminaries

We begin by a brief introduction on monoidal categories and morphisms of such categories, given by lax, colax and bilax monoidal functors. For these notions, see [2, 4, 5, 14]. We then give two examples: the monoidal categories of graded vector spaces and of species. We recall the construction of the cofree comonoid on a species, as done by Aguiar and Mahajan [2]. For more details on species, see [2, 7, 9, 13].

2.1. **Monoidal categories.** For a definition of monoidal category, see MacLane [15]. For the definitions of monoids, comonoids, bimonoids and Hopf monoids in monoidal categories, see Aguiar and Mahajan [2].

Definition 1 (Bénadou [4] and Aguiar and Mahajan [2])**.** Let $(C, \bullet, e)$ and $(D, \diamond, e')$ be two monoidal categories.

(1) A lax monoidal functor (F, ϕ, ϕ_0) between $(C, \bullet, e)$ and $(D, \diamond, e')$ is a functor $F\colon C \to D$ with a morphism $\phi_{a,b}\colon Fa \diamond Fb \to F(a \bullet b)$ in D, natural in a and b for each pair a, b of objects in C, and a morphism $\phi_0\colon e' \to Fe$ in D such that ϕ is associative and left and right unital in the usual sense.
(2) A colax monoidal functor (F, ψ, ψ_0) is a functor $F\colon C \to D$ with a morphism $\phi_{a,b}\colon F(a \bullet b) \to Fa \diamond Fb$ in D, natural in a and b for each pair of objects a, b in C, and a morphism $\psi_0\colon Fe \to e'$ in D such that ψ is coassociative and left and right counital in the usual sense.
(3) A bilax monoidal functor $(F, \phi, \phi_0, \psi, \psi_0)$ between two braided monoidal categories $(C, \bullet, e, \beta)$ and $(D, \diamond, e', \beta)$ is a lax monoidal functor (F, ϕ, ϕ_0) and a colax monoidal functor (F, ψ, ψ_0) satisfying the braiding condition and unitality conditions.
(4) A bilax monoidal functor with ϕ, ϕ_0, ψ and ψ_0 invertible is a bistrong monoidal functor.

The composites of lax, colax, bilax and bistrong monoidal functors are lax, colax, bilax and bistrong monoidal functors respectively. A morphism of bilax monoidal functors between (F, ϕ, ψ) and (G, γ, δ) is a natural transformation $\alpha\colon F \to G$ which commutes with ϕ, ψ, γ and δ. See [2] for more details.

Proposition 2 (Bénabou [5] and Aguiar and Mahajan [2])**.** (1) *If F is a lax (colax, bilax) monoidal functor from $(C, \bullet)$ to $(D, \diamond)$ and h is a monoid (comonoid, bimonoid) in C then Fh is a monoid (comonoid, bimonoid) in D. Furthermore, a morphism of lax (colax, bilax) monoidal functors $\Theta\colon F \to G$ yields a morphism of monoids (comonoids, bimonoids) $\Theta_h\colon Fh \to Gh$ if h is a monoid (comonoid, bimonoid) in C.*
(2) *If F is a bistrong monoidal functor from $(C, \bullet)$ to $(D, \diamond)$ and h is a Hopf monoid in C with antipode S then Fh is a Hopf monoid in D with antipode FS. Moreover, a morphism $\Theta\colon F \to G$ of bistrong monoidal functors yields a morphism $\Theta_h\colon Fh \to Gh$ of Hopf monoids if h is a Hopf monoid in C.*

For example, the category **gVec** of graded vector spaces is monoidal. Objects and morphisms of this category are as follow. A vector space V over $\mathbb{K}$ is said to be *graded* if it is a sequence $V = (V_n)_{n\geq 0}$ of vector spaces over $\mathbb{K}$. In that case, we often write $V = \bigoplus_{n\geq 0} V_n$, where V_n is the homogeneous component of degree n in V. A morphism of graded vector spaces $f\colon V \to W$ is a sequence of linear maps $(f_n\colon V_n \to W_n)_{n\geq 0}$. We often write $f = \bigoplus_{n\geq 0} f_n$. This category is monoidal with tensor product defined on the component of degree n by

$$(V \cdot W)_n = \bigoplus_{i=0}^{n} V_i \otimes W_{n-i}.$$

2.2. **Species.** We first recall the definitions of decomposition, set composition and set partition of finite sets and some operations on set compositions. Let I be a finite set. A *decomposition* of I is an ordered sequence $G = (G_1, \dots, G_k)$ of disjoint subsets of I such that $G_1 \cup G_2 \cup \cdots \cup G_k = I$. We write $I = G_1 \sqcup \cdots \sqcup G_k$. A *set composition* of I is a decomposition $G = (G_1, G_2, \dots, G_k)$ of I in which each G_i, $1 \leq i \leq k$, is nonempty. We write $G \models I$ or $G = G_1|G_2|\dots|G_k$. A *set partition* $F = \{F_1, F_2, \dots, F_l\}$ of I is an unordered collection of disjoint nonempty subsets of I such that $F_1 \cup \cdots \cup F_l = I$. We write $F \vdash I$. The subsets G_i (F_i) are the parts of the composition (partition). For example, $34|1|25$ is a set composition of $[5]$ and $\{15, 234\}$ is a set partition of $[5]$. We omit brackets around each part for clarity. There is one (empty) set composition (set partition) of the empty set, denoted by $\emptyset$.

Let $F = F_1|\dots|F_k$ and $G = G_1|\dots|G_l$ be two set compositions on disjoint sets S and T, respectively. Note that $S \sqcup T$ denotes the disjoint union of S and T. The *restriction* of F to a set $S' \subseteq S$ is the set composition

$$F|_{S'} = F_1 \cap S'|\dots|F_k \cap S',$$

where we delete any empty parts.

The *concatenation* of F and G is the set composition of $S \sqcup T$ defined by

$$F|G = F_1|\dots|F_k|G_1|\dots|G_l.$$

A *shuffle* of F and G is a set composition of $S \sqcup T$ whose parts are parts of F and G and whose restriction to S is F and whose restriction to T is G. Let $F \sqcup\!\sqcup G$ denote the set of all shuffles of F and G. For example, $12|3 \sqcup\!\sqcup 45 = \{12|3|45, 12|45|3, 45|12|3\}$.

A *quasishuffle* of F and G is a set composition of $S \sqcup T$ whose restriction to S is F and whose restriction to T is G. In other words a quasishuffle is a shuffle such that two adjacent parts $F_i|G_j$ can be replaced by $F_i \sqcup G_j$. Let $F \,\underline{\sqcup\!\sqcup}\, G$ denote the set of all quasishuffles of F and G. For example,

$$12|3 \,\underline{\sqcup\!\sqcup}\, 45 = \{12|3|45, 12|45|3, 45|12|3, 12|345, 1245|3\}.$$

Definition 3. A *species* p with values in $\mathbf{Vec}_{\mathbb{K}}$, the category of vector spaces over a field $\mathbb{K}$ of characteristic zero, is a functor

$$\mathsf{p} \colon \mathbf{B} \to \mathbf{Vec}_{\mathbb{K}},$$

where $\mathbf{B}$ is the category of finite sets with bijections. A morphism of species is a natural transformation $\alpha \colon \mathsf{p} \to \mathsf{q}$. Let $\mathbf{Sp}$ denote the category of species.

A species is a family of vector spaces $\mathsf{p}[I]$, one for each finite set I, together with linear maps $\mathsf{p}[f] \colon \mathsf{p}[I] \to \mathsf{p}[J]$, one for each bijection $f \colon I \to J$. A morphism of species $\alpha\colon \mathsf{p} \to \mathsf{q}$ is a family of linear maps $\alpha_I\colon \mathsf{p}[I] \to \mathsf{q}[I]$, one for each finite set I, such that for each bijection $f\colon I \to J$, the following diagram commutes

$$\begin{tikzcd} \mathsf{p}[I] \arrow[r, "\alpha_I"] \arrow[d, "\mathsf{p}[f]"'] & \mathsf{q}[I] \arrow[d, "\mathsf{q}[f]"] \\ \mathsf{p}[J] \arrow[r, "\alpha_J"'] & \mathsf{q}[J]. \end{tikzcd} \tag{2.1}$$

We denote by $\mathsf{p}[n]$ the vector space $\mathsf{p}[\{1,2,\ldots,n\}]$, $n \geq 0$.

Each permutation $\sigma \in \mathcal{S}_n$ induces a map

$$\mathsf{p}[\sigma]\colon \mathsf{p}[n] \to \mathsf{p}[n],$$

which turns $\mathsf{p}[n]$ into an $\mathcal{S}_n$-module. A species p can then equivalently be defined by a sequence of $\mathcal{S}_n$-modules.

The category of species is monoidal with tensor product defined by

$$(\mathsf{p} \cdot \mathsf{q})[I] = \bigoplus_{S \sqcup T = I} \mathsf{p}[S] \otimes \mathsf{q}[T],$$

and with unit

$$\mathsf{o}[I] = \begin{cases} \mathbb{K} & \text{if } I = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Descriptions of monoids, comonoids, bimonoids and Hopf monoids are found in [2]. They are the same as the ones for $\mathcal{H}$-species in Section 4, by omitting the involution.

Example 4. Two Hopf monoids in $(\mathbf{Sp}, \cdot)$ will be considered in this paper. We give their definitions along with their products and coproducts.

(1) **Exponential species.**

$$\mathsf{e}[I] := \mathbb{K}, \text{ for each finite set } I.$$

Let I denote the only basis element of $\mathsf{e}[I]$. The components $\mu_{S,T}$ and $\Delta_{S,T}$ of the product and the coproduct are given by

$$\begin{aligned} \mathsf{e}[S] \otimes \mathsf{e}[T] &\to \mathsf{e}[I] & \mathsf{e}[I] &\to \mathsf{e}[S] \otimes \mathsf{e}[T] \\ S \otimes T &\mapsto S \sqcup T, & I &\mapsto S \otimes T. \end{aligned} \tag{2.2}$$

It is a commutative and a cocommutative Hopf monoid. Its antipode is given for any finite I by

$$S(I) = (-1)^{|I|} I. \tag{2.3}$$

(2) **Linear order species.**

$$\ell^*[I] := \mathbb{K}\text{-span of all linear orders on } I.$$

The components $\mu_{S,T}$ and $\Delta_{S,T}$ of the product and coproduct are

$$\begin{aligned} &\ell^*[S]\otimes\ell^*[T]\to\ell^*[I] && \ell^*[I]\to\ell^*[S]\otimes\ell^*[T] \\ &l_1\otimes l_2\mapsto\sum_{l\in l_1\sqcup l_2} l, && l\mapsto\begin{cases} l|_S\otimes l|_T & \text{if } S \text{ is an initial segment of } l,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}\tag{2.4}$$

The direct sum of the product is over all shuffles l of l_1 and l_2.

Note that we use here the dual of the linear order species. For more details, see [2].

A *positive species* is a species p such that $\mathsf{p}[\emptyset]=0$. Denote by $\mathbf{Sp}_+$ the subcategory of positive species. Let

$$(-)_+\colon \mathbf{Sp}\to\mathbf{Sp}_+,$$

be the functor defined by

$$\mathsf{p}_+[I]=\begin{cases}\mathsf{p}[I] & \text{if } I\neq\emptyset,\\ 0 & \text{otherwise.}\end{cases}$$

Remark 5 (Aguiar and Mahajan [2])**.** Any positive species has a monoid structure. Since the functor $(-)_+$ preserves monoids, if p is a monoid then p_+ is a (nonunital) monoid. Any other positive species q can be endowed with the trivial product (the zero map).

The *substitution* of a species p and a positive species q is defined by

$$(\mathsf{p}\circ\mathsf{q})[I]:=\bigoplus_{X\vdash I}\mathsf{p}[X]\otimes\Big(\bigotimes_{S\in X}\mathsf{q}[S]\Big).\tag{2.5}$$

For any positive species p,

$$\ell^*\circ\mathsf{p}=\bigoplus_{n\geq 0}\mathsf{p}^{\cdot n},\tag{2.6}$$

where $\mathsf{p}^{\cdot 0}=\mathsf{o}$. Furthermore, for any finite set I,

$$(\ell^*\circ\mathsf{p})[I]=\bigoplus_{F\models I}\mathsf{p}(F),$$

where for a composition $F=F_1|\dots|F_l$, $\mathsf{p}(F)=\mathsf{p}[F_1]\otimes\cdots\otimes\mathsf{p}[F_l]$.

Proposition 6 (Aguiar and Mahajan [2])**.** *The species $\ell^*\circ\mathsf{q}$ is the cofree comonoid on a positive species q.*

The component $\Delta_{S,T}$ of the coproduct of $\ell^*\circ\mathsf{q}$ is the direct sum of the following maps. For each set composition $F_1|\dots|F_k\models I$ for which S is the union of the first i parts, take the identity map

$$\mathsf{q}(F_1|\dots|F_k)\to\mathsf{q}(F_1|\dots|F_i)\otimes\mathsf{q}(F_{i+1}|\dots|F_k).\tag{2.7}$$

The comonoid $\ell^* \circ \mathsf{q}$ also carries a canonical Hopf monoid structure. The product is quasishuffle. Fix a decomposition $I = S \sqcup T$, compositions $F \models S$, $G \models T$ and a quasishuffle H of F and G. There is a unique map

$$\mathsf{q}(F) \otimes \mathsf{q}(G) \to \mathsf{q}(H)$$

obtained by reordering the factors according to the shuffle and then taking the tensor product of the following maps:

$$\mathsf{q}[H_k] \leftarrow \begin{cases} \mathsf{q}[H_k] & \text{if } H_k \text{ is a block of } F \text{ or of } G, \\ \mathsf{q}[F_i] \otimes \mathsf{q}[G_j] & \text{if } H_k = F_i \sqcup G_j. \end{cases} \tag{2.8}$$

If H_k is a block of F or of G, use the identity map and if $H_k = F_i \sqcup G_j$, use the appropriate component of the product of q. Since every positive species is a monoid, see Remark 5, this map is well-defined. In the case that q is the trivial monoid, the product is the shuffle.

Example 7. Consider the Hopf monoid e. Let e_+ be the positive Hopf monoid defined by

$$\mathsf{e}_+[I] = \begin{cases} \mathsf{e}[I] & \text{if } I \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

Then $(\ell^* \circ \mathsf{e}_+)[I] = \bigoplus_{F \models I} \mathsf{e}_+(F)$, and denote by $F_1|F_2|\dots|F_k$ the only basis element in $\mathsf{e}_+(F) = \mathsf{e}_+[F_1] \otimes \cdots \otimes \mathsf{e}_+[F_k]$. Therefore, $\ell^* \circ \mathsf{e}_+$ has a basis given by set compositions.

The product and the coproduct are given by quasishuffle and deconcatenation of set compositions. For example,

$$\mu(1|2,3) = 1|2|3 + 1|3|2 + 3|1|2 + 1|23 + 13|2,$$
$$\Delta(4|13|2) = 4|13|2 \otimes \emptyset + 4|13 \otimes 2 + 4 \otimes 13|2 + \emptyset \otimes 4|13|2.$$

3. $\mathcal{H}$-Species

We introduce the category of $\mathcal{H}$-sets along with set compositions of $\mathcal{H}$-sets. We then define hyperoctahedral species, $\mathcal{H}$-species, which are sequences of modules for the hyperoctahedral groups.

Definition 8. An $\mathcal{H}$-*set* (A, σ) is a finite set A together with an involution σ on A, where σ is without fixed points. A bijection of $\mathcal{H}$-sets between (A, σ) and (B, τ), called an $\mathcal{H}$-*bijection*, is a bijection $f\colon A \to B$ of finite sets such that this diagram commutes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\sigma"'] & B \arrow[d, "\tau"] \\ A \arrow[r, "f"'] & B. \end{tikzcd} \tag{3.1}$$

We write $f\colon (A, \sigma) \to (B, \tau)$ for such an $\mathcal{H}$-bijection.

Let $\mathbf{B}^{\mathcal{H}}$ be the category of $\mathcal{H}$-sets. There is a natural involution $\sigma_0\colon \mathbb{N}\cup\overline{\mathbb{N}} \to \mathbb{N}\cup\overline{\mathbb{N}}$, where $\overline{\mathbb{N}} = \{\bar{1}, \bar{2}, \dots\}$, defined for $i \in \mathbb{N}$ and $\bar{i} \in \overline{\mathbb{N}}$ by $\sigma_0(i) = \bar{i}$ and $\sigma_0(\bar{i}) = i$. Each time we consider a set $S = \{s_1, \dots, s_k\} \subset \mathbb{N}$ and its negative $\overline{S} = \{\overline{s_1}, \dots, \overline{s_k}\}$, we endow it with the natural involution σ_0. In particular, we use σ_0 for the sets

$$[\bar{n}, n] = \{\bar{n}, \dots, \bar{1}, 1, \dots, n\},$$

$n \geq 0$, and for the sets

$$[\pm(a, b)] = \{\bar{b}, \overline{b-1}, \dots, \bar{a}, a, a+1, \dots b\},$$

for $a, b \in \mathbb{N}$ and $a \leq b$. We omit the involution when using σ_0. For example, $\overline{S} \sqcup S := (\overline{S} \cup S, \sigma_0)$ and $[\bar{n}, n] := ([\bar{n}, n], \sigma_0)$.

Consider the quotient set of A by σ:

$$A/\sigma = \{[a] : a \in A\}. \tag{3.2}$$

A *section* is a map

$$s\colon A/\sigma \to A$$

which is a right inverse for the projection $A \to A/\sigma$. In particular, $s([a]) \in \{a, \sigma(a)\}$. For the sets $[\bar{n}, n]$, we identify $[\bar{n}, n]/\sigma_0$ with $[n]$, for each $n \geq 0$, and we denote a section s by the list of its images $(s(1), s(2), \dots, s(n))$ or equivalently by omitting commas and brackets: $s(1)s(2)\dots s(n)$.

We define next two order-preserving $\mathcal{H}$-bijections on $\mathcal{H}$-sets of integers of the form $\bar{S} \sqcup S$ for $S \subset \mathbb{N}$. The natural order on the sets $[\bar{n}, n]$ is

$$\bar{n} < \dots < \bar{1} < 1 < \dots < n,$$

and any subset S of $[n]$ inherits this order.

The *canonical map*, can, and the *standardization map*, st, are defined to be the only order-preserving maps between sets of same cardinality. Explicitly, if $S, T \subset \mathbb{N}$ with $|S| = |T| = n$ then

$$\mathrm{st}_S\colon \bar{S} \sqcup S \to [\bar{n}, n] \quad \text{and} \quad \mathrm{can}\colon \bar{S} \sqcup S \to \bar{T} \sqcup T. \tag{3.3}$$

We denote by can_s the map $\mathrm{can}_s\colon [\bar{n}, n] \to [\pm(s+1, s+n)]$.

We define next two analogous notions of set partitions on $\mathcal{H}$-sets. The first one is a set partition for which each part of the partition is an $\mathcal{H}$-set and the second is a set partition on particular subsets of an $\mathcal{H}$-set. For the following definitions, let (A, σ) be an $\mathcal{H}$-set.

Definition 9. An $\mathcal{H}$-subset (S, σ_S) of (A, σ) is a set S, such that

$$S \subset A \quad \text{and} \quad \sigma_S(S) = S,$$

where $\sigma_S\colon S \to S$ is the restriction of σ to the set S and is defined by $\sigma_S(s) = \sigma(s)$, for all $s \in S$. If S is an $\mathcal{H}$-subset of $[\bar{n}, n]$, we use σ_0 for the restriction.

Definition 10. (1) An $\mathcal{H}$*-decomposition* F of (A,σ) is an ordered sequence $F = (F_1,\dots,F_l)$ of disjoint $\mathcal{H}$-subsets of (A,σ) such that $\bigcup_{i=1}^{l} F_i = A$. We write $F_1 \sqcup \dots \sqcup F_l = A$.

(2) An $\mathcal{H}$*-set composition* F of (A,σ) is an $\mathcal{H}$-decomposition $F = (F_1,\dots,F_l)$ of (A,σ) in which each F_i is nonempty. We write $F = F_1|F_2|\dots|F_l$ or $F \models A$. The $\mathcal{H}$-subsets F_i are the *parts* of the $\mathcal{H}$-set composition.

(3) An $\mathcal{H}$*-set partition* P of (A,σ) is an unordered collection of disjoint nonempty $\mathcal{H}$-subsets of (A,σ) such that $\bigcup_{S\in P} S = A$. We write $P \vdash A$. The $\mathcal{H}$-subsets S are the *parts* of the $\mathcal{H}$-set partition.

There is one (empty) $\mathcal{H}$-set composition ($\mathcal{H}$-set partition) of the empty set, denoted by $\emptyset$.

Remark 11. The $\mathcal{H}$-decompositions of $\mathcal{H}$-sets of the form $[\bar{n}, n]$ are in bijection with decompositions of the finite sets $[n]$, $n \geq 0$.

Definition 12. (1) An $\mathcal{H}_{\mathcal{S}}$*-set composition* F of (A,σ) is a set composition of $s(A/\sigma)$, for any section s. We write $F \models_{\mathcal{S}} (A,\sigma)$.

(2) An $\mathcal{H}_{\mathcal{S}}$*-set partition* P of (A,σ) is a set partition of $s(A/\sigma)$ for any section s. We write $P \vdash_{\mathcal{S}} (A,\sigma)$.

Definition 13. A *hyperoctahedral species*, or *species of type B* ($\mathcal{H}$*-species*), with values in $\mathbf{Vec}_{\mathbb{K}}$, the category of vector spaces over a field $\mathbb{K}$ of characteristic zero with linear maps, is a functor

$$\mathsf{p} : \mathbf{B}^{\mathcal{H}} \to \mathbf{Vec}_{\mathbb{K}}.$$

A morphism of $\mathcal{H}$-species is a natural transformation $\alpha\colon \mathsf{p} \to \mathsf{q}$. Let $\mathbf{Sp}^{\mathcal{H}}$ denote the category of $\mathcal{H}$-species.

An $\mathcal{H}$-species consists of a family of vector spaces $\mathsf{p}[A,\sigma]$, one for each $\mathcal{H}$-set (A,σ), and a family of linear maps

$$\mathsf{p}[f]\colon \mathsf{p}[A,\sigma] \to \mathsf{p}[B,\tau],$$

one for each $\mathcal{H}$-bijection $f\colon (A,\sigma) \to (B,\tau)$. The space $\mathsf{p}[A,\sigma]$ is the space of p*-structures* on (A,σ). For the sets $[\bar{n}, n]$, $n \geq 0$, we write $\mathsf{p}[\bar{n}, n]$ instead of $\mathsf{p}[([\bar{n}, n], \sigma_0)]$.

A morphism of $\mathcal{H}$-species consists of a family of linear maps

$$\alpha_{A,\sigma}\colon \mathsf{p}[A,\sigma] \to \mathsf{q}[A,\sigma],$$

one for each $\mathcal{H}$-set (A,σ), such that for each $\mathcal{H}$-bijection $f\colon (A,\sigma) \to (B,\tau)$, this diagram

$$\begin{tikzcd} \mathsf{p}[A,\sigma] \arrow[r, "\alpha_{A,\sigma}"] \arrow[d, "\mathsf{p}[f]"'] & \mathsf{q}[A,\sigma] \arrow[d, "\mathsf{q}[f]"] \\ \mathsf{p}[B,\tau] \arrow[r, "\alpha_{B,\tau}"'] & \mathsf{q}[B,\tau] \end{tikzcd} \tag{3.4}$$

commutes. We may abbreviate $\alpha_{[\bar{n},n]}$ by α_n or $\alpha_{\bar{S}\sqcup S}\colon \mathsf{p}[\bar{S}\sqcup S] \to \mathsf{q}[\bar{S}\sqcup S]$ by α_S. Also, the identity 1_n may either denote the identity linear map $1_n\colon \mathsf{p}[\bar{n},n] \to \mathsf{p}[\bar{n},n]$ or the identity $\mathcal{H}$-bijection $1_n\colon [\bar{n},n] \to [\bar{n},n]$.

Remark 14. The group of automorphisms of the $\mathcal{H}$-set $[\bar{n},n]$ is the hyperoctahedral group or the group of signed permutations, $\mathcal{B}_n$. More generally, the group of $\mathcal{H}$-bijections taking (A,σ) to itself is denoted by $\mathcal{B}_{A,\sigma}$, and it is isomorphic to $\mathcal{B}_n$, if $|A| = 2n$.

Proposition 15. *An $\mathcal{H}$-species p can be defined as a sequence*

$$V_0, V_1, V_2, \dots$$

of modules of the hyperoctahedral groups, and morphisms of $\mathcal{H}$-species are degree preserving maps of $\mathcal{B}_n$-modules.

Proof. Since p is a functor, each $\pi \in \mathcal{B}_n$ induces a map

$$\mathsf{p}[\pi]\colon \mathsf{p}[\bar{n},n] \to \mathsf{p}[\bar{n},n]$$

of vector spaces, which turns $\mathsf{p}[\bar{n},n]$ into a $\mathcal{B}_n$-module. A morphism of $\mathcal{H}$-species $\alpha\colon \mathsf{p} \to \mathsf{q}$ gives rise to degree-preserving maps $\alpha_n\colon \mathsf{p}[\bar{n},n] \to \mathsf{q}[\bar{n},n]$, $n \geq 0$, of $\mathcal{B}_n$-modules.

Conversely, let $V_0, V_1, V_2, \dots$ be a sequence of modules of the hyperoctahedral groups, with action $\pi.V_n$, $\pi \in \mathcal{B}_n$. Define an $\mathcal{H}$-species p as $\mathsf{p}[A,\sigma] = V_n$, for any $\mathcal{H}$-set (A,σ) of cardinality $2n$. For any $\mathcal{H}$-bijection $f\colon (A,\sigma) \to (B,\tau)$, consider the composite $g = \mathrm{st}_B \circ f \circ \mathrm{st}_A^{-1}$, where st_A and st_B are any bijections $(A,\sigma) \to [\bar{n},n]$ and $(B,\tau) \to [\bar{n},n]$ respectively. Then $g \in \mathcal{B}_n$ and let $\mathsf{p}[f] := g.V_n$. $\square$

Remark 16. The category of cubical species, defined by Hetyei *et al.* in [13], is equivalent to the category of $\mathcal{H}$-species. See [8] for more details.

Example 17. (1) **Exponential $\mathcal{H}$-species.**

$$\mathsf{e}_{\mathcal{H}}[A,\sigma] := \mathbb{K}, \text{ for each } \mathcal{H}\text{-set } (A,\sigma).$$

Let A denote the only basis element of $\mathsf{e}_{\mathcal{H}}[A,\sigma]$. This is the trivial representation.

(2) **$\mathcal{H}$-species of linear orders.**

$$\begin{aligned}\ell_{\mathcal{H}}[A,\sigma] &= \mathbb{K}\text{-span of linear orders on } s(A/\sigma), \text{ for all sections } s\colon A/\sigma \to A\\ &= \bigoplus_{s\colon A/\sigma \to A} \ell[s(A/\sigma)],\end{aligned}$$

where ℓ is the usual species of linear order. It is the regular representation. For example,

$$\ell_{\mathcal{H}}[\bar{2},2] = \ell[\{1,2\}] \oplus \ell[\{\bar{1},2\}] \oplus \ell[\{1,\bar{2}\}] \oplus \ell[\{\bar{1},\bar{2}\}].$$

(3) **$\mathcal{H}$-species of sections.**

$$\mathsf{s}[A,\sigma] = \mathbb{K}[s(A/\sigma) \mid s\colon A/\sigma \to A].$$

(4) **$\mathcal{H}$-species of sets of sections.** $\mathcal{F}[A,\sigma]$ is the $\mathbb{K}$-span of the following sets of sections: for each $\mathcal{H}$-subset (T,σ_T) of (A,σ), let $S=\{s_1,\ldots,s_k\}$ be the set of sections on A/σ with $s_i(T/\sigma_T)$ fixed, $1\le i\le k$, where $k=2^{|(A\setminus T)/\sigma|}$. Each $\mathcal{H}$-subset (T,σ_T) give rise to $2^{|T/\sigma_T|}$ basis elements. For example,

$$\begin{aligned}\mathcal{F}[\bar{2},2]=\mathbb{K}[&\{12\},\{\bar{1}2\},\{1\bar{2}\},\{\bar{1}\bar{2}\},\{12,1\bar{2}\},\\&\{\bar{1}2,\bar{1}\bar{2}\},\{\bar{1}2,12\},\{\bar{1}\bar{2},1\bar{2}\},\{12,\bar{1}2,1\bar{2},\bar{1}\bar{2}\}].\end{aligned}\tag{3.5}$$

4. Tensor product and Hopf monoids

We set the definitions for monoids, comonoids, bimonoids and Hopf monoids in the monoidal category of $\mathcal{H}$-species.

Proposition 18. *$(\mathbf{Sp}^{\mathcal{H}},\cdot,\mathsf{o},\beta)$ is a symmetric monoidal category, with tensor product*

$$(\mathsf{p}\cdot\mathsf{q})[A,\sigma]=\bigoplus_{S\sqcup T=A}\mathsf{p}[S,\sigma_S]\otimes\mathsf{q}[T,\sigma_T],\tag{4.1}$$

where the direct sum is over $\mathcal{H}$-decompositions, see Definition 10. The unit for this product is

$$\mathsf{o}[A,\sigma]=\begin{cases}\mathbb{K} & \text{if } A=\emptyset,\\ 0 & \text{otherwise.}\end{cases}$$

The braiding $\beta_{\mathsf{p},\mathsf{q}}\colon\mathsf{p}\cdot\mathsf{q}\to\mathsf{q}\cdot\mathsf{p}$ has components given by

$$\begin{aligned}\mathsf{p}[S,\sigma_S]\otimes\mathsf{q}[T,\sigma_T]&\to\mathsf{q}[T,\sigma_T]\otimes\mathsf{p}[S,\sigma_S]\\ x\otimes y&\mapsto y\otimes x.\end{aligned}\tag{4.2}$$

The axioms, found in [15], are straightforward to check.

A *monoid* (p,μ,η) in $\mathbf{Sp}^{\mathcal{H}}$ is an $\mathcal{H}$-species p with product μ and unit η,

$$\mu\colon\mathsf{p}\cdot\mathsf{p}\to\mathsf{p},\quad \eta\colon\mathsf{o}\to\mathsf{p},$$

associative and unital in the usual sense. There is one linear map $\mu_{A,\sigma}$ for each $\mathcal{H}$-set (A,σ) and let

$$\mu_{S,T}\colon\mathsf{p}[S,\sigma_S]\otimes\mathsf{p}[T,\sigma_T]\to\mathsf{p}[A,\sigma]$$

be the components of the product, one for each $\mathcal{H}$-decomposition $S\sqcup T=A$. There is one linear map for the unit

$$\eta_\emptyset\colon\mathbb{K}\to\mathsf{p}[\emptyset].$$

A *comonoid* $(\mathsf{p},\Delta,\epsilon)$ in $\mathbf{Sp}^{\mathcal{H}}$ is an $\mathcal{H}$-species p with a coproduct Δ and counit ϵ,

$$\Delta\colon\mathsf{p}\to\mathsf{p}\cdot\mathsf{p},\quad \epsilon\colon\mathsf{p}\to\mathsf{o},$$

coassociative and counital in the usual sense. There is one map $\Delta_{A,\sigma}$ for each $\mathcal{H}$-set (A,σ) and let

$$\Delta_{S,T}\colon\mathsf{p}[A,\sigma]\to\mathsf{p}[S,\sigma_S]\otimes\mathsf{p}[T,\sigma_T]$$

be the components of the coproduct, one for each $\mathcal{H}$-decomposition $S \sqcup T = A$. Only one map is non-trivial for the counit, namely when $A = \emptyset$:

$$\epsilon_\emptyset \colon \mathsf{p}[\emptyset] \to \mathbb{K}.$$

A *bimonoid* $(\mathsf{h}, \mu, \eta, \Delta, \epsilon)$ is a monoid (h, μ, η) and a comonoid $(\mathsf{h}, \Delta, \epsilon)$ such that Δ and ϵ are morphisms of monoids. A *Hopf monoid* $(\mathsf{h}, \mu, \eta, \Delta, \epsilon, s)$ is a bimonoid $(\mathsf{h}, \mu, \eta, \Delta, \epsilon)$ with an $\mathcal{H}$-species morphism $s \colon \mathsf{h} \to \mathsf{h}$, the antipode, such that for each nonzero $\mathcal{H}$-set (A, σ), the composites

$$\mathsf{h}[A,\sigma] \xrightarrow{\oplus \Delta_{S,T}} \bigoplus \mathsf{h}[S,\sigma_S] \otimes \mathsf{h}[T,\sigma_T] \xrightarrow{1_S \otimes s_T} \bigoplus \mathsf{h}[S,\sigma_S] \otimes \mathsf{h}[T,\sigma_T] \xrightarrow{\oplus \mu_{S,T}} \mathsf{h}[A,\sigma], \tag{4.3}$$

$$\mathsf{h}[A,\sigma] \xrightarrow{\oplus \Delta_{S,T}} \bigoplus \mathsf{h}[S,\sigma_S] \otimes \mathsf{h}[T,\sigma_T] \xrightarrow{s_S \otimes 1_T} \bigoplus \mathsf{h}[S,\sigma_S] \otimes \mathsf{h}[T,\sigma_T] \xrightarrow{\oplus \mu_{S,T}} \mathsf{h}[A,\sigma]$$

are zero, where the direct sums are over $\mathcal{H}$-decompositions $S \sqcup T = A$, and such that the following equation for the empty set is satisfied:

$$\mu_{\emptyset,\emptyset} \circ (1 \otimes s_\emptyset) \circ \Delta_{\emptyset,\emptyset} = \eta_\emptyset \circ \epsilon_\emptyset = \mu_{\emptyset,\emptyset} \circ (s_\emptyset \otimes 1) \circ \Delta_{\emptyset,\emptyset}. \tag{4.4}$$

As is the case for graded bialgebras, if a bimonoid h is connected, i.e., $\mathsf{h}[\emptyset] = \mathbb{K}$, then h is a Hopf monoid. See [2] for details.

Example 19. (1) The $\mathcal{H}$-species $\mathsf{e}_\mathcal{H}$ is a Hopf monoid. Recall that A denotes the only basis element in $\mathsf{e}_\mathcal{H}[A,\sigma]$. The components of the product and the coproduct are given by

$$\begin{array}{cc} \mathsf{e}_\mathcal{H}[S,\sigma_S] \otimes \mathsf{e}_\mathcal{H}[T,\sigma_T] \to \mathsf{e}_\mathcal{H}[S \sqcup T] & \mathsf{e}_\mathcal{H}[A,\sigma] \to \mathsf{e}_\mathcal{H}[S,\sigma_S] \otimes \mathsf{e}_\mathcal{H}[T,\sigma_T] \\ S \otimes T \mapsto S \sqcup T, & A \mapsto S \otimes T. \end{array}$$

Solving (4.3) recursively, the antipode is found to be

$$S(A) = (-1)^{|A/\sigma|} A. \tag{4.5}$$

(2) The $\mathcal{H}$-species $\ell_\mathcal{H}$ is a Hopf monoid. The components of the product are

$$\begin{aligned} \ell_\mathcal{H}[S,\sigma_S] \otimes \ell_\mathcal{H}[T,\sigma_T] &\to \ell_\mathcal{H}[A,\sigma] \\ l_1 \otimes l_2 &\mapsto l_1 l_2, \end{aligned} \tag{4.6}$$

where for $l_1 = l_1^1 \dots l_1^i$ a linear order on $s(S/\sigma_S)$ and $l_2 = l_2^1 \dots l_2^j$ a linear order on $s'(T/\sigma_T)$ for some sections s and s',

$$l_1 l_2 = l_1^1 \dots l_1^i l_2^1 \dots l_2^j$$

is the concatenation. Then $l_1 l_2$ is a linear order on $s''(A/\sigma)$ where

$$s''([a]) = \begin{cases} s([a]) & \text{if } [a] \in S/\sigma_S, \\ s'([a]) & \text{if } [a] \in T/\sigma_T. \end{cases}$$

The components of the coproduct are given by deshuffling:

$$\begin{aligned} \ell_\mathcal{H}[A,\sigma] &\to \ell_\mathcal{H}[S,\sigma_S] \otimes \ell_\mathcal{H}[T,\sigma_T] \\ l &\mapsto l_S \otimes l_T, \end{aligned} \tag{4.7}$$

where for $l = l_1 \dots l_k$ a linear order on $s(A/\sigma)$, for some section s, l_S is the sublist of l consisting of elements of S. Then l_S and l_T are linear orders on $s'(S/\sigma_S)$ and $s''(T/\sigma_T)$ respectively, where $s' = s|_{S/\sigma_S}$ and $s'' = s|_{T/\sigma_T}$.

Since this species is connected, it is a Hopf monoid. By solving recursively (4.3), one can compute the antipode $S\colon \ell_{\mathcal{H}} \to \ell_{\mathcal{H}}$ for any linear order l on $s(A/\sigma)$ to be

$$S(l) = (-1)^{|l|}\widetilde{l},$$

where $|l|$ is the length of l and if $l = l_1 \dots l_k$ then $\widetilde{l} = l_k \dots l_2 l_1$.

(3) The $\mathcal{H}$-species $\mathcal{F}$ of faces of sections (see Example 17(4)) is a Hopf monoid. Let $F = \{s_1, \dots, s_k\}$ and $G = \{s'_1, \dots, s'_l\}$ be two sets of sections on S and T, where $S \sqcup T = A$. Each s_i and s'_j, for $1 \le i \le k$ and $1 \le j \le l$, is a set. The components of the product are given by

$$\mu_{S,T}(\{s_1, \dots, s_k\}, \{s'_1, \dots, s'_l\}) = \{s_i \sqcup s'_j : \ 1 \le i \le k,\ 1 \le j \le l\}. \tag{4.8}$$

Let $H = \{s_1, \dots, s_m\}$ be a set of sections of (A, σ). The components of the coproduct are given by

$$\Delta_{S,T}(H) = H|_S \otimes H|_T, \tag{4.9}$$

where $H|_S = \{s_1 \cap S, \dots, s_k \cap S\}$. We delete any empty sets. For example,

$$\mu(\{13, \bar{1}3\}, \{\bar{2}\}) = \{1\bar{2}3, \bar{1}\bar{2}3\},$$

$$\Delta(\{12, 1\bar{2}\}) = \{12, 1\bar{2}\} \otimes \emptyset + \{1\} \otimes \{2, \bar{2}\} + \{2, \bar{2}\} \otimes \{1\} + \emptyset \otimes \{12, 1\bar{2}\}.$$

5. Functors

In this section, we define a functor $\mathcal{S}\colon \mathbf{Sp} \to \mathbf{Sp}^{\mathcal{H}}$ and two functors

$$K^{\mathcal{H}}, \widetilde{K^{\mathcal{H}}}\colon \mathbf{Sp}^{\mathcal{H}} \to \mathbf{gVec}.$$

We also make explicit the natural transformation $\widetilde{\alpha^{\mathcal{S}}}$ of (1.1).

Definition 20. The functor

$$\mathcal{S} : \mathbf{Sp} \to \mathbf{Sp}^{\mathcal{H}}$$

is defined, for a species p, an $\mathcal{H}$-set (A, σ) and an $\mathcal{H}$-bijection $f\colon (A, \sigma) \to (B, \tau)$ by

$$\begin{aligned} \mathcal{S}\mathsf{p}[A, \sigma] &= \bigoplus_{s\colon A/\sigma \to A} \mathsf{p}[s(A/\sigma)], \\ \mathcal{S}\mathsf{p}[f] &= \bigoplus_{s\colon A/\sigma \to A} \mathsf{p}[f|_{s(A/\sigma)}]. \end{aligned} \tag{5.1}$$

Proposition 21. *$\mathcal{S}$ is a bistrong monoidal functor.*

Verifying the axioms is straightforward in this case. See [8] for details.

Example 22. The $\mathcal{H}$-species $\mathcal{S}(\ell^* \circ \mathsf{e}_+)$ has a basis given by the set of $\mathcal{H}_{\mathcal{S}}$-set compositions, see Definition 10. Since $\ell^* \circ \mathsf{e}_+$ is a Hopf monoid by the discussion following Proposition 6 and $\mathcal{S}$ is bistrong, $\mathcal{S}(\ell^* \circ \mathsf{e}_+)$ is a Hopf monoid. The coproduct of an $\mathcal{H}_{\mathcal{S}}$-set composition is the deconcatenation and the product of two $\mathcal{H}_{\mathcal{S}}$-set compositions is the quasishuffle, as defined by the maps (2.7) and (2.8).

Let G be a group and V be a G-module. Then $V_G := V/\{x - gx : g \in G,\, x \in V\}$ is the space of G-coinvariants of V.

Proposition 23 (Aguiar and Mahajan [2])**.** *The functors $K, \overline{K} \colon \mathbf{Sp} \to \mathbf{gVec}$ defined by*

$$K\mathsf{p} = \bigoplus_{n \geq 0} \mathsf{p}[n] \quad \text{and} \quad \overline{K}\mathsf{p} = \bigoplus_{n\geq 0} \mathsf{p}[n]_{\mathcal{S}_n} \tag{5.2}$$

are such that K is bilax monoidal and $\overline{K}$ is bistrong monoidal.

The analogous functors $\mathbf{Sp}^{\mathcal{H}} \to \mathbf{gVec}$ for $\mathcal{H}$-species are as follows.

Definition 24. Let

$$K^{\mathcal{H}}, \widetilde{K^{\mathcal{H}}} \colon \mathbf{Sp}^{\mathcal{H}} \to \mathbf{gVec}$$

be defined, for $\mathsf{p} \in \mathbf{Sp}^{\mathcal{H}}$, by

$$\begin{aligned} K^{\mathcal{H}}\mathsf{p} &= \bigoplus_{n\geq 0} \mathsf{p}[\bar{n}, n], \\ \widetilde{K^{\mathcal{H}}}\mathsf{p} &= \bigoplus_{n \geq 0} \mathsf{p}[\bar{n}, n]_{\mathcal{S}_n}. \end{aligned} \tag{5.3}$$

Proposition 25. *$K^{\mathcal{H}}$ is a bilax monoidal functor with natural transformations ϕ and ψ:*

$$K^{\mathcal{H}}\mathsf{p} \cdot K^{\mathcal{H}}\mathsf{q} \underset{\psi_{\mathsf{p},\mathsf{q}}}{\overset{\phi_{\mathsf{p},\mathsf{q}}}{\rightleftarrows}} K^{\mathcal{H}}(\mathsf{p}\cdot \mathsf{q}), \tag{5.4}$$

defined respectively by the direct sums over $s + t = n$ and $S \sqcup T = A$ of these maps:

$$\begin{aligned} 1_s \otimes \mathsf{q}[\mathrm{can}_s] \colon \mathsf{p}[\bar{s}, s] \otimes \mathsf{q}[\bar{t}, t] &\to \mathsf{p}[\bar{s}, s] \otimes \mathsf{q}[\pm(s+1, s+t)], \\ \mathsf{p}[\mathrm{st}_S] \otimes \mathsf{q}[\mathrm{st}_T] \colon \mathsf{p}[\bar{S} \sqcup S] \otimes \mathsf{q}[\bar{T} \sqcup T] &\to \mathsf{p}[\overline{|S|}, |S|] \otimes \mathsf{q}[\overline{|T|}, |T|]. \end{aligned} \tag{5.5}$$

Proof. We give an idea of the proof for ϕ and the same can be done for ψ using the dual axioms. For a detailed proof, see [8]. First, we verify that ϕ is a natural transformation. Let $\alpha \colon \mathsf{p} \to \mathsf{p}'$ and $\beta \colon \mathsf{q} \to \mathsf{q}'$ be two morphisms of $\mathcal{H}$-species. Then, by fixing s and t, we have

$$\begin{array}{ccc} \mathsf{p}[\bar{s}, s] \otimes \mathsf{q}[\bar{t}, t] & \xrightarrow{1_s \otimes \mathsf{q}[\mathrm{can}_s]} & \mathsf{p}[\bar{s}, s] \otimes \mathsf{q}[\pm(s+1, s+t)] \\ {\scriptstyle \alpha_s \otimes \beta_t}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_s \otimes \beta_{[\pm(s+1,s+t)]}} \\ \mathsf{p}'[\bar{s}, s] \otimes \mathsf{q}'[\bar{t}, t] & \xrightarrow{1_s \otimes \mathsf{q}'[\mathrm{can}_s]} & \mathsf{p}'[\bar{s}, s] \otimes \mathsf{q}'[\pm(s+1, s+t)]. \end{array} \tag{5.6}$$

The diagram commutes since α and β are natural transformations and can_s is an $\mathcal{H}$-bijection. We check next that ϕ is associative and that ϕ and ψ satisfy the braiding condition. All the unitality conditions follow, since the maps involved are isomorphisms.

Associativity. The map ϕ leads to an unambiguous map

$$K^{\mathcal{H}}\mathsf{p} \cdot K^{\mathcal{H}}\mathsf{q} \cdot K^{\mathcal{H}}\mathsf{r} \longrightarrow K^{\mathcal{H}}(\mathsf{p} \cdot \mathsf{q} \cdot \mathsf{r})$$

defined, by fixing s, t and u, by

$$\begin{array}{c} \mathsf{p}[\bar{s}, s] \otimes \mathsf{q}[\bar{t}, t] \otimes \mathsf{r}[\bar{u}, u] \\ \Big\downarrow{\scriptstyle 1_s \otimes \mathsf{q}[\mathrm{can}_s] \otimes \mathsf{r}[\mathrm{can}_{s+t}]} \\ \mathsf{p}[\bar{s}, s] \otimes \mathsf{q}[\pm(s+1, s+t)] \otimes \mathsf{r}[\pm(s+t+1, s+t+u)]. \end{array}$$

Braiding. The following equality must be satisfied, for all $\mathcal{H}$-species $\mathsf{p}, \mathsf{q}, \mathsf{r}, \mathsf{s}$,

$$(\phi_{\mathsf{p},\mathsf{r}} \cdot \phi_{\mathsf{q},\mathsf{s}}) \circ (1 \cdot \beta \cdot 1) \circ (\psi_{\mathsf{p},\mathsf{q}} \cdot \psi_{\mathsf{r},\mathsf{s}}) = \psi_{\mathsf{p}\cdot\mathsf{r},\mathsf{q}\cdot\mathsf{s}} \circ K^{\mathcal{H}}(1 \cdot \beta \cdot 1) \circ \phi_{\mathsf{p}\cdot\mathsf{q},\mathsf{r}\cdot\mathsf{s}}, \tag{5.7}$$

which are two maps

$$K^{\mathcal{H}}(\mathsf{p} \cdot \mathsf{q}) \cdot K^{\mathcal{H}}(\mathsf{r} \cdot \mathsf{s}) \to K^{\mathcal{H}}(\mathsf{p} \cdot \mathsf{r}) \cdot K^{\mathcal{H}}(\mathsf{q} \cdot \mathsf{s}). \tag{5.8}$$

Fix degrees n and m along with decompositions $S \sqcup T = [n]$ and $A \sqcup B = [m]$ on the left-hand side of (5.8). Then the left-hand side of (5.7) can be written as

$$(\phi_{\mathsf{p},\mathsf{r}} \cdot \phi_{\mathsf{q},\mathsf{s}}) \circ (1 \cdot \beta \cdot 1) \circ (\psi_{\mathsf{p},\mathsf{q}} \cdot \psi_{\mathsf{r},\mathsf{s}}) = \mathsf{p}[\mathrm{st}_S] \otimes \beta(\mathsf{q}[\mathrm{st}_T] \otimes \mathsf{r}[\mathrm{can}_{|S|} \circ \mathrm{st}_A]) \circ \mathsf{s}[\mathrm{can}_{|T|} \circ \mathrm{st}_B]. \tag{5.9}$$

Let

$$S = \{s_1, \dots, s_p\} \quad \text{and} \quad A = \{a_1, \dots, a_r\} \tag{5.10}$$

be ordered. Then

$$\mathrm{st}_S \colon \overline{S} \sqcup S \to [|\overline{S}|, |S|]$$

is such that $\mathrm{st}_S(s_i) = i$ and $\mathrm{st}_S(\overline{s_i}) = \bar{i}$. Also,

$$\overline{A} \sqcup A \xrightarrow{\mathrm{st}_A} [|\overline{A}|, |A|] \xrightarrow{\mathrm{can}_{|S|}} [\pm(|S|+1, |S|+|A|)],$$

is such that $(\mathrm{can}_{|S|} \circ \mathrm{st}_A)(a_i) = \mathrm{can}_{|S|}(i) = i + |S|$ and $(\mathrm{can}_{|S|} \circ \mathrm{st}_A)(\overline{a_i}) = \mathrm{can}_{|S|}(\bar{i}) = \overline{i + |S|}$. A similar description can be made for the sets T and B. The right-hand side of (5.7) can be written as

$$\mathsf{p}[\mathrm{st}_{|S|+|A|}] \otimes \beta(\mathsf{q}[\mathrm{st}_{|T|+|B|}] \otimes \mathsf{r}[\mathrm{st}_{|S|+|A|} \circ \mathrm{can}_n]) \otimes \mathsf{s}[\mathrm{st}_{|T|+|B|} \circ \mathrm{can}_n]. \tag{5.11}$$

Consider the same sets S and A of (5.10) and the following composites

$$\overline{S} \sqcup S \xrightarrow{\mathrm{st}_{|S|+|A|}} [\overline{(|S|+|A|)}, (|S|+|A|)],$$

$$\overline{A} \sqcup A \xrightarrow{\mathrm{can}_n} \overline{A^{+n}} \sqcup A^{+n} \xrightarrow{\mathrm{st}_{|S|+|A|}} [\overline{(|S|+|A|)}, (|S|+|A|)],$$

where $A^{+n} = \{a+n : a \in A\}$. The domain of the bijection $\mathrm{st}_{|S|+|A|}$ is $(\overline{S} \sqcup S) \sqcup (\overline{A^{+n}} \sqcup A^{+n})$ and it preserves the relative order. Since $S \subset [n]$ and $a+n > n$ for all $a \in A$, we have $\mathrm{st}_{|S|+|A|}(s_i) = i$, $\mathrm{st}_{|S|+|A|}(\overline{s_i}) = \overline{i}$, $(\mathrm{st}_{|S|+|A|} \circ \mathrm{can}_n)(a_i) = \mathrm{st}_{|S|+|A|}(a_i + n) = i + |S|$ and $(\mathrm{st}_{|S|+|A|} \circ \mathrm{can}_n)(\overline{a_i}) = \mathrm{st}_{|S|+|A|}(\overline{a_i + n}) = \overline{i + |S|}$. A similar description can be made for the sets T and B. Hence, the two maps of (5.7) are indeed identical and the braiding condition is satisfied.

□

Proposition 26. *$\widetilde{K^{\mathcal{H}}}$ is a bistrong monoidal functor.*

Proof. Define $\widetilde{\phi}_{\mathsf{p},\mathsf{q}}$ and $\widetilde{\psi}_{\mathsf{p},\mathsf{q}}$ with the following commutative diagrams:

(5.12)
$$\begin{tikzcd} K^{\mathcal{H}}\mathsf{p} \cdot K^{\mathcal{H}}\mathsf{q} \arrow[r, "\phi_{\mathsf{p},\mathsf{q}}"] \arrow[d] & K^{\mathcal{H}}(\mathsf{p} \cdot \mathsf{q}) \arrow[d] \\ \overline{K^{\mathcal{H}}}\mathsf{p} \cdot \overline{K^{\mathcal{H}}}\mathsf{q} \arrow[r, dashed, "\widetilde{\phi}_{\mathsf{p},\mathsf{q}}"'] & \overline{K^{\mathcal{H}}}(\mathsf{p} \cdot \mathsf{q}), \end{tikzcd} \qquad \begin{tikzcd} K^{\mathcal{H}}\mathsf{p} \cdot K^{\mathcal{H}}\mathsf{q} \arrow[d] & K^{\mathcal{H}}(\mathsf{p} \cdot \mathsf{q}) \arrow[l, "\psi_{\mathsf{p},\mathsf{q}}"'] \arrow[d] \\ \overline{K^{\mathcal{H}}}\mathsf{p} \cdot \overline{K^{\mathcal{H}}}\mathsf{q} & \overline{K^{\mathcal{H}}}(\mathsf{p} \cdot \mathsf{q}). \arrow[l, dashed, "\widetilde{\psi}_{\mathsf{p},\mathsf{q}}"] \end{tikzcd}$$

We prove that ψ is well-defined, and the same can be done for ϕ. For more details, see [8].

Consider the component of degree n of $K^{\mathcal{H}}(\mathsf{p} \cdot \mathsf{q})$, along with a decomposition $S \sqcup T = [n]$ with $|S| = s$ and $|T| = t$. Let $\tau \in S_n$ and suppose that $\tau(S) = U$ and that $\tau(T) = V$. Then τ send $\bar{S}$ to $\bar{U}$ and $\bar{T}$ to $\bar{V}$.

Let $\tau_S = \tau|_{\bar{S} \sqcup S} \colon \bar{S} \sqcup S \to \bar{U} \sqcup U$ and $\tau_T = \tau|_{\bar{T} \sqcup T} \colon \bar{T} \sqcup T \to \bar{V} \sqcup V$. These two $\mathcal{H}$-bijections induce two other $\mathcal{H}$-bijections on $[\bar{s}, s]$ and on $[\bar{t}, t]$. Let $\tau_s \colon [\bar{s}, s] \to [\bar{s}, s]$ be defined by $\tau_s = \mathrm{st}_U \circ \tau_S \circ \mathrm{st}_S^{-1}$ and let $\tau_t \colon [\bar{t}, t] \to [\bar{t}, t]$ be defined by $\tau_t = \mathrm{st}_V \circ \tau_T \circ \mathrm{st}_T^{-1}$. Since ψ is a natural transformation, the following diagram commutes

$$\begin{tikzcd} \mathsf{p}[\bar{S} \sqcup S] \otimes \mathsf{q}[\bar{T} \sqcup T] \arrow[r, "{\mathsf{p}[\mathrm{st}_S] \otimes \mathsf{q}[\mathrm{st}_T]}"] \arrow[d, "{\mathsf{p}[\pi_S] \otimes \mathsf{q}[\pi_T]}"'] & \mathsf{p}[\bar{s}, s] \otimes \mathsf{q}[\bar{t}, t] \arrow[d, "{\mathsf{p}[\pi_s] \otimes \mathsf{q}[\pi_t]}"] \\ \mathsf{p}[\bar{U} \sqcup U] \otimes \mathsf{q}[\bar{V} \sqcup V] \arrow[r, "{\mathsf{p}[\mathrm{st}_U] \otimes \mathsf{q}[\mathrm{st}_V]}"'] & \mathsf{p}[\bar{s}, s] \otimes \mathsf{q}[\bar{t}, t] \end{tikzcd}$$

which means that ψ factors through the space of $\mathcal{B}_n$-coinvariants of $(\mathsf{p} \cdot \mathsf{q})[\bar{n}, n]$.

Furthermore, since $\widetilde{\phi}$ and $\widetilde{\psi}$ are invertible, $\widetilde{K^{\mathcal{H}}}$ is a bistrong monoidal functor. □

Since $K^{\mathcal{H}}$ is a bilax monoidal functor, $K^{\mathcal{H}}\mathsf{h}$ is a graded bialgebra for any bimonoid h in $\mathbf{Sp}^{\mathcal{H}}$, by Proposition 2. Furthermore, since $\widetilde{K^{\mathcal{H}}}$ is bistrong, $\widetilde{K^{\mathcal{H}}}\mathsf{h}'$ is a graded Hopf algebra for any Hopf monoid h' in $\mathbf{Sp}^{\mathcal{H}}$, by Proposition 2.

Example 27. Consider $K^{\mathcal{H}}\mathcal{F}$, where $\mathcal{F}$ is the Hopf monoid of sets of sections of Example 19(3). Let $F = \{s_1, \ldots, s_k\}$ be a set of sections on $[n]$. So each s_j is a section $[n] \to [\bar{n}, n]$. Define

$$F^+ := \{i : s_j(i) = i,\ 1 \le j \le k\}, \quad F^- := \{i : s_j(i) = \bar{i},\ 1 \le j \le k\}$$

$$\text{and } F^* = [n] \backslash (F^+ \cup F^-).$$

Now, encode F by a word w_F in letters a, b, c, where the i-th letter of w_F is

$$w_F(i) = \begin{cases} a & \text{if } i \in F^+, \\ b & \text{if } i \in F^-, \\ c & \text{otherwise.} \end{cases}$$

For example, the word associated to $F = \{(1,2,\bar{3},4), (1,\bar{2},\bar{3},4), (1,2,\bar{3},\bar{4}), (1,\bar{2},\bar{3},\bar{4})\}$ is $w_F = acbc$ and the one for $G = \{(1,\bar{2})\}$ is $w_G = ab$. The product of $K^{\mathcal{H}}\mathcal{F}$ is given by concatenation and the coproduct by deshuffle of words. Therefore, $K^{\mathcal{H}}\mathcal{F} \cong \mathbb{K}\langle a, b, c\rangle$ as bialgebras.

Consider the composites $K^{\mathcal{H}}\mathcal{S}$ and $\widetilde{K^{\mathcal{H}}}\mathcal{S}$ of the functors $\mathcal{S} \colon \mathbf{Sp} \to \mathbf{Sp}^{\mathcal{H}}$ and $K^{\mathcal{H}}, \widetilde{K^{\mathcal{H}}} \colon \mathbf{Sp}^{\mathcal{H}} \to \mathbf{gVec}$. The composite $K^{\mathcal{H}}\mathcal{S}$ is a bilax monoidal functor and the composite $\widetilde{K^{\mathcal{H}}}\mathcal{S}$ is a bistrong monoidal functor.

Example 28. The $\mathcal{H}$-species $\mathcal{S}\mathsf{e}$ has a basis given by sections. Since $\mathcal{S}$ is bistrong and e is a Hopf monoid, $\mathcal{S}\mathsf{e}$ is a Hopf monoid. Therefore $\widetilde{K^{\mathcal{H}}}\mathcal{S}\mathsf{e}$ is a graded Hopf algebra described as follow.

Each element of the basis of $\widetilde{K^{\mathcal{H}}}\mathcal{S}\mathsf{e}$ is an equivalence class of sections under the action of $\mathcal{S}_n$. Two sections are in the same class if the number of negative integers (and hence the number of positive integers) is the same. Such a class $[s]$ can be represented as a polynomial $p(x, y) = x^{s^+}y^{s^-}$, where s^+ and s^- are respectively the number of positive and negative integers in s. Furthermore, the product, coming from the product of e, is concatenation of polynomials, and the coproduct is deshuffle. As a Hopf algebra, $\widetilde{K^{\mathcal{H}}}\mathcal{S}\mathsf{e}$ is isomorphic to $\mathbb{K}[x, y]$.

Example 29. Recall from Example 22 that $\mathcal{S}(\ell^* \circ \mathsf{e}_+)$ is the $\mathcal{H}$-species of $\mathcal{H}_{\mathcal{S}}$-set compositions. Therefore, $K^{\mathcal{H}}\mathcal{S}(\ell^* \circ \mathsf{e}_+)$ is the graded Hopf algebra of $\mathcal{H}_{\mathcal{S}}$-set compositions with quasishuffle product and deconcatenation coproduct.

The relation between the functor $K^{\mathcal{H}}\mathcal{S}$ and the functor K, i.e., the natural transformation $\alpha^{\mathcal{S}}$ in this diagram

(5.13)
$$\begin{tikzcd} & \mathbf{Sp}^{\mathcal{H}} \arrow[dr, "K^{\mathcal{H}}"] \arrow[d, phantom, "\Downarrow \alpha^{\mathcal{S}}"] & \\ \mathbf{Sp} \arrow[ur, "\mathcal{S}"] \arrow[rr, "K"'] & {} & \mathbf{gVec}, \end{tikzcd}$$

is given in the next proposition.

Proposition 30. *The map $\alpha^{\mathcal{S}} : K^{\mathcal{H}}\mathcal{S} \to K$, defined, for any $\mathsf{p} \in \mathbf{Sp}$, by*

$$\alpha^{\mathcal{S}}_{p} = \bigoplus_{\substack{n\geq 0\\ s\colon [n]\to[\bar{n},n]}} p[s^{-1}], \tag{5.14}$$

where $s^{-1}\colon s([n]) \to [n]$ is the unique bijection $s^{-1}(i) = |i|$, $i \in s([n])$, where for $j \in [n]$, $|j| = |\bar{j}| = j$, is a morphism of bilax monoidal functors.

Proof. We prove that $\alpha^{\mathcal{S}}$ is a natural transformation. The map $\alpha^{\mathcal{S}}$ assigns a linear map $\alpha^{\mathcal{S}}_{\mathsf{p}}$ for each object $\mathsf{p} \in \mathbf{Sp}$. Furthermore, for each arrow $\beta : \mathsf{p} \to \mathsf{q}$ in $\mathbf{Sp}$, by fixing n, we get the following diagram

$$\begin{tikzcd} \bigoplus \mathsf{p}[s([n])] \arrow[r, "\bigoplus \mathsf{p}[s^{-1}]"] \arrow[d, "\bigoplus \beta_{s([n])}"'] & \mathsf{p}[n] \arrow[d, "\beta_n"] \\ \bigoplus \mathsf{q}[s([n])] \arrow[r, "\bigoplus \mathsf{q}[s^{-1}]"'] & \mathsf{q}[n], \end{tikzcd} \tag{5.15}$$

where the direct sums are over sections $s\colon [n] \to [\bar{n}, n]$. Since β is a natural transformation and $s^{-1}\colon s([n]) \to [n]$ is a bijection, it commutes.

For the proof that $\alpha^{\mathcal{S}}$ is a morphism of bilax monoidal functors, see [8]. □

The natural transformation $\alpha^{\mathcal{S}}$ factors through the space of $\mathcal{S}_n$-coinvariants of $\mathcal{S}\mathsf{p}[\bar{n}, n]$:

$$\begin{tikzcd} \bigoplus \mathsf{p}[s([n])] \arrow[r, "\alpha^{\mathcal{S}}_{\mathsf{p}}"] \arrow[d, two heads] & \mathsf{p}[n] \arrow[d, two heads] \\ \left(\bigoplus \mathsf{p}[s([n])]\right)_{\mathcal{S}_n} \arrow[r, dashed, "\widetilde{\alpha^{\mathcal{S}}_{\mathsf{p}}}"'] & \mathsf{p}[n]_{\mathcal{S}_n}. \end{tikzcd}$$

Therefore, $\widetilde{\alpha^{\mathcal{S}}} : \widetilde{K^{\mathcal{H}}}\mathcal{S} \to \overline{K}$ is the map induced by $\alpha^{\mathcal{S}}$ on the space of $\mathcal{S}_n$-coinvariants.

Since $\alpha^{\mathcal{S}}$ is a morphism of bilax monoidal functor and $\widetilde{\alpha^{\mathcal{S}}}$ is a morphism of bistrong monoidal functor, for any bimonoid h in $\mathbf{Sp}$, $\alpha^{\mathcal{S}}_{\mathsf{h}}$ is a morphism of graded bialgebras and for any Hopf monoid h' in $\mathbf{Sp}$, $\widetilde{\alpha^{\mathcal{S}}_{\mathsf{h}'}}$ is a morphism of graded Hopf algebras, by Proposition 2.

Example 31. Consider the linear order species ℓ^*. On the one hand $K^{\mathcal{H}}\mathcal{S}\ell^*$ is isomorphic to the graded bialgebra of signed permutations $\mathbb{K}\mathcal{B}$, and on the other hand $K\ell^*$ is isomorphic to the graded bialgebra of Malvenuto and Reutenauer $\mathbb{K}\mathcal{S}$ (see [2]). See [1] for details on these Hopf algebras.

In this case, $\alpha^{\mathcal{S}}_{\ell^*}\colon \mathbb{K}\mathcal{B} \to \mathbb{K}\mathcal{S}$ is the graded bialgebra morphism forgetting the signs of a signed permutation. It is defined by $\alpha^{\mathcal{S}}(\pi) = |\pi|$, where, if $\pi = \pi_1\pi_2 \ldots \pi_n$ is a signed permutation, then $|\pi| = |\pi_1||\pi_2| \ldots |\pi_n|$.

6. Hopf algebra of set compositions

As a final word, we study the species $\ell^* \circ \mathsf{e}_+$ under the composite of functors

$$\mathbf{Sp} \xrightarrow{\mathcal{S}} \mathbf{Sp}^{\mathcal{H}} \xrightarrow{\widetilde{K^{\mathcal{H}}}} \mathbf{gVec},$$

and we give the Hopf algebra isomorphism $\widetilde{K^{\mathcal{H}}}\mathcal{S}(\ell^* \circ \mathsf{e}_+) \simeq \mathrm{DQSym}$. Let $DQ\Lambda = \widetilde{K^{\mathcal{H}}}\mathcal{S}(\ell^* \circ \mathsf{e}_+)$. Explicitly,

$$DQ\Lambda := \bigoplus_{n\geq 0}\Big(\bigoplus_{s:[n]\to[\bar{n},n]}\ \bigoplus_{F\models s([n])} \mathsf{e}_+(F)\Big)_{S_n}. \tag{6.1}$$

Recall from Example 22 that $\mathcal{S}(\ell^* \circ \mathsf{e}_+)$ has a basis given by $\mathcal{H}_{\mathcal{S}}$-set compositions.

Definition 32. A *bicomposition* $\mathbf{a} = \left(\begin{smallmatrix}\alpha_1 & \cdots & \alpha_k\\ \beta_1 & \cdots & \beta_k\end{smallmatrix}\right)$ of $n \neq 0$ is an ordered list of 1×2 vectors $\left(\begin{smallmatrix}\alpha_i\\ \beta_i\end{smallmatrix}\right) \neq \left(\begin{smallmatrix}0\\ 0\end{smallmatrix}\right)$, called *biparts*, such that $\sum_{i=1}^k(\alpha_i+\beta_i) = n$. If $n = 0$, there is one (empty) bicomposition, denoted by 0 or $()$. The length of a bicomposition is the number of biparts, denoted by $\ell(\mathbf{a})$. Let $\mathbf{a} \models n$ denote a bicomposition of n.

For example, the bicompositions of $n = 2$ are

$$\binom{2}{0}, \begin{pmatrix}1\,1\\0\,0\end{pmatrix}, \binom{1}{1}, \begin{pmatrix}1\,0\\0\,1\end{pmatrix}, \begin{pmatrix}0\,1\\1\,0\end{pmatrix}, \begin{pmatrix}0\,0\\1\,1\end{pmatrix}, \binom{0}{2}.$$

Let $\mathbf{a} = \left(\begin{smallmatrix}\alpha_1 & \cdots & \alpha_k\\ \beta_1 & \cdots & \beta_k\end{smallmatrix}\right)$ and $\mathbf{b} = \left(\begin{smallmatrix}\alpha'_1 & \cdots & \alpha'_l\\ \beta'_1 & \cdots & \beta'_l\end{smallmatrix}\right)$ be two bicompositions. The *concatenation* $\mathbf{a}.\mathbf{b}$, the *addition* $\mathbf{a}+\mathbf{b}$ and the *quasishuffle* $\mathbf{a} \uplus \mathbf{b}$ are given respectively by

$$\begin{aligned}
\mathbf{a}.\mathbf{b} &= \begin{pmatrix}\alpha_1 & \cdots & \alpha_k & \alpha'_1 & \cdots & \alpha'_l\\ \beta_1 & \cdots & \beta_k & \beta'_1 & \cdots & \beta'_l\end{pmatrix},\\
\mathbf{a}+\mathbf{b} &= \begin{pmatrix}\alpha_1 & \cdots & \alpha_k+\alpha'_1 & \cdots & \alpha'_l\\ \beta_1 & \cdots & \beta_k+\beta'_1 & \cdots & \beta'_l\end{pmatrix},\\
\mathbf{a} \uplus \mathbf{b} &= \begin{cases}\mathbf{a} & \text{if } \mathbf{b}=0,\\ \mathbf{b} & \text{if } \mathbf{a}=0,\\ \displaystyle\bigcup_{\mathbf{a}_1.\mathbf{a}_2=\mathbf{a}}\ \bigcup_{\mathbf{c}\in\mathbf{a}_2\uplus\mathbf{b}_2} \{\mathbf{a}_1.\mathbf{b}_1.\mathbf{c}, (\mathbf{a}_1+\mathbf{b}_1).\mathbf{c}\} & \text{otherwise,}\end{cases}
\end{aligned}$$

where $\mathbf{b} = \mathbf{b}_1.\mathbf{b}_2$ is the unique factorization of $\mathbf{b}$ such that $\ell(\mathbf{b}_1) = 1$. In other words $\mathbf{c} \in \mathbf{a} \uplus \mathbf{b}$ if $\mathbf{c}$ can be obtained by shuffling the columns of $\mathbf{a}$ and $\mathbf{b}$ and two adjacent columns coming from different bicompositions can be added using $\left(\begin{smallmatrix}\alpha_i\\ \beta_i\end{smallmatrix}\right) + \left(\begin{smallmatrix}\alpha'_j\\ \beta'_j\end{smallmatrix}\right) = \left(\begin{smallmatrix}\alpha_i+\alpha'_j\\ \beta_i+\beta'_j\end{smallmatrix}\right)$. For example,

$$\binom{2}{0} \uplus \begin{pmatrix}1\,0\\0\,1\end{pmatrix} = \left\{\begin{pmatrix}2\,1\,0\\0\,0\,1\end{pmatrix}, \begin{pmatrix}1\,2\,0\\0\,0\,1\end{pmatrix}, \begin{pmatrix}1\,0\,2\\0\,1\,0\end{pmatrix}, \begin{pmatrix}3\,0\\0\,1\end{pmatrix}, \begin{pmatrix}1\,2\\0\,1\end{pmatrix}\right\}.$$

Definition 33. DQSym is the Hopf algebra of diagonally quasisymmetric functions linearly spanned by

$$\{M_{\mathbf{a}} : \mathbf{a} \text{ bicomposition}\},$$

where $M_{\mathbf{a}}$ are the *bimonomial functions.* The product and the coproduct are given respectively by quasishuffle and deconcatenation of bicompositions.

See [3] for more detailed information about this Hopf algebra.

Proposition 34. *There is a bijection between basis elements of $DQ\Lambda$ and bicompositions.*

Proof. Basis elements of $DQ\Lambda$ are classes of $\mathcal{H}_{\mathcal{S}}$-set compositions. For an $\mathcal{H}_{\mathcal{S}}$-set composition F, the number of positive integers in the i-th part will be denoted by $|F_i|^+$ and the number of negative integers in the i-th part will denoted by $|F_i|^-$. Let F and G be two $\mathcal{H}_{\mathcal{S}}$-set compositions. Suppose that for all i, $|F_i|^+ = |G_i|^+$ and $|F_i|^- = |G_i|^-$. In particular $|F_i| = |G_i|$ and $\ell(F) = \ell(G)$. Build a permutation π by setting $\pi(F_i^+) = G_i^+$ and $\pi(F_i^-) = G_i^-$ for each i, $1 \leq i \leq \ell(F)$. Therefore $\pi(F) = G$ and F and G are in the same class. Form a bicomposition $\mathbf{a} = \left(\begin{smallmatrix} \alpha_1 & \cdots & \alpha_k \\ \beta_1 & & \beta_k \end{smallmatrix}\right)$ from an $\mathcal{H}_{\mathcal{S}}$-set composition F of n integers, by setting α_i and β_i to be respectively $|F_i|^+$ and $|F_i|^-$. It is a bicomposition as each vector is non-zero and the sum of its integers is n.

Conversely, form an $\mathcal{H}_{\mathcal{S}}$-set composition F from a bicomposition $\mathbf{a} = \left(\begin{smallmatrix} \alpha_1 & \cdots & \alpha_k \\ \beta_1 & & \beta_k \end{smallmatrix}\right)$ by building each part of F, where the i-th part is given by the positive integers:

$$\alpha_1 + \beta_1 + \cdots + \alpha_{i-1} + \beta_{i-1} + 1, \ldots, \alpha_1 + \beta_1 + \cdots + \alpha_{i-1} + \beta_{i-1} + \alpha_i,$$

and the negative integers:

$$\overline{\alpha_1 + \beta_1 + \cdots + \alpha_i + 1}, \ldots, \overline{\alpha_1 + \beta_1 + \cdots + \alpha_i + \beta_i}.$$

This $\mathcal{H}_{\mathcal{S}}$-set composition is a representative of a class of $DQ\Lambda$ with α_i negative integers and β_i positive integers in the i-th part, $1 \leq i \leq k$. Hence for each bicomposition, there is a unique class of $DQ\Lambda$ associated to it. $\square$

By Example 22, the product of $\mathcal{S}(\ell^* \circ \mathsf{e}_+)$ is the quasishuffle and the coproduct is deconcatenation of $\mathcal{H}_{\mathcal{S}}$-set compositions. Since $\widetilde{K^{\mathcal{H}}}$ is bistrong by Proposition 26, the product and coproduct of $DQ\Lambda$ is given by quasishuffle and by deconcatenation of bicompositions. For example, using the product and coproduct of $\mathcal{S}(\ell^* \circ \mathsf{e}_+)$, the natural transformations (5.5) of the bistrong monoidal functor $\widetilde{K^{\mathcal{H}}}$ and the bijection and representatives described in Proposition 34, we have

$$\begin{aligned}
\mu([\bar{1}\bar{2}],[\mathrm{can}(\bar{1}|2)]) &= \mu([\bar{1}\bar{2}],[\bar{3}|4]) = [\bar{1}\bar{2}|\bar{3}|4] + [\bar{3}|\bar{1}\bar{2}|4] + [\bar{3}|4|\bar{1}\bar{2}] + [\bar{1}\bar{2}\bar{3}|4] + [\bar{3}|\bar{1}\bar{2}4] \\
&= [\bar{1}\bar{2}|\bar{3}|4] + [\bar{1}|\bar{2}\bar{3}|4] + [\bar{1}|2|\bar{3}\bar{4}] + [\bar{1}\bar{2}\bar{3}|4] + [\bar{1}|\bar{2}\bar{3}4] \\
&= \begin{pmatrix} 0\,0\,1 \\ 2\,1\,0 \end{pmatrix} + \begin{pmatrix} 0\,0\,1 \\ 1\,2\,0 \end{pmatrix} + \begin{pmatrix} 0\,1\,0 \\ 1\,0\,2 \end{pmatrix} + \begin{pmatrix} 0\,1 \\ 3\,0 \end{pmatrix} + \begin{pmatrix} 0\,1 \\ 1\,2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 2 \end{pmatrix} \sqcup\!\sqcup \begin{pmatrix} 01 \\ 10 \end{pmatrix}, \\
\Delta([\bar{1}|\bar{2}3]) &= [\bar{1}|\bar{2}3] \otimes [\emptyset] + [\bar{1}] \otimes [\mathrm{st}(\bar{2}3)] + [\emptyset] \otimes [\bar{1}|\bar{2}3] \\
&= [\bar{1}|\bar{2}3] \otimes [\emptyset] + [\bar{1}] \otimes [\bar{1}2] + [\emptyset] \otimes [\bar{1}|\bar{2}3] \\
&= \begin{pmatrix} 0\,1 \\ 1\,1 \end{pmatrix} \otimes () + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} + () \otimes \begin{pmatrix} 0\,1 \\ 1\,1 \end{pmatrix} \\
&= \Delta(\begin{pmatrix} 1\,1 \\ 0\,1 \end{pmatrix}).
\end{aligned}$$

Theorem 35.

$$DQ\Lambda \simeq \mathrm{DQSym} \textit{ as Hopf algebras.} \tag{6.2}$$

Proof. The isomorphism sending a bicomposition $\mathbf{a} = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_k \\ \beta_1 & \beta_2 & & \beta_k \end{pmatrix}$ to the bimonomial basis $M_{\mathbf{a}}$ is one of bialgebras as it preserves product and coproduct. Since $\mathcal{S}(\ell^* \circ \mathtt{e}_+)$ is a Hopf monoid and by Proposition 26, $\widetilde{K^{\mathcal{H}}}$ is a bistrong monoidal functor, the isomorphism is one of Hopf algebras. $\square$

Example 36. Since $\widetilde{K^{\mathcal{H}}}\mathcal{S}(\ell^* \circ \mathtt{e}_+) \cong \mathrm{DQSym}$ and $\overline{K}(\ell^* \circ \mathtt{e}_+) \cong \mathrm{QSym}$, the Hopf algebra of quasisymmetric functions (see [2] for the isomorphism and [11] for QSym), the map $\widetilde{\alpha^{\mathcal{S}}}_{\ell^* \circ \mathtt{e}_+}$ is the morphism of graded Hopf algebras:

$$\begin{aligned}
\mathrm{DQSym} &\to \mathrm{QSym} \\
M_{\begin{pmatrix} \alpha_1 \alpha_2 \cdots \alpha_k \\ \beta_1 \beta_2 \cdots \beta_k \end{pmatrix}} &\mapsto M_{(\alpha_1+\beta_1, \alpha_2+\beta_2, \ldots, \alpha_k+\beta_k)}.
\end{aligned}$$

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