Piecewise quasipolynomial formulas for Kronecker coefficients indexed by two-row shapes

SLC 61

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The Clebsch–Gordan Problem

Decompose into irreducible the tensor product of two irreducible (polynomial, finite–dimensional) representations of a semi–simple group G:

$$V_\mu \otimes V_\nu = \bigoplus_\lambda m^\lambda_{\mu,\nu} V_\lambda$$

We consider the two basic families of groups:

- **The general linear groups groups $GL_n(\mathbb{C})$:** the multiplicities $m^\lambda_{\mu,\nu}$ are the Littlewood–Richardson coefficients $c^\lambda_{\mu,\nu}$.

- **The symmetric groups $\mathfrak{S}_n$:** the multiplicities $m^\lambda_{\mu,\nu}$ are the Kronecker coefficients $g^\lambda_{\mu,\nu}$.

Computation problem and decision problem

	Computation	Decision (? $m^{\lambda}_{\mu,\nu} = 0$)
LR coeffs $c^{\lambda}_{\mu,\nu}$	#P-complete (Narayanan, 2006) $\Rightarrow$ not computable in polynomial time if $P \neq NP$.	P (Tao+Knutson, 2001; Mulmuley+Sohoni, 2005; De Loera+McAllister, 2006)
Kronecker coeffs $g^{\lambda}_{\mu,\nu}$	• ? $\in$ #P • GapP (Bürgisser+Ikenmeyer 2008)	? P Mulmuley's Geometric Complexity Theory (Attack for ?$P \neq NP$ over $\mathbb{C}$)

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Deciding whether or not $c_{\mu,\nu}^{\lambda} = 0$ is in P because . . .

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- Positive Formulas (PH1): LR coefficients $c^{\lambda}_{\mu,\nu}$ counts the integral points of a polytope Hive(λ, μ, ν), described by linear constraints:

  $$A \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{pmatrix} \leq B \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \nu_n \end{pmatrix}$$

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- Saturation property (SH) (Knutson+Tao, 1999):

  $$c^{\lambda}_{\mu,\nu} = 0 \quad \Leftrightarrow \quad c^{N\lambda}_{N\mu,N\nu} = 0 \text{ for all } N > 0$$

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- Linear programming (deciding nonemptyness of a polytyope defined by linear constraints) $\in P$.

Mulmuley's Positivity and Saturation hypotheses

PH1 (Positivity): The Kronecker coefficients $g_{\mu,\nu}^{\lambda}$ count the integral points of a polytope $\mathrm{Kron}(\lambda,\mu,\nu)$ defined by linear constraints.

Fix λ, μ, ν. The "stretching function" $N \mapsto g_{(N\mu)(N\nu)}^{(N\lambda)}$ is a (univariate) quasi–polynomial (Mulmuley, 2007)

i.e. there exist $k > 0$ and polynomials F_i such that

$$g_{(N\mu)(N\nu)}^{(N\lambda)} = \begin{cases} F_1(N) \text{ if } N \equiv 1 \mod k \\ F_2(N) \text{ if } N \equiv 2 \mod k \\ \vdots \\ F_k(N) \text{ if } N \equiv k \mod k \end{cases}$$

SH (Saturation): $g_{\mu,\nu}^{\lambda} = 0 \Leftrightarrow g_{N\mu,N\nu}^{N\lambda} = 0$ for all $N \equiv 1 \mod k$.

? Check SH for Kron in simple cases.

Check SH for $g^{(\lambda_1,\lambda_2,\lambda_3)}_{(\mu_1,\mu_2)(\nu_1,\nu_2)}$

Simplest non–trivial case: coefficients $g^{(\lambda_1,\lambda_2,\lambda_3)}_{(\mu_1,\mu_2)(\nu_1,\nu_2)}$.

Find explicit formulas for them to check SH.

Explicit formulas ?

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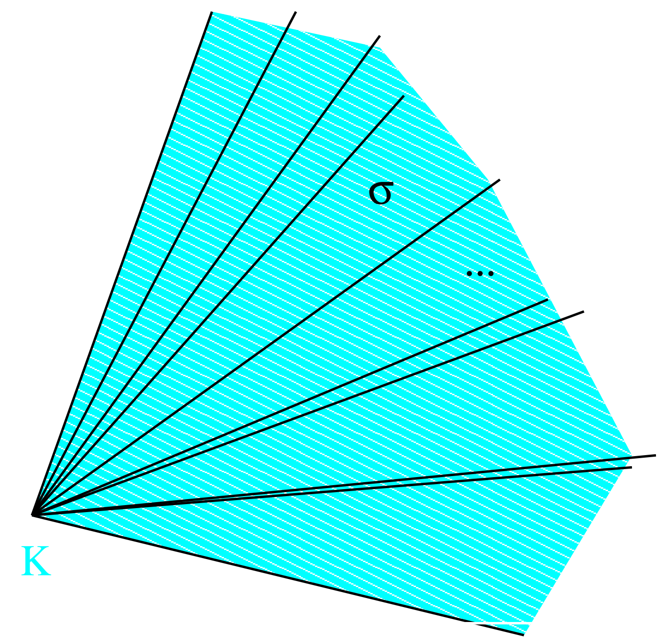
Explicit formulas ?

Assume PH1 holds: then $(\lambda_1,\dots,\nu_2) \mapsto g^{(\lambda_1,\lambda_2,\lambda_3)}_{(\mu_1,\mu_2)(\nu_1,\nu_2)}$ is piecewise quasi–polynomial: there exists

- a convex rational polyhedral cone $K \subset \mathbb{R}^7$, such that outside K there is $g^{(\lambda_1,\lambda_2,\lambda_3)}_{(\mu_1,\mu_2)(\nu_1,\nu_2)} = 0$.
- a fan $\mathcal{F}$ subdividing K
- on each of its maximal cells σ a (multi-variate) quasi–polynomial q_σ such that on σ: $g^{(\lambda_1,\lambda_2,\lambda_3)}_{(\mu_1,\mu_2)(\nu_1,\nu_2)} = q_\sigma(\lambda_1,\dots,\nu_2)$



Computing Kronecker coeffs through reduced Kronecker coeffs

The Reduced Kronecker coefficients $\overline{g}_{\alpha\beta}^{\gamma}$: limits of certain stationary sequences of Kronecker coefficients (Murnaghan, 1938)

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We established a formula to recover the Kronecker coeffs $g_{\mu,\nu}^{\lambda}$ from the reduced Kronecker coeffs $\overline{g}_{\alpha,\beta}^{\gamma}$.

$$g_{(\mu_1,\mu_2)(\nu_1,\nu_2)}^{(\lambda_1,\lambda_2,\lambda_3)} = \overline{g}_{(\mu_2)(\nu_2)}^{(\lambda_2,\lambda_3)} - \overline{g}_{(\mu_2)(\nu_2)}^{(\lambda_1+1,\lambda_3)} + \overline{g}_{(\mu_2)(\nu_2)}^{(\lambda_1+1,\lambda_2+1)}$$

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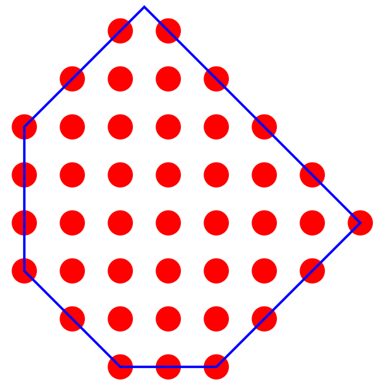
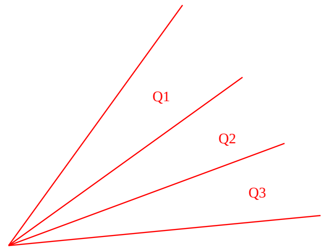
From results by M.R., 2002, we showed that $\overline{g}_{(r)(s)}^{(\gamma_1\gamma_2)}$ counts the integral point of a polygon:

$$\begin{cases} X \geq \max(r,s) \\ Y \geq 0 \\ r+s-\gamma_2 \geq X+Y \geq r+s-\gamma_1 \\ \gamma_1+\gamma_2 \geq X-Y \geq \gamma_1 \end{cases}$$

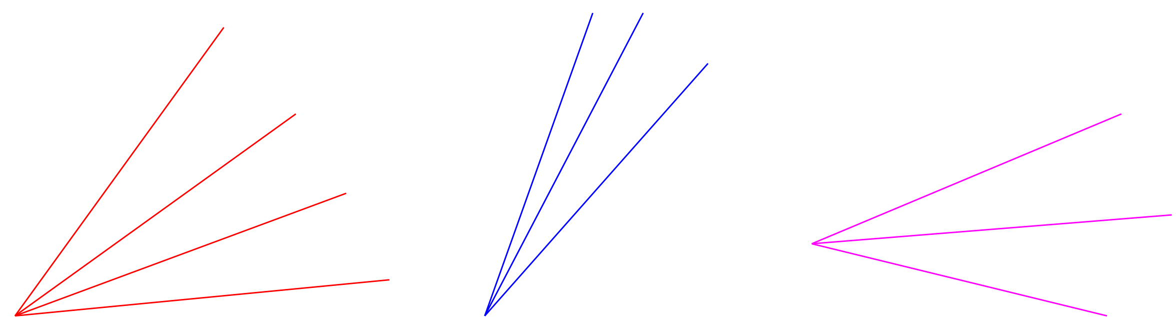


and from this we obtained a description for $\overline{g}_{(r)(s)}^{(\gamma_1\gamma_2)}$ as a piecewise quasi–polynomial supported by a fan $\mathcal{F}_0$.



From reduced to non–reduced Kronecker coefficients

$$g^{(\lambda_1,\lambda_2,\lambda_3)}_{(\mu_1,\mu_2)(\nu_1,\nu_2)} = \overline{g}^{(\lambda_2,\lambda_3)}_{(\mu_2)(\nu_2)} - \overline{g}^{(\lambda_1+1,\lambda_3)}_{(\mu_2)(\nu_2)} + \overline{g}^{(\lambda_1+1,\lambda_2+1)}_{(\mu_2)(\nu_2)}$$



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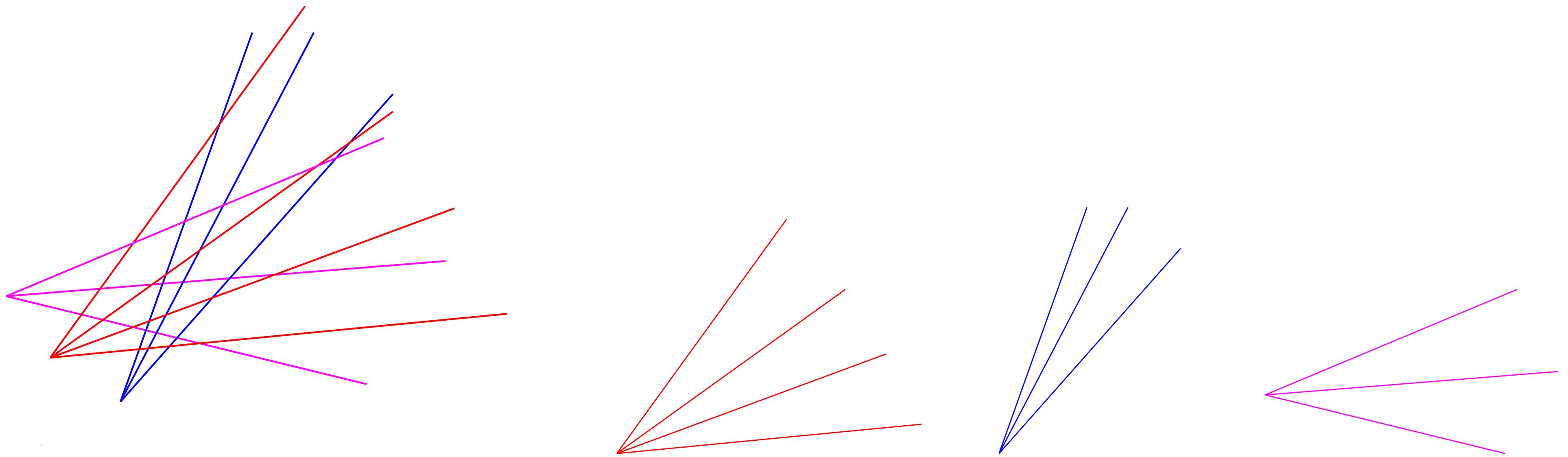

Pieces of quasipolynomiality for $g^{(\lambda_1,\lambda_2,\lambda_3)}_{(\mu_1,\mu_2)(\nu_1,\nu_2)}$: polyhedral cells, but not cones.

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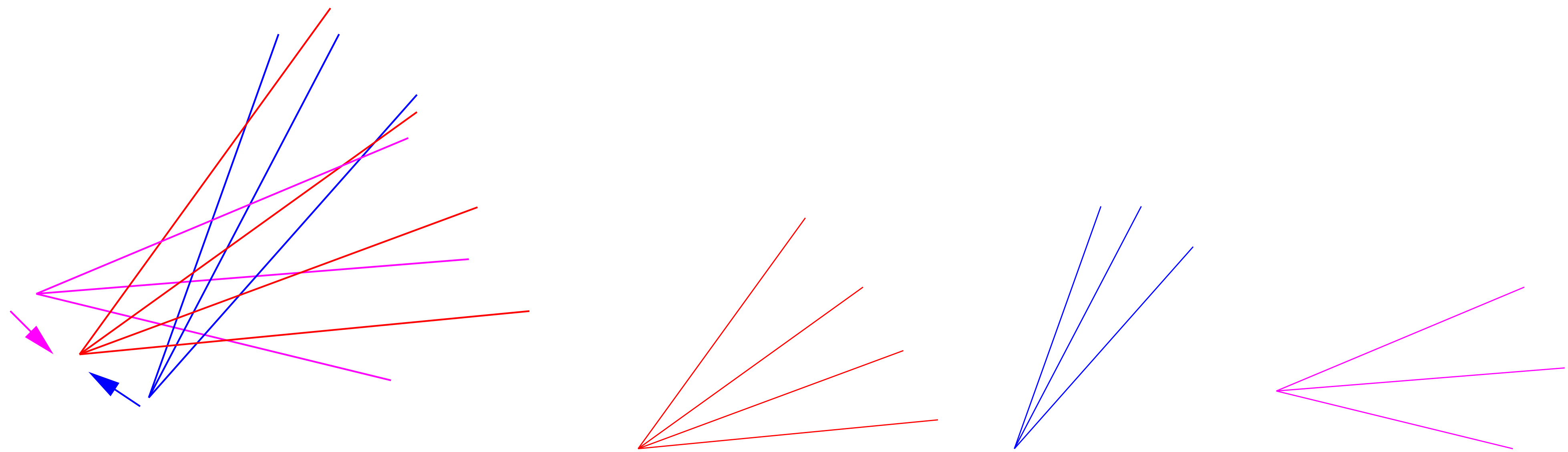

The quasi–polynomial formulas for $\overline{g}^{(\gamma_1\gamma_2)}_{(r)(s)}$ are still valid on some shifts of the cells of the fan $\mathcal{F}_0$,

so that the pieces for $g^{(\lambda_1,\lambda_2,\lambda_3)}_{(\mu_1,\mu_2)(\nu_1,\nu_2)}$ are cones.

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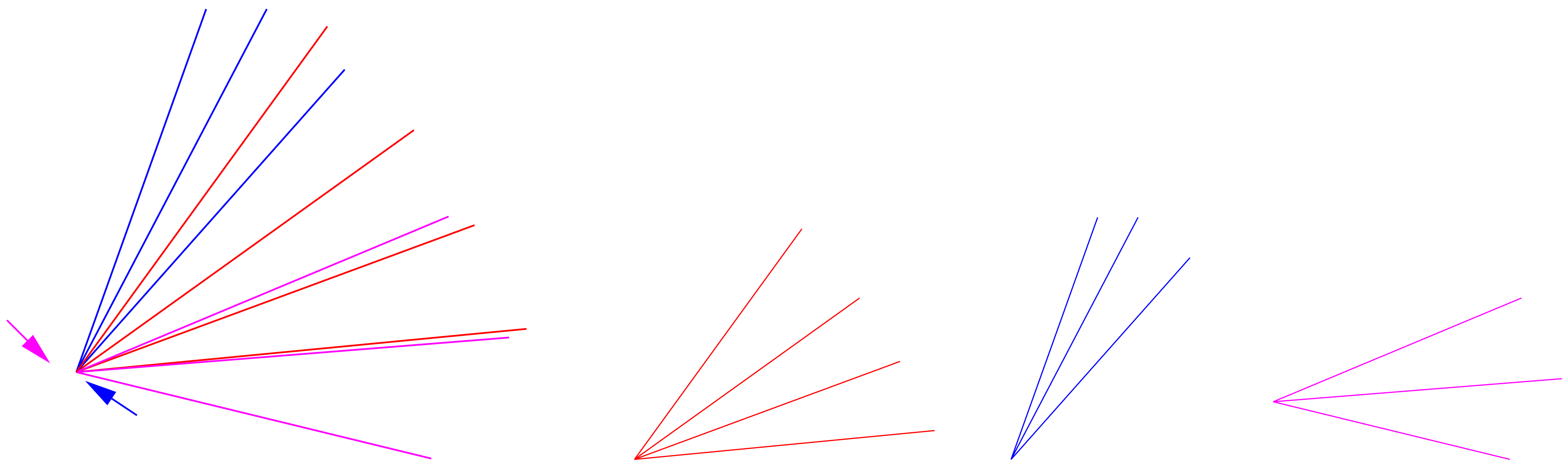

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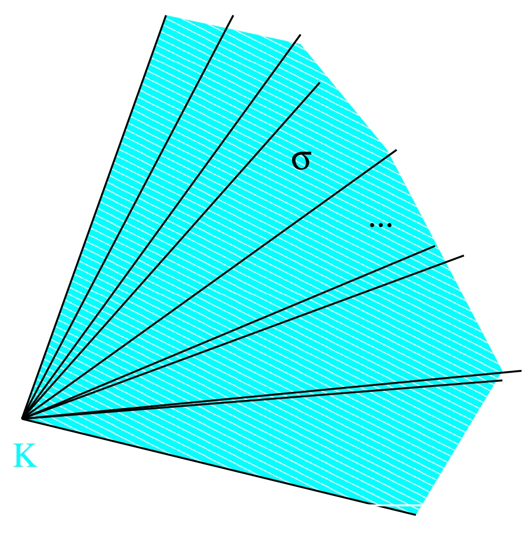
Explicit formulas

With the help of the Maple Package convex (Matthias Franz) we obtain the description of the fan $\mathcal{F}$ associated to the coefficients $g^{(\lambda_1,\lambda_2,\lambda_3)}_{(\mu_1,\mu_2)(\nu_1,\nu_2)}$:

it has 74 maximal cells which are the domains of quasi–polynomiality for $g^{(\lambda_1,\lambda_2,\lambda_3)}_{(\mu_1,\mu_2)(\nu_1,\nu_2)}$.



Checking SH

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Wanted:

Mulmuley's Hypothesis SH holds for the Kronecker coefficients $g^{(\lambda_1,\lambda_2,\lambda_3)}_{(\mu_1,\mu_2)(\nu_1,\nu_2)}$.

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Obtained:

Mulmuley's Hypothesis SH does not hold for the Kronecker coefficients $g_{(\mu_1,\mu_2)(\nu_1,\nu_2)}^{(\lambda_1,\lambda_2,\lambda_3)}$.

Indeed, if SH holds then

$$g_{\mu,\nu}^{\lambda} = 0 \Rightarrow g_{N\mu,N\nu}^{N\lambda} = 0 \quad \text{for infinitely many } N > 0$$

but !!!

$$g_{(6N,6N)(7N,5N)}^{(6N,4N,2N)} \begin{cases} = 0 \text{ for } N = 1 \\ > 0 \text{ for } N > 1 \end{cases}$$

Conclusion

This is part of a more general work about Reduced Kronecker Coefficients.

Other results:

- we gave simpler proofs of some properties of the reduced Kronecker coefficients using *vertex operators* acting on symmetric functions.

- we obtained new bounds for the so–called *stability of the Kronecker product* considered earlier by Ernesto Vallejo.