PERMUTATIONS, $(\mathbf{2}+\mathbf{2})$-FREE POSETS AND ASCENT SEQUENCES

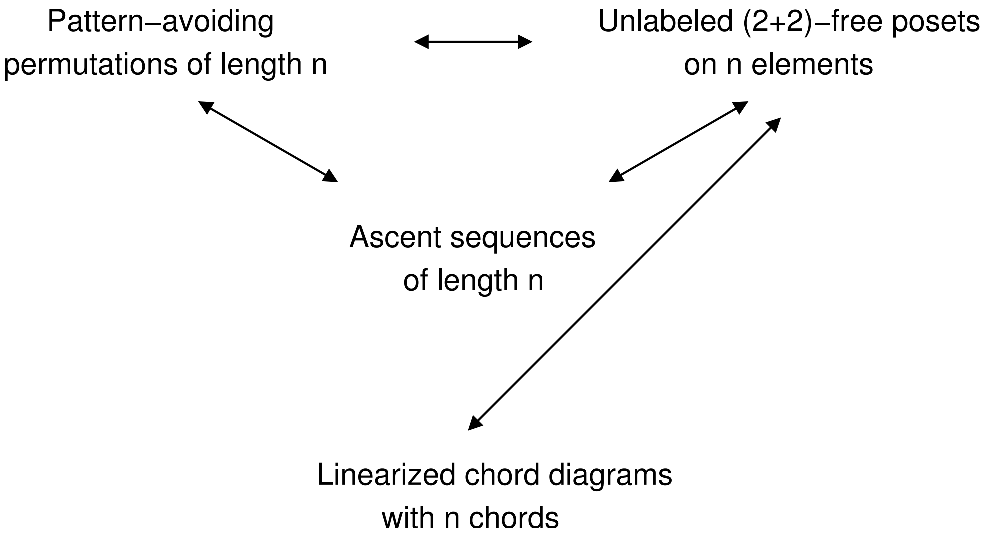
Mark Dukes, Science Institute, University of Iceland

joint work with

Mireille Bousquet-Mélou   Anders Claesson   Sergey Kitaev

Overview of results

Bijections between the following objects



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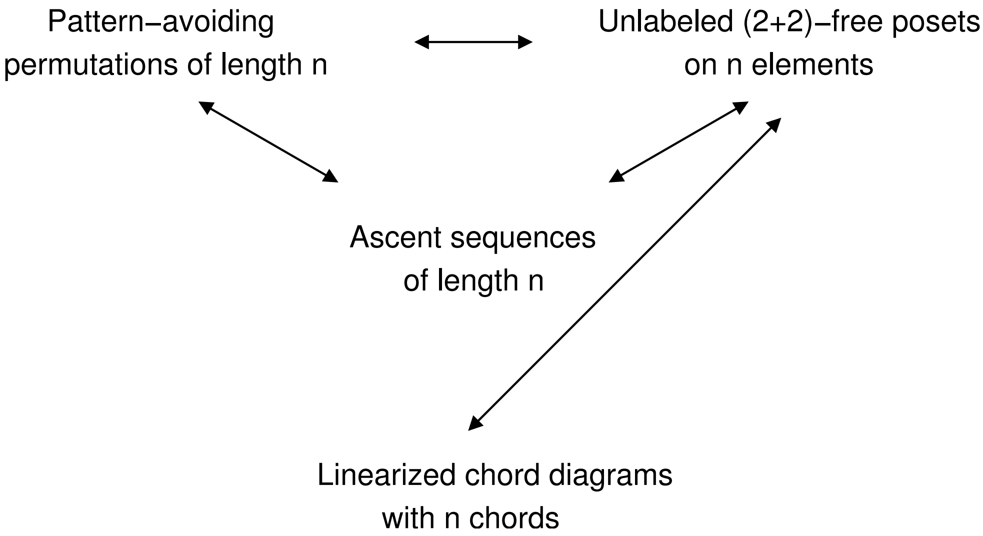

- Bijections respect several statistics on the structures
- Closed form for generating function
- Modified ascent sequences and Pudwell's conjecture.

Ascent sequences

We call a sequence $(x_1, \ldots, x_n)$ of non-negative integers an **ascent sequence** if

- $x_1 = 0$, and
- $x_i \in [0, 1 + \mathrm{asc}(x_1, \ldots, x_{i-1})]$ for all $1 < i \leq n$.

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$\mathrm{asc}(0, 0, 1, 0, 1, 2, 0) = 3$

All ascent sequences of length 4:

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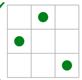

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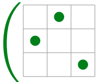

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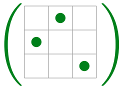

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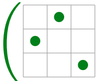

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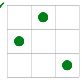

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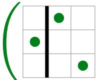

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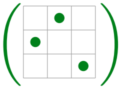

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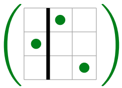

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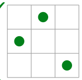

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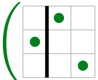

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A new class of pattern avoiding permutations:

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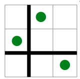
$$R_n = S_n\left(\right.$$  $$\left.\right) \quad ?$$

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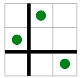
$$R_n = S_n\left(\right.$$  $$\left.\right)$$?

Formally a permutation is in the class if there do not exist indices i and k satisfying $1 \leq i < i+1 < k \leq n$ and such that

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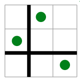
$$R_n = S_n\Big($$  $$\Big)$$?

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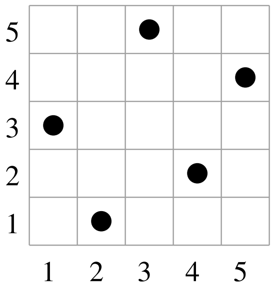

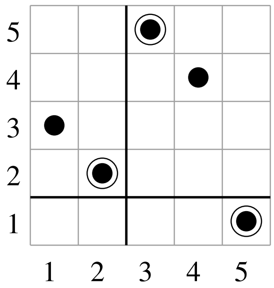

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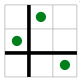
$$R_n = S_n \left(\right)$$  ?

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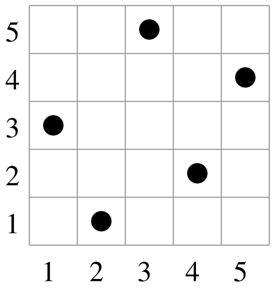

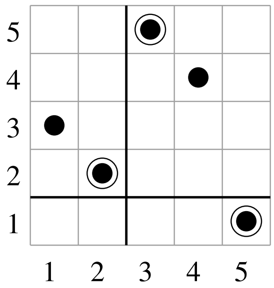

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Theorem 1 There is a 1-1 correspondence between ascent sequences of length n and permutations in R_n.

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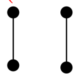
A partially ordered set P is called $(\mathbf{2}+\mathbf{2})$-free if it contains no induced sub-poset isomorphic to $(\mathbf{2}+\mathbf{2}) =$ 

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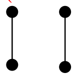
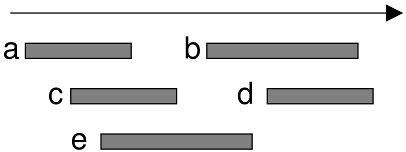
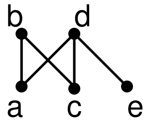
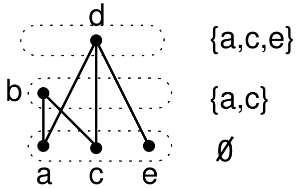
A partially ordered set P is called $(\mathbf{2}+\mathbf{2})$-free if it contains no induced sub-poset isomorphic to $(\mathbf{2}+\mathbf{2}) =$ 

Such posets arise as interval orders:







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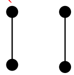
A partially ordered set P is called $(\mathbf{2}+\mathbf{2})$-free if it contains no induced sub-poset isomorphic to $(\mathbf{2}+\mathbf{2})=$ 

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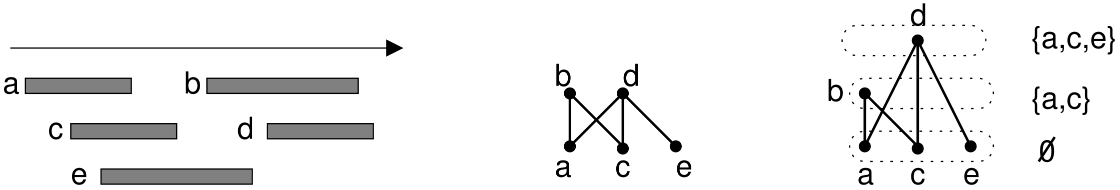

Theorem 2 (Not ours!) A poset P is $(\mathbf{2}+\mathbf{2})$-free iff the collection of strict order ideals $\{D(x)=\{y<x\}: x\in P\}$ may be linearly ordered by inclusion.

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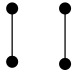
A partially ordered set P is called $(\mathbf{2}+\mathbf{2})$-free if it contains no induced sub-poset isomorphic to $(\mathbf{2}+\mathbf{2}) =$ 

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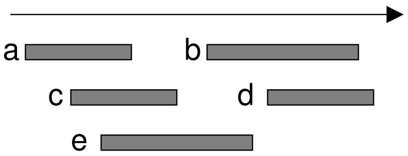

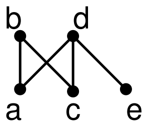

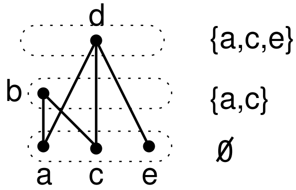

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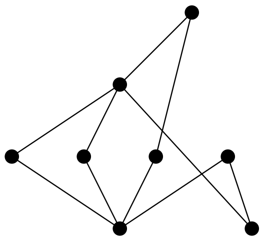
Clearly $D(a) \subseteq D(c) \subseteq D(e) \subseteq D(b) \subseteq D(d)$.

- P. C. Fishburn, *Interval Graphs and Interval Orders*, Wiley, New York, 1985.
- P. C. Fishburn, Intransitive indifference with unequal indifference intervals, *J. Math. Psych.* **7** (1970) 144–149.

How can one decompose such posets?



$x = (x_1, \ldots, x_8)$?

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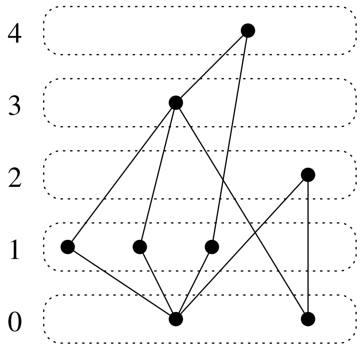

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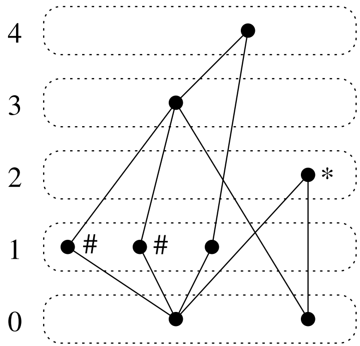

$x_8 = 2$

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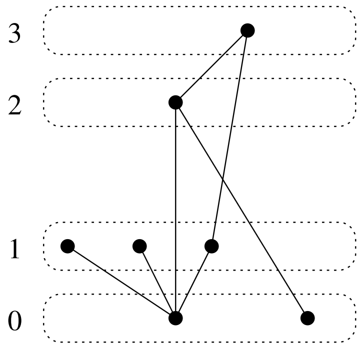

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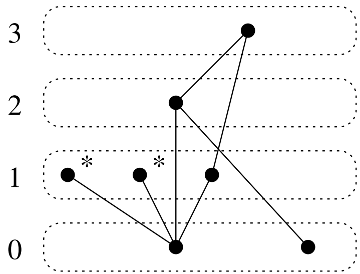

$$x_7 = 1$$

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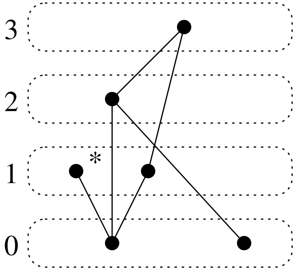

$x_6 = 1$

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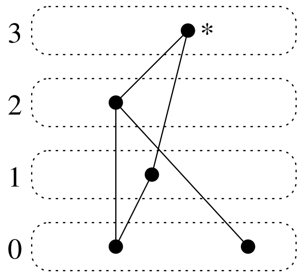
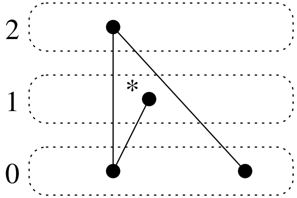


$$x_5 = 3$$

How can one decompose such posets?



$$x_4 = 1$$

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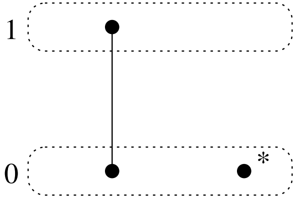
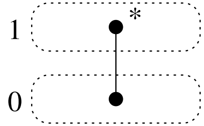


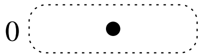
$x_3 = 0$

How can one decompose such posets?



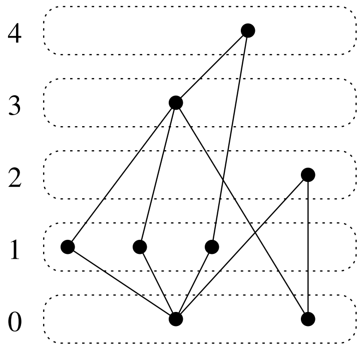
$$x_2 = 1$$

How can one decompose such posets?



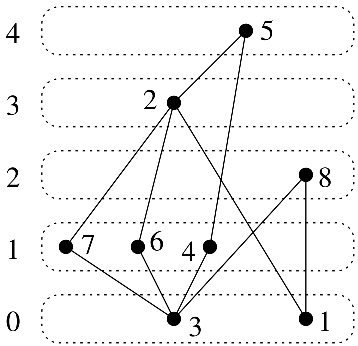
$x_1 = 0$

How can one decompose such posets?



$$x = (0, 1, 0, 1, 3, 1, 1, 2)$$

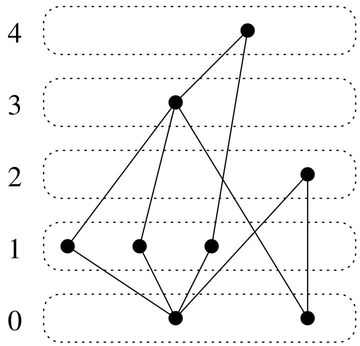
Recording the order in which elements were removed:



From bottom to top list the elements in decreasing order:

$$\pi = 31764825.$$

How can one decompose such posets?



$x = (0, 1, 0, 1, 3, 1, 1, 2)$

Theorem 3 There is a 1-1 correspondence between unlabeled $(\mathbf{2}+\mathbf{2})$-free posets on n elements and ascent sequences of length n.

Theorem 4 There is a 1-1 correspondence between unlabeled $(\mathbf{2}+\mathbf{2})$-free posets on n elements and permutations in R_n.

Statistics: For a $(\mathbf{2}+\mathbf{2})$-free poset P, a sequence x and a permutation $\pi \in R_n$, define:

$$\lambda(P,q)=\sum_{v\in P} q^{\ell(v)}, \quad \chi(x,q)=\sum_{i=1}^{|x|} q^{x_i}, \quad \delta(\pi,q)=\sum_{i=0}^{s(\pi)} d_i q^i,$$

where d_i is the number of entries of π between the active site labeled i and the active site labeled $i+1$.

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Theorem 5 Given an ascent sequence x of length n, let P and π be the poset and permutation corresponding to x under the operations described. Then

$$\begin{aligned}
\min(P) &= \mathrm{zeros}(x) = \mathrm{ldr}(\pi);\\
\ell^\star(P) &= \mathrm{last}(x) = b(\pi);\\
\ell(P) &= \mathrm{asc}(x) = \mathrm{asc}(\pi^{-1});\\
\max(P) &= \mathrm{rmax}(\widehat{x}) = \mathrm{rmax}(\pi);\\
\mathrm{comp}(P) &= \mathrm{comp}(\widehat{x}) = \mathrm{comp}(\pi);\\
\lambda(P,q) &= \chi(\widehat{x},q) = \delta(\pi,q);\\
\overline{\lambda}(P,q) &= \overline{\chi}(\widehat{x},q) = \overline{\delta}(\pi,q).
\end{aligned}$$

The generating function

Generating function $P(t)$ of unlabeled $(\mathbf{2}+\mathbf{2})$-free posets:

$$P(t) = \sum_{n\geq 0} p_n t^n$$
$$= 1 + t + 2t^2 + 5t^3 + 15t^4 + 53t^5 + 217t^6 + 1014t^7 + 5335t^8 + O(t^9),$$

where p_n is the number of $(\mathbf{2}+\mathbf{2})$-free posets of cardinality n.

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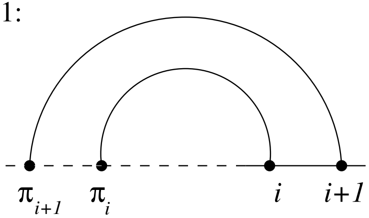
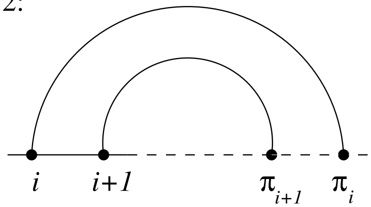
Regular Linearized Chord Diagrams (RLCD's)

A regular linearized chord diagram with n chords is a matching of $2n$ points such that the chords extending from two adjacent points are not nested (i.e. avoid Type 1 and Type 2 below):

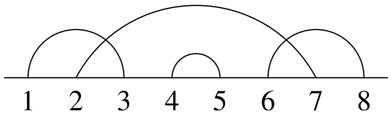
Type 1:



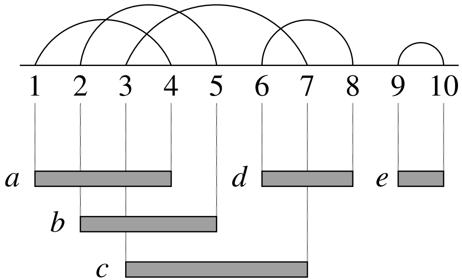
Type 2:

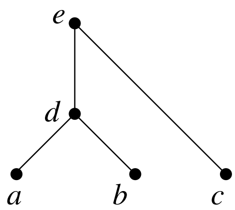


Example:



RLCD’s may be mapped to $(\mathbf{2}+\mathbf{2})$-free posets via the following map:





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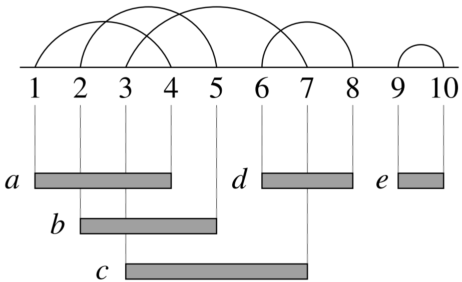

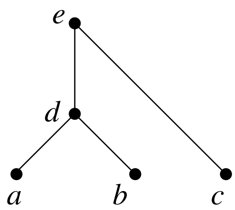

Theorem 7 There is a 1-1 correspondence between RLCD's with n chords and $(\mathbf{2}+\mathbf{2})$-free posets on n elements.

A conjecture of Lara Pudwell

A permutation π avoids the barred pattern $3\bar{1}52\bar{4}$ if every occurrence of the (classical) pattern 231 plays the role of 352 in an occurrence of the (classical) pattern 31524.

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Conjecture (Lara Pudwell, PhD thesis, 2008) The length generating function of $3\bar{1}52\bar{4}$-avoiding permutations is

$$\sum_{k\geq 1} \frac{t^k}{(1-t)^{\binom{k+1}{2}}}.$$

Equivalently, the number of such permutations of length n is

$$\sum_{k=1}^{n} \binom{\binom{k}{2}+n-1}{n-k}.$$

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Theorem 8 The ascent sequence x is self modified if and only if the corresponding permutation π avoids $3\bar{1}52\bar{4}$.