ORIENTABILITY OF CUBES

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Content

I - Matroids and Oriented Matroids

A very brief introduction.

II - Orientability of cubes.

How many cubes are orientable?

For proofs:

IS08 - Ilda P.F. da Silva, Orientability of Cubes, *Discrete Maths.*, **308** (2008), 3574-3585.

IS07 - Ilda P. F. da Silva, On Minimal nonorientable matroids with $2n$-elements and rank n, preprint 2007, to appear in *Europ. J. Comb.*

Matroid (Whitney 35's ...)

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Matroid over a (finite) set E - $M(E)$

$$M = M(E) = (E, \mathcal{H}) \simeq (E, \mathcal{C})$$

$\mathcal{H} \subset 2^E$ satisfying the axioms of hyperplanes of a matroid.
$\mathcal{C} \subset 2^E$ satisfying the axioms of circuits of a matroid.

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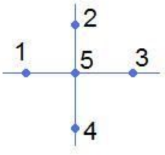
A Matroid is representable over a field K - $\mathbf{Aff}_K(E)$ - when:

E is a (finite) set of points of some affine space K^n.

a hyperplane - is a subset of E lying in an affine hyperplane spanned by points of E.

a circuit - is subset of E which is minimal affine dependent.

Example: $\mathrm{Aff}_{\mathbb{R}}(E)$ and its dual matroid

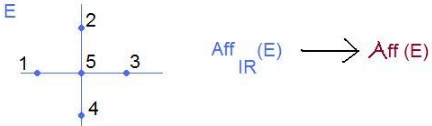


$\mathcal{H}$ – *hyperplanes*	*compl.*	$\mathcal{C}^*$ – *cocircuits*
12	–	**345**
135	–	**24**
14	–	**235**
23	–	**145**
245	–	**13**
34	–	**125**
$\mathcal{C}$ – *circuits*		$\mathcal{H}^*$ – *cohyperplanes*
135	–	**24**
245	–	**13**
1234	–	**5**

Oriented Matroid (Bland, Las Vergnas, Folkman-Lawrence 75's...)

Oriented Matroid $\mathcal{M}(E)$ = Matroid $M(E)$ + Orientation

Realizable OM= **Matroid representable over $\mathbb{R}$ + Canonical Orientation.**

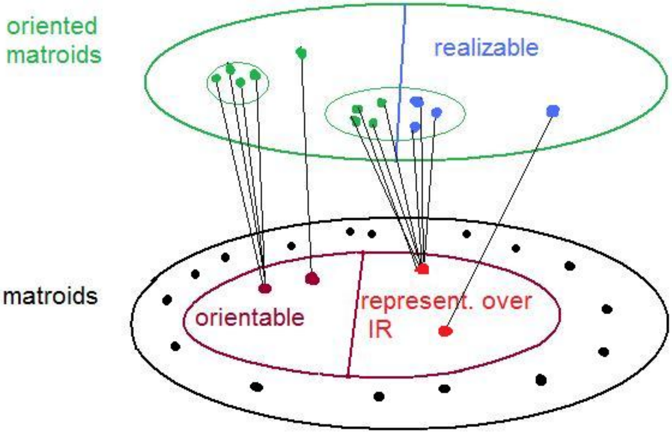


$\mathcal{C}^{\perp}$ – *or.cocircuits* *comb.orthog.* $\mathcal{C}$ – *or.circuits*

$$\begin{array}{ccc} 345, \overline{345} & \longleftrightarrow & \\ 2\bar{4}, \bar{2}4 & & 13\bar{5}, \overline{13}5 \\ 235, \overline{235} & & 24\bar{5}, \overline{24}5 \\ 145, \overline{145} & & 13\overline{24}, \overline{13}24 \\ 1\bar{3}, \bar{1}3 & & \\ 125, \overline{125} & & \end{array}$$

Convexity in oriented Matroids (Las Vergnas 80) - *Face lattice of a polytope conv(E)* $\longrightarrow$ LV- face lattice of an (acyclic) oriented matroid.

Matroids and Oriented Matroids



Representation Theorems for Oriented Matroids

Topological Representation Theorem. (Folkman/Lawrence 78)

Oriented Matroid over $[n]$ and rank $d \iff$ cell complex of a (signed) arrangement of n pseudo-spheres of the unit sphere S^{d-1}.

Euclidean Representation Theorem. (IS 98)

Oriented Matroid over $[n]$ (without loops) $\iff$ subset $\mathcal{T} \subseteq \{-1,1\}^n$ of vertices of the real cube $[-1,1]^n$ of $\mathbb{R}^n$ satisfying symmetry conditions - centers of faces and orthogonal projections onto faces .

This is a representation theorem for oriented matroids on the LV-face lattice of the oriented real affine cube $\mathcal{A}ff(C^n)$, $C^n = \{-1,1\}^n$.

Why this particular orientation? Why this particular cube?

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What happens if we choose another <u>orientable cube</u> ?

II. Orientability of Cubes

What is a cube:

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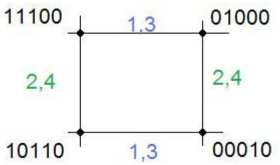
Definition(IS 08) **A Cubic Matroid (or cube)** is a matroid M over $C^n = \{0,1\}^n$ that satisfies the following two conditions:

(i) Every facet and skew-facet of C^n is a hyperplane of M.

$2n$ facets : $x_i = 0, 1$

$\binom{n}{2}$ skew facets: $x_i + x_j = 1,\ x_i - x_j = 0.$

(ii) Every rectangle of C^n is a circuit of M.



A rectangle of C^5

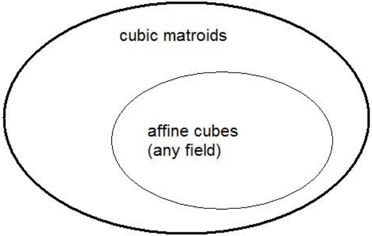
Lots of rectangles! Each vertex is contained in 3^n rectangles!

Properties of Cubic Matroids IS08

For every field K the matroid $\mathbf{Aff}_K(C^n)$ is a cubic matroid.

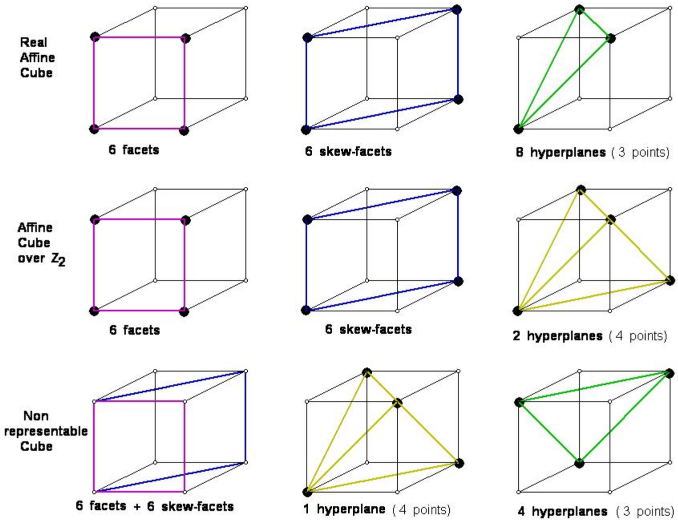
All cubes have 2^n points and rank $n+1$.

Theorem 1. *The class of cubic matroids remains invariant under certain perturbations of matroids: pushing an element onto a hyperplane.*



a "large" class of matroids

Cubic Matroids over $C^3 = \{0,1\}^3$:



Invariants of ALL Orientable Cubes

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Theorems 2,3 (IS 08) - Topologic version

Every arrangement of 2^n *pseudospheres of the sphere* S^n *representing an oriented cubic matroid* $\mathcal{M}(C^n)$ *has the following properties:*

1) $(n+1)$*-pairs of opposite regions which are "n-cross-polytopes" bounded by the* 2^n *pseudospheres .*

2) The relative position of these $2(n+1)$ *regions is the same as in the arrangement of spheres representing the real oriented cube* $\mathcal{A}ff(C^n)$*.*

In particular,

Every orientable cube has exactly one orientation with the same LV-face lattice then the oriented real n-cube. Good!

How many cubes are Orientable?

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1. **Non-orientability results**

How many cubes are Orientable?

1. Non-orientability results

From Theorems 2 and 3, with a very short proof:

Theorem 4. (implicit in B-LV 78, IS08) *If K is a field of prime characteristic p then K-affine cube $Aff_K(C^n)$ is not orientable for $n \geq p+1$.*

2. Perturbations of matroids and orientability

Alternative proof for Theorem 4: for $n \geq q+1$, $Aff_K(C^n)$ contains a Bland-Las Vergnas minimal non-orientable matroid M_{n+1}.

In IS 07 we use the operation of pushing an element onto a hyperplane to obtain **NEW minimal non-orientable matroids** - minors of perturbations of the real affine cube.

Conjectures

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Conjecture 1. (Las Vergnas Cube Conjecture) **The real affine cube has a unique orientation (class).**

Conjecture 2. **The real affine cube is the unique orientable cube.**

Both Conjectures are true for small dimensions - $n \leq 7$**:**
(Bokowski, Guedes de Oliveira etal 96, da Silva 06) *Las Vergnas Cube Conjecture is true for* $n \leq 7$*.*
(da Silva 06)*Conjecture 2 is true for* $n \leq 7$*.*

All proofs use explicit descriptions of the real affine cube. **The real afine cube is a difficult object.**
Recall that:
Determining whether a linear equation $\mathbf{h}.\mathbf{x} = b$*, with* $(\mathbf{h}, b) \in \mathbb{Z}^{n+1}$*, has a* ± 1 *solution is* $\mathcal{NP} -$ *complete.*

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Let M_n be a random $n \times n$, ± 1-matrix (random with respect to the uniform distribution) = Bernoulli matrix

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Results.

(Khan, Komlós, Szemerédi, 95) *There is a positive constant* $\epsilon > 0$ *for which* $P_n < (1 - \epsilon)^n$. *True for* $\epsilon = 0.001$.

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This last result strengthens our results and conjecture because:

The last Conjecture 3 is equivalent to the following one:(KKS 95, Tao/Vu 06, Voigt/Ziegler 06)

Conjecture 3'. For $\mathbf{v_1}, \ldots, \mathbf{v_r}$ chosen randomly from $\{\pm 1\}^n$, $r \leq n-1$, *$Pr(lin(\mathbf{v_1}, \ldots, \mathbf{v_r}) \cap \{\pm 1\}^n \neq \{\pm \mathbf{v_1}, \ldots \pm \mathbf{v_r}\}) \simeq$ the probability that 3 of the $\pm \mathbf{v_j'}\mathbf{s}$ span a rectangle with the fourth vertex different from any $\pm \mathbf{v_j}$.*

This means essentially that rectangles determine the closure operator of the real affine cube.

and

This is exactly our proof of Conjectures 1 and 2 for small dimensions!

Final Remark.

Conjecture 1 and 2 TRUE imply :

No need of numbers to define the affine/linear dependencies of C^n over the REALS!

Orientability of Cubes: picture - September 08