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TWISTED LIE ALGEBRAS AND IDEMPOTENT OF DYNKIN

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Abstract. In this paper we define a Dynkin idempotent for twisted Hopf algebras and generalize the results of Patras and Reutenauer in the classical case. We treat as a special case the free Lie algebra and so generalize the results of Waldenfels.

1. Introduction

These pages are intended to be the initial part of a larger program: to spread the light of twisted Lie algebras onto the James–Hopf and the Hilton–Hopf invariants. This is certainly presumptuous, but presently it seems a promising direction to go.

Actually the starting post was planted by Barratt in his well-known article [3]: there he announced his intention to look inside twisted structures "where the milling crowd of generalised Hopf invariants may be reduced to order or at least quieted." But this remained almost without posterity; nevertheless, we should mention the work of Goerss [7] in this direction.

Except for this topological context, twisted algebraic structures grew in interest in their own right as a combinatorial subject of study. General framework studies were proposed: Stover [16] set up definitions directly modelled on the classical ones; they are effective and very useful for elementary calculations (we shall briefly recall them and use them in our proofs). Joyal [8] gave an abstract and very synthetic approach by the species of structures; Bergeron, Labelle and Leroux [4] devote a book to this subject. In this vein, we can cite Patras and Reutenauer [12] and Patras and Schocker [13, 14]. Operadic results on bi- and Hopf algebras enriched this combinatorial point of view (see [1, 5, 6, 9, 10]). An extensive account of all these subjects by Aguiar and Mahajan is in press [2].

After all these developments on twisted algebraic structures, it is perhaps worthwhile revisiting Barratt's pages on the subject.

First, at the end of [3], Barratt gives a description of a linear basis of the free twisted Lie algebra, but without proof. As a quick check reveals (just consider the action of the symmetric group by the sign of a permutation — what topologists call a graded Lie algebra — and the element $[[x, y], [x, y]]$), Barratt's basis is not a generating set. So the first task arises by itself: determine a basis for the free twisted Lie algebra (and prove that it really is one!).

This can be done using the Hall basis as in the classical case — this is the concern of a forthcoming paper. Major tools for that are the Dynkin and Klyachko idempotents for Lie algebras. As an introductory chapter, this article deals with the idempotent of Dynkin in a more general setting, namely the setting of Hopf algebras. Following

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Patras and Reutenauer [11], we do that abstractly on morphisms, not on elements. But since our next task is to deal with the Hall basis (elements in Lie algebras), we think that it is not too redundant to also translate the description of Waldenfels [18], which explicitly uses elements.

The paper is organized as follows.

We recall some definitions about twisted algebraic structures in Section 2. We also set up notations once for all.

In Section 3, we discuss free twisted objects and adapt the notion of associative and Lie polynomials to the twisted case.

Section 4 deals with the case of free Lie algebras and reproduces the proofs of Waldenfels and Reutenauer giving various characterizations of Lie polynomials. We shall directly use this approach to construct the Hall basis.

In Section 5, we adapt the notion of pseudo-coproduct for the twisted case.

Making a large use of pseudo-coproducts, in Section 6 we prove the properties of the Dynkin idempotent for Hopf Algebras. This forms a generalization of the results obtained in Section 4.

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2. Twisted algebras

We briefly review all algebraic structures we shall use in their twisted version. To this end, we follow and refer to Stover [16]. This presentation is very explicit on elements and so immediately manageable when we construct the Hall basis.

First we fix some notations for the permutation group.

2.1. **Permutation groups.** Let us denote by $\mathfrak{S}_n$ the group of all bijections of n objects; in the following it is understood (unless otherwise specified) that these objects are the set of integers $\{1, \ldots, n\}$; we also explicitly denote a permutation σ by its image $(\sigma(1), \ldots \sigma(n))$. We compose permutations as usual for maps by acting on the left: $\sigma \circ \tau(i) = \sigma(\tau(i))$.

Given a decomposition (all integers in the following are non-negative) (p_1, p_2) of $n = p_1 + p_2$ (by abuse of terminology, we shall say: a decomposition $n = p_1 + p_2$), we define the inclusion $\mathfrak{S}_{p_1} \times \mathfrak{S}_{p_2} \subset \mathfrak{S}_n$ as reflecting the inclusion given on objects by the map preserving order from left to right:

$$\{1, \ldots, p_1\} \amalg \{1, \ldots, p_2\} \subset \{1, \ldots, n\}$$

$$i \mapsto i \text{ on the first factor}$$

$$i \mapsto p_1 + i \text{ on the second factor}$$

(Clearly, for $\amalg$ order matters). One immediately extends this to the case $n = p_1 + \cdots + p_k$, $k \geq 2$, to define $\mathfrak{S}_{p_1} \times \cdots \times \mathfrak{S}_{p_k} \subset \mathfrak{S}_n$. If Φ_i is a permutation in $\mathfrak{S}_i$, $i = 1, 2, \ldots, k$, we denote by $(\Phi_1, \ldots, \Phi_k)$ the image in $\mathfrak{S}_n$ of $(\Phi_1 \times \cdots \times \Phi_k) \in \mathfrak{S}_{p_1} \times \cdots \times \mathfrak{S}_{p_k}$.

We now define permutations acting on blocks. Let $n = p_1 + \cdots + p_k$ be some decomposition and $\sigma \in \mathfrak{S}_k$. We define the permutation of $\mathfrak{S}_n$ acting on the k-blocks $p_1, \ldots, p_k$ by the following composition:

$$\begin{aligned}\mathcal{C}_{p_{\sigma^{-1}(1)},\ldots,p_{\sigma^{-1}(k)}}(\sigma) : \{1,\ldots,n\} &\to \{1,\ldots,p_1\} \amalg \cdots \amalg \{1,\ldots,p_k\} \\ &\to \{1,\ldots,p_{\sigma^{-1}(1)}\} \amalg \cdots \amalg \{1,\ldots,p_{\sigma^{-1}(k)}\} \to \{1,\ldots,n\}\end{aligned}$$

where the first and last arrows preserve the order from left to right, and the second one preserves the elements (i.e., if $\sigma(j) = i$, at the l-th spot of the i-th block of the image you find the element that was at the l-th spot of the j-th block in the preimage).

We also recall the following facts.

Proposition 2.1.1. 1) *For all $\sigma, \tau \in \mathfrak{S}_k$, we have*

$$\mathcal{C}_{p_{(\sigma\circ\tau)^{-1}(1)},\ldots,p_{(\sigma\circ\tau)^{-1}(k)}}(\sigma \circ \tau) = \mathcal{C}_{p_{(\sigma\circ\tau)^{-1}(1)},\ldots,p_{(\sigma\circ\tau)^{-1}(k)}}(\sigma) \circ \mathcal{C}_{p_{\tau^{-1}(1)},\ldots,p_{\tau^{-1}(k)}}(\tau).$$

2) *For all $\sigma \in \mathfrak{S}_k$, $\Phi_1 \in \mathfrak{S}_{p_1}$, $\ldots$, $\Phi_k \in \mathfrak{S}_{p_k}$, we have*

$$\mathcal{C}_{p_{\sigma^{-1}(1)},\ldots,p_{\sigma^{-1}(k)}\circ(\Phi_1\times\cdots\times\Phi_k)=(\Phi_{\sigma^{-1}(1)}\times\cdots\times\Phi_{\sigma^{-1}(k)})} \circ \mathcal{C}_{p_{\sigma^{-1}(1)},\ldots,p_{\sigma^{-1}(k)}}(\sigma)$$

2.2. **Twisted modules and tensor products.** Let R be a ring. A graded R-module X is a collection $(X_n)_{n\in\mathbb{N}}$ of R-modules X_n indexed by non-negative integers n.

Twisted modules. A twisted module M is a graded module together with a right $\mathfrak{S}_n$-action (a right $R(\mathfrak{S}_n)$-module structure on X_n for each n). Morphisms of graded R-modules and of twisted modules are defined as one can imagine; we shall only consider morphisms of degree 0. A twisted module M is connected if $M_0 = 0$. R is canonically given the structure of a twisted module.

Twisted tensor product. The twisted tensor product of k twisted modules $M_1, \ldots, M_k$ is defined by its nth-term

$$(M_1 \otimes \cdots \otimes M_k)_n = \sum_{\substack{p_1+\cdots+p_k=n \\ p_i \geq 0}} ((M_1)_{p_1} \otimes_R \cdots \otimes_R (M_k)_{p_k}) \otimes_{R(\mathfrak{S}_{p_1}\times\ldots\mathfrak{S}_{p_k})} R(\mathfrak{S}_n).$$

Notice that in what follows $M \otimes N$ will always denote this product when M and N are twisted modules. The canonical element $(m_1 \otimes \cdots \otimes m_k) \otimes \sigma$ will be denoted by $(m_1 \otimes \cdots \otimes m_k) \circ \sigma$ unless $\sigma \in \mathfrak{S}_n$ is the identity, in which case we simplify the notation to $m_1 \otimes \cdots \otimes m_k$.

Because of their importance, in the following we specify the notions of associativity and commutativity for tensor products of twisted modules on elements.

Associativity. There is a natural morphism of twisted modules

$$(M_{1,1} \otimes \cdots \otimes M_{1,k_1}) \otimes \cdots \otimes (M_{l,1} \otimes \cdots \otimes M_{l,k_l}) \to M_{1,1} \otimes \cdots \otimes M_{1,k_1} \otimes \cdots \otimes M_{l,1} \otimes \cdots \otimes M_{l,k_l}$$

which sends the element

$$((m_{1,1} \otimes \cdots \otimes m_{1,k_1}) \circ \sigma_1 \otimes \cdots \otimes (m_{l,1} \otimes \cdots \otimes m_{l,k_l}) \circ \sigma_l) \circ \tau$$

to the element

$$(m_{1,1} \otimes \cdots \otimes m_{1,k_1}) \otimes \cdots \otimes (m_{l,1} \otimes \cdots \otimes m_{l,k_l}) \circ (\sigma_1 \times \cdots \times \sigma_l) \circ \tau.$$

Commutativity. Let $\alpha \in \mathfrak{S}_k$. There is an isomorphism of twisted modules

$$\alpha_{\#} : M_1 \otimes \cdots \otimes M_k \to M_{\alpha^{-1}(1)} \otimes \cdots \otimes M_{\alpha^{-1}(k)}$$

which sends the element

$$(m_1 \otimes \cdots \otimes_k) \circ \sigma$$

to the element

$$(m_{\alpha^{-1}(1)} \otimes \cdots \otimes m_{\alpha^{-1}(k)}) \circ \mathcal{C}_{p_{\alpha^{-1}(1)},\ldots p_{\alpha^{-1}(k)}} \circ \sigma,$$

where p_i is the degree of m_i; to avoid too many definitions, we shall often write $p_i = |m_i|$.

We denote by T the swapping map: $(2,1)_{\#} : M \otimes N \to N \otimes M$.

Remark. In the following sections, and to avoid confusions, we shall omit the symbol $\circ$ for denoting the action of $R(\mathfrak{S})$. It will be reserved to the composition of maps.

2.3. **Twisted algebras, coalgebras, bialgebras, Hopf algebras.** We refer to [16] for the definitions of twisted algebras, coalgebras and bialgebras. Formally they reproduce the definition diagrams of the classical case. We go into details only for the case of Hopf algebras because we want to prove a twisted version of a certain useful proposition concerning the antipode.

Definition 2.3.1. A twisted Hopf algebra is a twisted bialgebra A together with a morphism of twisted modules $S : A \to A$ such that following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "\epsilon"] & A \otimes A \arrow[r, "S \otimes A"] & A \otimes A \arrow[dd, "\mu"] \\
 & R \arrow[dr, "\eta"] & \\
A \otimes A \arrow[r, "A \otimes S"] & A \otimes A \arrow[r, "\mu"] & A
\end{tikzcd}$$

where $\epsilon : A \to R$ (respectively $\eta : R \to A$) is the counit (respectively unit) of the coalgebra A.

Convolution. At this point it seems judicious to introduce an operation we shall use very often in the next sections.

Proposition 2.3.2. *Let C be a twisted coalgebra and A a twisted algebra. The set of morphisms of twisted modules $Hom_{R(\mathfrak{S})}(C, A)$ is an associative monoid with product, called the convolution and denoted by $\star$:*

$$f \star g : C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu} A$$

Now, by definition the antipode is the inverse of the identity under the convolution product; it is thus unique. Like in the classical case, there is a canonical way to define an antipode on a twisted connected bialgebra, and thus to give the latter the structure of a twisted Hopf algebra.

We conclude this subsection with a proposition which has not yet been published in the twisted case; here we follow Sweedler [17]. We shall use it in Section 6.

Convention. For any object O of any category, we denote by the same symbol O the identity: $O \to O$.

Proposition 2.3.3. *Let H be a Hopf algebra with antipode S. Then the following statements hold.*

1) $S \circ \pi = \pi \circ T \circ (S \otimes S) : H \otimes H \to H.$
2) $S \circ \eta = \eta : R \to H.$
3) $\epsilon \circ S = \epsilon : H \to R.$
4) $T \circ (S \otimes S) \circ \Delta = \Delta \circ S : H \to H \otimes H.$
5) *The following conditions are equivalent:*
    a) $\pi \circ (S \otimes H) \circ T \circ \Delta = \eta \circ \epsilon : H \to H.$
    b) $\pi \circ (H \otimes S) \circ T \circ \Delta = \eta \circ \epsilon : H \to H.$
    c) $S \circ S = H.$
6) *If H is commutative or cocommutative then $S \circ S = H$.*

Proof. 6) We prove that 5) implies 6).

If H is cocommutative, then $T \circ \Delta = \Delta : H \to H$, and a) merely says that S is the left convolution inverse of the identity H, and equivalently b) says that S is the right convolution inverse of H. Similarly we check $\pi \circ (S \otimes H) \circ T = \pi \circ T \circ (H \otimes S)$ and $\pi \circ (H \otimes S) \circ T = \pi \circ T \circ (S \otimes H)$. If H is commutative, then $\pi \circ T = \pi$, and again we recover that S is the inverse of the identity H. So in each case the conclusion of 6) holds.

1) Consider the three $R(\mathfrak{S})$ linear maps : $M, N, P : H \otimes H \to H$ defined by $M = \pi$, $N = \pi \circ (S \otimes S)$ and $P = S \circ \pi$.

Let us endow $H \otimes H$ with the structure of a coalgebra given by the coalgebra structure of H, and consider $\mathrm{Hom}(H \otimes H, H)$, where the target space is H with its algebra structure. $\mathrm{Hom}(H \otimes H, H)$ is an algebra under the convolution product (cf. Proposition 2.4.3). Let us denote the counit of $H \otimes H$ by ϵ'. Then $\eta \circ \epsilon'$ is the unit of the algebra $\mathrm{Hom}(H \otimes H, H)$. We shall now prove that $P \star M = \eta \circ \epsilon' = M \star N$, that is, that P is a left inverse and N a right inverse of M. As we know, this implies that $P = N$, which had to be proved. Let us now proceed.

Let us make some abuse of notation: if $x \in A \otimes B$ we write $x = a \otimes b$ for the expression that, more accurately, should be written as $x = \sum_{i \in I} a^i \otimes b^i$. Similarly, if Δ is a coproduct of C, let us write $\Delta(c) = c_{(1)} \otimes c_{(2)}$ for $\Delta(c) = \sum_{i \in I} c_1^i \otimes c_2^i$.

Let g, h be two elements of H. We calculate first:

$$\begin{aligned}
P \star M(g \otimes h) &= P((g \otimes h)_{(1)}) M((g \otimes h)_{(2)}) \\
&= P(g_{(1)} \otimes h_{(1)}) M((g_{(2)} \otimes h_{(2)}) \mathcal{C}_{|g_{(1)}|,|h_{(1)}|,|g_{(2)}|,|h_{(2)}|}((1,3,2,4)) \\
&= S(g_{(1)} h_{(1)}) g_{(2)} h_{(2)} \mathcal{C}_{|g_{(1)}|,|h_{(1)}|,|g_{(2)}|,|h_{(2)}|}((1,3,2,4)) \\
&= S((gh)_{(1)})(gh)_{(2)}.
\end{aligned}$$

Remark. For convenience, when the symmetric group $\mathfrak{S}_n$ acts on a factor of degree n of an element a of degree N, $n \leq N$, we shall write this as an action of $\mathfrak{S}_N$ on a via the corresponding canonical embedding $\mathfrak{S}_n \subset \mathfrak{S}_N$.

The last equation uses the statement that Δ is an algebra map, i.e., that

$$\Delta \circ M(g \otimes h) = (gh)_{(1)} \otimes (gh)_{(2)} = g_{(1)} h_{(1)} \otimes g_{(2)} h_{(2)} \mathcal{C}_{|h_{(1}|,|g_{(2)}|}((1,3,2,4)).$$

The last term is $(S \star H)(gh)$ (and thus $\epsilon(gh)$ or $\epsilon(g)\epsilon(h)$) by definition.

On the other hand, we have

$$\begin{aligned}
M \star N(g \otimes h) &= M(g_{(1)} \otimes h_{(1)}) N(g_{(2)} \otimes h_{(2)}) \mathcal{C}_{|g_{(1)}|,|h_{(1)}|,|g_{(2)}|,|h_{(2)}|}((1,3,2,4)) \\
&= g_{(1)} h_{(1)} S(h_{(2)}) S(g_{(2)}) \mathcal{C}_{|g_{(1)}|,|h_{(1)}|,|h_{(2)}|,|g_{(2)}|}((1,2,4,3)) \mathcal{C}_{|g_{(1)}|,|h_{(1)}|,|g_{(2)}|,|h_{(2)}|}((1,3,2,4)) \\
&= g_{(1)} \epsilon(h) S((g)_{(2)}) \mathcal{C}_{|g_{(1)}|,|h|,|g_{(2)}|}((1,3,2)),
\end{aligned}$$

where we used $H \star S(h) = \epsilon(h)$ by definition of S. Since the above factor is 0 if h is not a coefficient, we can continue this computation by

$$\begin{aligned}
M \star N(g \otimes h) &= g_{(1)} S((g)_{(2)}) \epsilon(h) \\
&= \epsilon(g)\epsilon(h),
\end{aligned}$$

where we used $H \star S(g) = \epsilon(g)$, again by definition of S. This ends the proof of 1).

2) and 3) are straightforward and similar to the classical case.

4) can be proved similarly to 1).

5) a) $\Rightarrow$ c) It suffices to prove that $S \circ S$ is a convolution inverse to S itself like the identity. Let us proceed:

$$\begin{aligned}
S \star (S \circ S)(g) &= S(g_{(1)})(S \circ S)(g_{(2)}) \\
&= S(S(g_{(2)}) g_{(1)}) \mathcal{C}_{|g_{(2)}|,|g_{(1)}|}((2,1)) \\
&= S \circ \epsilon(g) \\
&= \epsilon(g).
\end{aligned}$$

Here, the second line follows by part 1) of this proposition, the third line by hypothesis, and the last line by part 2) of this proposition.

c)$\Rightarrow$ b) We have

$$\begin{aligned}
\eta\epsilon(g) = H \star S(g) &= g_{(1)} S(g_{(2)}) \\
&= (S \circ S)(g_{(1)}) S(g_{(2)}) \\
&= S(g_{(2)} S(g_{(1)}) \mathcal{C}_{|g_{(2}|,|g_{(1)}|}((2,1)).
\end{aligned}$$

Here, the second line follows by hypothesis, while the last line follows by part 1) of this proposition.

Applying S to both sides, we get

$$S \circ \eta \circ \epsilon(g) = (S \circ S)(g_{(2)} S(g_{(1)}))\mathcal{C}_{|g_{(2}|,|g_{(1)}|}((2,1)) = g_{(2)} S(g_{(1)}))\mathcal{C}_{|g_{(2)}|,|g_{(1)}|}((2,1)).$$

Then b) follows by 2) of the proposition.

The other implications can be proved along the same lines. This completes the proof of the proposition. □

2.4. **Lie algebras.**

Definition 2.4.1. A twisted Lie algebra L is a twisted module together with a morphism of twisted modules $\beta : L \otimes L \to L$, called the bracket, which satisfies the traditional anticommutativity

$$\beta + \beta \circ T = 0 \quad \text{in } \mathrm{Hom}_{R(\mathfrak{S})}(L \otimes L, L)$$

and the Jacobi identities

$$\beta \circ (\beta \otimes L) + \beta \circ (\beta \otimes L) \circ (2,3,1)_{\#} + \beta \circ (\beta \otimes L) \circ (2,3,1)_{\#}^2 = 0 \quad \text{in } \mathrm{Hom}_{R(\mathfrak{S})}(L \otimes L \otimes L, L),$$

where $(2,3,1)_{\#}$ acts on $L \otimes L \otimes L$ by $x \otimes y \otimes z \mapsto y \otimes z \otimes x$.

Let us be redundant and transcribe this definition on elements. As usual we write the bracket β in the form $\beta = [\ ,\ ]$, and the identities are written with explicit elements $u_i \in L_{p_i}$ for $i = 1,2,3$ as

$$[u_1, u_2] = [u_2, u_1] \circ \mathcal{C}_{p_2,p_1}((2,1))$$

and

$$[[u_1, u_2], u_3] + [[u_2, u_3], u_1]\mathcal{C}_{p_2,p_3,p_1}((2,3,1)) + [[u_3, u_1], u_2]\mathcal{C}_{p_3,p_1,p_2}((3,1,2)) = 0.$$

As in the classical case, we can define a Lie bracket on each twisted algebra A by $\beta = \mu - \mu \circ T$, or, on elements, by $[x, y] = xy - yx\mathcal{C}_{q,p}((2,1))$ for x and y elements of A of respective degrees p and q.

We conclude here the review on generalities about twisted algebraic structures. It gives a convenient framework to understand the notations of the coming sections. The paper of Stover [16] continues with enveloping algebras and the Milnor–Moore theorem, to which we shall not refer for the moment.

Let us now try to fix some ideas about free twisted objects.

3. Free twisted objects and twisted Lie polynomials

In this section, we follow again Stover [16] and adapt the first chapter of Reutenauer's book [15] to the case of twisted structures.

First let us recall some basic definitions and properties of free monoids. Here no twisting occurs, and we shall be brief.

3.1. **Words and free monoids.** Let X be a set, finite or infinite, and denote by x or x_i, $i \in X$, its elements. A juxtaposition (or concatenation) of a finite number of ordered letters, e.g., $x_1x_2 \ldots x_n$, is called a word. The collection of all words generated by X, denoted by $W(X)$, comes with an obvious embedding of sets $\imath : X \to W(X)$. Moreover $W(X)$ admits a product, called the concatenation product, defined as in the following example: $(x_1 \ldots x_n)(x'_1 \ldots x'_{n'}) = x_1 \ldots x_n x'_1 \ldots x'_{n'}$. $W(X)$ with this product is a free monoid. This definition is justified by the following result.

Proposition 3.1.1. *For any monoid M and any map of sets $f : X \to M$, there is a unique map of monoids $\overline{f} : W(X) \to M$ such that the following diagram in the category of sets commutes:*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\imath"'] & & M \\ & W(X) \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

3.2. **Definitions of free twisted objects and polynomials.** Let X be a graded set (each $x \in X$ is equipped with a positive integer $|x|$ called the degree) and R be a ring. The twisted free module over R generated by X is any twisted module isomorphic to $\bigoplus_{x\in X} xR(\mathfrak{S}_{|x|})$ and is denoted by $R(\mathfrak{S})(X)$. Again, there is an obvious embedding of sets $\imath : X \to R(\mathfrak{S})(X)$.

Proposition 3.2.1. *For any twisted module M and any map of graded sets $f : X \to M$, there is a unique map of twisted module $\overline{f} : R(\mathfrak{S})(X) \to M$ such that the following diagram in the category of graded sets commutes:*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\imath"'] & & M \\ & R(\mathfrak{S})(X) \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

Proof. Given any element $x_1r_1 + \cdots + x_nr_n$, $x_i \in X$, $r_i \in R(\mathfrak{S})$, the commutation of the diagram implies that $\overline{f}(x_i) = f(x_i)$ and, by linearity, that

$$\overline{f}(x_1r_1 + \cdots + x_nr_n) = f(x_1)r_1 + \cdots + f(x_n)r_n.$$

Thus $\overline{f}$, if existing, is unique. Moreover the preceding formula is precisely a definition of $\overline{f}$ once f is given. □

Now let M be a twisted module over R. Let us denote by $M^{\otimes n}$ the twisted module given by the tensor product of n copies of M, and by $T(M)$ the direct sum $\bigoplus_{n>1} M^{\otimes n}$ (see Subsection 2.2 for the definition of the twisted tensor product). The associativity formula of Subsection 2.2 defines a (product) map $M^{\otimes n} \otimes M^{\otimes m} \to M^{\otimes(n+m)}$ which, by linearity, extends to $T(M)$ and endows it with a structure of an associative twisted algebra. This is called the free twisted (associative) algebra generated by the twisted module M. There is an obvious embedding of twisted modules $\imath : M \to T(M)$. This terminology is justified by the following result.

Proposition 3.2.2. *For any twisted (associative) algebra A and any map of twisted modules $f : M \to A$, there is a unique map of twisted algebras $\overline{f} : T(M) \to A$ such that the following diagram in the category of twisted modules commutes:*

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "\imath"'] & & A \\ & T(M) \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

Proof. Given an element $(m_1 \otimes \cdots \otimes m_i)\sigma$, define

$$\overline{f}((m_1 \otimes \cdots \otimes m_i)\sigma) = f(m_1) \otimes \cdots \otimes f(m_i) \otimes \sigma.$$

A straightforward inspection shows that

$$\begin{aligned} \overline{f}((m_1\sigma_1 \otimes \cdots \otimes m_i\sigma_i)\sigma) &= f(m_1)\sigma_1 \otimes \cdots \otimes f(m_i)\sigma_i \otimes \sigma \\ &= f(m_1) \otimes \cdots \otimes f(m_i)(\sigma_i \times \sigma_1 \times \cdots \times \sigma_i)\sigma. \end{aligned}$$

This proves, first, that $\overline{f}$ is well-defined on $T(M)$ as a twisted module map and, secondly, that $\overline{f}$ is multiplicative. Moreover, and by definition, $\overline{f} = f$ on the twisted module M. This completes the proof. □

We briefly pause here to emphasize an important point we shall only use in the next subsection. Consider the map of twisted modules $\Delta : M \to T(M) \otimes T(M)$ given by $\Delta(m) = m \otimes 1 + 1 \otimes m$, and extend it to obtain a map of twisted algebras $\Delta : T(M) \to T(M) \otimes T(M)$. This process endows $T(M)$ with a structure of a twisted bialgebra. Actually, this bialgebra is connected; indeed it is easy to check that the anti-automorphism $S : T(M) \to T(M)$ defined by $S(m) = -m$ (just apply the universal property of Proposition 3.2.2 to the algebra opposite to $T(M)$) satisfies the axioms of an antipode for $T(M)$. In other words, we just defined the structure of a twisted Hopf algebra for $T(M)$.

Now let us specialize to the free twisted module generated by a graded set X. Let us denote by $\mathcal{F}(X)$ the free twisted associative algebra $T(R(\mathfrak{S})(X))$. Combining Propositions 3.2.1 and 3.2.2 , we readily obtain the following result.

Proposition 3.2.3. *For any twisted associative algebra A and any map of graded sets $f : X \to A$, there is a unique map of twisted algebras $\overline{f} : \mathcal{F}(X) \to A$ such that the following diagram in the category of graded sets commutes:*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\imath"'] & & A \\ & \mathcal{F}(X) \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

We end this subsection by introducing polynomials in the twisted case. A typical element of $R(\mathfrak{S})(X)$ may be written as $\sum_{i\in I} x_i \circ \sigma_i$, for a finite indexing set I. Thus $\mathcal{F}(X)$ is linearly generated by elements of the type $\otimes_{j\in J} x_j$, where J runs through all finite tuples of elements of X. Such an element is also written $x_1 \dots x_j$ for a j-tuple $(x_1, \dots, x_j)$, and it is called a monomial of $\mathcal{F}(X)$. In the algebra $\mathcal{F}(X)$ the product of polynomials follows the rules of the product in a free twisted algebra discussed in Subsection 2.2:

$$\begin{aligned}
&((x_{1,1}\sigma_{1,1} \otimes \cdots \otimes x_{1,k}\sigma_{1,k})\tau_1 \times (x_{2,1}\sigma_{2,1} \otimes \cdots \otimes x_{2,l})\sigma_{2,l})\tau_2 \\
&= (x_{1,1}\sigma_{1,1} \otimes \cdots \otimes x_{1,k}\sigma_{1,k}) \otimes (x_{2,1}\sigma_{2,1} \otimes \cdots \otimes x_{2,l}\sigma_{2,l})\tau_1 \times \tau_2 \\
&= (x_{1,1} \otimes \cdots \otimes x_{1,k} \otimes x_{2,1} \otimes \cdots \otimes x_{2,l})(\sigma_{1,1} \times \cdots \times \sigma_{1,k} \times \sigma_{2,1} \times \cdots \times \sigma_{2,l})(\tau_1 \times \tau_2).
\end{aligned}$$

Notation. We denote by $T_R(M)(\subset T(M))$ the obvious embedding (of R-modules) of the classical free tensor algebra generated by M over R in the twisted one; i.e., a typical element of $T_R(M)$ is $m_1 \otimes \cdots \otimes m_k$, where the $m_1, \dots, m_k$ are elements of M.

3.3. **Free twisted Lie algebras and Lie polynomials.** We refer here again to Stover [16], especially for the proofs.

Consider a non-associative abstract operation on symbols, and write it as a bracketing. Starting with a unique symbol — say x — the bracketing operation gives rise to an infinite set $\mathcal{N}(x)$ — the free non-associative monoid generated by x. Given a twisted module M and an element b of $\mathcal{N}(x)$, we define $M^{\otimes b}$ as the twisted module $M^{\otimes \# b}$, where $\# b$ denotes the number of occurrences of x in b, and the twisted structure is similar to the twisted structure of the ordinary tensor product. The bracketing operation in the monoid $\mathcal{N}(x)$ induces an obvious bracketing operation $M^{\otimes b} \otimes M^{\otimes c} \to M^{\otimes (bc)}$. Let us define the twisted module $\mathcal{T}(M) = \bigoplus_{b\in\mathcal{N}(x)} M^{\otimes b}$. The bracketing operation just defined extends to $\mathcal{T}(M)$ by linearity. Call it β.

Define $\mathcal{I}(M)$ as the two sided twisted ideal of $\mathcal{T}(M)$ generated by the images of

$$\beta + \beta \circ (2,1) : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$$

$$\begin{aligned}
&\beta \circ (\beta \times \mathcal{I}(M)) + \beta \circ (\beta \times \mathcal{I}(M))(2,3,1) \\
&\qquad + \beta \circ (\beta \times \mathcal{I}(M))(2,3,1)^2 : \mathcal{T}(M) \times \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M).
\end{aligned}$$

The quotient $\mathcal{L}(M) = \mathcal{T}(M)/\mathcal{I}(M)$ is equipped with the map induced by β (denoted as usual by [,]), and it is called the free twisted Lie algebra generated by M. This denomination is justified by the following proposition proved by Stover [16]. There is an obvious embedding of twisted modules $\imath : M \to \mathcal{L}(M)$.

Proposition 3.3.1. *For any twisted Lie algebra L and any map of twisted modules $f : M \to L$, there is a unique map of twisted Lie algebra $\overline{f} : \mathcal{L}(M) \to L$ such that the following diagram in the category of twisted modules commutes:*

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "\imath"'] & & L \\ & \mathcal{L}(M) \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

Let us end this subsection with some lines on twisted universal algebras.

If L is a twisted Lie algebra, consider $T(L)$, the free (associative) twisted algebra generated by the twisted module L, with the linear embedding $\imath : L \to T(L)$. Now let IL be the two-sided twisted Lie ideal generated in $T(L)$ by elements of the form $[\imath(x), \imath(y)] - \imath[x, y]$.

The enveloping algebra of L is the quotient (associative) algebra $T(L)/I(L)$. We denote it by UL. It satisfies the following property.

Proposition 3.3.2. *For any twisted Lie algebra L and any map of twisted Lie algebras $f : L \to A$, there is a unique map of twisted algebras $\overline{f} : UL \to A$ such that the following diagram in the category of twisted Lie algebras commutes:*

$$\begin{tikzcd} L \arrow[rr, "f"] \arrow[dr, "\imath"'] & & A \\ & UL \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

Now we can phrase the twisted version of Milnor–Moore theorem given in [16].

Proposition 3.3.3. *Let M be a twisted module. The twisted algebra map*

$$T(M) \to U\mathcal{L}(M)$$

induced by the composition of maps of twisted modules

$$M \to \mathcal{L}(M) \to U\mathcal{L}(M)$$

is an isomorphism.

As in the preceding subsection, we can introduce Lie polynomials. Recall that a typical element of $R(\mathfrak{S})(X)$ may be written in the form $\sum_{i \in I} x_i \sigma_i$, for an indexing set I, and that a (non-commutative) polynomial in $\mathcal{F}(X)$ is a linear combination of elements such as $x_1\sigma_1 \otimes \cdots \otimes x_i\sigma_j$. By Subsection 2.4 and Proposition 3.3.1, we define an embedding of twisted Lie algebras: $\mathcal{L}(X) \subset \mathcal{F}(X)$. A polynomial in $\mathcal{F}(X)$ is called a Lie polynomial if it is in the image of $\mathcal{L}(X)$ by this embedding.

Remark. If we suppose that all elements of X are of degree 1, we recover Barratt's definition of free twisted (associative) algebras and Lie algebras.

4. Characterization of Lie polynomials

This section and the following ones examine the notion of Dynkin idempotent. The last one gives a more abstract and general presentation. However, when we shall be facing the problem of finding a basis for the free twisted Lie algebra, we shall need to work more concretely with elements.

So, as a first result, we prove Friedrich's theorem in the twisted case. We partly follow Waldenfels [18] and rewrite the proofs in the context of the free twisted Lie algebras; we shall also use Reutenauer's presentation [15]. We emphasize that all the characteristic properties of the "classical" Lie polynomials remain valid in the twisted version. From now on the ground ring (R in the preceding section) is a a field $\mathbb{F}$ of characteristic 0, and so twisted algebraic objects are $\mathbb{F}(\mathfrak{S})$-modules with possibly more structure.

Let us first introduce some definitions in the framework of twisted algebras.

1. If A is a twisted module (in the following A will always be an algebra), we define an endomorphism D by $D(a) = na$ for any (homogeneous) element a of degree n.
2. If A is a twisted algebra and $a \in A$, we define a map of $\mathbb{F}$-modules $ad(\ ) : A \to \mathrm{E}nd_{\mathbb{F}(\mathfrak{S})}(A)$ by $ad(a)(b) = ab - ba\mathcal{C}_{|b|,|a|}((2,1))$ for any $b \in A$.
3. If M is a twisted module, we define a map of $\mathbb{F}$-modules $Ad(\ ) : T_{\mathbb{F}}(M) \to \mathrm{E}nd_{\mathbb{F}(\mathfrak{S})}(T(M))$ by $Ad(m_1 \otimes \cdots \otimes m_k)(p) = [m_1, [\dots[m_k, p]\dots]]$, where $T_{\mathbb{F}}(M)$ is defined before Subsection 3.3.

Notation. For elements $x_i \in X$, $i \in I$, we abbreviate $[\dots[x_1, x_2], \dots, x_n]$ to $[x_1 x_2 \dots x_n]$ in $\mathcal{L}(X)$.

Theorem 4.0.4. *For a polynomial P of $\mathcal{F}(X)$ the following statements are equivalent:*

i) *P is a Lie polynomial,*
iii) *$(S \otimes \mathcal{F}(X))(\Delta)(P) = -P \otimes 1 + 1 \otimes P$,*
iv) *$\Delta(P) = P \otimes 1 + 1 \otimes P$,*
v) *$P(0) = 0$ and $D(P) = [P]$.*

In the case where $P \in T_{\mathbb{F}}(M)$ i), iii), iv), v) are also equivalent to

ii) *$ad(P) = Ad(P)$ and $P(0) = 0$.*

For any free twisted module M, let us define the $\mathbb{F}$-linear mapping

$$\zeta : T_{\mathbb{F}}(M) \otimes_{\mathbb{F}} T_{\mathbb{F}}(M) \to \mathrm{E}nd_{\mathbb{F}(\mathfrak{S})}(T(M))$$
$$\zeta(P_1 \otimes P_2))(Q) = P_1 \otimes Q \otimes P_2 \mathcal{C}_{|P_1|,|Q|,|P_2|}((1,3,2))$$

for any monomials P_1, P_2 and Q. The main arguments for proving the theorem are contained in the following two lemmas.

Lemma 4.0.5. i) *For any polynomial P, the following identities are satisfied:*

$$ad(P) = \zeta(-P \otimes 1 + 1 \otimes P) \text{ and } Ad(P) = \zeta(T(M) \otimes S)\Delta(P).$$

ii) ζ *is injective.*

Proof. i) The first identity only reflects the definitions of ad and ζ.

The second one is also clear for monomials of degree 1 (elements of $M \subset T_{\mathbb{F}}(M)$). By definition, Δ is multiplicative. So let us prove that $\zeta \circ (S \otimes T(M))$ is also multiplicative, and we will be done for part i) of the lemma. We compute (P_1, P_2, Q are monomials)

$$\begin{aligned}
&\zeta(T_{\mathbb{F}}(M) \otimes S)((U_1 \otimes U_2)(V_1 \otimes V_2))(Q) \\
&\quad = \zeta(T_{\mathbb{F}}(M) \otimes S)(U_1V_1 \otimes U_2V_2)(Q)\mathcal{C}_{|U_1|,|V_1|,|U_2|,|V_2|,|Q|}((1,3,2,4,5)) \\
&\quad = \zeta(U_1V_1 \otimes S(V_2)S(U_2))(Q)\mathcal{C}_{|U_1|,|V_1|,|V_2|,|U_2|,|Q|}((1,2,4,3,5) \\
&\qquad\qquad\qquad\qquad \mathcal{C}_{|U_1|,|V_1|,|U_2|,|V_2|,|Q|}((1,3,2,4,5)) \\
&\quad = U_1V_1QS(V_2)S(U_2)\mathcal{C}_{|U_1|,|V_1|,|Q|,|V_2|,|U_2|}((1,2,5,3,4)) \\
&\qquad\quad \mathcal{C}_{|U_1|,|V_1|,|V_2|,|U_2|,|Q|}((1,2,4,3,5)\mathcal{C}_{|U_1|,|V_1|,|U_2|,|V_2|,|Q|}((1,3,2,4,5)) \\
&\quad = \zeta(U_1 \otimes S(U_2))(V_1QS(V_2))\mathcal{C}_{|U_1|,|U_2|,|V_1|,|Q|,|V_2|}((1,5,2,3,4)) \\
&\qquad\quad \mathcal{C}_{|V_1|,|U_1|,|U_2|,|V_2|,|Q|}((1,2,5,3,4)) \\
&\qquad\quad \mathcal{C}_{|U_1|,|V_1|,|V_2|,|U_2|,|Q|}((1,2,4,3,5)\mathcal{C}_{|U_1|,|V_1|,|U_2|,|V_2|,|Q|}((1,3,2,4,5)) \\
&\quad = \zeta(U_1 \otimes S(U_2))\zeta(V_1 \otimes S(V_2))(Q) \\
&\qquad\quad \mathcal{C}_{|U_1|,|U_2|,|V_1|,|V_2|,|Q|}((1,2,3,5,4))\mathcal{C}_{|V_1|,|V_2|,|U_1|,|Q|,|U_2|}((1,5,2,3,4)) \\
&\qquad\quad \mathcal{C}_{|V_1|,|U_1|,|U_2|,|V_2|,|Q|}(1,2,5,3,4)) \\
&\qquad\quad \mathcal{C}_{|U_1|,|V_1|,|V_2|,|U_2|,|Q|}((1,2,4,3,5)\mathcal{C}_{|U_1|,|V_1|,|U_2|,|V_2|,|Q|}((1,3,2,4,5)) \\
&\quad = \zeta(T_{\mathbb{F}}(M) \otimes S)(V_1 \otimes V_2)\zeta(T_{\mathbb{F}}(M) \otimes S)(U_1 \otimes U_2)(Q),
\end{aligned}$$

where the last identity only reflects the obvious composition of permutations.

ii) Write $M = \mathbb{F}(\mathfrak{S}(X))$. When X is not a singleton, the proof runs exactly as in [15]. Let us recall it for completeness. Following our notation introduced after Proposition 3.2.3, a typical element of $T_{\mathbb{F}}(M)$ is $x_1 \circ \sigma_1 \otimes \cdots \otimes x_l \circ \sigma_l$, where $x_k \in X$ and $\sigma_k \in \mathbb{F}(\mathfrak{S}_k))$, for $k = 1, \ldots, l$. In the twisted algebra $T(M)$, it should be written $x_1 \otimes \cdots \otimes x_l \circ (\sigma_1 \times \cdots \times \sigma_l)$ or $u \circ \sigma$ in abridged form. Thus we can write an element of $T_{\mathbb{F}}(M) \otimes_{\mathbb{F}} T_{\mathbb{F}}(M)$ as a finite linear combination of homogeneous degree: $\sum_i u_i \circ \sigma_i \otimes v_i \circ \tau_i$, where u_i and v_i are words in X. Suppose u_1 is of minimal length among all the u_i; choose two distinct letters x and y in X and an integer N strictly greater than the length of u_1. Let us now examine $\zeta(\sum_i u_i \circ \sigma_i \otimes v_i \circ \tau_i)(x^N y)$ (call this element Σ for convenience). Focus our attention on $u_1 \circ \sigma_1 \otimes x^N y \otimes v_i \circ \tau_i \mathcal{C}_{|u_i|,|x^N y|,|v_j|}((1,3,2))$ in Σ. The assertion $\Sigma = 0$ implies that there is an index $i \neq 1$ with $u_1 x^N y v_1 = u_i x^N y v_i$. Because of the minimality of u_1, u_1 is a left factor of u_i, and we can write $u_1 w = u_i$, from which we deduce $x^N y v_1 = w x^N y v_i$. As N is large, this implies that w is a left factor of x^N, say $w = x^n$, for $n \leq N$. So the above equation turns into $x^N y v_1 = x^{n+N} y v_i$; consequently $n = 0$, $u_1 = u_i$ (and thus $v_1 = v_i$) in contradiction with our hypothesis that all pairs (u_i, v_i) are pairwise distinct.

Suppose now that $X = \{x\}$. An element in degree $N|x|$ of $T_{\mathbb{F}}(M) \otimes_{\mathbb{F}} T_{\mathbb{F}}(M)$ is a finite linear combination $\sum_{n,N} x^n \circ \sigma_n \otimes x^{N-n} \circ \tau_{N-n}$ (σ_n and τ_{N-n} are supposed to be non-zero in $\mathbb{F}(\mathfrak{S})$). We want to prove that $\zeta(\sum_n x^n \circ \sigma_n \otimes x^{N-n} \circ \tau_{N-n})(x)$ is non-zero (again, let us call this sum Σ, for convenience),

$$\Sigma = x^{N+1} \sum_n (\sigma_n \times 1_x \times \tau_{N-n}) \mathcal{C}_{n|x|,|x|,(N-n)|x|}((1,3,2)).$$

For two different n's the corresponding block permutations are different and remain different after left multiplication by any $\sigma_n \times 1_x \times \tau_{N-n}$. This observation proves that $\Sigma \neq 0$. $\square$

Lemma 4.0.6. *Let μ and $\lambda : \mathcal{F}(X) \otimes \mathcal{F}(X) \to \mathcal{F}(X)$ be the $\mathbb{F}(\mathfrak{S})$-module morphisms $\mu = \pi \circ (S \otimes \mathcal{F}(X))$ and $\lambda = \pi \circ (S \otimes D)$. Then*

$$\mu \circ \Delta(P) = P(0)$$

and

$$\lambda \circ \Delta(P) = [P]$$

Proof. First it is immediate to check that

$$\Delta 1 = 1 \otimes 1; \text{ whence } \mu \circ \Delta(1) = 1 = 1(0) \text{ and } \lambda \circ \Delta(1) = 0 = [1].$$

Let us now examine the case of a non-constant polynomial. By linearity, it is enough to consider the monomial $P = x_1 \dots x_n$.

Let us proceed by induction. The step $n = 1$ is obvious. So, let us suppose that both equations of the lemma are established until step n. We now examine step $n+1$.

Let us write $\Delta(x_1 \dots x_n) = f_i \otimes g_i$ (as above, this is an abbreviation for $\sum_i f_i \otimes g_i$). By induction hypothesis, we have

$$S(f_i)D(g_i) = [x_1 \dots x_n] \text{ and } S(f_i)g_i = 0$$

We calculate first:

$$\begin{aligned}
&(S \otimes \mathcal{F}(X)) \circ \Delta(x_1 \dots x_n x_{n+1}) \\
&\quad = (S \otimes \mathcal{F}(X))(f_i \otimes g_i)(x_{n+1} \otimes 1 + 1 \otimes x_{n+1}) \\
&\quad = (S \otimes \mathcal{F}(X))(f_i x_{n+1} \otimes g_i \mathcal{C}_{|f_i|,|x_{n+1}|,|g_i|}((1,3,2)) + f_i \otimes g_i x_{n+1}) \\
&\quad = (S(x_{n+1})S(f_i) \otimes g_i \mathcal{C}_{|x_{n+1}|,|f_i|,|g_i|}((2,1,3)) \mathcal{C}_{|f_i|,|x_{n+1}|,|g_i|}((1,3,2)) + S(f_i) \otimes g_i x_{n+1}).
\end{aligned}$$

We immediately observe that

$$\mathcal{C}_{|x_{n+1}|,|f_i|,|g_i|}((2,1,3)) \mathcal{C}_{|f_i|,|x_{n+1}|,|g_i|}((1,3,2)) = \mathcal{C}_{|x_{n+1}|,|f_i|+|g_i|}((3,1,2)).$$

Hence, we have

$$\mu(x_1 \dots x_n x_{n+1}) = S(x_{n+1} S(f_i) g_i \mathcal{C}_{|x_{n+1}|,|f_i|+|g_i|}((3,1,2)) + (f_i g_i) x_{n+1} = 0$$

and

$$\begin{aligned}
\lambda(x_1 \dots x_n x_{n+1}) &= -S(x_{n+1}) f_i D(g_i) \mathcal{C}_{|x_{n+1}|,|f_i|+|g_i|}((3,1,2)) + S(f_i) D(g_i x_{n+1}) \\
&= -x_{n+1}(f_i D(g_i)) \mathcal{C}_{|x_{n+1}|,|f_i|+|g_i|}((3,1,2)) + (S(f_i) D(g_i)) x_{n+1} \\
&\qquad\qquad + (S(f_i) g_i) x_{n+1} \\
&= [[x_1 \dots x_n], x_{n+1}].
\end{aligned}$$

Here we used the induction hypothesis on μ to cancel the third term and the induction hypothesis on λ to form the Lie bracket out of the first two terms. This establishes the $(n+1)$-step. $\square$

Let us now turn to the proof of Theorem 4.0.4.

Proof of Theorem 4.0.4. The implications (i) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv) are straightforward (see [18]); they just reformulate that Lie elements are primitive.

Concerning (iv) $\Rightarrow$(v), let us apply $\lambda = \pi \circ (S \otimes D)$ to both sides of $\Delta(P) = P \otimes 1 + 1 \otimes P$. As $D(1) = 0$, Lemma 4.0.6 gives $[P] = D(P)$. Similarly, apply $\mu = \pi \circ (S \otimes \mathcal{F}(X))$. We obtain $P(0) = S(P) + P$. Obviously, S is the identity on the constant $P(0)$, which leads to $P(0) = 0$.

As to (iii)$\Rightarrow$ (v), let us apply $S \otimes \mathcal{F}(X)$ to both sides of equation iii). As $S \circ S = \mathcal{F}(X)$, we get $\Delta(P) = -S(P) \otimes 1 + 1 \otimes P$.

To begin with, applying now μ to both sides, we obtain $P(0) = 0$, with the help of Lemma 4.0.6. Applying next λ, we obtain $[P] = 0$, by using the same arguments as in the proof of the implication iv) $\Rightarrow$ v).

For (v) $\Rightarrow$(i) we just observe that it suffices to prove (i) for every homogeneous $f \in \mathcal{F}(X)$. Thus assume f is of degree $n > 0$ (the case $n = 0$ is obvious). Then by hypothesis $D(f) = nf = [f]$, which is equivalent to $f = [f]/n$ as the characteristic of $\mathbb{F}$ is 0.

Let us now examine the particular case (ii).

(i)$\Rightarrow$ (ii). As observed above, (ii) is satisfied by the elements of X. Thus, by induction, it suffices to prove that (ii) is closed under the Lie bracket. We notice that the bracket operations distribute the sum of each factor, so we may examine the induction on monomials instead of polynomials.

Suppose that we are given two monomials P_1, P_2 in $T_{\mathbb{F}}(M)$ such that $ad(P_i) = Ad(P_i)$, $i = 1, 2$.

For any monomial Q, we have

$$\begin{aligned} ad([P_1, P_2])(Q) &= ad(P_1P_2 - P_2P_1\mathcal{C}_{|P_2|,|P_1|}((2,1))Q) \\ &= P_1P_2Q - P_2P_1Q\mathcal{C}_{|P_2|,|P_1|,|Q|}((2,1,3)) - QP_1P_2\mathcal{C}_{|P_1|,|P_2|,|Q|}((3,1,2)) \\ &+ QP_2P_1\mathcal{C}_{|Q|,|P_2|,|P_1|}((3,1,2))\mathcal{C}_{|P_2|,|P_1|,|Q|}((2,1,3)) \end{aligned}$$

and

$$
\begin{aligned}
&Ad([P_1, P_2])(Q)\\
&\qquad = Ad(P_1P_2 - P_2P_1\mathcal{C}_{|P_2|,|P_1|}((2,1)))(Q)\\
&\qquad = (Ad(P_1)\circ Ad(P_2) - Ad(P_2)\circ Ad(P_1)\mathcal{C}_{|P_2|,|P_1|}((2,1)))(Q)\\
&\qquad = (ad(P_1)\circ ad(P_2) - ad(P_2)\circ ad(P_1)\mathcal{C}_{|P_2|,|P_1|}((2,1)))(Q)\\
&\qquad = [P_1,[P_2,Q]] - [P_2,[P_1,Q]]\mathcal{C}_{|P_2|,|P_1|,|Q|}((2,1,3))\\
&\qquad = P_1P_2Q - P_1QP_2\mathcal{C}_{|P_1|,|Q|,|P_2|}((1,3,2)) - P_2QP_1\mathcal{C}_{|P_2|,|Q|,|P_1|}((2,3,1))\\
&\qquad + QP_2P_1\mathcal{C}_{|P_2|,|P_1|,|Q|}((2,1,3))\mathcal{C}_{|Q|,|P_2|,|P_1|}((2,3,1))\\
&\qquad P_2P_1Q\mathcal{C}_{|P_2|,|P_1|,|Q|}((2,1,3)) - P_2QP_1\mathcal{C}_{|P_2|,|Q|,|P_1|}((1,3,2))\mathcal{C}_{|P_2|,|P_1|,|Q|}((2,1,3))\\
&\qquad - P_1QP_2\mathcal{C}_{|P_1|,|Q|,|P_2|}((1,3,2))\mathcal{C}_{|P_2|,|P_1|,|Q|}((2,1,3))\\
&\qquad + QP_1P_2\mathcal{C}_{|Q|,|P_1|,|P_2|}((2,3,1))\mathcal{C}_{|P_2|,|Q|,|P_1|}((1,3,2))\mathcal{C}_{|P_2|,|P_1|,|Q|}((2,1,3))\\
&\qquad = P_1P_2Q - P_2P_1Q\mathcal{C}_{|P_2|,|P_1|,|Q|}((2,1,3)) - QP_1P_2\mathcal{C}_{|P_1|,|P_2|,|Q|}((3,1,2))\\
&\qquad + QP_2P_1\mathcal{C}_{|Q|,|P_2|,|P_1|}((3,2,1)),
\end{aligned}
$$

which proves the identity $ad([P_1, P_2])(Q) = Ad([P_1, P_2])(Q)$ and the induction.

(ii)⇒ (iv) By hypothesis and Lemma 4.0.5 (i), we have

$$\zeta(-P \otimes 1 + 1 \otimes P) = \zeta(S \otimes \mathcal{F}(X)) \circ \Delta(P).$$

The injectivity of ζ (part (ii) of the same lemma) gives

$$-P \otimes 1 + 1 \otimes P = (S \otimes \mathcal{F}(X)) \circ \Delta(P).$$

□

5. Pseudo-coproduct

As we already saw in the preceding sections, when dealing with twisted structures we have only to pay attention to the action of the symmetric group. Most definitions and proofs seem to be just a rewriting of the classical ones. However, automatism in this process could be a source of errors, as we shall see when we build the Hall basis. So it is worthwhile to have a look at each case when we translate from classical to twisted context.

Let A be a cocommutative bialgebra. We use the notations of Section 2 and denote by π, Δ, η and ϵ its product, coproduct, unit and counit, respectively. Let $\nu = \eta \circ \epsilon$. Formally, the same definition as in [11] works.

Definition 5.0.7. An endomorphism f of A (here and in what follows, endomorphism means $\mathbb{F}(\mathfrak{S})$-module endomorphism, and we denote the corresponding set — $\mathbb{F}$-module — by $\mathrm{End}(A)$) admits $F \in \mathrm{End}(A \otimes A)$ as a pseudo-coproduct if $F \circ \Delta = \Delta \circ f$. If f admits the pseudo-coproduct $f \otimes \nu + \nu \otimes f$, we say that f is pseudo-primitive.

Like in the classical case we prove the following facts.

Theorem 5.0.8. *If f and g in End(A) admit the pseudo-coproducts F and G, then $f+g$, $f\alpha$ (where $\alpha \in \mathbb{F}(\mathfrak{S})$), $f \star g$, $f \circ g$ admit the coproducts $F+G$, $F\alpha$, $F \star G$, $F \circ G$, respectively, where the products $\circ$ and $\star$ are naturally extended to End$(A) \otimes$ End(A).*

An element $f \in$ End(A) takes values in the primitives of A if and only if it is pseudo-primitive.

Proof. Most proofs work exactly as in the classical case. We have to be more cautious when we exchange factors; this occurs particularly for the product of the tensor product of algebras or the coproduct of the tensor product of coalgebras. So we develop the case of the convolution.

Let us write $F = f_1 \otimes f_2$ (i.e., $F = \sum_i f_1^i \otimes f_2^i$) and $G = g_1 \otimes g_2$. We are now ready to compute

$$\begin{aligned}
&F \star G(a_1 \otimes a_2)\\
&\quad = (f_1 \otimes f_2) \star (g_1 \otimes g_2)(a_1 \otimes a_2)\\
&\quad = (f_1 \star g_1) \otimes (f_2 \star g_2)(a_1 \otimes a_2)\\
&\quad = (\pi \circ (f_1 \otimes g_1) \circ \Delta \otimes \pi \circ (f_2 \otimes g_2) \circ \Delta)(a_1 \otimes a_2)\\
&\quad = (\pi \otimes \pi) \circ (f_1 \otimes g_1 \otimes f_2 \otimes g_2) \circ (\Delta \otimes \Delta)(a_1 \otimes a_2)\\
&\quad = (\pi \otimes \pi) \circ (A \otimes T \otimes A) \circ (f_1 \otimes f_2 \otimes g_1 \otimes g_2) \circ (A \otimes T \otimes A) \circ (\Delta \otimes \Delta)(a_1 \otimes a_2),
\end{aligned}$$

where A denotes the identity of A and T the swap map $T(a \otimes b) = (b \otimes a)\mathcal{C}_{|b|,|a|}((2,1))$. For the sake of completeness, we have to point out that the introduction of the first swap on the right leads to a factor $\mathcal{C}_{|a_{2(1)}|,|a_{1(2)}|}((2,1))$ and the second one to a factor $\mathcal{C}_{|a_{1(2)}|,|a_{2(1)}|}((2,1))$ the product of which is 1 of course. Thus going back to F and G:

$$F \star G(a_1 \otimes a_2) = (\pi \otimes \pi) \circ (A \otimes T \otimes A) \circ (F \otimes G) \circ (A \otimes T \otimes A) \circ (\Delta \otimes \Delta)(a_1 \otimes a_2).$$

We can now continue as in the classical case:

$$\begin{aligned}
(F \star G) \circ \Delta(a) &= (\pi \otimes \pi) \circ (A \otimes T \otimes A) \circ (F \otimes G) \circ (A \otimes T \otimes A) \circ (\Delta \otimes \Delta) \circ \Delta(a)\\
&= (\pi \otimes \pi) \circ (A \otimes T \otimes A) \circ ((F \circ \Delta) \otimes (G \circ \Delta)) \circ \Delta(a),
\end{aligned}$$

where we have simplified $(A \otimes T \otimes A) \circ (\Delta \otimes \Delta) \circ \Delta = (\Delta \otimes \Delta) \circ \Delta$ since A is cocommutative. For this very property, Δ is an algebra endomorphism of A, and taking the definition of the pseudo-coproducts F and G in account, we conclude

$$\begin{aligned}
(F \star G) \circ \Delta(a) &= (\pi \otimes \pi) \circ (A \otimes T \otimes A) \circ ((\Delta \circ f) \otimes (\Delta \circ g)) \circ \Delta(a)\\
&= (\pi \otimes \pi) \circ (A \otimes T \otimes A) \circ ((\Delta \otimes \Delta) \circ (f \otimes g)) \circ \Delta(a)\\
&= \Delta \circ \pi \circ (f \otimes g) \circ \Delta(a).
\end{aligned}$$

Moreover, by definition of the convolution, we have

$$(F \star G) \circ \Delta = \Delta \circ (f \star g),$$

as requested.

Remark. In the preceding calculations, we need to work with the elements a_1, a_2, a for reminding us that the formulas generate coefficients in $\mathbb{F}(\mathfrak{S})$.

For the second part of the theorem no swaps occur, and the classical proof runs through automatically. □

6. The Dynkin idempotent

Once again we shall follow the classical case, going into details only when swaps occur.

In this section, the cocommutative bialgebra A is supposed to be connected, and so it is a Hopf algebra with an antipode denoted by S. In the preceding section, we defined an endomorphism D of A by $Da = na$, for $a \in A$ of degree n. Let us consider the convolution $l = S \star D$. The following theorems identify the primitives of A with the image of l ($\mathbb{F}$ has characteristic 0).

Theorem 6.0.9. *If $a \in A$ is primitive, then $l(a) = D(a)$.*

Theorem 6.0.10. *$l(a)$ is primitive for all $a \in A$.*

Proof of Theorems 6.0.9 and 6.0.10. For the first theorem, the classical proof can be reproduced word by word.

As for the second one, the classical proof may also be reproduced up to a point. So we quickly develop the argument again.

We are going to use Theorem 6.0.8 and prove that l is pseudo-primitive. Again, by the same theorem and by the definition $l = S \star D$, we need to find pseudo-coproducts of S and D such that their convolution product is $l \otimes \nu + \nu \otimes l$.

One still immediately checks that $D \otimes A + A \otimes D$ is a pseudo-coproduct for D.

The fact that $S \otimes S$ is a pseudo-coproduct for S is an immediate consequence of Proposition 2.3.3, item 4), and of the cocommutativity of the coalgebra A.

The end of the proof is straightforward : $(S\otimes S)\star(D\otimes A+A\otimes D) = l\otimes\nu+\nu\otimes l$. □

We now proceed to the main result of this section. At this point we emphasize the first occurrence (in Corollary 6.0.13) of a result (not only a proof) which differs in the twisted case from the classical one.

Theorem 6.0.11. *For elements a and b of non-zero degree in A, the following identity holds: $l(al(b)) = [l(a), l(b)]$.*

We derive two corollaries right away.

Corollary 6.0.12. *If $a_1, \dots, a_n$ are homogeneous primitive elements of A, then*

$$l(a_1, \dots, a_n) = deg(a_1)[\dots[a_1, a_2], \dots, a_n].$$

Corollary 6.0.13. *As an $\mathbb{F}(\mathfrak{S})$-module, the kernel of l is spanned by 1 and the elements $al(b) + bl(a)\mathcal{C}_{|b|,|a|}((2,1))$, $a, b \in A$.*

Proof of Corollaries 6.0.12 and 6.0.13. Nothing is to be changed in the proof of Corollary 6.0.12.

Theorem 6.0.11 shows that $al(b)+bl(a)\mathcal{C}_{|b|,|a|}((2,1))$ is in the kernel of l. Conversely, consider $a \in A$ of degree n. By abuse of notation and as already mentioned, we write $\Delta(a) = a_{(1)} \otimes a_{(2)}$. By the cocommutativity of A, we can be more precise:

$$\Delta(a) = a \otimes 1 + 1 \otimes a + a'_{(1)} \otimes a'_{(2)} + a'_{(2)} \otimes a'_{(1)}\mathcal{C}_{|a'_{(2)}|,|a'_{(1)}|}((2,1)).$$

Now $l = S \star D$; hence $D = l \star S = \pi \circ (A \otimes l) \circ \Delta$. Let us apply this identity to a; we get:

$$\begin{aligned} D(a) &= na \\ &= \pi \circ (A \otimes l)(a \otimes 1 + 1 \otimes a + a'_{(1)} \otimes a'_{(2)} + a'_{(2)} \otimes a'_{(1)} \mathcal{C}_{|a'_{(2)}|,|a'_{(1)}|}((2,1))) \\ &= \pi(A(a) \otimes l(1) + A(1) \otimes l(a) + A(a'_{(1)}) \otimes l(a'_{(2)}) \\ &\qquad\qquad\qquad + A(a'_{(2)}) \otimes l(a'_{(1)}) \mathcal{C}_{|a'_{(2)}|,|a'_{(1)}|}((2,1)) \\ &= l(a) + a'_{(1)} l(a'_{(2)}) + a'_{(2)} l(a'_{(1)}) \mathcal{C}_{|a'_{(2)}|,|a'_{(1)}|}((2,1)), \end{aligned}$$

and the corollary follows since $n \neq 0$. □

We turn now towards the proof of Theorem 6.0.11.

Proof of Theorem 6.0.11. Let us recall once more that we denote by $T : A \otimes A \to A \otimes A$ the $\mathbb{F}(\mathfrak{S})$-module homomorphism ("the swap map") given by

$$T(a \otimes b) = b \otimes a \mathcal{C}_{|b|,|a|}((2,1)),$$

$a, b \in A$.

We have to show that $l \circ \pi \circ (A \otimes l)$ and $\pi \circ (l \otimes l)(A \otimes A - T)$ coincide on $A_+ \otimes A_+$.

We use several facts. The first three ones are direct consequences of the structure of a Hopf algebra and are listed in Section 2, the fourth one is obvious, and the last one is proved in Theorem 6.0.10.

1) The coproduct Δ is an algebra endomorphism.
2) The antipode S is an antiautomorphism.
3) S sends each primitive element to its additive inverse: $S \circ A = -A$.
4) D is a derivative of the algebra A, i.e., $D \circ \pi = \pi \circ (A \otimes D + D \circ A)$ and $D \circ \nu = 0$.
5) l is pseudo-primitive, i.e., $\Delta \circ l = (l \otimes \nu + \nu \otimes l) \circ \Delta$.

The calculation goes as follows:

$$\begin{aligned} & l \circ \pi \circ (A \otimes l) \\ &= \pi \circ (S \otimes D) \circ \Delta \circ \pi \circ (A \otimes l) \\ &= \pi \circ (S \otimes D) \circ (\pi \otimes \pi) \circ (A \otimes T \otimes A) \circ (\Delta \otimes \Delta) \circ (A \otimes l) \\ &= \pi \circ (\pi \otimes \pi) \circ (T \otimes A \otimes A) \circ (S \otimes S \otimes A \otimes D + S \otimes S \otimes D \otimes A) \\ &\qquad \circ (A \otimes T \otimes A) \circ (A \otimes A \otimes l \otimes \nu + A \otimes A \otimes \nu \otimes A) \circ (\Delta \otimes \Delta) \\ &= \pi \circ (\pi \otimes \pi) \circ (T \otimes A \otimes A) \circ (A \otimes T \otimes A) \circ (S \otimes A \otimes S \otimes D + S \otimes D \otimes S \otimes A) \\ &\qquad \circ (A \otimes A \otimes l \otimes \nu + A \otimes A \otimes \nu \otimes A) \circ (\Delta \otimes \Delta) \\ &= \pi \circ (\pi \otimes \pi) \circ (T \otimes A \otimes A) \circ (A \otimes T \otimes A) \\ &\qquad \circ (S \otimes A \otimes \nu \otimes (D \circ l) + S \otimes D \otimes (-l) \otimes \nu + S \otimes D \otimes \nu \otimes l) \circ (\Delta \otimes \Delta). \end{aligned}$$

We used that $D \circ \nu = 0$ to delete one term in the preceding formula; then three terms remain. We provide the details for the computation of the first and the second one. The third one is similar to the first one and is left to the reader. Notice that, as A is a bialgebra, $(\nu \otimes A) \circ \Delta = 1 \otimes A$.

Let us consider the first term:

$$\begin{aligned}\pi \circ (\pi \otimes \pi) \circ (T \otimes A \otimes A) \circ (A \otimes T \otimes A) \circ (S \otimes A \otimes \nu \otimes (D \circ l)) \circ (\Delta \otimes \Delta)\\ = \pi \circ (\pi \otimes \pi) \circ (T \otimes A \otimes A) \circ (A \otimes T \otimes A) \circ ((S \otimes A \circ \Delta) \otimes 1 \otimes (D \circ l)).\end{aligned}$$

Let us have a special look at the compositions of swaps:

$$\begin{aligned}\pi \circ (\pi \otimes \pi) \circ (T \otimes A \otimes A) \circ (A \otimes T \otimes A)(a \otimes b \otimes c \otimes d)\\ = \pi \circ (\pi \otimes \pi)(c \otimes a \otimes b \otimes d)\mathcal{C}_{|c|,|a|,|b|,|d|}((2,1,3,4))\mathcal{C}_{|a|,|c|,|b|,|d|}((1,3,2,4)).\end{aligned}$$

Now — a simple, but fundamental remark — in the first term as in the third one the element at the third spot, namely c, is equal to 1, and thus

$$\mathcal{C}_{|c|,|a|,|b|,|d|}((2,1,3,4))\mathcal{C}_{|a|,|c|,|b|,|d|}((1,3,2,4)) = 1$$

and

$$\pi \circ (\pi \otimes \pi) \circ (T \otimes A \otimes A) \circ (A \otimes T \otimes A)(a \otimes b \otimes 1 \otimes d) = \pi \circ (A \otimes \pi)(a \otimes b \otimes d).$$

Consequently, we have

$$\begin{aligned}&\pi \circ (\pi \otimes \pi) \circ (T \otimes A \otimes A) \circ (A \otimes T \otimes A) \circ ((S \otimes A \circ \Delta) \otimes 1 \otimes (D \circ l))\\ &= \pi \circ (\pi \otimes \pi)((S \otimes A) \circ \Delta) \otimes 1 \otimes (D \circ l))\\ &= (S \otimes A \otimes (D \circ l)) \circ (\Delta \otimes A).\end{aligned}$$

For the second term we calculate:

$$\begin{aligned}&(S \otimes D \otimes (-l) \otimes \nu) \circ (\Delta \otimes \Delta)(a \otimes b)\\ &\quad = (S \otimes D) \circ \Delta(a) \otimes (-l)(b) \otimes 1\\ &\quad = ((S \otimes D) \otimes (-l)) \circ (\Delta \otimes A \otimes 1)(a \otimes b)\\ &\quad = (T' \otimes 1)((-l) \otimes S \otimes D)) \circ (A \otimes \Delta \otimes 1)T(a \otimes b)\\ &\qquad\qquad \mathcal{C}_{|a_{(1)}|,|a_{(2)}|,|b|}((3,1,2))\mathcal{C}_{|b|,|a_{(1)}|,|a_{(2)}|}((3,1,2)),\end{aligned}$$

where $T'(a \otimes b \otimes c) = b \otimes c \otimes a\mathcal{C}_{|b|,|c|,|a|}((3,1,2))$. Of course, the product of the two coefficients in the last equation is $\mathcal{C}_{|a_{(1)}|,|a_{(2)}|,|b|}((3,1,2))\mathcal{C}_{|b|,|a_{(1)}|,|a_{(2)}|}((3,1,2)) = 1$.

Now, the second term is

$$\begin{aligned}&\pi \circ (\pi \otimes \pi) \circ (T \otimes A \otimes A)\\ &\qquad \circ (A \otimes T \otimes A)(T' \otimes 1)((-l) \otimes S \otimes D)) \circ (A \otimes \Delta \otimes 1)T(a \otimes b)\\ &= -\pi \circ (A \otimes \pi)(l \otimes S \otimes D) \circ (A \otimes \Delta) \circ T.\end{aligned}$$

We go back to our main computation:

$$\begin{aligned} l \circ \pi \circ (A \otimes l) &= \pi \circ (A \otimes \pi) \circ ((S \otimes A \otimes (D \circ l)) \circ (\Delta \otimes A) \\ &\qquad - (l \otimes S \otimes D) \circ (A \otimes \Delta) \circ T \\ &\qquad + (S \otimes D \otimes l)) \circ (\Delta \otimes A)) \\ &= \pi \circ (\pi \otimes A) \circ (S \otimes A \otimes (D \circ l)) \circ (\Delta \otimes A) \\ &\qquad - \pi \circ (A \otimes \pi) \circ (l \otimes S \otimes D) \circ (A \otimes \Delta) \circ T \\ &\qquad + \pi \circ (\pi \otimes A) \circ (S \otimes D \otimes l) \circ (\Delta \otimes A) \\ &= \pi \circ (\pi \circ ((S \otimes A) \circ \Delta) \otimes (D \circ l)) \\ &\qquad - \pi \circ ((l \otimes (\pi \circ (S \otimes D) \circ \Delta)) \circ T) \\ &\qquad + \pi \circ ((\pi \circ (S \otimes D) \circ \Delta) \otimes l) \\ &= \pi \circ (\nu \otimes (D \circ l) - (l \otimes l) \circ T + l \otimes l), \end{aligned}$$

where we used the associativity of the product $\pi \circ (A \otimes \pi) = \pi \circ (\pi \otimes A)$ and the defining identities of S and l, namely $S \star A = \nu$ and $S \star D = l$. □

References

[1] Aguiar, M., *Infinitesimal bialgebras, pre-Lie and dendriform algebras.* Hopf algebras, 1–33, Lecture Notes in Pure and Appl. Math., 237, Dekker, New York, 2004.

[2] Aguiar, M.; Mahajan, S., *Monoidal functors, species and Hopf algebras.* 672 pages. To appear in the CRM Monograph Series of the AMS.

[3] Barratt, M. G., *Twisted Lie algebras.* Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II, pp. 9–15, Lecture Notes in Math., 658, Springer, Berlin, 1978.

[4] Bergeron, F.; Labelle, G.; Leroux, P., *Combinatorial species and tree-like structures.* Translated from the 1994 French original by Margaret Readdy. With a foreword by Gian-Carlo Rota. Encyclopedia of Mathematics and its Applications, 67. Cambridge University Press, Cambridge, 1998. xx+457 pp.

[5] Fresse, B.,*On the homotopy of simplicial algebras over an operad.* Trans. Amer. Math. Soc. **352** (2000), 4113–4141.

[6] Fresse, B., *Koszul duality of operads and homology of partition posets.* Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory, pp. 115–215, Contemp. Math., 346, Amer. Math. Soc., Providence, RI, 2004.

[7] Goerss, P. G., *Barratt's desuspension spectral sequence and the Lie ring analyzer. Quart. J. Math. Oxford Ser. (2)* **44** (1993), no. 173, 43–85.

[8] Joyal, A., *Foncteurs analytiques et espèces de structures.* Combinatoire énumérative (Montréal, Québec, 1985), pp. 126–159, Lecture Notes in Math., 1234, Springer, Berlin, 1986.

[9] Livernet, M.; Patras, F., *Lie theory for Hopf operads.* J. Algebra **319** (2008), 4899–4920.

[10] Markl, M.; Shnider, S.; Stasheff, J., *Operads in algebra, topology and physics.* Mathematical Surveys and Monographs, 96. American Mathematical Society, Providence, RI, 2002. x+349 pp.

[11] Patras, F.; Reutenauer, C., *On Dynkin and Klyachko idempotents in graded bialgebras.* Special issue in memory of Rodica Simion, Adv. in Appl. Math. **28** (2002), 560–579.

[12] Patras, F.; Reutenauer, C., *On descent algebras and twisted bialgebras.* Mosc. Math. J. **4** (2004), 199–216, 311.

[13] Patras, F., Schocker, M., *Twisted descent algebras and the Solomon-Tits algebra.* Adv. in Math. **199** (2006), 151–184.

[14] Patras, F., Schocker, M., *Trees, set compositions and the twisted descent algebra.* J. Algebraic Combin. **28** (2008), 3–23.

[15] Reutenauer, C., *Free Lie algebras.* London Mathematical Society Monographs. New Series, 7. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993. xviii+269 pp.

[16] Stover, C. R., *The equivalence of certain categories of twisted Lie and Hopf algebras over a commutative ring.* J. Pure Appl. Algebra **86** (1993), 289–326.

[17] Sweedler, M. E., *Hopf algebras.* Mathematics Lecture Note Series, W. A. Benjamin Inc., New York, 1969. vii+336 pp.

[18] von Waldenfels, W., *Zur Charakterisierung Liescher Elemente in freien Algebren.* Arch. Math. **17** (1966), 44–48.

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