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COMBINATORIAL DEFORMATIONS OF ALGEBRAS: TWISTING AND PERTURBATIONS

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Abstract. The framework used to prove the multiplicative deformation of the algebra of Feynman–Bender diagrams is that of a *twisted shifted dual law* (in fact, doubly twisted). We give here a clear interpretation of its two parameters. The crossing parameter is a deformation of the tensor structure, whereas the superposition parameter is a perturbation of the shuffle coproduct which, in its turn, can be interpreted as the diagonal restriction of a superproduct. Here, we systematically explain these constructions in detail.

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1. Introduction

In [1], Bender, Brody, and Meister introduced a special field theory, then called "Quantum Field Theory of Partitions". This theory is based on the bilinear product defined by

$$\mathcal{H}(F, G) = F\left(z\frac{d}{dx}\right) G(x)\bigg|_{x=0}. \tag{1.1}$$

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If one expands this formula in the case when F and G are free exponentials, one obtains a summation over all the (finite) bipartite[1] graphs with multiple edges and no isolated point [6] (the set of these diagrams will be called **diag**), a data structure which is equivalent to classes of packed matrices [8] under permutations of rows and columns.

So, one has a Feynman-type expansion of the product formula

$$\mathcal{H}\left(\exp\left(\sum_{n=1}^{\infty} L_n\frac{z^n}{n!}\right), \exp\left(\sum_{n=1}^{\infty} V_n\frac{z^n}{n!}\right)\right) = \sum_{n\geq 0}\frac{z^n}{n!}\sum_{\substack{d\in\mathbf{diag}\\ |d|=n}} \mathrm{mult}(d)\mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)}, \qquad (1.2)$$

where $\mathrm{mult}(d)$ is the number of pairs (P_1, P_2) of (ordered) set partitions of $\{1,\ldots,n\}$ which correspond to a diagram d, where $|d|$ the number of edges in d, and

$$\mathbb{L}^{\alpha(d)} = L_1^{\alpha_1}L_2^{\alpha_2}\cdots\ ;\ \mathbb{V}^{\beta(d)} = V_1^{\beta_1}V_2^{\beta_2}\cdots \qquad (1.3)$$

is the multiindex notation for the monomials in $\mathbb{L}\cup\mathbb{V}$, i.e., $\alpha_i = \alpha_i(d)$ (respectively $\beta_j = \beta_j(d)$) is the number of white (respectively black) spots of degree i (respectively j) in d.

The set **diag** endowed with the operation of disjoint union inherits the structure of a monoid such that the arrow $d \mapsto \mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)}$ is a homomorphism (of monoids), and then, by linear extension, one deduces an algebra homomorphism

$$\mathbb{C}[\mathbf{diag}] \to \mathrm{Pol}(\mathbb{C};\mathbb{L}\cup\mathbb{V})\ , \qquad (1.4)$$

where $\mathrm{Pol}(\mathbb{C};\mathbb{L}\cup\mathbb{V})$ is the Hopf algebra of (commutative) polynomials with complex coefficients generated by the alphabet $\mathbb{L}\cup\mathbb{V}$. For at least three models of Physics, one can specialize $\mathbb{L}$ so that the canonical Hopf algebra structure of $\mathrm{Pol}(\mathbb{C};\mathbb{L}\cup\mathbb{V})$ can be lifted, through (1.4). The resulting Hopf algebra (based on $\mathbb{C}[\mathbf{diag}]$) has been denoted **DIAG**. To our great surprise, this Hopf algebra structure could be lifted at the (noncommutative) level of the objects themselves instead of classes, resulting in the construction of a Hopf algebra on (linear combinations of) "labelled diagrams" (the monoid **ldiag**, see [6]). As these "labelled diagrams" are in one-to-one correspondence with the packed matrices of **MQSym**, we get two (combinatorially natural) structures of algebra (and coalgebra) on the vector space $\mathbb{C}[\mathbf{ldiag}]$, and one could raise the question of the existence of a continuous deformation between the two. The answer is positive and can be performed through a three-parameter (two formal, or continuous, and one boolean) Hopf deformation[2] of **LDIAG** called $\mathbf{LDIAG}(q_c, q_s, q_t)$ [6] such that

$$\mathbf{LDIAG}(0,0,0) \simeq \mathbf{LDIAG}\ ;\ \mathbf{LDIAG}(1,1,1) \simeq \mathbf{MQSym}\ . \qquad (1.5)$$

The rôle of the two parameters q_c, q_s (algebra parameters, whereas q_t is a coalgebra parameter) was discovered just counting crossings and superpositions in the twisted labelled diagrams (see [6] for details). This simple statistics (counting crossings and superpositions) yields an associative product on the diagrams. The first proof given for the associativity was mainly computational, and it was actually a surprise that associativity held. This raised the need to understand this phenomenon in a deeper way and the question whether the two parameters (q_c and q_s) would be of different nature. The aim of this paper is to answer this question and give a conceptual proof of

[1]The (bi)-partition of the vertices is understood to be ordered. In this case, the term *bicoloured* can also be found in the literature.

[2]This algebra deformation has received recently another realisation in terms of biwords [9].

associativity by developing four building blocks which are general and separately easy to test: addition of a group-like element to a coassociative coalgebra, shifting lemma, codiagonal deformation of a semigroup, and extension of a colour factor to words.

The essential ingredient in the two last operations is what has become nowadays a useful tool, the coloured product of algebras, for which we give some new results.

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2. The deformed algebra $\mathbf{LDIAG}(q_c, q_s)$

2.1. **Review of the construction of the algebra.** The complete story of the algebra of Feynman–Bender diagrams which arose in Combinatorial Physics in 2005 can be found in [6], and a fragment of it, as well as a realisation using an alternative data structure, in [9].

Recall that (classical) shuffle products (of words) can be expressed in two ways:

a) by a recursion;
b) by a summation on (and by means of) some permutations.

Here, we will trace back the construction of the deformed product between two diagrams, starting from an analog of (b) (using however the symmetric semigroup instead of the symmetric group, see below) and going gradually to (a) following the first description of the deformed case which was graphical (and was discovered as such in [6]).

The diagrams on which the product has to be performed are plane bipartite graphs (vertices being called black and white spots) with multiple ordered edges; they look as follows:

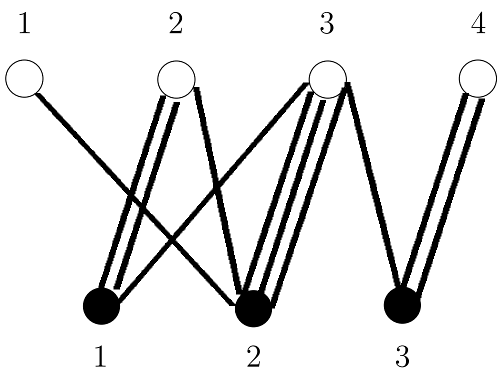

Fig 1. — *Labelled diagram of format* 3×4.

One can define more formally this data structure using the equivalent notion of a weight function. Here, it is a function $\omega : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}$ (as in [9]) with support

$$\mathrm{supp}(\omega) = \{(i,j) \in \mathbb{N}^+ \times \mathbb{N}^+ \mid w(i,j) \neq 0\} \tag{2.1}$$

having projections of the form $pr_1(\mathrm{supp}(\omega)) = [1 \dots p]$; $pr_2(\mathrm{supp}(\omega)) = [1 \dots q]$ for some $p, q \in \mathbb{N}^+$. This last description can be rephrased without pr_i by observing that p (respectively q) is the last i such that there exists a $j \in \mathbb{N}^+$ with $\omega(i,j) \neq 0$ (respectively q is the last j such that there exists an $i \in \mathbb{N}^+$ with $\omega(i,j) \neq 0$). In this way our graphs are in one-to-one correspondence with such weight functions.

j	2	3	1	2	3	3	4
i	1	1	2	2	2	3	3
$\omega(i,j)$	2	1	1	1	3	1	2

Fig 2. — *The weight function (when not* 0*) of the diagram in Fig 1. Here* $p = 3$ *and* $q = 4$.

We are now in the position to describe the (deformed) product of our diagrams by means of the symmetric semigroup (whereas the symmetric group would only provide crossings as it occurs with the shuffle product).

The symmetric semigroup on a finite set F (denoted here SSG_F) is the set of endofunctions $F \to F$. In order to preserve the requirement that black spots are labelled from 1 to some integer, we have to require that the mapping acting on the diagram d with n black spots has an image of the type $[1 \dots m]$ for some $m \leq n$. The result, denoted by $d.f$, has m black spots such that the black spot of (former) label "i" bears the new label $f(i)$.

If we consider any onto mapping $[1 \dots p] \to [1 \dots r]$, the diagram $d.f = d'$ has the weight function ω' defined by

$$\omega'(i,k) = \sum_{f(j)=i} \omega(j,k) \ , \tag{2.2}$$

which can be easily checked to be admissible in our context.

Before giving the expression of the deformed product, we must define local partial degrees. For a black spot with label "l", we denote by $bks(d,l)$ its degree (number of adjacent edges). For d_1 (respectively d_2) with p (respectively q) black spots, let $[d_1|d_2]_L$ be their concatenation (non-deformed product). Then the deformed product of d_1 by d_2 reads

$$[d_1|d_2]_{L(q_c,q_s)} = \sum_{f \in Shs(p,q)} \Big(\prod_{\substack{i<j \\ f(i)>f(j)}} q_c^{bks(d_1,i).bks(d_2,j)} \Big) \Big(\prod_{\substack{i<j \\ f(i)=f(j)}} q_s^{bks(d_1,i).bks(d_2,j)} \Big) [d_1|d_2]_L.f, \tag{2.3}$$

where $Shs(p,q)$ is the set of mappings $f \in SSG_{[1 \dots p+q]}$ with image an interval of type $[1 \dots m]$ (with $\max\{p,q\} \leq m \leq p+q$), and such that

$$f(1) < f(2) < \cdots < f(p) \ ; \ f(p+1) < f(p+2) < \cdots < f(p+q) \ . \tag{2.4}$$

This condition, similar to that of the shuffle product, guarantees that the black spots of the diagrams are kept in order during the process of shuffling with superposition (hence the name Shs).

2.2. **Coding and the recursive definition.** The graphical and symmetric-semigroup-indexed description of the deformed product neither gives immediately a recursive definition nor an explanation of "why" the product is associative. We will, on our way to understand this (as well as the different natures of its parameters), proceed in three steps:

- encoding the diagrams by words of monomials,
- presenting the product as a shifted law,
- give a recursive definition of the (non-shifted) law.

The code used here relies on monomials over a commutative alphabet of variables $\mathbb{X} = \{x_i\}_{i\geq 1}$. As in [6], we let $\mathfrak{MON}(\mathbb{X})$ denote the monoid of monomials $\{\mathbb{X}^\alpha\}_{\alpha\in\mathbb{N}^{(\mathbb{X})}}$ (indeed, the free commutative monoid over $\mathbb{X}$) and $\mathfrak{MON}^+(X) = \{\mathbb{X}^\alpha\}_{\alpha\in\mathbb{N}^{(\mathbb{X})}-\{0\}}$ the semigroup of its non-unit elements (the free commutative semigroup over $\mathbb{X}$).

Note that each weight function $\omega \in \mathbb{N}^{(\mathbb{N}^+\times\mathbb{N}^+)}$ yields an equivalent "word of monomials[3]" $W(\omega) = w_1.w_2.\cdots.w_p$ such that

$$W(\omega)[i] = w_i = \prod_{j=1}^{\infty} x_j^{\omega(i,j)} \ . \tag{2.5}$$

The correspondence $code : \mathbb{N}^{(\mathbb{N}^+\times\mathbb{N}^+)} \to (\mathfrak{MON}^+(X))^*$ is one-to-one and provides at once a way to code each labelled diagram through its weight function as a word of monomials. Conversely, a word $W \in (\mathfrak{MON}^+(X))^*$ is the code of a diagram (i.e., the image by *code* of the weight function of a diagram) if and only if

$$indexes(Alph(W)) = [1 \dots m] \tag{2.6}$$

(where $indexes(Alph(W))$ is the set of $i \in \mathbb{N}^+$ such that an x_i is involved in W). Due to the special indexation of its alphabet, the monoid $(\mathfrak{MON}^+(X))^*$ comes equipped with a set of endomorphisms, the translations T_n defined on the variables by $T_n(x_i) = x_{i+n}$ and extended to $\mathfrak{MON}^+(X)$, to $(\mathfrak{MON}^+(X))^*$ and then to $K\langle\mathfrak{MON}^+(X)\rangle$.

Note that the code of a concatenation reads

$$code([d_1|d_2]_L) = code(d_1).T_{max(indexes(Alph(code(d_1))))}(code(d_2)) \ . \tag{2.7}$$

Therefore, the function "*code*" being below extended by linearity, the reader may check easily that one can compute recursively the deformed product on the codes by

$$code([d_1|d_2]_L) = code(d_1) \uparrow T_{max(indexes(Alph(code(d_1))))}(code(d_2)), \tag{2.8}$$

where the bilinear product $\uparrow$ is recursively defined on the words by

$$\begin{cases} 1_{(\mathfrak{MON}^+(X))^*} \uparrow w &= w \uparrow 1_{(\mathfrak{MON}^+(X))^*} = w \\ au \uparrow bv &= a(u \uparrow bv) + q_c^{|au||b|} b(au \uparrow v) + q_c^{|u||b|} q_s^{|a||b|} (a \cdot b)(u \uparrow v) \end{cases} , \tag{2.9}$$

where $a \cdot b$ (medium $\cdot$ dot) denotes the (monomial, commutative) product of a and b within $\mathfrak{MON}^+(X)$.

It is this last recursion that we will decompose and analyse below in order to get a better understanding of the parameters.

The associativity of the product (2.9) is a consequence of the following proposition.

[3]The low point . here is used to emphasize concatenation which is elsewhere denoted by simple juxtaposition of letters.

Proposition 2.1 ([6, Prop. 5.1])**.** *Let $(S,.)$ be a semigroup graded by a degree function $|\ |_d : S \to \mathbb{N}$ (i.e., a homomorphism to $(\mathbb{N},+)$) and S^* the set of lists (denoted by words $a_1a_2\cdots a_k$) with letters in S (including the empty list 1_{S^*}).*

Let $q_c, q_s \in K$ be two elements in a (commutative) ring K. We define on $K\langle S\rangle = K[S^*]$ a new product $\uparrow$ by

$$\begin{aligned} w \uparrow 1_{S^*} &= 1_{S^*} \uparrow w = w \\ au \uparrow bv &= a(u \uparrow bv) + q_c^{|au|_d|b|_d} b(au \uparrow v) + q_c^{|u|_d|b|_d} q_s^{|a|_d|b|_d} (a.b)(u \uparrow v), \end{aligned} \tag{2.10}$$

where the weights are extended additively to lists (words) by

$$\Big|a_1a_2\cdots a_k\Big|_d = \sum_{i=1}^{k} |a_i|_d\ .$$

Then the new product $\uparrow$ is graded, associative, with 1_{S^*} as its unit.

The questions that have arisen in the introduction can now be reformulated as follows:

Q1) Are q_c and q_s of the same nature?

Q2) If no, can the associativity be explained, step by step, by constructions which will show their different natures?

Here, by *"nature,"* it is understood that q_c and q_s, although at the level of statistics they seem to play a similar rôle, could be distinguished by general algebra. Indeed, in the sequel, we attempt to show that q_c is of geometric nature (deformation at the level of the tensor structure), whereas q_s is of perturbative nature (perturbation of the Lie coproduct).

With this aim in mind, we need to recall a now classical tool, the coloured product of two algebras.

3. Colour factors and products

Colour factors were introduced by R. Ree [14], and the theory was developed or used in [7, 5, 12, 16]. See also [2][4].

Let $\mathcal{A} = \bigoplus_{\alpha\in\mathcal{D}} \mathcal{A}_\alpha$ and $\mathcal{B} = \bigoplus_{\beta\in\mathcal{D}} \mathcal{B}_\beta$ be two $\mathcal{D}$-graded associative K-algebras[5] ($\mathcal{D}$ is a commutative semigroup whose law is denoted additively). Readers that are not familiar with graded algebras can think of $\mathcal{D} = \mathbb{N}^{(X)}$, the free commutative monoid over X, and $\mathcal{A}_\alpha = K[X]_\alpha$, the space of homogeneous polynomials of multidegree α.

Given a mapping $\chi : \mathcal{D}\times\mathcal{D} \longrightarrow K$, we define an algebra product on $\mathcal{A}\otimes\mathcal{B}$ by

$$(x_1 \otimes y_1)(x_2 \otimes y_2) \;=\; \chi(\beta_1, \alpha_2)(x_1x_2 \otimes y_1y_2) \tag{3.1}$$

for $(x_i) \in \mathcal{A}_{\alpha_i}$ and $(y_i) \in \mathcal{B}_{\beta_i}$ $(i = 1, 2)$.

Equating the expressions that we obtain for $((x_1 \otimes y_1)(x_2 \otimes y_2))\,(x_3 \otimes y_3)$ and $(x_1 \otimes y_1)\,((x_2 \otimes y_2)(x_3 \otimes y_3))$ by using (3.1), we are led to the following proposition.

Proposition 3.1 ([16])**.** *Let $\chi : \mathcal{D}\times\mathcal{D} \longrightarrow K$. The following are equivalent:*

[4] In fact, some of them (*"Facteurs de commutation"*, with values in $\{-1, 1\}$ and an [anti]symmetry condition) are already considered in the edition of 1970 of [2]. See Section 10 (*Dérivations*) of Chapter III.

[5] Not necessarily with unit.

i) *For $\mathcal{A}$, $\mathcal{B}$ $\mathcal{D}$-graded associative algebras, the product defined by* (3.1) *is associative.*
ii) *For all* $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \in \mathcal{D}$ *we have*

$$\chi(\beta_1, \alpha_2)\chi(\beta_1 + \beta_2, \alpha_3) = \chi(\beta_2, \alpha_3)\chi(\beta_1, \alpha_2 + \alpha_3). \tag{3.2}$$

Definition 3.2. Every mapping $\chi : \mathcal{D}\times\mathcal{D} \longrightarrow K$ which fulfills the equivalent conditions of Proposition 3.1 will be called a colour (twisting) factor.

Remarks 3.3. I) If χ is bilinear, which means in this context that the following equations are satisfied (for all $\alpha, \alpha', \beta, \beta' \in \mathcal{D}$):

$$\begin{aligned}\chi(\alpha + \alpha', \beta) &= \chi(\alpha, \beta)\chi(\alpha', \beta)\\ \chi(\alpha, \beta + \beta') &= \chi(\alpha, \beta)\chi(\alpha, \beta'),\end{aligned} \tag{3.3}$$

then the two sides of (3.2) amount to

$$\chi(\beta_1, \alpha_2)\chi(\beta_1, \alpha_3)\chi(\beta_2, \alpha_3) = \prod_{1\le i<j\le 3} \chi(\beta_i, \alpha_j), \tag{3.4}$$

and hence χ is a colour factor[6]. But the full class of colour factors is much larger than solutions of Eq. (3.3). Just observe that Eq. (3.2) is homogeneous in the classical sense, i.e., for all $\lambda \in K$, if χ fulfills (3.2), then after rescaling by λ it still does. Hence, for example, any constant function on $\mathcal{D} \times \mathcal{D}$ is a colour factor. This shows the existence of colour factors that are not bilinear.

II) The converse part of Proposition 3.1 (i.e., ii) $\Longrightarrow$ i)) can be easily proved by considering (free) semigroup algebras $K[\mathcal{D}]$.

Notes 3.4. I) The colour product of two algebras $\mathcal{A} = \bigoplus_{\alpha\in\mathcal{D}} \mathcal{A}_\alpha$ and $\mathcal{B} = \bigoplus_{\beta\in\mathcal{D}} \mathcal{B}_\beta$ comes also as a graded algebra by

$$(\mathcal{A} \otimes \mathcal{B})_\gamma = \bigoplus_{\alpha+\beta=\gamma} \mathcal{A}_\alpha \otimes \mathcal{B}_\beta. \tag{3.5}$$

The usual identification

$$(\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} \simeq \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C}) \tag{3.6}$$

holds for coloured products.

II) Moreover, if $\mathcal{A} \xrightarrow{f} \mathcal{A}'$ (respectively $\mathcal{B} \xrightarrow{g} \mathcal{B}'$) are two homomorphisms of (graded) algebras (over the same semigroup of degrees $\mathcal{D}$), then $\mathcal{A}\otimes\mathcal{B} \xrightarrow{f\otimes g} \mathcal{A}'\otimes\mathcal{B}'$ is an algebra homomorphism (the colour products being taken with respect to the same colour factor).

[6]These bilinear mappings are also called bicharacters in the literature [14].

4. Special classes of laws

4.1. Dual laws.

4.1.1. *Algebras and coalgebras in duality.* An algebra $(\mathcal{A},\mu)$ and a coalgebra (C,Δ) are said to be in duality if and only if there is a non-degenerate pairing $\langle -|-\rangle$ such that for all $x,y\in\mathcal{A},\ z\in C$

$$\langle \mu(x,y)|z\rangle = \langle x\otimes y|\Delta(z)\rangle^{\otimes 2}. \tag{4.1}$$

In the following, a product $K\langle A\rangle\otimes K\langle A\rangle \xrightarrow{*} K\langle A\rangle$ on the free algebra which is the dual of a comultiplication will be called *dual law*, the pairing being given on the basis of words by $\langle u|v\rangle=\delta_{u,v}$.

Our first examples are essential in modern and not-so-modern research (cf. [13, 15]). First of all, we have the dual of the Cauchy product

$$\Delta_{Cauchy}(w)=\sum_{uv=w} u\otimes v\ . \tag{4.2}$$

In contrast to (4.2), which is not an algebra homomorphism[7]

$$K\langle A\rangle \longrightarrow K\langle A\rangle\otimes K\langle A\rangle\ , \tag{4.3}$$

one has three very well-known examples being so, namely duals of the shuffle ш, the Hadamard $\odot$ and the infiltration product $\uparrow$. As they are homomorphisms between the algebras (4.3), they are well defined by their values on the letters; namely, we have

$$\Delta_{ш}(x)=x\otimes 1+1\otimes x\ ;\ \Delta_{\odot}(x)=x\otimes x\ ;\ \Delta_{\uparrow}(x)=x\otimes 1+1\otimes x+x\otimes x\ , \tag{4.4}$$

respectively. One can prove that the deformations $\Delta_q=\Delta_{ш}(x)+q\Delta_{\odot}(x)$ are also coassociative, and that they are the unique solutions of the problem of bialgebra comultiplications on $K\langle A\rangle$ that are compatible with subalphabets [10].

In the sequel, we will make use of the following lemma several times, the proof of which is left to the reader.

Lemma 4.1. *Let $\mathcal{A}$ be an algebra and $\mathcal{C}$ be a coalgebra in (non-degenerate) duality. Then $\mathcal{A}$ is associative if and only if $\mathcal{C}$ is coassociative.*

4.1.2. *Duality between group-like elements and units.* Let $(\mathcal{C},\Delta)$ be a coalgebra with counit $\epsilon:\mathcal{C}\to K$. We call an element u *group-like* if it satisfies

$$\epsilon(u)=1\ ;\ \Delta(u)=u\otimes u\ . \tag{4.5}$$

One then has $\mathcal{C}=ker(\epsilon)\oplus K.u$ and

$$\Delta(y)=\Delta^+(y)+y\otimes u+u\otimes y-\epsilon(y)u\otimes u\ , \tag{4.6}$$

where Δ^+ is a comultiplication on $\mathcal{C}$ for which $ker(\epsilon)=\mathcal{C}^+$ is a subcoalgebra (i.e., $\Delta^+(\mathcal{C}^+)\subset\mathcal{C}^+\otimes\mathcal{C}^+$; cf. [3]).

Proposition 4.2. *Let $(\mathcal{C},\Delta,\epsilon)$ be a coalgebra with counit, u a group-like element in $\mathcal{C}$, and $(\mathcal{C}^+,\Delta^+)$ be as in (4.6). On the other hand, let $\mathcal{A}$ be an algebra and $\mathcal{A}^{(1)}=\mathcal{A}\oplus K.v$ be the algebra with unit constructed from $\mathcal{A}$ by adjunction of the unit v. Then, if $\mathcal{C}^+$ and $\mathcal{A}$ are in duality by $\langle\ |\ \rangle$, so are $\mathcal{C}$ and $\mathcal{A}^{(1)}$ by $\langle\ |\ \rangle_\bullet$, the latter being defined by*

$$\langle x+\alpha v|y+\beta u\rangle_\bullet=\langle x|y\rangle+\beta\alpha \tag{4.7}$$

[7] Unless $A=\emptyset$.

for $x \in \mathcal{A}$ *and* $y \in \mathcal{C}^+ = ker(\epsilon)$.

Proof. Let

$$\langle (x_1+\alpha_1 v)\otimes(x_2+\alpha_2 v)|\Delta(y+\beta u)\rangle_\bullet^{\otimes 2} = \langle (x_1+\alpha_1 v)\otimes(x_2+\alpha_2 v)|\Delta^+(y)+y\otimes u+u\otimes y+\beta u\otimes u)\rangle_\bullet^{\otimes 2}. \quad (4.8)$$

However, according to the fact that

$$\langle x_i|u\rangle = \langle x_1\otimes v|\Delta^+(y)\rangle = \langle v\otimes x_2|\Delta^+(y)\rangle = \langle v\otimes v|\Delta^+(y)\rangle = \langle v|y\rangle = 0,$$

one concludes from (4.8) that

$$\begin{aligned}\langle (x_1+\alpha_1 v)\otimes(x_2+\alpha_2 v)|\Delta(y+\beta u)\rangle_\bullet^{\otimes 2} &= \langle x_1\otimes x_2|\Delta^+(y)\rangle^{\otimes 2} + \alpha_2\langle x_1|y\rangle + \alpha_1\langle x_2|y\rangle + \alpha_1\alpha_2\beta \\ &= \langle x_1x_2+\alpha_2x_1+\alpha_1x_2+\alpha_1\alpha_2 v|y+\beta u\rangle_\bullet = \langle (x_1+\alpha_1 v)(x_2+\alpha_2 v)|y+\beta u\rangle_\bullet,\end{aligned}$$

which proves the claim. □

4.2. **Deformed laws.** Let S be a semigroup graded by a semigroup of degrees $\mathcal{D}$ and $\mathcal{A} = K[S]$ its algebra. A colour factor $\chi : \mathcal{D}\times\mathcal{D} \to K$ being given, we endow the algebra $\mathcal{A}\otimes\mathcal{A}$ with the coloured tensor product structure. Notice that the diagonal subspace $D_S = \bigoplus_{x\in S} Kx\otimes x$ is a subalgebra since

$$(x\otimes x)(y\otimes y) = \chi(x,y)\ xy\otimes xy\ . \quad (4.9)$$

Transferring (4.9) back to $\mathcal{A}$ by means of the isomorphism of vector spaces, $\mathcal{A}\to D_S$, one sees immediately that the deformed product on $\mathcal{A}$ given by

$$x._\chi y = \chi(x,y)\ xy \quad (4.10)$$

(for $x, y \in \mathcal{D}$) is associative.

From now on, we suppose that the semigroup $\mathcal{D}$ satisfies condition [D] of Bourbaki [2], which means that, for all $z\in\mathcal{D}$, the number of solutions $(x,y)\in\mathcal{D}^2$ of the equation $xy = z$ is finite. This condition is satisfied by almost all the graded semigroups used by combinatorialists, in particular by the semigroups $(\mathbb{N},+)$, $(\mathbb{N}^+,\times)$, $(\mathbb{N}^{(X)},+)$.

If $\mathcal{A}$ is endowed with the scalar product for which the basis $(s)_{s\in S}$ is orthonormal, the pairing is non-degenerate and the dual comultiplication is given by

$$\Delta(z) = \sum_{xy=z}\chi(x,y)\ x\otimes y\ . \quad (4.11)$$

The construction together with Lemma 4.1 proves that this comultiplication on $\mathcal{A}$ is coassociative.

4.3. **Shifted laws.** We begin by a very general version of the "shifting lemma" (more general than the one given and needed in [6]).

We start from an algebra $\mathcal{A}$ decomposed (as a vector space) into the direct sum

$$\mathcal{A} = \bigoplus_{\alpha\in\mathcal{D}}\mathcal{A}_\alpha$$

over $\mathcal{D}$, a semigroup. We denote by $\mathrm{End}^\alpha(\mathcal{A})$ the algebra homomorphisms $\mathcal{A}\to\mathcal{A}$ (then multiplicative) which "shift by α" (i.e., $\phi\in\mathrm{End}^\alpha(\mathcal{A})$ if and only if, for all $\beta\in\mathcal{D}$, one has $\phi(\mathcal{A}_\beta)\subset\mathcal{A}_{\alpha+\beta}$). This situation is typical of "shift of indices" in free algebras.

For example, let $Y = \{y_j\}_{j\geq 1} = \{y_1, y_2, \ldots, y_k, \ldots\}$ be an infinite alphabet, $\mathcal{D}$ be its indexing semigroup $(\mathbb{N}^+, +)$ and $\mathcal{A}$ be the algebra $K\langle Y\rangle$. For every monomial (word) w let $d(w)$ be the maximal index j of a letter y_j occurring in w. With $K\langle Y\rangle_j := \bigoplus_{d(w)=j} K.w$, one gets a direct sum decomposition

$$K\langle Y\rangle = \bigoplus_{j\geq 1} K\langle Y\rangle_j \ , \tag{4.12}$$

for which $K\langle Y\rangle$ is not a graded algebra (it is, in fact for the *sup* law in $\mathbb{N}^+$, but we aim here at constructing a graded algebra for the addition of degrees). The change of variables $T_n(y_j) := y_{j+n}$ defines an algebra homomorphism $T_n \in \mathrm{End}^n(\mathcal{A})$. The algebra product is the usual concatenation, whereas the shifted law reads

$$w_1 \ \overline{conc}\ w_2 := w_1 T_n(w_2),$$

where $n = max\{j \geq 1 || w_1|_{y_j} \neq 0\}$.

One can easily check that the following spaces are subalgebras of $(K\langle Y\rangle, \overline{conc})$:

(1) the space generated by packed words (i.e., the words whose alphabet indices are of the form $[1 \ldots q]$);
(2) the space generated by injective words (each letter occurs at most once);
(3) the space generated by permutation words (packed and injective, see [7]);
(4) the space generated by increasing (respectively strictly increasing) words; i.e., $w = y_{j_1} y_{j_2} \cdots y_{j_k}$ such that the function $r \to j_r$ is increasing (respectively strictly increasing);
(5) the space generated by disconnected words (i.e., $w = y_{j_1} y_{j_2} \cdots y_{j_k}$ such that there exists an index $r < k$ with y_{j_r+1} not occurring in w).

The following lemma gives general conditions for such shifted laws to be associative.

Lemma 4.3. *Let $\mathcal{A}$ be an algebra (whose multiplicative law will be denoted by $\star$) and $\mathcal{A} = \bigoplus_{\alpha\in\mathcal{D}} \mathcal{A}_\alpha$ a decomposition of $\mathcal{A}$ as a direct sum over $\mathcal{D}$, a semigroup ($\mathcal{A}$ is then graded, but only as a vector space). Let $\alpha \mapsto T_\alpha$: $\mathcal{D} \to \mathrm{End}^{gr}(\mathcal{A})$ be a semigroup homomorphism such that $T_\alpha \in \mathrm{End}^\alpha(\mathcal{A})$. Explicitly, for all $\alpha, \beta \in \mathcal{D}$, $x \in \mathcal{A}_\beta$, we have*

$$T_\alpha(x) \in \mathcal{A}_{\alpha+\beta} \text{ and } T_\alpha \circ T_\beta = T_{\alpha+\beta} \ . \tag{4.13}$$

We suppose that the shifted law defined for $x \in \mathcal{A}_\alpha$ and $y \in \mathcal{A}$ by

$$x \ \bar{\star}\ y = x \star T_\alpha(y) \tag{4.14}$$

is graded for the decomposition

$$\mathcal{A} = \bigoplus_{\alpha\in\mathcal{D}} \mathcal{A}_\alpha$$

(i.e., if $x \in \mathcal{A}_\alpha$ and $y \in \mathcal{A}_\beta$ then $x \ \bar{\star}\ y \in \mathcal{A}_{\alpha+\beta}$). Then, if the law $\star$ is associative, so is the law $\bar{\star}$.

Proof. One only needs to prove the identity of associativity of $\bar{\star}$ for homogeneous elements. Suppose that $\star$ is associative. Then, for $x \in \mathcal{A}_\alpha$, $y \in \mathcal{A}_\beta$, and $z \in \mathcal{A}$, one has

$$x\bar{\star}(y\bar{\star}z) = x \star T_\alpha(y\bar{\star}z) = x \star T_\alpha(y \star T_\beta(z))) = x \star (T_\alpha(y) \star T_\alpha(T_\beta(z)))$$
$$= x \star (T_\alpha(y) \star T_{\alpha+\beta}(z)) = (x \star T_\alpha(y)) \star T_{\alpha+\beta}(z) = \underbrace{(x \bar{\star} y)}_{\in \mathcal{A}_{\alpha+\beta}} \star T_{\alpha+\beta}(z) = (x\bar{\star}y)\bar{\star}z.$$

□

5. Application to the structure of LDIAG(q_c, q_s)

5.1. **Associativity of LDIAG(q_c, q_s) using the previous tools.** As was stated in Section 2.2, we just have to prove Proposition 2.1, and we keep the notations of it. We first observe, from Section 4.2, that, for a semigroup S of type (D)[8], graded by a degree function $|\ |_d :\ S \to \mathbb{N}$, the comultiplication $\Delta_1 :\ K[S] \to K[S] \otimes K[S]$ given for $s \in S$ by

$$\Delta_1(s) = \sum_{rt=s} q_s^{|r|_d|t|_d} r \otimes t \tag{5.1}$$

is coassociative.

Now we endow $K\langle S\rangle \otimes K\langle S\rangle$ with the structure of coloured product given by the bicharacter on S^*

$$\chi(u, v) = \prod_{\substack{1\le i\le |u|\\ 1\le j\le |v|}} q_c^{|u[i]|_d|v[j]|_d}. \tag{5.2}$$

One defines a mapping $\Delta :\ S \to K\langle S\rangle \otimes K\langle S\rangle$ by

$$\Delta(s) = s \otimes 1_{S^*} + 1_{S^*} \otimes s + \Delta_1(s), \tag{5.3}$$

which is extended to an algebra homomorphism $\Delta : K\langle S\rangle \to K\langle S\rangle \otimes K\langle S\rangle$. Note that $V = \bigoplus_{x\in S\cup\{1_{S^*}\}} K.x = KS \oplus K.1_{S^*}$ is a subcoalgebra for Δ, and the coalgebra V is, by Section 4.1.2, still coassociative. Now one has to prove that the following rectangle is commutative:

$$\begin{array}{ccc} K\langle S\rangle & \xrightarrow{\Delta} & K\langle S\rangle \otimes K\langle S\rangle \\ \Delta\downarrow & & \downarrow Id\otimes\Delta \\ K\langle S\rangle \otimes K\langle S\rangle & \xrightarrow{\Delta\otimes Id} & K\langle S\rangle \otimes K\langle S\rangle \otimes K\langle S\rangle \end{array} \quad . \tag{5.4}$$

By Note (3.4).II), all the arrows are algebra homomorphisms, and, in particular, the composites $(Id \otimes \Delta) \circ \Delta$ and $(\Delta \otimes Id) \circ \Delta$, which, as we just proved, coincide on S (coassociativity of the subcoalgebra V). This shows that the rectangle (5.4) is commutative.

End of the duality. We denote by $\downarrow$ the law which is dual to Δ. This law, being dual to a coassociative comultiplication, is associative. We prove that it satisfies the

[8] After [2], a semigroup S of type (D) is such that the product mapping $S \times S \to S$ has finite fibers.

same recursion as in Proposition 2.1, so $\downarrow=\uparrow$. It is sufficient to prove the recursion for non-empty factors. One has

$$\begin{aligned}
au\downarrow bv &= \sum_{w\in S^+}\langle au\downarrow bv|w\rangle w\\
&= \sum_{\substack{x\in S\\ w_1\in S^*}}\langle au\downarrow bv|xw_1\rangle xw_1\\
&= \sum_{\substack{x\in S\\ w_1\in S^*}}\langle au\otimes bv|\Delta(x)\Delta(w_1)\rangle xw_1\\
&= \sum_{\substack{x\in S\\ w_1\in S^*}}\langle au\otimes bv|(x\otimes 1+1\otimes x+\sum_{yz=x}\chi(y,z)y\otimes z)\Delta(w_1)\rangle xw_1\\
&= \sum_{\substack{x\in S\\ w_1\in S^*}}\langle au\otimes bv|(x\otimes 1)\Delta(w_1)\rangle xw_1+\sum_{\substack{x\in S\\ w_1\in S^*}}\langle au\otimes bv|(1\otimes x)\Delta(w_1)\rangle xw_1\\
&\qquad +\sum_{\substack{x\in S\\ w_1\in S^*}}\langle au\otimes bv|(\sum_{yz=x}\chi(y,z)y\otimes z)\Delta(w_1)\rangle xw_1\\
&= \sum_{\substack{x=a\\ w_1\in S^*}}\langle au\otimes bv|(x\otimes 1)\Delta(w_1)\rangle xw_1+\sum_{\substack{x\in S\\ w_1\in S^*}}\langle au\otimes bv|(1\otimes x)\sum_{i,j}\beta_{ij}w_i\otimes w_j\rangle xw_1\\
&\qquad +\sum_{\substack{x\in S\\ w_1\in S^*}}\langle au\otimes bv|(\sum_{yz=x}\chi(y,z)y\otimes z)\sum_{i,j}\beta_{ij}w_i\otimes w_j\rangle xw_1\\
&= a(u\downarrow bv)+q_c^{|au|_d|b|_d}b(au\downarrow v)+q_c^{|u|_d|b|_d}q_s^{|a|_d|b|_d}(a.b)(u\downarrow v)\ ,
\end{aligned}$$

which proves the claim.

5.2. **Structure of LDIAG(q_c,q_s).** This section is devoted to the thorough study of the structure of **LDIAG**(q_c,q_s), using that of the algebra of $(\mathfrak{MON}^+(X))^*$ endowed with the shifted law $\bar{\uparrow}$.

We first investigate the structure of the monoid $((\mathfrak{MON}^+(X))^*,\bar{\star})$, extending to some extent Proposition 3.1 of [6]. For a general monoid, $(M,\star,1_M)$, the irreducible elements are the elements $x\neq 1_M$ such that $x=y\star z$ implies $1_M\in\{y,z\}$. The set of these elements will be denoted by $irr(M)$. For convenience, in the following statement, M stands for the monoid $((\mathfrak{MON}^+(X))^*,\bar{\star})$, $M^+=M-\{1_M\}$, and M_c is the submonoid of codes of diagrams (i.e., words which satisfy Eq. (2.6)).

The monoid M is free. An element $w=m_1.m_2.\cdots.m_l\in M^+$ (hence $l=|w|>0$) is reducible if and only if there exists $0<k<l$ such that

$$\Big(indices(Alph(m_1.m_2.\cdots.m_k))\Big)\prec\Big(indices(Alph(m_{k+1}.m_{k+2}.\cdots.m_l))\Big), \tag{5.5}$$

where, for two nonempty subsets $X,Y\subset\mathbb{N}^+$, one writes $\prec$ for the relation of majoration, i.e.,

$$(\forall(x,y)\in X\times Y)(x<y)\ . \tag{5.6}$$

One checks readily that the monoid M_c is generated by the subalphabet $irr(M) \cap M_c$, and therefore it is free.

Now we need a classical tool of general algebra (see [4, Ch. III] for details). Let $(\mathcal{A}, \mu)$ be an algebra endowed with an increasing exhaustive filtration $(\mathcal{A}_n)_{n\in\mathbb{N}}$ (i.e., two-sided ideals such that $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ and $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n = \mathcal{A}$). It is classical to construct the associated graded algebra $Gr(\mathcal{A}) = \bigoplus_{n\geq 0} \mathcal{A}_n/\mathcal{A}_{n-1}$ by passing the law to quotients, i.e., $\bar{\mu}_{p,q} : \mathcal{A}_p/\mathcal{A}_{p-1} \otimes \mathcal{A}_q/\mathcal{A}_{q-1} \to \mathcal{A}_{p+q}/\mathcal{A}_{p+q-1}$ (one sets $\mathcal{A}_{-1} = \{0\}$). A classical lemma (and easy exercise) states that, if the associated graded algebra is free, so is $\mathcal{A}$.

Now, returning to $(K\langle \mathfrak{MON}^+(X)\rangle, \bar{\uparrow})$ ($\bar{\uparrow}$ is the shifted deformed law), one constructs a filtration by the number of irreducible components of a word of monomials (call it $l(w)$ for $w \in \mathfrak{MON}^+(X)$). From (2.10), one gets

$$w_1 \bar{\uparrow} w_2 = w_1 \bar{\star} w_2 + \sum_{l(w)<l(w_1)+l(w_2)} P_w(q_c, q_s) w, \tag{5.7}$$

with $P_w \in K[q_c, q_s]$ (indeed, $\bar{\uparrow}$ is the same law as in (2.10) but shifted). Consequently, the associated graded algebra is, by a triangularity argument, free. One can then state the following structure theorem.

Theorem 5.1. *The algebra $K\langle \mathfrak{MON}^+(X)\rangle$, endowed with the shifted deformed law $\bar{\uparrow}$, is free on the irreducible words. Furthermore, the algebra* **LDIAG**(q_c, q_s)*, isomorphic to a subalgebra generated by irreducible words, is free for every choice of* (q_c, q_s).

6. Conclusion

To summarize, we can state that the deformed algebra **LDIAG**(q_c, q_s), which originates from a special quantum field theory [1], is free, and its law can be constructed from very general procedures: it is a shifted twisted law. Before shifting, one can observe that the law is, in fact, dual to a comultiplication on a free algebra. This comultiplication is a perturbation, with q_s (the superposition parameter) of the shuffle comultiplication on this free algebra. The parameter q_s is obtained by addition of a perturbating factor which is just dual to a (diagonally) deformed law of a semigroup, whereas the crossing parameter q_c is obtained by extending a colour factor of an algebra to the tensor structure (i.e., to words).

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