ASYMPTOTICS ON RANDOM DISCRETE OBJECTS

Michael Drmota

Institute of Discrete Mathematics and Geometry

Vienna University of Technology

A 1040 Wien, Austria

michael.drmota@tuwien.ac.at

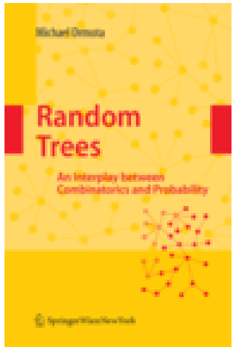
http://www.dmg.tuwien.ac.at/drmota/

SLC 62, Heilsbronn, February 22–25, 2009

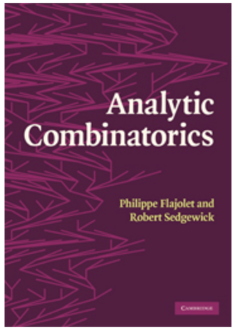
Contents

References

Books



Michael Drmota,
Random Trees, Springer, Wien-New York, 2009.



Philippe Flajolet and Robert Sedgewick,
Analytic Combinatorics, Cambridge University Press, 2009.
(http://algo.inria.fr/flajolet/Publications/books.html)

Asymptotic analysis of random objects

Levels of complexity:

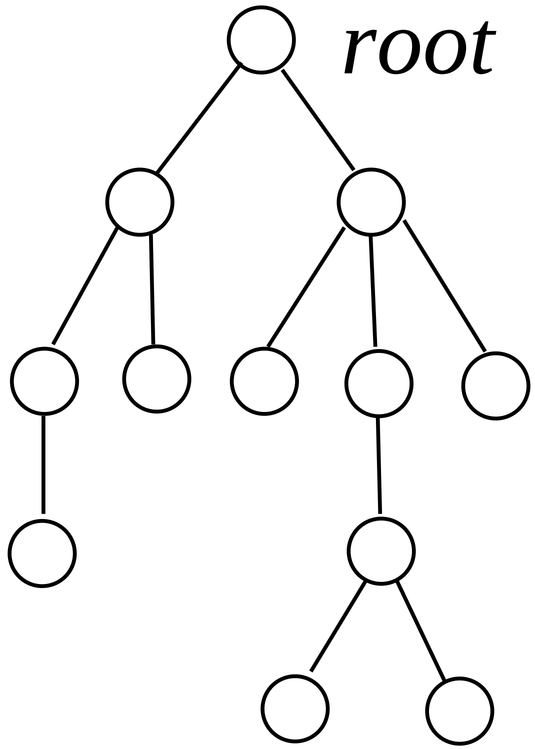
1. Asymptotic enumeration
2. Distribution of (shape) parameters
3. Asymptotic shape (= continuous limiting object)

Contents 1

I. RANDOM TREES

Random Trees

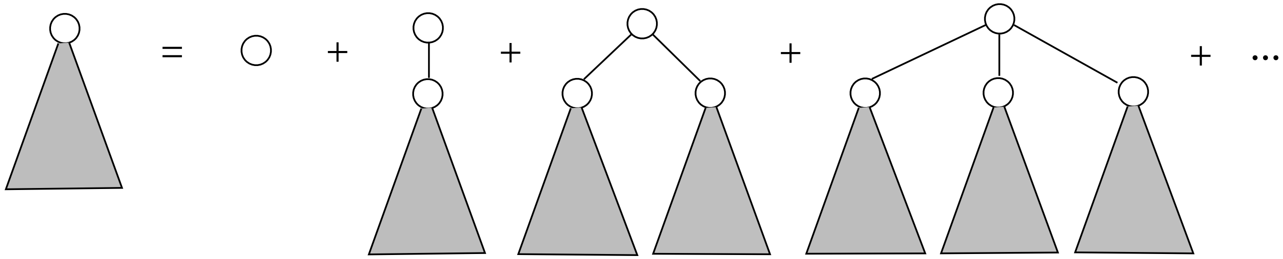
Catalan trees



rooted, ordered (or plane) tree

Random Trees

Catalan trees. g_n = number of Catalan trees of size n; $\boxed{G(x) = \sum_{n\geq 1} g_n x^n}$



$$\boxed{G(x) = x(1 + G(x) + G(x)^2 + \cdots) = \frac{x}{1 - G(x)}}$$

$$G(x) = \frac{1 - \sqrt{1 - 4x}}{2} \quad \Longrightarrow \quad \boxed{g_n = \frac{1}{n}\binom{2n-2}{n-1}} \sim \frac{4^{n-1}}{\sqrt{\pi} \cdot n^{3/2}}$$

(Catalan numbers)

Random Trees

Catalan trees with singularity analysis (to be discussed later)

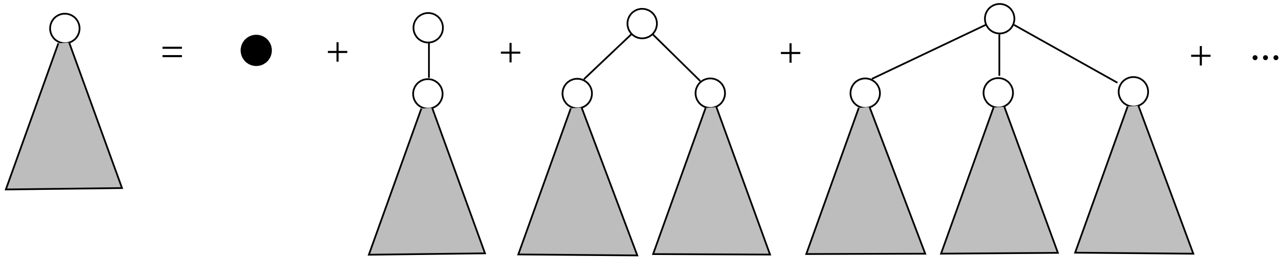
$$G(x) = \frac{1 - \sqrt{1-4x}}{2} = \frac{1}{2} - \frac{1}{2}\sqrt{1-4x}$$

$$\Longrightarrow \quad g_n \sim -\frac{1}{2} \cdot \frac{4^n n^{-3/2}}{\Gamma(-\frac{1}{2})} = \frac{4^{n-1}}{\sqrt{\pi} \cdot n^{3/2}}$$

Random Trees

Number of leaves of Catalan trees

$g_{n,k}$ = number of Catalan trees of size n with k leaves.



$$G(x,u) = xu + x(G(x,u) + G(x,u)^2 + \cdots = xu + \frac{xG(x,u)}{1 - G(x,u)}$$

$$\Longrightarrow \quad G(x,u) = \frac{1}{2}\left(1 + (u-1)x - \sqrt{1 - 2(u+1)x + (u-1)^2x^2}\right)$$

$$\Longrightarrow \quad g_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n-1}{k} \sim \frac{4^n}{\pi n^2}\exp\left(-\frac{(k-\frac{n}{2})^2}{\frac{1}{4}n}\right) \quad \text{for } k \approx \frac{n}{2}$$

Random Trees

Number of leaves of Catalan trees

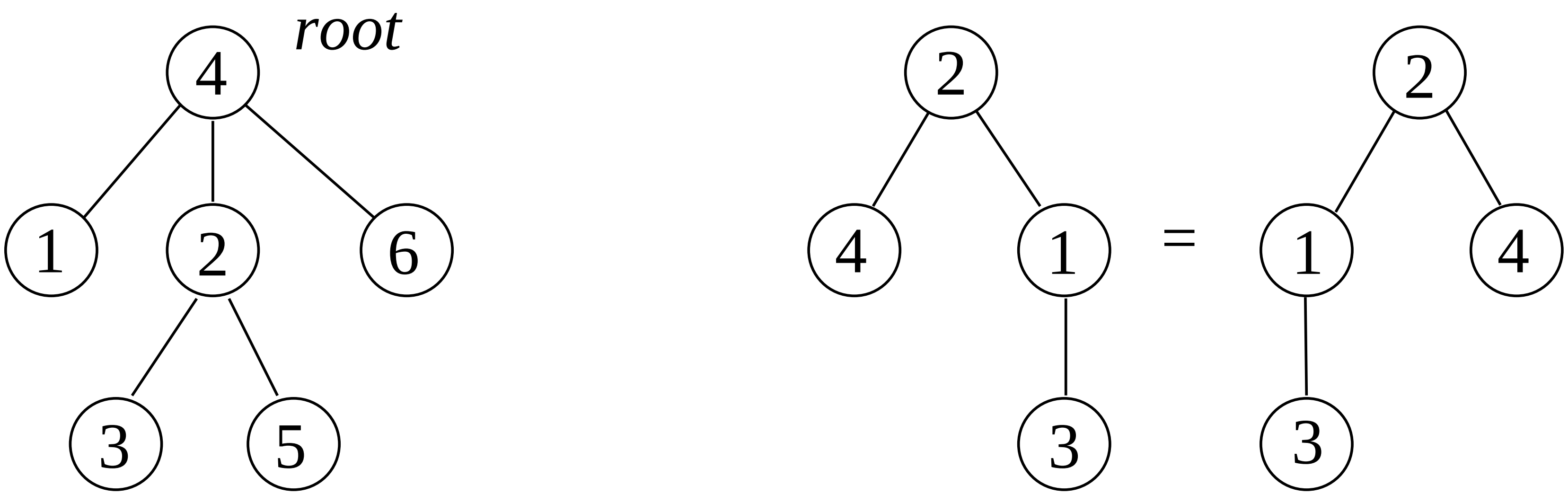
$$\boxed{G(x,u) = g(x,u) - h(x,u)\sqrt{1 - \frac{x}{\rho(u)}}}$$

for certain analytic function $g(x,u), h(x,u)$, and $\rho(u)$.

$$\Longrightarrow \quad g_{n,k} = ???$$

Random Trees

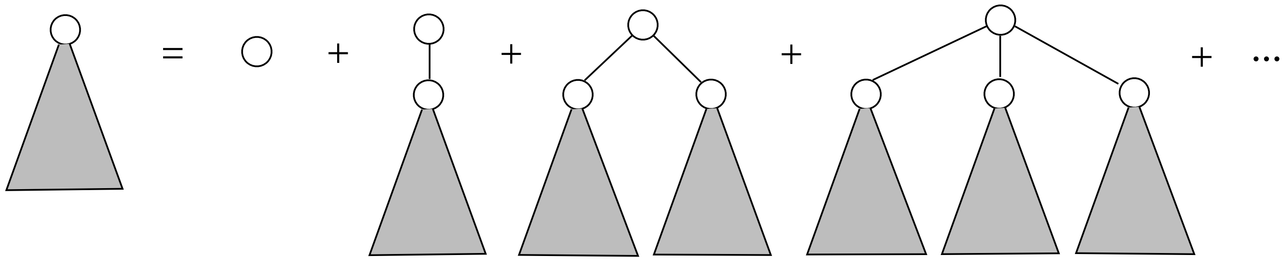
Cayley Trees:



labelled, rooted, unordered (or non-plane) tree

Random Trees

Cayley Trees. r_n =number of Cayley trees of size n; $\boxed{R(x) = \sum_{n\geq 1} r_n \frac{x^n}{n!}}$



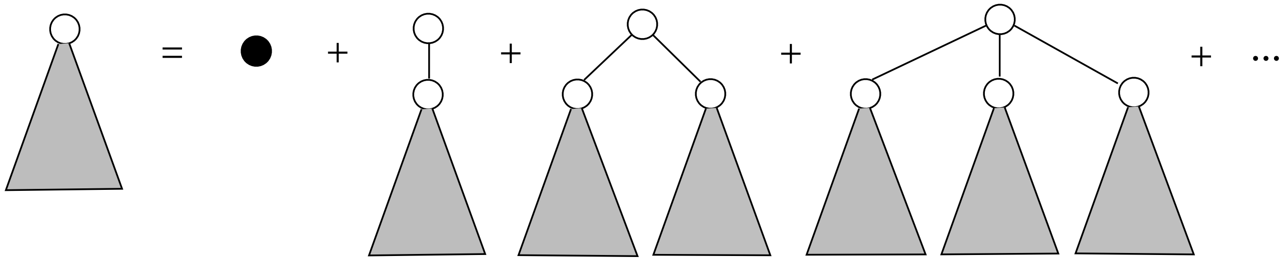
$$\boxed{R(x) = x\left(1 + R(x) + \frac{R(x)^2}{2!} + \frac{R(x)^3}{3!} + \cdots\right) = x\,e^{R(x)}}$$

$\Longrightarrow$ $\boxed{r_n = n^{n-1}}$ ... by Lagrange inversion

Random Trees

Number of leaves of Cayley trees

$r_{n,k}$ = number of Cayley trees of size n with k leaves.



$$R(x,u) = xu + x\left(R(x,u) + \frac{R(x,u)^2}{2!} + \frac{R(x,u)^3}{3!} + \cdots\right) = xe^{R(x,u)} + x(u-1)$$

$$\Longrightarrow \quad R(x,u) = ???$$

Functional equations

Catalan trees: $G(x,u) = xu + xG(x,u)/(1 - G(x,u))$

Cayley trees: $R(x,u) = xe^{R(x,u)} + x(u-1)$

Recursive structure leads to functional equation for gen. func.:

$$\boxed{\boxed{A(x,u) = \Phi(x,u,A(x,u))}}$$

Functional equations

Linear functional equation: $\Phi(x,u,a) = \Phi_0(x,u) + a\Phi_1(x,u)$

$$\Longrightarrow \quad A(x,u) = \frac{\Phi_0(x,u)}{1-\Phi_1(x,u)}$$

Usually these kinds of generating functions are easy to handle, since they are explicit.

Functional equations

Non-linear functional equations: $\Phi_{aa}(x,u,a) \neq 0$.

Suppose that $\boxed{A(x,u) = \Phi(x,u,A(x,u))}$, where $\Phi(x,u,a)$ has a power series expansion at $(0,0,0)$ with non-negative coefficients and $\Phi_{aa}(x,u,a) \neq 0$.

Let $x_0 > 0$, $a_0 > 0$ (inside the region of convergence) satisfy the system of equations:

$$\boxed{a_0 = \Phi(x_0, 1, a_0), \quad 1 = \Phi_a(x_0, 1, a_0)}.$$

Then there exists analytic function $g(x,u), h(x,u)$, and $\rho(u)$ such that locally

$$\boxed{A(x,u) = g(x,u) - h(x,u)\sqrt{1 - \frac{x}{\rho(u)}}}.$$

Functional equations

Idea of the Proof.

Set $\boxed{F(x,u,a) = \Phi(x,u,a) - a}$. Then we have

$$
\begin{aligned}
F(x_0, 1, a_0) &= 0 \\
F_a(x_0, 1, a_0) &= 0 \\
F_x(x_0, 1, a_0) &\neq 0 \\
F_{aa}(x_0, 1, a_0) &\neq 0.
\end{aligned}
$$

Weierstrass preparation theorem implies that there exist analytic functions $H(x,u,a)$, $p(x,u)$, $q(x,u)$ with $H(x_0,1,a_0) \neq 0$, $p(x_0,1) = q(x_0,1) = 0$ and

$$
\boxed{F(x,u,a) = H(x,u,a)\Big((a-a_0)^2 + p(x,u)(a-a_0) + q(x,u)\Big)}.
$$

Functional equations

$$F(x,u,a) = 0 \quad \Longleftrightarrow \quad \boxed{(a-a_0)^2 + p(x,u)(a-a_0) + q(x,u) = 0}.$$

Consequently

$$\begin{aligned} A(x,u) &= a_0 - \frac{p(x,u)}{2} \pm \sqrt{\frac{p(x,u)^2}{4} - q(x,u)} \\ &= \boxed{g(x,u) - h(x,u)\sqrt{1 - \frac{x}{\rho(u)}}}, \end{aligned}$$

where we write

$$\frac{p(x,u)^2}{4} - q(x,u) = K(x,u)(x-\rho(u))$$

which is again granted by the Weierstrass preparation theorem and we set

$$g(x,u) = a_0 - \frac{p(x,u)}{2} \quad \text{and} \quad h(x,u) = \sqrt{-K(x,u)\rho(u)}.$$

Random Trees

Catalan Trees $G(x,u) = xu + \frac{xG(x,u)}{1-G(x,u)}$

$$\Longrightarrow \quad G(x,u) = g(x,u) - h(x,u)\sqrt{1 - \frac{x}{\rho(u)}}$$

$$G(x,1) = G(x) = g(x,1) - h(x,1)\sqrt{1 - \frac{x}{\rho(1)}}, \quad \rho(1) = \frac{1}{4}$$

Cayley Trees $T(x,u) = xe^{T(x,u)} + x(u-1)$

$$\Longrightarrow \quad T(x,u) = g(x,u) - h(x,u)\sqrt{1 - \frac{x}{\rho(u)}}$$

$$T(x,1) = T(x) = g(x,1) - h(x,1)\sqrt{1 - \frac{x}{\rho(1)}}, \quad \rho(1) = \frac{1}{e}$$

Algebraic Singularities

Singular expansion

$$
\begin{aligned}
A(x) &= g(x) - h(x)\sqrt{1-\frac{x}{\rho}} \\
&= \left(g_0 + g_1(x-\rho) + g_2(x-\rho)^2 + \cdots\right) \\
&\quad + \left(h_0 + h_1(x-\rho) + h_2(x-\rho)^2 + \cdots\right)\sqrt{1-\frac{x}{\rho}} \\
&= a_0 + a_1\left(1-\frac{x}{\rho}\right)^{\frac{1}{2}} + a_2\left(1-\frac{x}{\rho}\right)^{\frac{2}{2}} + a_3\left(1-\frac{x}{\rho}\right)^{\frac{3}{2}} + \cdots \\
&= a_0 + a_1\left(1-\frac{x}{\rho}\right)^{\frac{1}{2}} + a_2\left(1-\frac{x}{\rho}\right) + O\left(\left(1-\frac{x}{\rho}\right)^{\frac{3}{2}}\right)
\end{aligned}
$$

Algebraic Singularities

Singular expansion

$$
\begin{aligned}
A(x) &= \boxed{g(x) - h(x)\sqrt{1 - \frac{x}{\rho}}} \\
&= \Big(g_0 + g_1(x-\rho) + g_2(x-\rho)^2 + \cdots\Big) \\
&\quad + \Big(h_0 + h_1(x-\rho) + h_2(x-\rho)^2 + \cdots\Big)\sqrt{1 - \frac{x}{\rho}} \\
&= a_0 + a_1\left(1 - \frac{x}{\rho}\right)^{\frac{1}{2}} + a_2\left(1 - \frac{x}{\rho}\right)^{\frac{2}{2}} + a_3\left(1 - \frac{x}{\rho}\right)^{\frac{3}{2}} + \cdots \\
&= a_0 + a_1\boxed{\left(1 - \frac{x}{\rho}\right)^{\frac{1}{2}}} + a_2\left(1 - \frac{x}{\rho}\right) + \boxed{O\left(\left(1 - \frac{x}{\rho}\right)^{\frac{3}{2}}\right)}
\end{aligned}
$$

Algebraic Singularities

Singularity Analysis

Lemma 1 *Suppose that*

$$\boxed{y(x) = \left(1 - \frac{x}{x_0}\right)^{-\alpha}}.$$

Then

$$\boxed{y_n = (-1)^n \binom{-\alpha}{n} x_0^{-n} = \frac{n^{\alpha-1}}{\Gamma(\alpha)} x_0^{-n} + \mathcal{O}\left(n^{\alpha-2} x_0^{-n}\right).}$$

Remark: This asymptotic expansion is uniform in α if α varies in a compact region of the complex plane.

Algebraic Singularities

Singularity Analysis

Lemma 2 (Flajolet and Odlyzko) *Let*

$$y(x) = \sum_{n \ge 0} y_n \, x^n$$

be analytic in a region

$$\Delta = \{x : |x| < x_0 + \eta, \, |\arg(x - x_0)| > \delta\},$$

$x_0 > 0$, $\eta > 0$, $0 < \delta < \pi/2$.

Suppose that for some real α

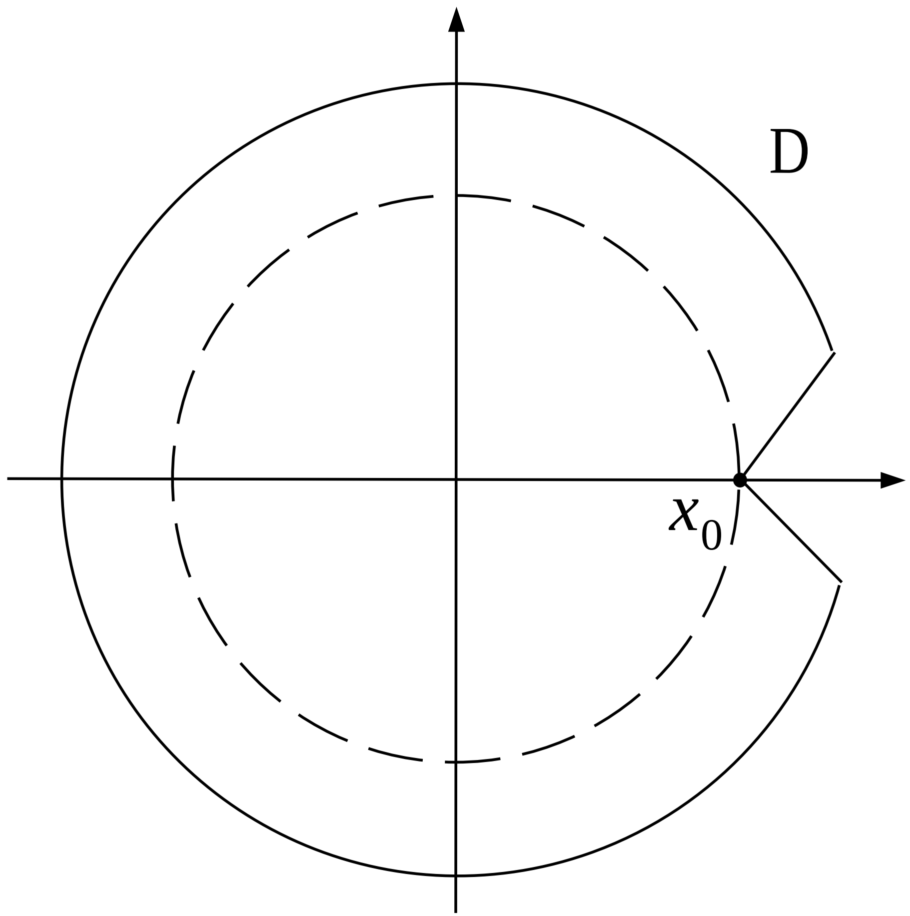
$$\boxed{y(x) = \mathcal{O}\left((1 - x/x_0)^{-\alpha}\right) \qquad (x \in \Delta).}$$

Then

$$\boxed{y_n = \mathcal{O}\left(x_0^{-n} n^{\alpha - 1}\right).}$$

Algebraic Singularities

Δ-region



Algebraic Singularities

Singularity Analysis

Suppose that

$$A(x) = g(x) - h(x)\sqrt{1 - \frac{x}{\rho}}$$
$$= a_0 + a_1\left(1 - \frac{x}{\rho}\right)^{\frac{1}{2}} + a_2\left(1 - \frac{x}{\rho}\right) + O\left(\left(1 - \frac{x}{\rho}\right)^{\frac{3}{2}}\right)$$

for $x \in \Delta$ then

$$\boxed{a_n = [x^n]\, A(x) = \frac{h(\rho)}{2\sqrt{\pi}}\rho^{-n} n^{-\frac{3}{2}}\left(1 + O\left(\frac{1}{n}\right)\right)}.$$

Algebraic Singularities

Singularity Analysis

Suppose that

$$A(x,u) = g(x,u) - h(x,u)\sqrt{1 - \frac{x}{\rho(u)}}$$

$$= a_0(u) + a_1(u)\left(1 - \frac{x}{\rho(u)}\right)^{\frac{1}{2}} + a_2(u)\left(1 - \frac{x}{\rho(u)}\right) + O\left(\left(1 - \frac{x}{\rho(u)}\right)^{\frac{3}{2}}\right)$$

for $x \in \Delta = \Delta(u)$ then

$$\boxed{a_n(u) = [x^n]\, A(x,u) = \frac{h(\rho(u),u)}{2\sqrt{\pi}}\rho(u)^{-n} n^{-\frac{3}{2}}\left(1 + O\left(\frac{1}{n}\right)\right)}.$$

Probabilistic Model

a_n ... number of objects of size n

$a_{n,k}$ ... number of objects of size n, where a certain **parameter** has value k

If all objects of size n are considered to be **equally likely** then the parameter can be considered as a random variable X_n with distribution

$$\boxed{\mathbb{P}\{X_n = k\} = \frac{a_{nk}}{a_n}}.$$

Probabilistic Model

Generating functions and the probability generating function

$$A(x,u) = \sum_{n,k} a_{n,k} x^n u^k$$

$$\Longrightarrow \quad \boxed{\mathbb{E}\, u^{X_n}} = \sum_{k\ge 0} \mathbb{P}\{X_n = k\}\, u^k$$

$$= \sum_{k\ge 0} \frac{a_{nk}}{a_n}\, u^k$$

$$= \boxed{\frac{[x^n]\, A(x,u)}{[x^n]\, A(x,1)}} = \frac{a_n(u)}{a_n}$$

Probabilistic Model

Generating functions and the probability generating function

$$A(x,u) = g(x,u) - h(x,u)\sqrt{1 - \frac{x}{\rho(u)}}$$

$$\Longrightarrow \quad \boxed{\mathbb{E}\, u^{X_n}} = \frac{[x^n]\, A(x,u)}{[x^n]\, A(x,1)}$$

$$= \frac{\frac{h(\rho(u),u)}{2\sqrt{\pi}}\rho(u)^{-n} n^{-\frac{3}{2}}\left(1 + O\left(\frac{1}{n}\right)\right)}{\frac{h(\rho(1),1)}{2\sqrt{\pi}}\rho(1)^{-n} n^{-\frac{3}{2}}\left(1 + O\left(\frac{1}{n}\right)\right)}$$

$$= \boxed{\frac{h(\rho(u),u)}{h(\rho(1),1)}\left(\frac{\rho(1)}{\rho(u)}\right)^{n}\left(1 + O\left(\frac{1}{n}\right)\right)}.$$

Probabilistic Model

Quasi-Power Theorem (Hwang)

Let X_n be a sequence of random variables with the property that

$$\boxed{\mathbb{E}\, u^{X_n} = A(u) \cdot B(u)^{\lambda_n} \cdot \left(1 + O\left(\frac{1}{\phi_n}\right)\right)}$$

holds uniformly in a complex neighborhood of $u = 1$, $\boxed{\lambda_n \to \infty}$ and $\boxed{\phi_n \to \infty}$, and $A(u)$ and $B(u)$ are analytic functions in a neighborhood of $u = 1$ with $A(1) = B(1) = 1$. Set

$$\mu = B'(1) \quad \text{and} \quad \sigma^2 = B''(1) + B'(1) - B'(1)^2.$$

$$\Longrightarrow \quad \boxed{\mathbb{E}\, X_n = \mu\lambda_n + O\left(1 + \lambda_n/\phi_n\right)}, \quad \boxed{\mathbb{V}\, X_n = \sigma^2\lambda_n + O\left(1 + \lambda_n/\phi_n\right)},$$

$$\boxed{\frac{X_n - \mathbb{E}\, X_n}{\sqrt{\mathbb{V}\, X_n}} \xrightarrow{\text{d}} N(0,1)} \quad (\sigma^2 \neq 0).$$

Probabilistic Model

Sums of independent random variables

$X_n = \xi_1 + \xi_2 + \cdots + \xi_n$, where ξ_j are i.i.d.

$$B(u) = \mathbb{E}\, u^{\xi_j}$$

$$\Longrightarrow \quad \mathbb{E}\, u^{X_n} = \mathbb{E}\, u^{\xi_1+\xi_2+\cdots+\xi_n} = \mathbb{E}\, u^{\xi_1} \cdot \mathbb{E}\, u^{\xi_2} \cdots \mathbb{E}\, u^{\xi_n} = B(u)^n.$$

Probabilistic Model

COMBINATORIAL CENTRAL LIMIT THEOREM

Suppose that a sequence of random variables X_n has distribution

$$\mathbb{P}\{X_n = k\} = \frac{a_{nk}}{a_n},$$

where the generating function $A(x,u) = \sum_{n,k} a_{n,k} x^n u^k$ satisfies a functional equation of the form $\boxed{A(x,u) = \Phi(x,u,A(x,u))}$, where $\Phi(x,u,a)$ has a power series expansion at $(0,0,0)$ with non-negative coefficients and $\Phi_{aa}(x,u,a) \neq 0$.

Let $x_0 > 0$, $a_0 > 0$ (inside the region of convergence) satisfy the system of equations:

$$\boxed{a_0 = \Phi(x_0, 1, a_0), \quad 1 = \Phi_a(x_0, 1, a_0)}.$$

Probabilistic Model

COMBINATORIAL CENTRAL LIMIT THEOREM (cont.)

Set

$$\mu = \frac{\Phi_u}{x_0 \Phi_x},$$

$$\sigma^2 = \mu + \mu^2 + \frac{1}{x_0 \Phi_x^3 \Phi_{aa}} \Big(\Phi_x^2 (\Phi_{aa}\Phi_{uu} - \Phi_{au}^2) - 2\Phi_x\Phi_u(\Phi_{aa}\Phi_{xu} - \Phi_{ax}\Phi_{au}) + \Phi_u^2(\Phi_{aa}\Phi_{xx} - \Phi_{ax}^2) \Big),$$

(where all partial derivatives are evaluated at the point $(x_0, a_0, 1)$)

Then we have

$$\boxed{\mathbb{E}\, X_n = \mu n + O(1)} \quad \text{and} \quad \boxed{\mathbb{V}\mathrm{ar}\, X_n = \sigma^2 n + O(1)}$$

and if $\sigma^2 > 0$ then

$$\boxed{\boxed{\frac{X_n - \mathbb{E}\, X_n}{\sqrt{\mathbb{V}\mathrm{ar}\, X_n}} \to N(0,1)}}.$$

Random Trees

Leaves in Catalan trees

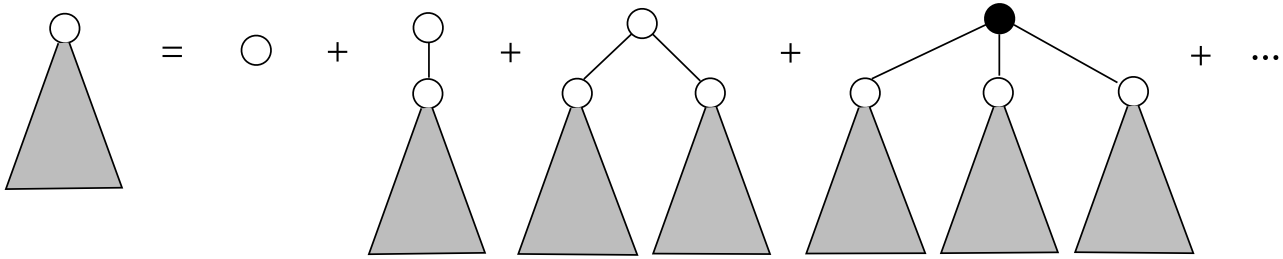
The number of leaves in Catalan trees of size n satisfy a **central limit theorem** with mean $\sim \frac{1}{2}n$ and variance $\sim \frac{1}{8}n$

Leaves in Cayley trees

The number of leaves in Cayley trees of size n satisfy a **central limit theorem** with mean $\sim \frac{1}{e}n$ and variance $\sim \left(\frac{1}{e^2} + \frac{1}{e}\right) n$

Random Trees

Nodes of out-degree d in Catalan trees



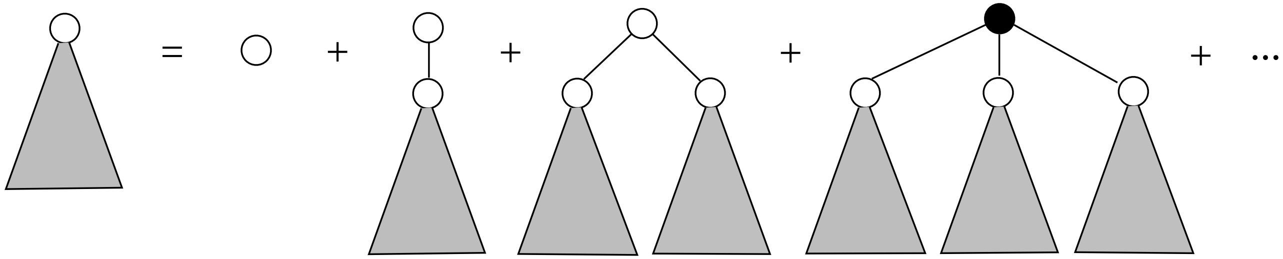
$$G(x,u) = \frac{x}{1 - G(x,u)} + x(u-1)G(x,u)^d$$

The number $X_n^{(d)}$ of nodes with out-degree d in Catalan trees of size n satisfy a **central limit theorem** with mean $\sim \mu_d n$ and variance $\sim \sigma_d^2 n$, where

$$\mu_d = \frac{1}{2^{d+1}} \quad \text{and} \quad \sigma_d^2 = \frac{1}{2^{d+1}} + \frac{1}{2^{2(d+1)}} - \frac{(d-1)^2}{2^{2d+3}}.$$

Random Trees

Nodes of out-degree d in Cayley trees



$$R(x,u) = xe^{R(x,u)} + x(u-1)\frac{R(x,u)^d}{d!}$$

The number of nodes with out-degree d in Cayley trees of size n satisfy a **central limit theorem** with mean $\sim \mu_d n$ and variance $\sim \sigma_d^2 n$, where

$$\mu_d = \frac{1}{e\,d!} \quad \text{and} \quad \sigma_d^2 = \frac{1+(d-1)^2}{e^2(d!)^2} + \frac{1}{e\,d!}$$

Random Trees

Degree distribution for Catalan trees

$p_{n,d}$ ... probability that a random node in a random Catalan tree of size n has out-degree d:

$$\mathbb{E}\, X_n^{(d)} = n\, p_{n,d}$$

$$p_d := \lim_{n\to\infty} p_{n,d} = \frac{1}{2^{d+1}} = \mu_d$$

Probability generating function of the out-degree distribution:

$$\boxed{p(w) := \sum_{d\ge 0} p_d w^d = \frac{1}{2-w}}$$

Random Trees

Degree distribution for Cayley trees

$p_{n,d}$ ... probability that a random node in a random Cayley tree of size n has out-degree d:

$$\mathbb{E}\, X_n^{(d)} = n\, p_{n,d}$$

$$p_d := \lim_{n\to\infty} p_{n,d} = \frac{1}{e\, d!} = \mu_d$$

Probability generating function of the out-degree distribution:

$$\boxed{p(w) := \sum_{d\geq 1} p_d w^d = e^{w-1}}$$

Random Trees

Maximum degree

Δ_n ... maximum out-degree

$X_n^{(>d)} = X_n^{(d+1)} + X_n^{(d+2)} + \cdots$ ... **number of nodes of out-degree** $> d$.

$$\boxed{\Delta_n > d \iff X_n^{(>d)} > 0}$$

Random Trees

First moment method

X ... a discrete random variable on non-negative integers.

$$\Longrightarrow \quad \boxed{\mathbb{P}\{X > 0\} \le \min\{1, \mathbb{E}\, X\}}$$

Proof

$$\mathbb{E}\, X = \sum_{k\ge 0} k\, \mathbb{P}\{X = k\} \ge \sum_{k\ge 1} \mathbb{P}\{X = k\} = \mathbb{P}\{X > 0\}.$$

Random Trees

Second moment method

X is a non-negative random variable with finite second moment.

$$\Longrightarrow \quad \boxed{\mathbb{P}\{X > 0\} \geq \frac{(\mathbb{E}\, X)^2}{\mathbb{E}\,(X^2)}}$$

Proof

$$\mathbb{E}\, X = \mathbb{E}\left(X \cdot \mathbf{1}_{[X>0]}\right) \leq \sqrt{\mathbb{E}\,(X^2)}\sqrt{\mathbb{E}\,(\mathbf{1}_{[X>0]}^2)} = \sqrt{\mathbb{E}(X^2)}\sqrt{\mathbb{P}\{X > 0\}}.$$

Random Trees

Tail estimates and expected value

- $\mathbb{P}\{\Delta_n > d\} \le \min\{1, \mathbb{E}\, X_n^{(>d)}\}$

- $\mathbb{P}\{\Delta_n > d\} \ge \dfrac{(\mathbb{E}\, X_n^{(>d)})^2}{\mathbb{E}\,(X_n^{(>d)})^2}$

$$\Longrightarrow \quad \mathbb{P}\{\Delta_n \le d\} \le 1 - \frac{(\mathbb{E}\, X_n^{(>d)})^2}{\mathbb{E}\,(X_n^{(>d)})^2} = \frac{\mathbb{V}\mathrm{ar}\, X_n^{(>d)}}{\mathbb{E}\,(X_n^{(>d)})^2}$$

- $\mathbb{E}\,\Delta_n = \displaystyle\sum_{d\ge 0} \mathbb{P}\{\Delta_n > d\}$

Random Trees

Maximum degree of Catalan trees

$$\mathbb{E}\, X_n^{(>d)} \sim \frac{n}{2^{d+1}}, \qquad \mathbb{V}\mathrm{ar}\,(X_n^{(>d)})^2 \sim n\left(\frac{1}{2^{d+1}} + \frac{1}{2^{2(d+1)}} - \frac{(d-1)^2}{2^{2d+3}}\right)$$

$$\Longrightarrow \quad \mathbb{P}\{\Delta_n > d\} \le \min\left\{1, \frac{n}{2^{d+1}}\right\},$$

$$\mathbb{P}\{\Delta_n \le d\} = 1 - \mathbb{P}\{\Delta_n > d\}$$

$$\le \frac{1}{n}\,\frac{\frac{1}{2^{d+1}} + \frac{1}{2^{2(d+1)}} - \frac{(d-1)^2}{2^{2d+3}}}{\frac{1}{2^{2(d+1)}}} \sim \frac{2^{d+1}}{n}$$

$$\Longrightarrow \quad \boxed{\Delta_n \text{ is concentrated at } \log_2 n}$$

Random Trees

Maximum degree of Catalan trees (Carr, Goh and Schmutz)

$$\boxed{\mathbb{P}\{\Delta_n \le k\} = \exp\left(-2^{-(k-\log_2 n+1)}\right) + o(1)}$$

$$\boxed{\mathbb{E}\,\Delta_n = \log_2 n + O(1)}$$

Random Trees

Unrooted trees

p_n ... number of different embeddings of **unrooted** trees of size n in the plane, $\boxed{P(x) = \sum_{n\ge 1} p_n x^n}$:

$$\boxed{P(x) = x \sum_{k\ge 0} Z_{\mathfrak{C}_k}(G(x), G(x^2), \ldots, G(x^k)) - \frac{1}{2}G(x)^2 + \frac{1}{2}G(x^2)},$$

where $G(x) = x/(1 - G(x)) = (1 - \sqrt{1-4x})/2$ and

$$Z_{\mathfrak{C}_k}(x_1, x_2, \ldots, x_k) = \frac{1}{k}\sum_{d|k} \varphi(d) x_d^{k/d}$$

is the cycle index of the cyclic group $\mathfrak{C}_k$ of k elements

Random Trees

Unrooted trees

Cancellation of the $\sqrt{1-4x}$-term:

$$G(x) = \frac{1-\sqrt{1-4x}}{2} \quad \Longrightarrow \quad P(x) = a_0 + a_2(1-4x) + \frac{1}{6}(1-4x)^{3/2} + \cdots$$

$$\Longrightarrow \quad \boxed{p_n = \frac{1}{8\sqrt{\pi}} 4^n n^{-5/2} \left(1 + O(n^{-1})\right)}$$

Random Trees

Degree distribution of unrooted trees

$X_n^{(d)}$ ... number of nodes of degree d in trees of size n

$$\begin{aligned} P(x,u) = x \sum_{k \neq d} & Z_{\mathfrak{C}_k}(G(x,u), G(x^2,u^2), \ldots, G(x^k,u^k)) \\ & + xu Z_{\mathfrak{C}_d}(G(x,u), G(x^2,u^2), \ldots, G(x^d,u^d)) \\ & - \frac{1}{2} G(x,u)^2 + \frac{1}{2} G(x^2,u^2), \end{aligned}$$

where

$$G(x,u) = \frac{x}{1 - G(x,u)} + x(u-1)G(x,u)^{d-1}.$$

Random Trees

Degree distribution of unrooted trees

Cancellation of the $\sqrt{1-4x}$-term:

$$G(x,u) = g(x,u) - h(x,u)\sqrt{1-\frac{x}{\rho(u)}}$$

$$\Longrightarrow \quad P(x,u) = a_0(u) + a_2(u)\left(1-\frac{x}{\rho(u)}\right) + a_3(u)\left(1-\frac{x}{\rho(u)}\right)^{\frac{3}{2}} + \cdots$$

$\Longrightarrow X_n^{(d)}$ satisfies a **central limit theorem** with mean $\sim \mu_{d-1} n$ and variance $\sim \sigma_{d-1}^2 n$, where

$$\mu_d = \frac{1}{2^{d+1}} \quad \text{and} \quad \sigma_d^2 = \frac{1}{2^{d+1}} + \frac{1}{2^{2(d+1)}} - \frac{(d-1)^2}{2^{2d+3}}.$$

Random Trees

Degree distribution of unrooted trees

$p_{n,d}$ ... probability that a random node in a tree of size n has degree d:

$$\mathbb{E}\, X_n^{(d)} = n\, p_{n,d}$$

$$p_d = \lim_{n\to\infty} p_{n,d} = \mu_{d-1} = \frac{1}{2^d}$$

Probability generating function of the degree distribution:

$$\boxed{p(w) = \sum_{d\ge 1} p_d w^d = \frac{w}{2-w}}$$

Random Trees

Maximum degree for unrooted trees

Δ_n ... maximum degree of unrooted trees of size n

$$\boxed{\Delta_n \text{ is concentrated at } \log_2 n}$$

$$\boxed{\mathbb{E}\,\Delta_n = \log_2 n + O(1)}$$

Random Trees

Unrooted labelled trees

$t_n = r_n/n = n^{n-2}$ ... number of different **unrooted** labelled trees of size n: $\boxed{T(x) = \sum_{n\ge 1} t_n \frac{x^n}{n!}}$:

$$\boxed{T(x) = xe^{R(x)} - \frac{1}{2}R(x)^2},$$

where $R(x) = xe^{R(x)}$

Cancellation of the $\sqrt{1-ex}$-term:

$$R(x) = g(x) - h(x)\sqrt{1-ex} \quad \Longrightarrow \quad T(x) = a_0 + a_2(1-4x) + \frac{1}{6}(1-ex)^{3/2} + \cdots$$

Random Trees

Degree distribution of unrooted labelled trees

$X_n^{(d)}$ ... number of nodes of degree d in trees of size n

$$T(x,u) = xe^{R(x,u)} + x(u-1)\frac{R(x,u)^d}{d!} - \frac{1}{2}R(x,u)^2,$$

where

$$R(x,u) = xe^{R(x,u)} + x(u-1)\frac{R(x,u)^{d-1}}{(d-1)!}.$$

Random Trees

Degree distribution of unrooted labelled trees

Cancellation of the $\sqrt{1-4x}$-term:

$$R(x,u) = g(x,u) - h(x,u)\sqrt{1 - \frac{x}{\rho(u)}}$$

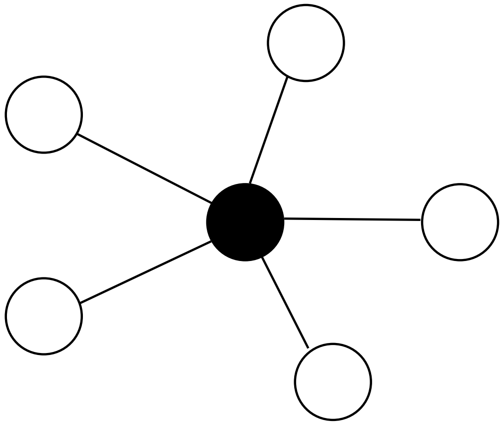
$$\Longrightarrow \quad T(x,u) = a_0(u) + a_2(u)\left(1 - \frac{x}{\rho(u)}\right) + a_3(u)\left(1 - \frac{x}{\rho(u)}\right)^{\frac{3}{2}} + \cdots$$

$\Longrightarrow X_n^{(d)}$ satisfies a **central limit theorem** with mean $\sim \mu_{d-1} n$ and variance $\sim \sigma_{d-1}^2 n$, where

$$\mu_d = \frac{1}{e\,d!} \quad \text{and} \quad \sigma_d^2 = \frac{1 + (d-1)^2}{e^2 (d!)^2} + \frac{1}{e\,d!}$$

Random Trees

Star pattern



$d = 5$

$$X_n^{(d)} = \text{number of nodes of degree } d \text{ in trees of size } n$$
$$= \text{number of star pattern with } d \text{ rays in trees of size } n$$

Systems of Functional equations

COMBINATORIAL CENTRAL LIMIT THEOREM II

Suppose that a sequence of random variables X_n has distribution

$$\mathbb{P}[X_n = k] = \frac{a_{nk}}{a_n},$$

where the generating function $A(x,u) = \sum_{n,k} a_{n,k} x^n u^k$ is given by

$$\boxed{A(x,u) = \Psi(x,u,A_1(x,u),\ldots,A_r(x,u))}$$

for an analytic function Ψ and the generating functions

$$A_1(x,u) = \sum_{n,k} a_{1;n,k} u^k x^n, \ \ldots \ , A_r(x,u) = \sum_{n,k} a_{r;n,k} u^k x^n$$

satisfy a **system of non-linear equations**

$$\boxed{A_j(x,u) = \Phi_j(x,u,A_1(x,u),\ldots,A_r(x,u))}, \quad (1 \le j \le r).$$

Systems of Functional equations

COMBINATORIAL CENTRAL LIMIT THEOREM II (cont.)

Suppose that at least one of the functions $\Phi_j(x, u, a_1, \ldots, a_r)$ is non-linear in $a_1, \ldots, a_r$ and they all have a power series expansion at $(0, 0, 0)$ with non-negative coefficients.

Let $x_0 > 0$, $\mathbf{a}_0 = (a_{0,0}, \ldots, a_{r,0}) > 0$ (inside the region of convergence) satisfy the system of equations: $(\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_r))$

$$\boxed{\mathbf{a}_0 = \mathbf{\Phi}(x_0, 1, \mathbf{a}_0), \quad 0 = \det(\mathbb{I} - \mathbf{\Phi}_{\mathbf{a}}(x_0, 1, \mathbf{a}_0)}.$$

Suppose further, that the **dependency graph** of the system $\mathbf{a} = \mathbf{\Phi}(x, u, \mathbf{a})$ is **strongly connected** (which means that no subsystem can be solved before the whole system).

Systems of Functional equations

COMBINATORIAL CENTRAL LIMIT THEOREM II (cont.)

Then there exists analytic function $g_j(x,u), h_j(x,u)$, and $\rho(u)$ (that is **independent of** j) such that locally

$$\boxed{A_j(x,u) = g_j(x,u) - h_j(x,u)\sqrt{1 - \frac{x}{\rho(u)}}}.$$

and consequently (for some $g(x,u)$, $h(x,u)$)

$$\boxed{A(x,u) = g(x,u) - h(x,u)\sqrt{1 - \frac{x}{\rho(u)}}}.$$

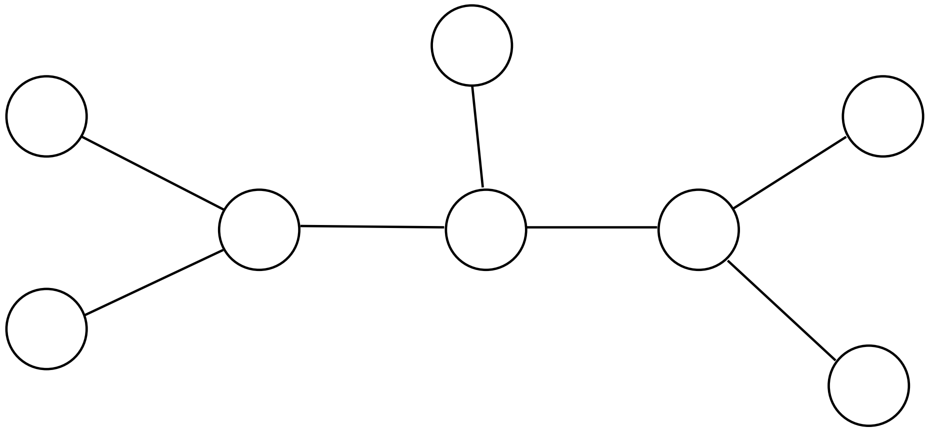
Consequently the random variable X_n satisfies a **central limit theorem** with

$$\boxed{\mathbb{E}\, X_n \sim n\mu} \quad \text{and} \quad \boxed{\mathbb{V}\mathrm{ar}\, X_n \sim n\sigma^2},$$

where μ and σ^2 can be computed.

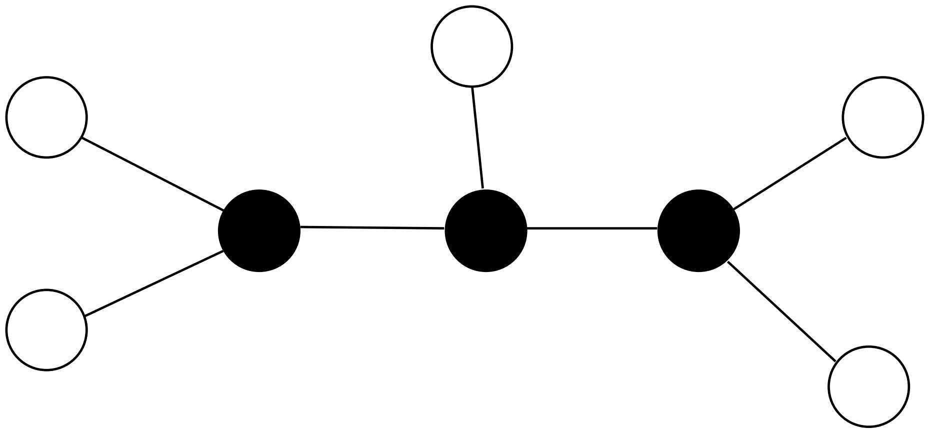
Patterns in Trees

Pattern $\mathcal{M}$

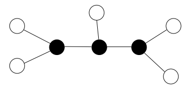


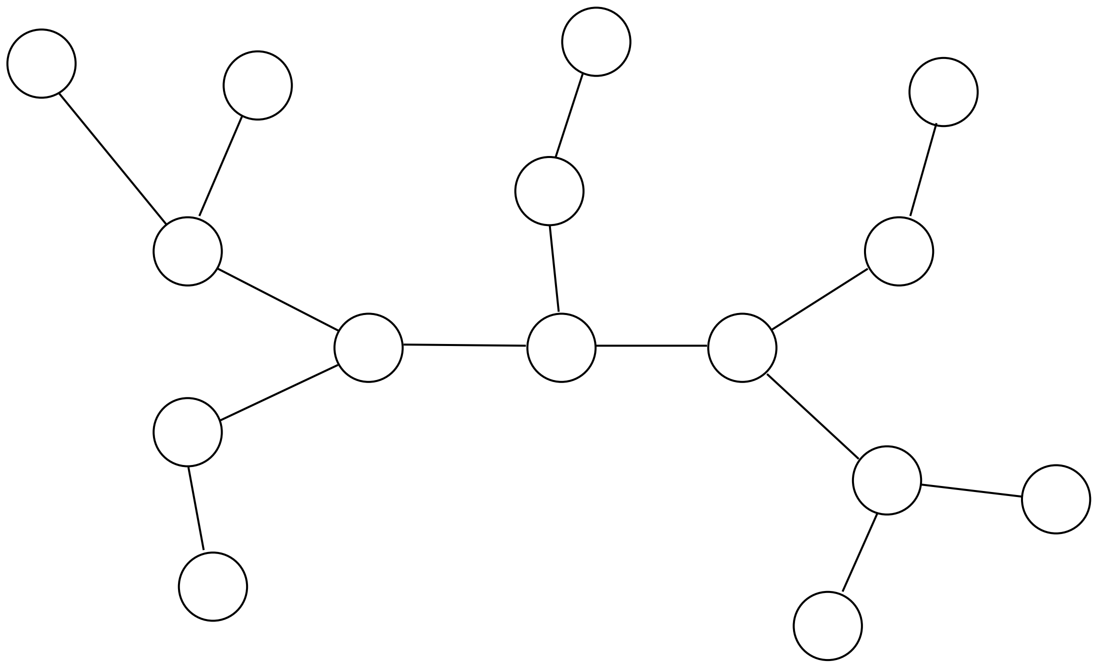
Patterns in Trees

Pattern $\mathcal{M}$

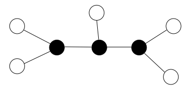


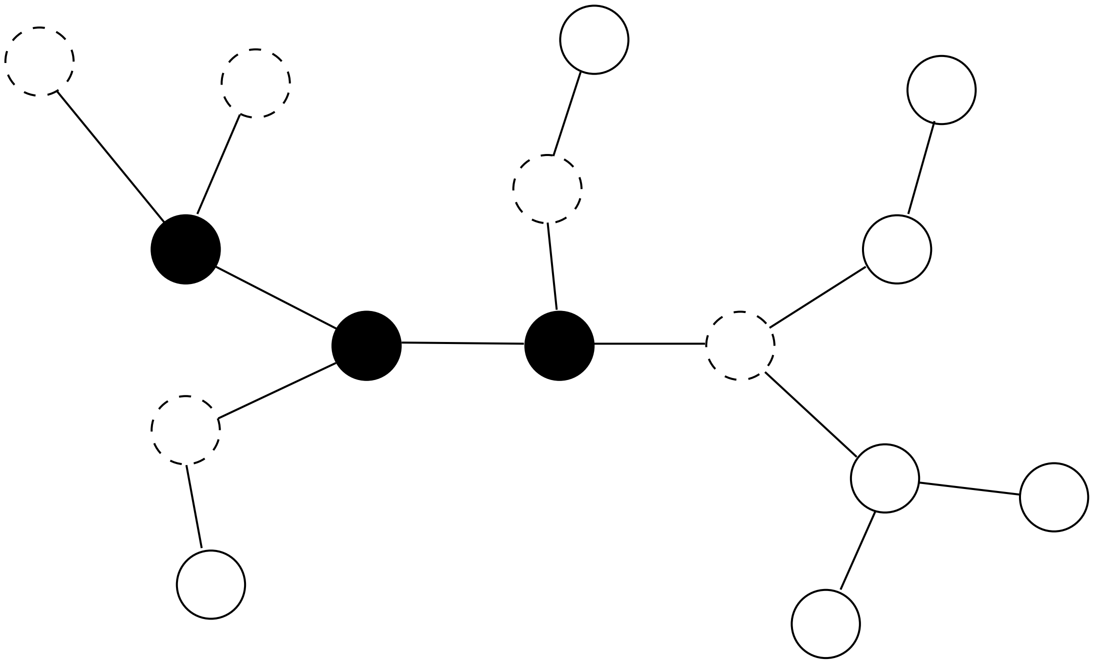
Patterns in Trees

Occurrence of a pattern $\mathcal{M}$ 

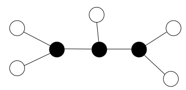


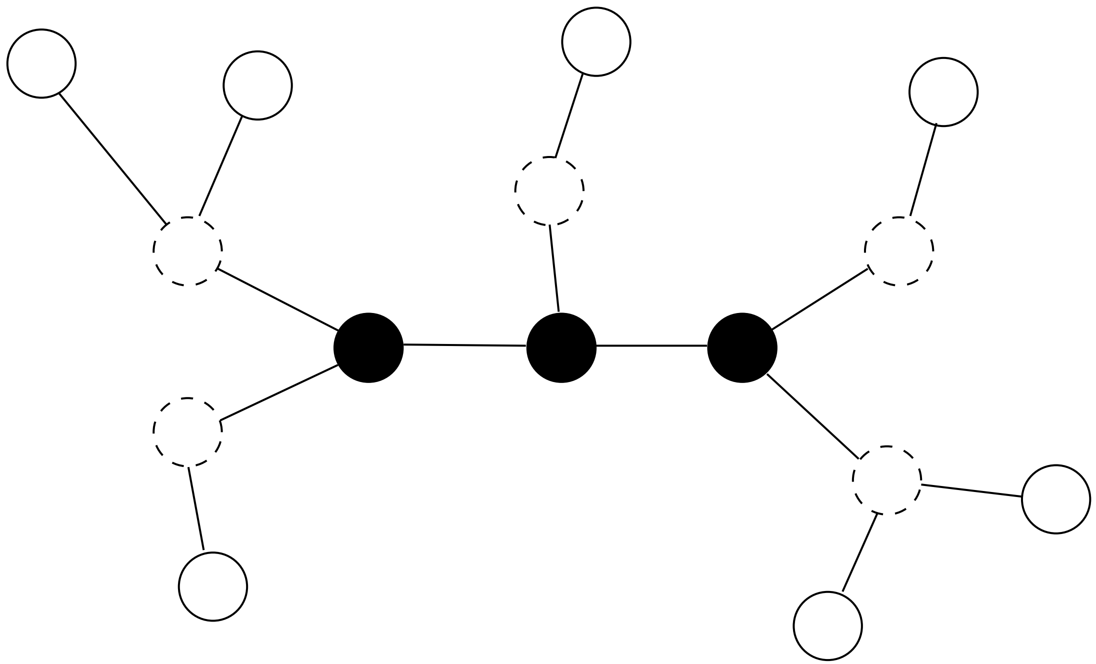
Patterns in Trees

Occurrence of a pattern $\mathcal{M}$ 

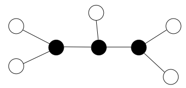


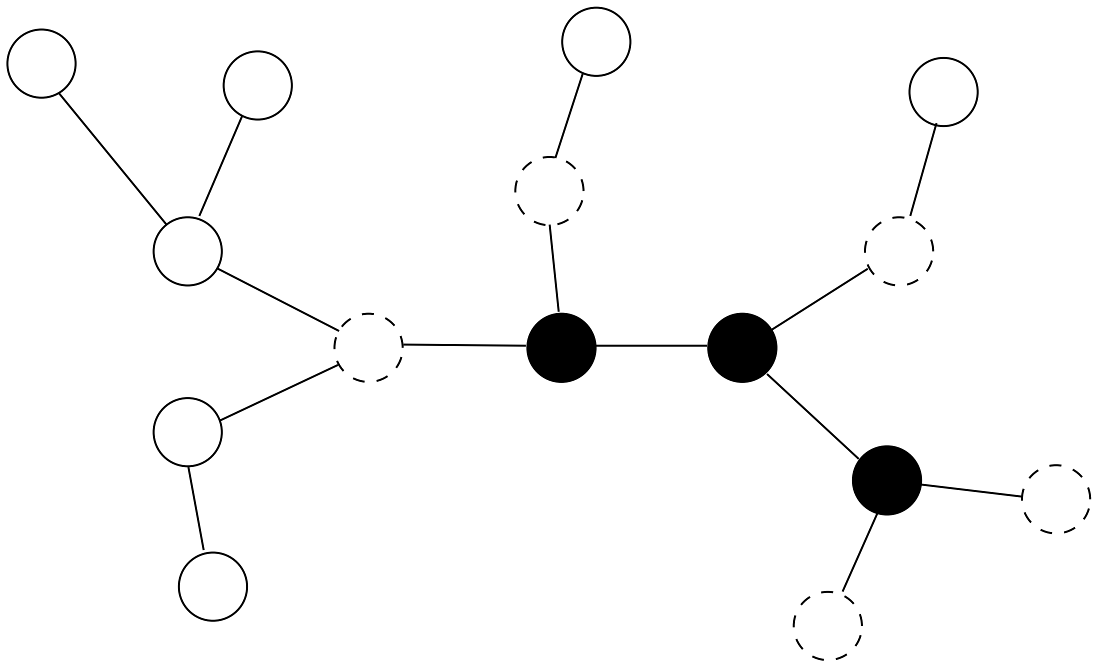
Patterns in Trees

Occurrence of a pattern $\mathcal{M}$ 

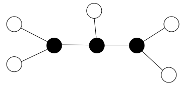


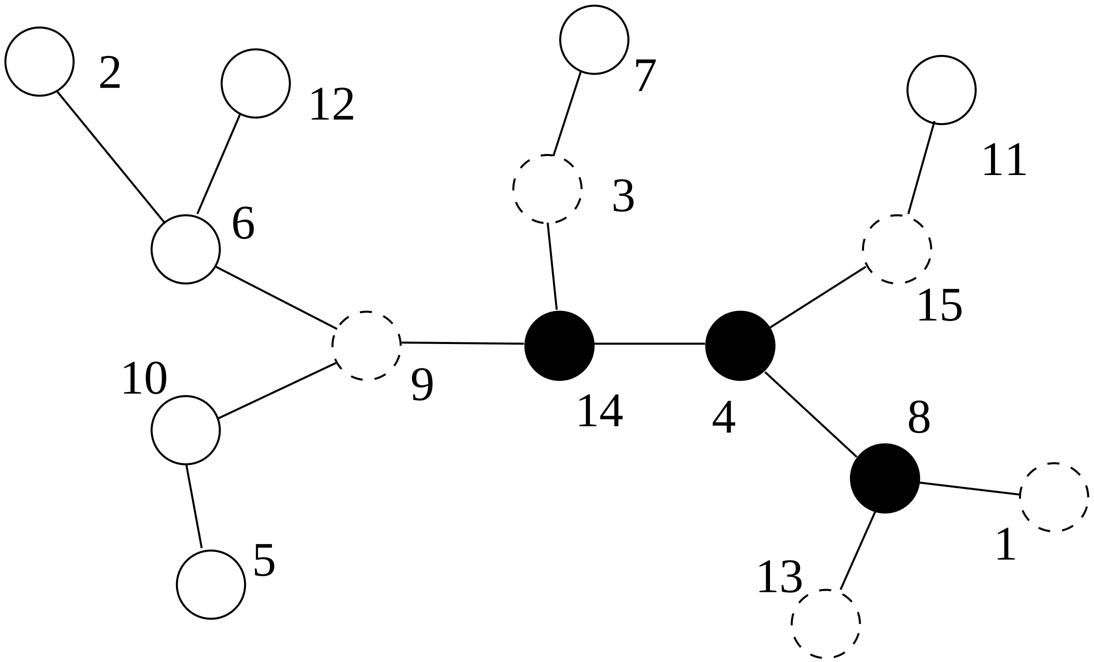
Patterns in Trees

Occurrence of a pattern $\mathcal{M}$ 



Patterns in Trees

Occurrence of a pattern $\mathcal{M}$  **in a labelled tree**



Patterns in Trees

Cayley's formula

$r_n = n^{n-1}$ ... number of **rooted** labelled trees with n nodes

$t_n = n^{n-2}$ ... number of labelled trees with n nodes

Generating functions

$$R(x) = \sum_{n\geq 1} r_n \frac{x^n}{n!}:$$

$$\boxed{R(x) = xe^{R(x)}}$$

$$t(x) = \sum_{n\geq 1} t_n \frac{x^n}{n!}:$$

$$\boxed{T(x) = R(x) - \frac{1}{2}R(x)^2}$$

Patterns in Trees

Theorem

$\mathcal{M}$ ... be a given finite tree.

X_n ... number of occurrences of of $\mathcal{M}$ in a labelled tree of size n

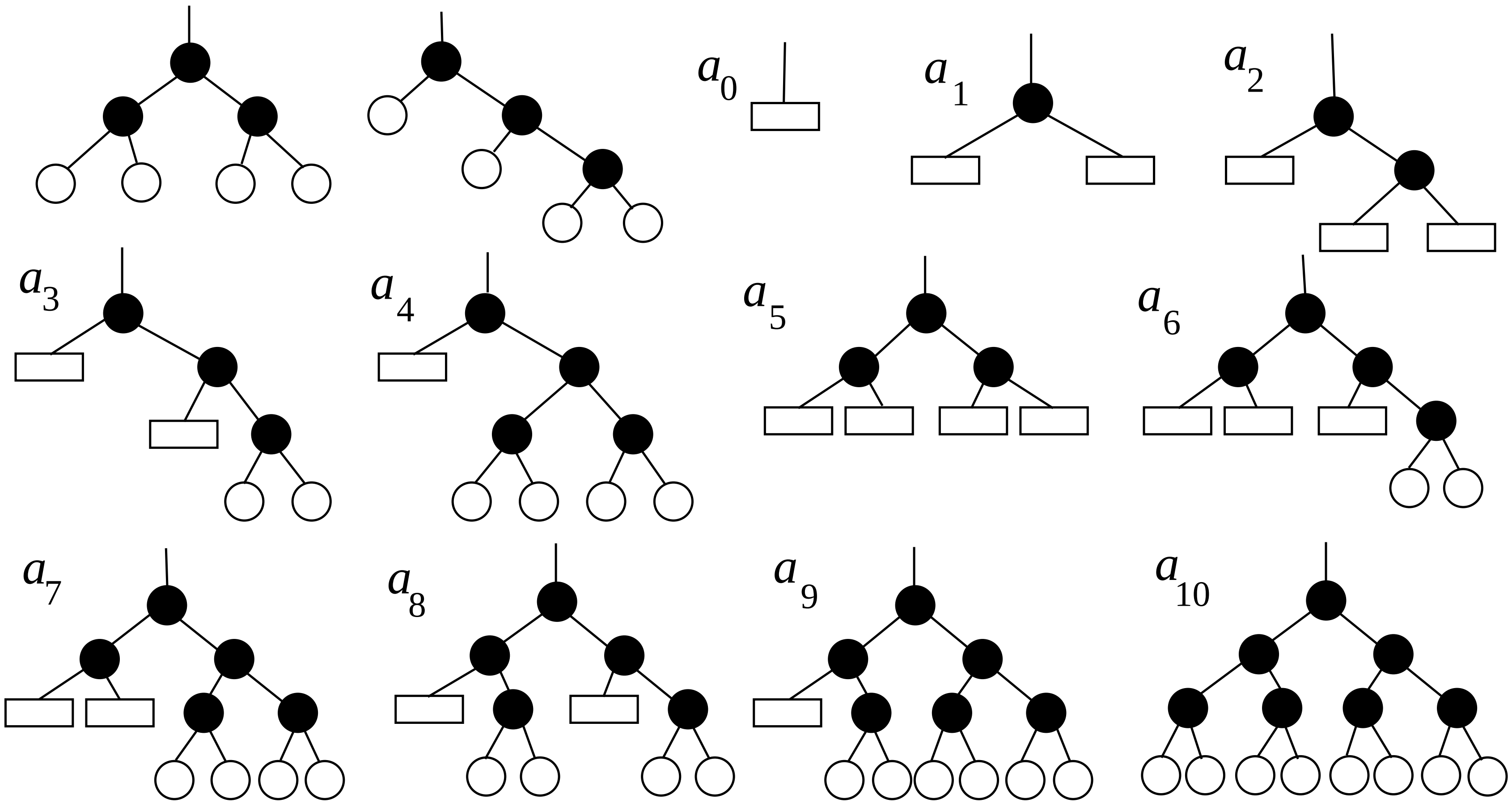
$\Longrightarrow$ X_n satisfies a **central limit theorem** with

$$\mathbb{E}\, X_n \sim \mu n \quad \text{and} \quad \mathbb{V}\, X_n \sim \sigma^2 n.$$

$\mu > 0$ and $\sigma^2 \geq 0$ depend on the pattern $\mathcal{M}$ and can be computed explicitly and algorithmically and can be represented as polynomials (with rational coefficients) in $1/e$.

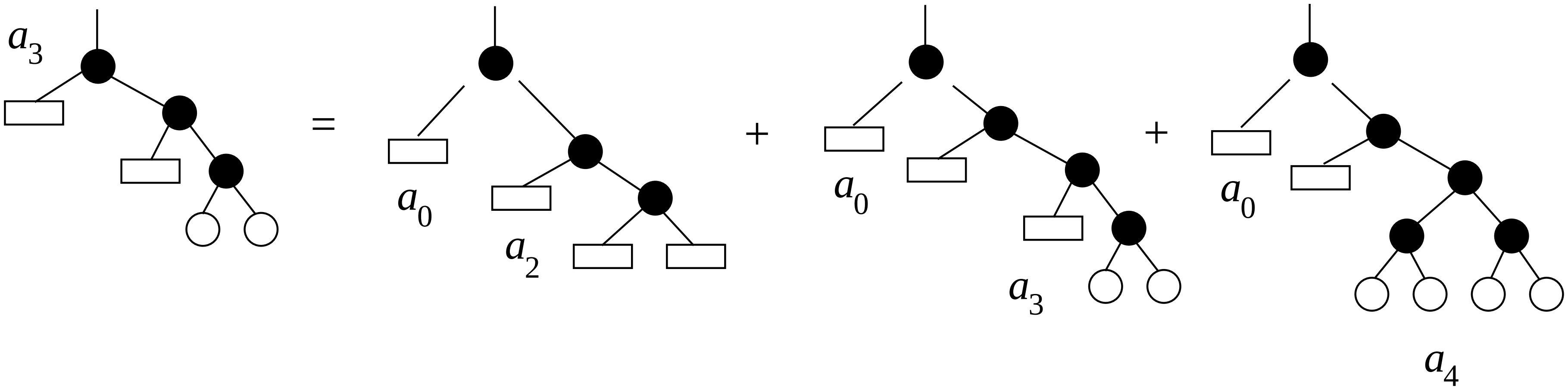
Patterns in Trees

Partition of trees in classes (▭ ... out-degree different from 2)



Patterns in Trees

Recurrences $\boxed{A_3 = xA_0A_2 + xA_0A_3 + xA_0A_4}$

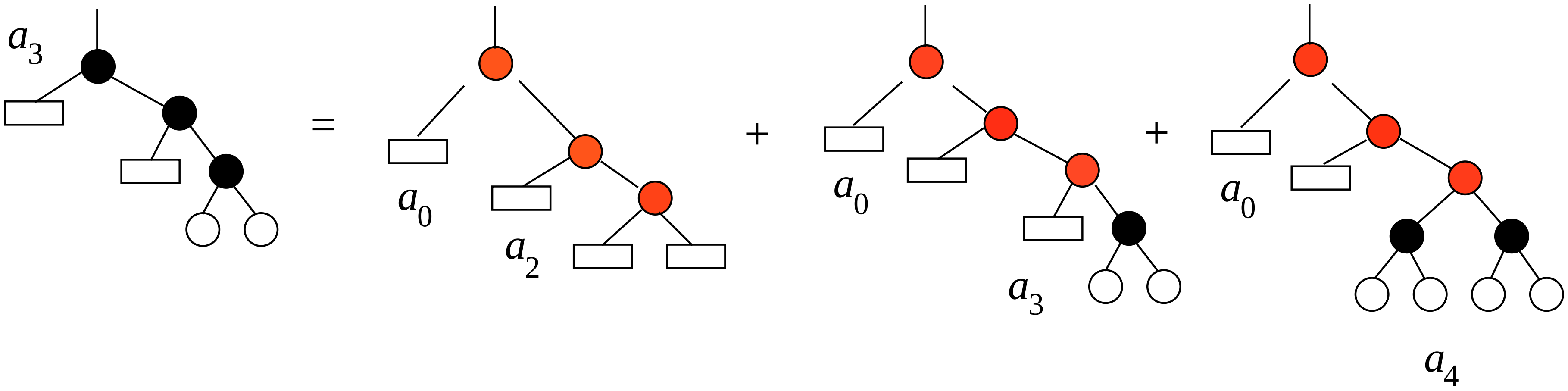


$$\boxed{A_j(x) = \sum_{n,k} a_{j;n} \frac{x^n}{n!}}$$

$a_{j;n}$ ... number of trees of size n in class j

Patterns in Trees

Recurrences $A_3 = xuA_0A_2 + xuA_0A_3 + xuA_0A_4$



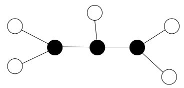
$$A_j(x,u) = \sum_{n,k} a_{j;n,k} \frac{x^n}{n!} u^k$$

$a_{j;n,k}$ ... number of trees of size n in class j with k occurrences of $\mathcal{M}$

Patterns in Trees

$$
\begin{aligned}
A_0 &= A_0(x,u) = x + x\sum_{i=0}^{10} A_i + x\sum_{n=3}^{\infty}\frac{1}{n!}\left(\sum_{i=0}^{10} A_i\right)^n, \\
A_1 &= A_1(x,u) = \frac{1}{2}xA_0^2, \\
A_2 &= A_2(x,u) = xA_0A_1, \\
A_3 &= A_3(x,u) = xA_0(A_2+A_3+A_4)u, \\
A_4 &= A_4(x,u) = xA_0(A_5+A_6+A_7+A_8+A_9+A_{10})u^2, \\
A_5 &= A_5(x,u) = \frac{1}{2}xA_1^2u, \\
A_6 &= A_6(x,u) = xA_1(A_2+A_3+A_4)u^2, \\
A_7 &= A_7(x,u) = xA_1(A_5+A_6+A_7+A_8+A_9+A_{10})u^3, \\
A_8 &= A_8(x,u) = \frac{1}{2}x(A_2+A_3+A_4)^2u^3, \\
A_9 &= A_9(x,u) = x(A_2+A_3+A_4)(A_5+A_6+A_7+A_8+A_9+A_{10})u^4, \\
A_{10} &= A_{10}(x,u) = \frac{1}{2}x(A_5+A_6+A_7+A_8+A_9+A_{10})^2u^5.
\end{aligned}
$$

Patterns in Trees

Final Result for $\mathcal{M} =$ 

Central limit theorem with

$$\mu = \frac{5}{8e^3} = 0.0311169177\ldots$$

and

$$\sigma^2 = \frac{20e^3 + 72e^2 + 84e - 175}{32e^6} = 0.0764585401\ldots.$$

Contents 2

II. RANDOM PLANAR GRAPHS

- Short history on random planar graphs
- Classes of planar graphs
- Asymptotic enumeration of planar graphs
- The number of edges of planar graphs
- Vertices of given degree
- The degree distribution of planar graphs
- The maximum degree of planar graphs

Several parts are joint work with Omer Giménez and Marc Noy (Barcelona).

Random Planar Graphs

“History”

$\mathcal{R}_n$ ... **labelled planar graphs** with n vertices with uniform distribution

X_n ... number of **edges** is a random planar graph with n vertices

Denise, Vasconcellos, Welsh (1996)

$$\boxed{\mathbb{P}\{X_n > \frac{3}{2}n\} \to 1, \quad \mathbb{P}\{X_n < \frac{5}{2}n\} \to 1}.$$

X_n ... **number of edges** in random planar graphs $\mathcal{R}_n$
(Note that $0 \le e \le 3n$ for all planar graphs.)

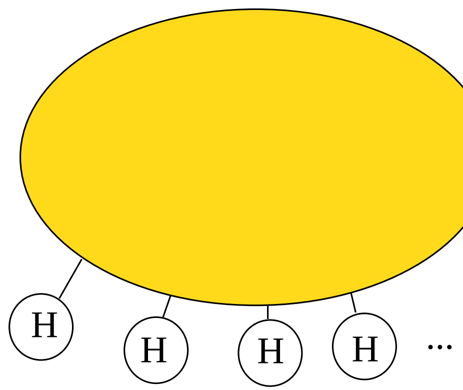
Random Planar Graphs

“History”

McDiarmid, Steger, Welsh (2005)

$$\boxed{\mathbb{P}\{H \text{ appears in } \mathcal{R}_n \text{ at least } \alpha n \text{ times}\} \to 1}$$

H ... any fixed planar graph, $\alpha > 0$ sufficiently small.



Random Planar Graphs

Consequences:

$$\boxed{\mathbb{P}\{\text{There are} \ge \alpha n \text{ vertices of degree } k\} \to 1}$$

$k > 0$ a given integer, $\alpha > 0$ sufficiently small.

$$\boxed{\mathbb{P}\{\text{There are} \ge C^n \text{ automorphisms}\} \to 1}$$

for some $C > 1$.

Random Planar Graphs

Further Results:

$$\boxed{\mathbb{P}\{\mathcal{R}_n \text{ is connected}\} \ge \gamma > 0}$$

[McDiarmid+Reed]

$$\boxed{\mathbb{E}\,\Delta_n = \Theta(\log n)}$$

Δ_n ... maximum degree in $\mathcal{R}_n$

Random Planar Graphs

The number of planar graphs

[Bender, Gao, Wormald (2002)]

b_n ... number of **2-connected** labelled planar graphs

$$\boxed{b_n \sim c \cdot n^{-\frac{7}{2}} \gamma_2^n \, n!}, \quad \gamma_2 = 26.18...$$

[Gimenez+Noy (2005)]

g_n .... number of all labelled planar graphs

$$\boxed{\boxed{g_n \sim c \cdot n^{-\frac{7}{2}} \gamma^n \, n!}}, \quad \gamma = 27.22...$$

Random Planar Graphs

Precise distributional results

[Gimenez+Noy (2005)]

- X_n satisfies a **central limit theorem**:

$$\mathbb{E}\, X_n \sim 2.21... \cdot n, \quad \mathbb{V}\, X_n \sim c \cdot n.$$

$$\mathbb{P}\{|X_n - 2.21... \cdot n| > \varepsilon n\} \le e^{-\alpha(\varepsilon)\cdot n}$$

- **Connectedness:**

$$\mathbb{P}\{\mathcal{R}_n \text{ is connected}\} \to e^{-\nu} = 0.96...$$

$$\text{number of components of } \mathcal{R}_n =: C_n \to 1 + Po(\nu).$$

Random Planar Graphs

Degree Distribution

Theorem [D.+Giménez+Noy]

Let $p_{n,k}$ be the probability that a random node in a random planar graph $\mathcal{R}_n$ has degree k. Then the limit

$$\boxed{p_k := \lim_{n\to\infty} p_{n,k}}$$

exists. The probability generating function

$$\boxed{p(w) = \sum_{k\ge 1} p_k w^k}$$

can be explicitly computed; $\boxed{p_k \sim c\, k^{-\frac{1}{2}} q^k}$ for some $c > 0$ and $0 < q < 1$.

p_1	p_2	p_3	p_4	p_5	p_6
0.0367284	0.1625794	0.2354360	0.1867737	0.1295023	0.0861805

Random Planar Graphs

Classes of planar graphs

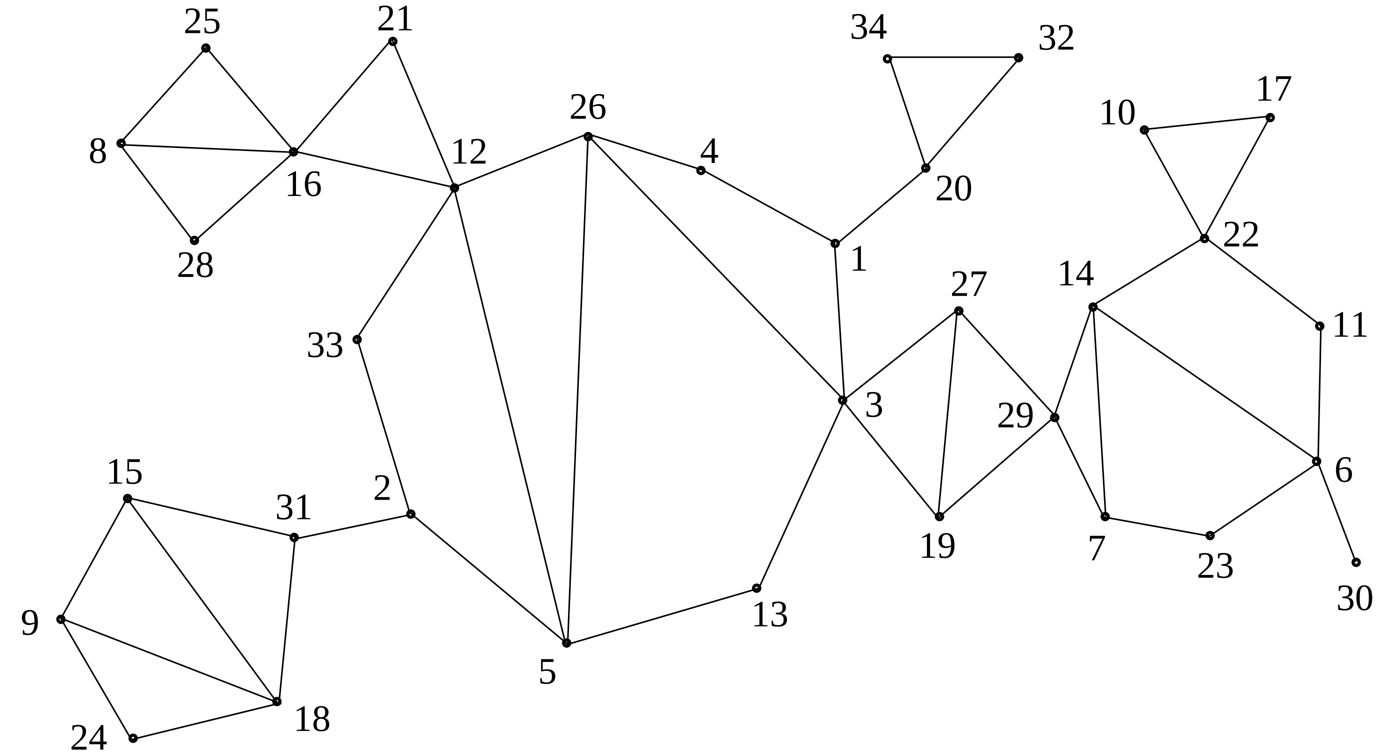
- **Outerplanar graph**: no K_4 and no $K_{2,3}$ as a minor.
- **Series-parallel graph**: no K_4 as a minor.
- **Planar graph**: no K_5 and no $K_{3,3}$ as a minor.

Remark.

$$\text{outerplanar} \subseteq \text{series-parallel} \subseteq \text{planar}$$

Random Planar Graphs

Outerplanar Graphs



All vertices are on the infinite face.

Random Planar Graphs

Outerplanar Graphs

b_n ... number of **2-connected labelled outer-planar** graphs with n vertices

c_n ... number of **connected labelled outer-planar** graphs with n vertices

g_n ... number of **labelled outer-planar** graphs with n vertices

$$B(x) = \sum_{n \ge 0} b_n \frac{x^n}{n!}, \quad C(x) = \sum_{n \ge 0} c_n \frac{x^n}{n!}, \quad G(x) = \sum_{n \ge 0} g_n \frac{x^n}{n!}$$

Random Planar Graphs

Outerplanar Graphs

$$G(x) = e^{C(x)},$$

$$C'(x) = e^{B'(xC'(x))},$$

$$B'(x) = x + \frac{1}{2}x\,A(x),$$

$$A(x) = x(1 + A(x))^2 + x(1 + A(x))A(x)$$

$$= \frac{1 - 3x - \sqrt{1 - 6x + x^2}}{4x}.$$

Random Planar Graphs

Outerplanar Graphs

$$b_n = b \cdot (3+2\sqrt{2})^n \, n^{-\frac{5}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right),$$

$$c_n = c \cdot \rho^{-n} \, n^{-\frac{5}{2}} \, n! \left(1 + O\left(\frac{1}{n}\right)\right),$$

$$g_n = g \cdot \rho^{-n} \, n^{-\frac{5}{2}} \, n! \left(1 + O\left(\frac{1}{n}\right)\right),$$

$$\rho = y_0 e^{-B'(y_0)} = 0.1365937...,$$

$$y_0 = 0.1707649... \text{satisfies } 1 = y_0 B''(y_0),$$

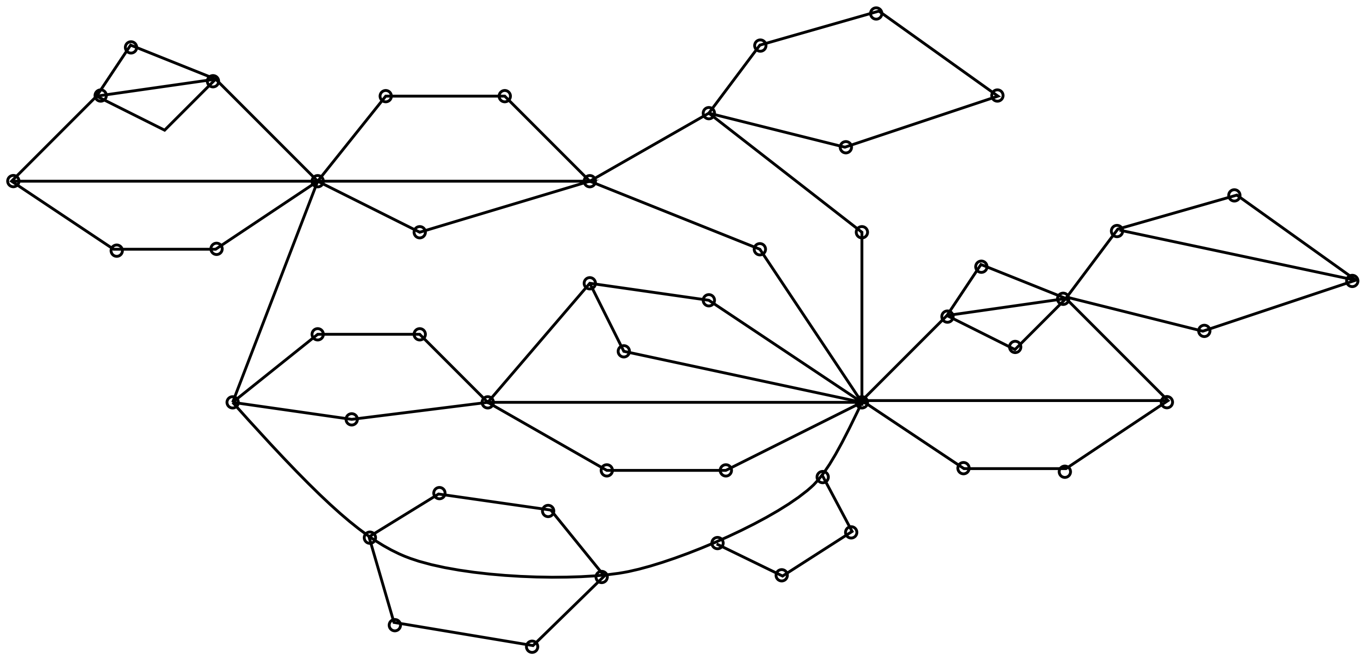
$$b = \frac{1}{8\sqrt{\pi}} \sqrt{114243\sqrt{2} - 161564} = 0.000175453...,$$

$$c = 0.0069760...,$$

$$g = 0.017657...$$

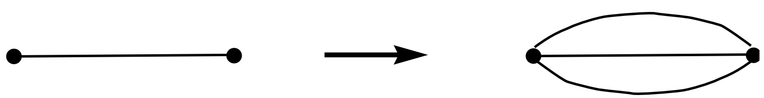
Random Planar Graphs

Series-Parallel Graphs



Series-parallel extension of a tree

Series-extension: 

Parallel-extension: 

Random Planar Graphs

Series-Parallel Graphs

$b_{n,m}$ ... number of **2-connected labelled series-parallel** graphs with n vertices and m edges, $b_n = \sum_m b_{n,m}$

$c_{n,m}$ ... number of **connected labelled series-parallel** graphs with n vertices and m edges, $c_n = \sum_m c_{n,m}$

$g_{n,m}$ ... number of **labelled series-parallel** graphs with n vertices and m edges, $g_n = \sum_m g_{n,m}$

$$B(x,y) = \sum_{n,m} b_{n,m} \frac{x^n}{n!} y^m, \quad C(x,y) = \sum_{n,m} c_{n,m} \frac{x^n}{n!} y^m, \quad G(x,y) = \sum_{n,m} g_{n,m} \frac{x^n}{n!} y^m$$

Random Planar Graphs

Series-Parallel Graphs

$$G(x,y) = e^{C(x,y)}$$

$$\frac{\partial C(x,y)}{\partial x} = \exp\left(\frac{\partial B}{\partial x}\left(x\frac{\partial C(x,y)}{\partial x}, y\right)\right),$$

$$\frac{\partial B(x,y)}{\partial y} = \frac{x^2}{2}\,\frac{1+D(x,y)}{1+y},$$

$$D(x,y) = (1+y)e^{S(x,y)} - 1,$$

$$S(x,y) = (D(x,y) - S(x,y))xD(x,y).$$

Random Planar Graphs

Series-Parallel Graphs

$$b_n = b \cdot \rho_1^{-n} n^{-\frac{5}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right),$$
$$c_n = c \cdot \rho_2^{-n} n^{-\frac{5}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right),$$
$$g_n = g \cdot \rho_2^{-n} n^{-\frac{5}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right),$$

$$\rho_1 = 0.1280038...,$$
$$\rho_2 = 0.11021...,$$
$$b = 0.0010131...,$$
$$c = 0.0067912...,$$
$$g = 0.0076388...$$

Random Planar Graphs

Planar Graphs

$b_{n,m}$ ... number of **2-connected labelled planar** graphs with n vertices and m edges, $b_n = \sum_m b_{n,m}$

$c_{n,m}$ ... number of **connected labelled planar** graphs with n vertices and m edges, $c_n = \sum_m c_{n,m}$

$g_{n,m}$ ... number of **labelled planar** graphs with n vertices and m edges, $g_n = \sum_m g_{n,m}$

$$B(x,y) = \sum_{n,m} b_{n,m} \frac{x^n}{n!} y^m, \; C(x,y) = \sum_{n,m} c_{n,m} \frac{x^n}{n!} y^m, \; G(x,y) = \sum_{n,m} g_{n,m} \frac{x^n}{n!} y^m$$

Random Planar Graphs

Planar Graphs

$$
\begin{aligned}
G(x,y) &= \exp\left(C(x,y)\right),\\
\frac{\partial C(x,y)}{\partial x} &= \exp\left(\frac{\partial B}{\partial x}\left(x\frac{\partial C(x,y)}{\partial x}, y\right)\right),\\
\frac{\partial B(x,y)}{\partial y} &= \frac{x^2}{2}\,\frac{1+D(x,y)}{1+y},\\
\frac{M(x,D)}{2x^2D} &= \log\left(\frac{1+D}{1+y}\right) - \frac{xD^2}{1+xD},\\
M(x,y) &= x^2y^2\left(\frac{1}{1+xy} + \frac{1}{1+y} - 1 - \frac{(1+U)^2(1+V)^2}{(1+U+V)^3}\right),\\
U &= xy(1+V)^2,\\
V &= y(1+U)^2.
\end{aligned}
$$

Random Planar Graphs

Planar Graphs

$$b_n = b \cdot \rho_1^{-n} n^{-\frac{7}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right),$$
$$c_n = c \cdot \rho_2^{-n} n^{-\frac{7}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right),$$
$$g_n = g \cdot \rho_2^{-n} n^{-\frac{7}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right),$$

$$\rho_1 = 0.03819...,$$
$$\rho_2 = 0.03672841...,$$
$$b = 0.3704247487... \cdot 10^{-5},$$
$$c = 0.4104361100... \cdot 10^{-5},$$
$$g = 0.4260938569... \cdot 10^{-5}$$

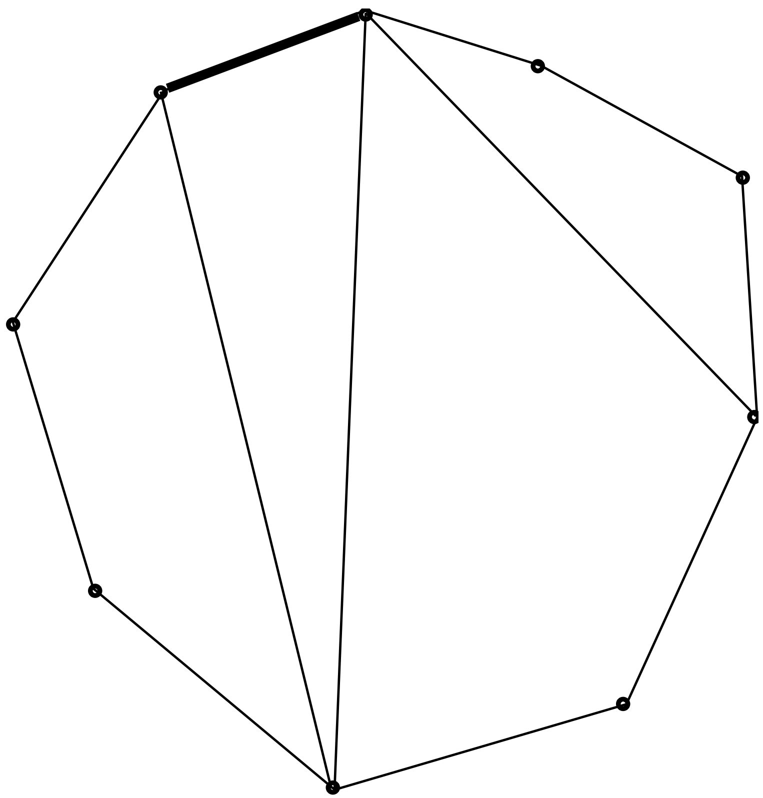
Outerplanar Graphs

Generating functions

$$
\begin{aligned}
G(x) &= e^{C(x)}, \\
C'(x) &= e^{B'(xC'(x))}, \\
B'(x) &= x + \frac{1}{2}x\,A(x), \\
A(x) &= x(1+A(x))^2 + x(1+A(x))A(x) \\
&= \frac{1-3x-\sqrt{1-6x+x^2}}{4x}.
\end{aligned}
$$

Outerplanar Graphs

Dissections



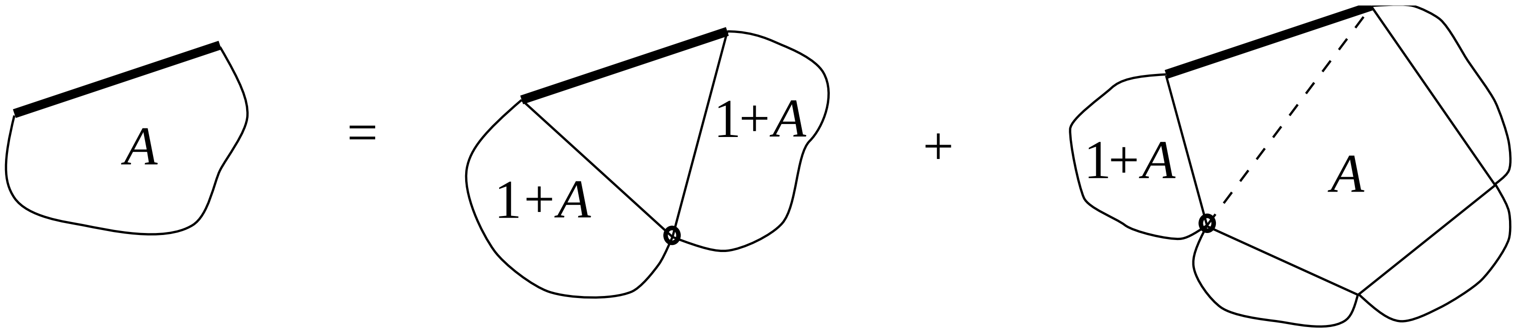
$\mathcal{A}$ ... set of dissections
(unlabelled planar graphs, where all nodes are on the outer face, one edge is marked, and there are at least 3 edges)

Outerplanar Graphs

Dissections

a_n ... number of dissections with $n+2$ nodes, $n \geq 1$,
(the nodes of the marked edge are not counted)

$A(x) = \sum_{n \geq 1} a_n x^n$ ... generating function of dissections



$$A(x) = x(1 + A(x))^2 + x(1 + A(x))A(x)$$

Outerplanar Graphs

Dissections

Quadratic equation:

$$A^2 + \frac{3x-1}{2x} A + \frac{1}{2} = 0$$

Solution:

$$\boxed{A(x) = \frac{1 - 3x - \sqrt{1 - 6x + x^2}}{4x}}$$

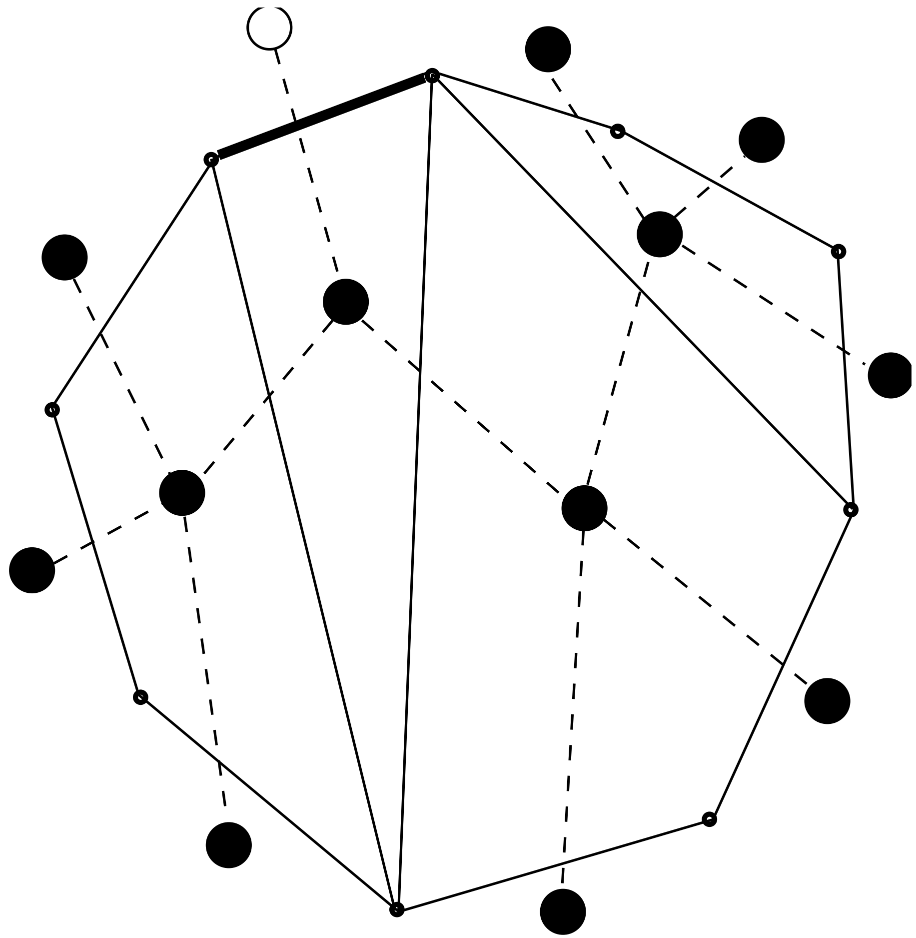
Radius of convergence: $\rho_1 = 3 - 2\sqrt{2}$.

Lagrange inversion formula:

$$a_n = \frac{1}{n} \sum_{\ell=0}^{n-1} \binom{n}{\ell} \binom{n}{\ell+1} 2^{\ell}.$$

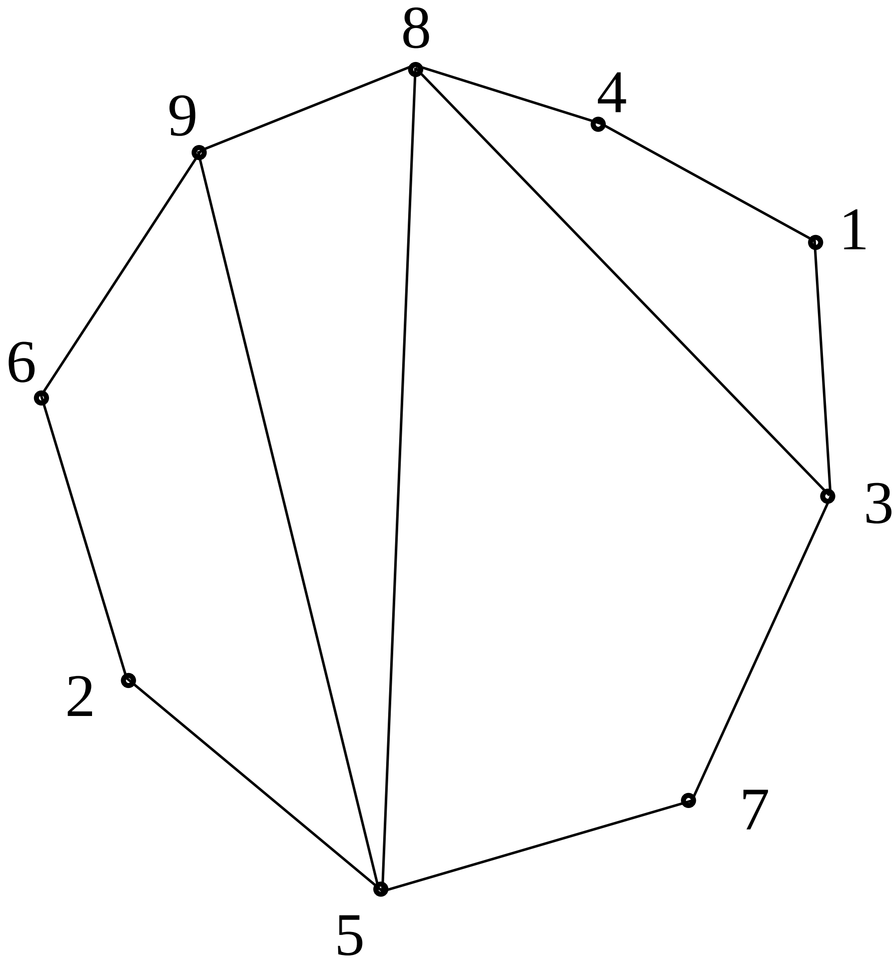
Outerplanar Graphs

Trees and outerplanar graphs



Outerplanar Graphs

2-Connected outerplanar graphs



b_n ... number of 2-connected vertex labelled outer planar graphs

Outerplanar Graphs

2-Connected outerplanar graphs

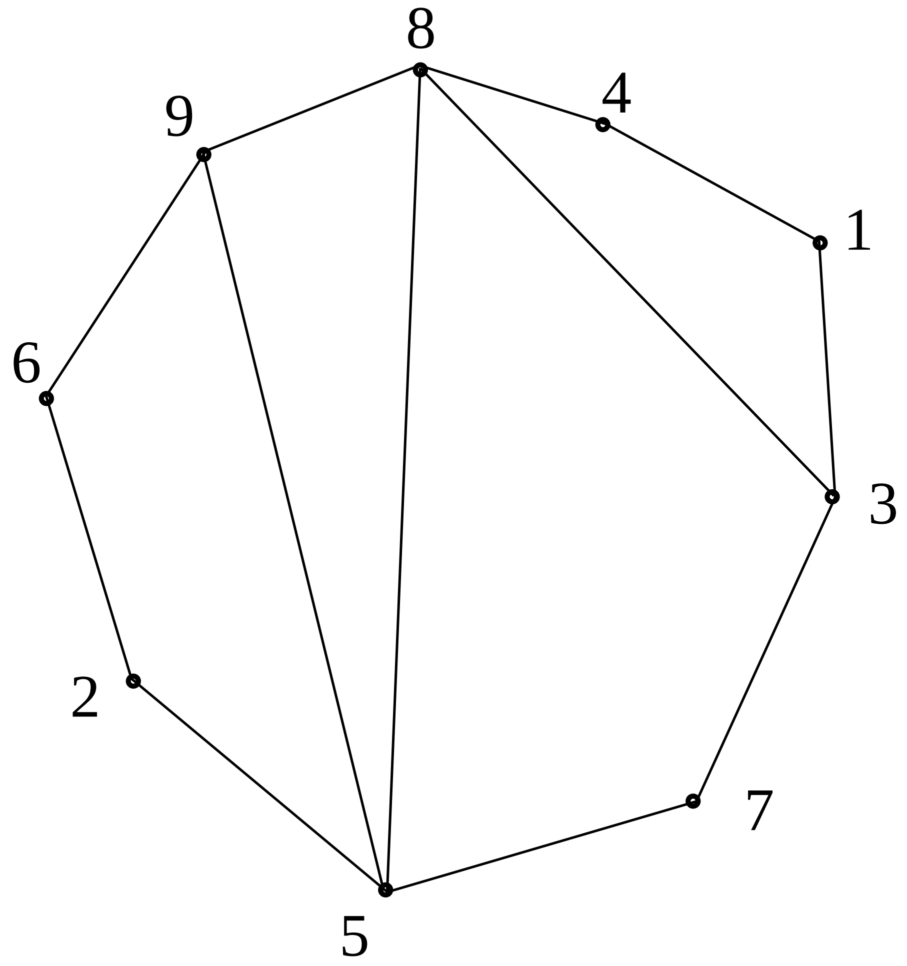
$B(x) = \sum_{n\geq 1} b_n \frac{x^n}{n!}$ ... exponential generating function of 2-connected labelled outer planar graphs:

$$\boxed{B'(x) = x + \frac{1}{2} x A(x)}$$

The derivative $B'(x)$ can be also interpreted as the exponential generating function $B^\bullet(x)$ of 2-connected labelled outer planar graphs, where one node is marked (and is not counted).

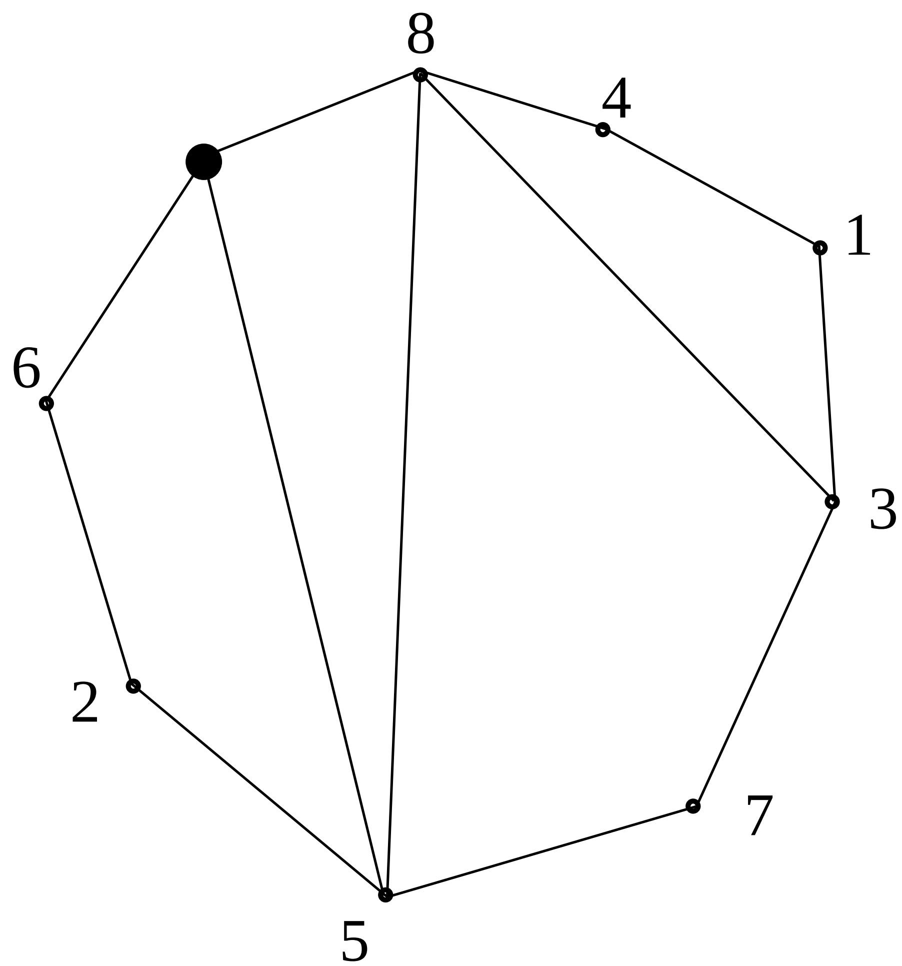
Outerplanar Graphs

2-Connected outerplanar graphs. $\boxed{b_n = \frac{1}{2} a_{n-2} \cdot (n-1)!}$ $(n \geq 3)$



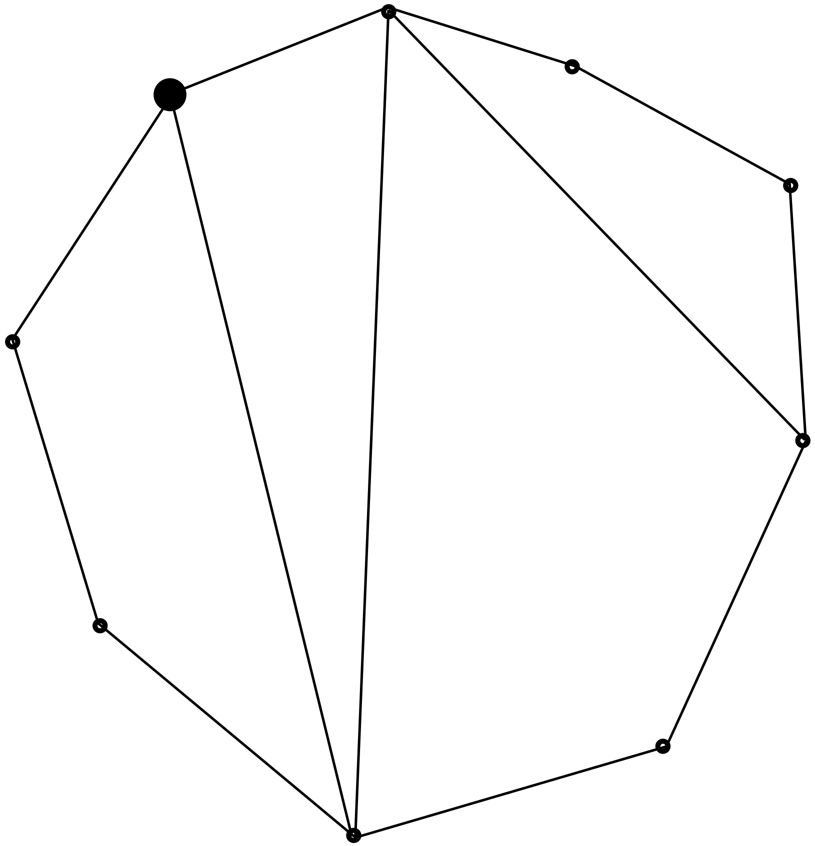
Outerplanar Graphs

2-Connected outerplanar graphs. $\boxed{b_n = \frac{1}{2} a_{n-2} \cdot (n-1)!}$ $(n \geq 3)$



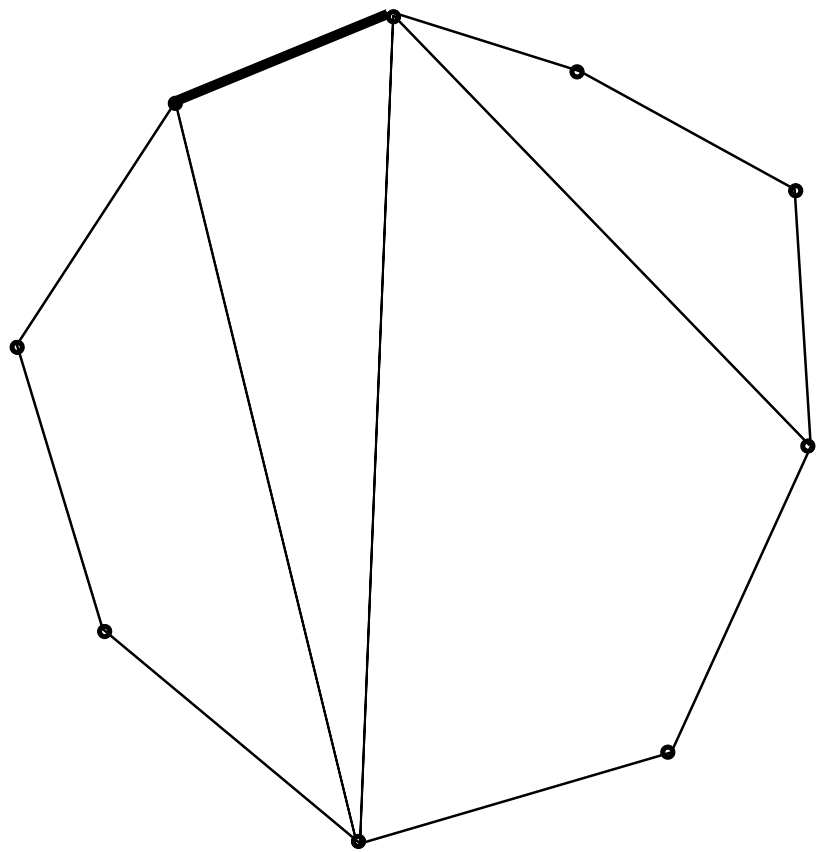
Outerplanar Graphs

2-Connected outerplanar graphs. $\boxed{b_n = \frac{1}{2} a_{n-2} \cdot (n-1)!}$ $\quad (n \geq 3)$



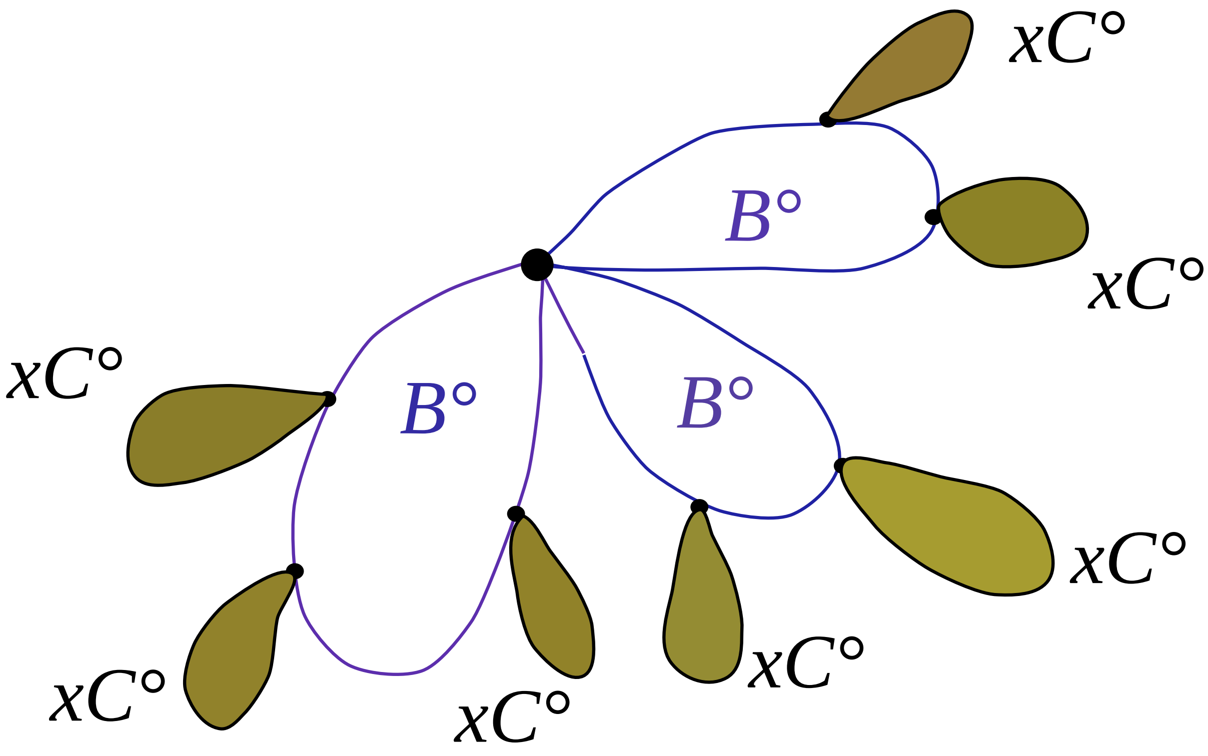
Outerplanar Graphs

2-Connected outerplanar graphs. $\boxed{b_n = \frac{1}{2} a_{n-2} \cdot (n-1)!}$ $(n \geq 3)$



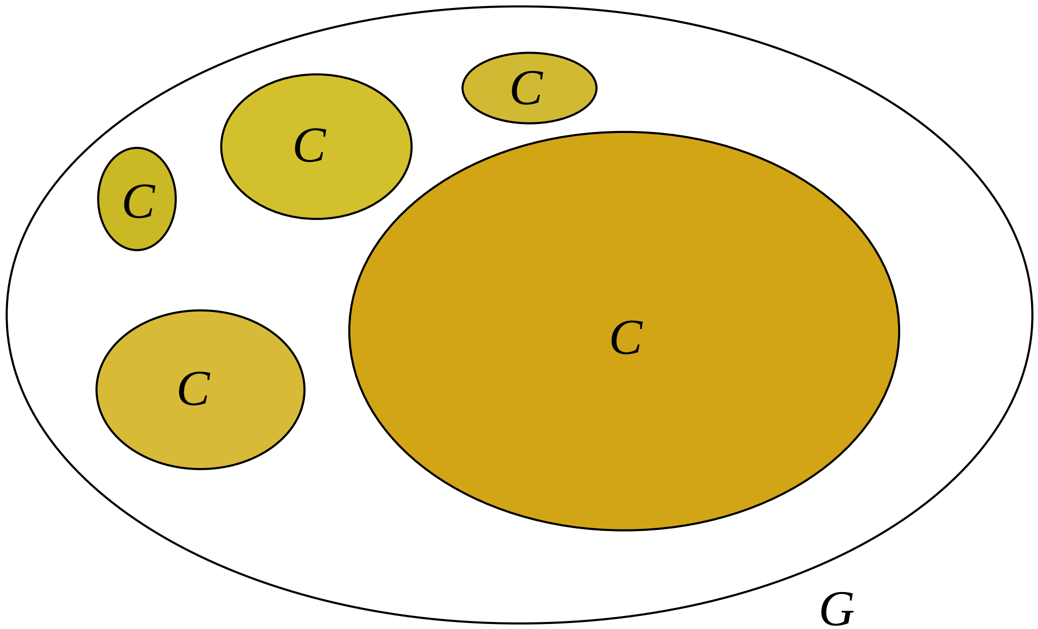
Outerplanar Graphs

Connected outerplanar graphs. $C^{\bullet}(x) = e^{B^{\bullet}(xC^{\bullet}(x))}$



Outerplanar Graphs

All outerplanar graphs. $\boxed{G(x) = \exp\left(C(x)\right)}$



Outerplanar Graphs

Asymptotics

$$A(x) = \frac{1 - 3x - \sqrt{1 - 6x + x^2}}{4x},$$

$$B'(x) = x + \frac{1}{2} x A(x) = \frac{1 + 5x - \sqrt{1 - 6x + x^2}}{8}.$$

$$\Longrightarrow \quad \boxed{b_n \sim b \cdot (3 + 2\sqrt{2})^n n^{-\frac{3}{2}} n!}$$

Outerplanar Graphs

Asymptotics

$C'(x) = e^{B'(xC'(x))}$, $v(x) = xC'(x)$, $\Phi(x,v) = xe^{B'v}$

$$\Longrightarrow \quad \boxed{v(x) = \Phi(x, v(x))}$$

$$\Longrightarrow \quad \boxed{v(x) = xC'(x) = g(x) - h(x)\sqrt{1 - \frac{x}{\rho}}}$$

with $\rho = 0.1365937....$ (Note that $v(\rho) = \rho C'(\rho) < 3 - 2\sqrt{2}$!!!)

$$\Longrightarrow \quad \boxed{C(x) = g_2(x) + h_2(x)\left(1 - \frac{x}{\rho}\right)^{\frac{3}{2}}}$$

$$\Longrightarrow \quad \boxed{c_n \sim c\,\rho^{-n} n^{-\frac{5}{2}} n!}$$

Outerplanar Graphs

Asymptotics

$$C(x) = g_2(x) + h_2(x)\left(1 - \frac{x}{\rho}\right)^{\frac{3}{2}}.$$

$$\Longrightarrow \quad G(x) = e^{C(x)} = g_3(x) + h_3(x)\left(1 - \frac{x}{\rho}\right)^{\frac{3}{2}}.$$

$$\Longrightarrow \quad \boxed{g_n \sim g \cdot \rho^{-n} n^{-\frac{5}{2}} n!}$$

Outerplanar Graphs

The number of edges $G(x,y) = \sum_{m,n} g_{n,m} \frac{x^n}{n!} y^m$ etc.

$$G(x,y) = e^{C(x,y)},$$

$$\frac{\partial C(x,y)}{\partial x} = \exp\left(\frac{\partial B}{\partial x}\left(x\frac{\partial C(x,y)}{\partial x}, y\right)\right),$$

$$\frac{\partial B(x,y)}{\partial x} = xy + \frac{1}{2}xy\, A(x,y),$$

$$A(x,y) = xy^2(1 + A(x,y))^2 + xy(1 + A(x,y))A(x,y)$$

$$= \frac{1 - xy2xy^2 - \sqrt{1 - 2xy - 4xy^2 + x^2y^2}}{2xy(1+y)}.$$

Outerplanar Graphs

The number of edges

$$G(x,y) = g_2(x,y) + h_2(x,y)\left(1 - \frac{x}{\rho(y)}\right)^{\frac{3}{2}}.$$

Theorem

The number of edges X_n in an outerplanar graph of size n satisfies a **central limit theorem** with

$$\mathbb{E}\, X_n = \mu n + O(1) \quad \text{and} \quad \mathbb{V}\, X_n = \sigma^2 n + O(1),$$

where $\mu = 1.56251...$ and $\sigma^2 = 0.22399...$

Series-Parallel Graphs

Generating functions

$$G(x,y) = e^{C(x,y)}$$

$$\frac{\partial C(x,y)}{\partial x} = \exp\left(\frac{\partial B}{\partial x}\left(x\frac{\partial C(x,y)}{\partial x}, y\right)\right),$$

$$\frac{\partial B(x,y)}{\partial y} = \frac{x^2}{2}\,\frac{1+D(x,y)}{1+y},$$

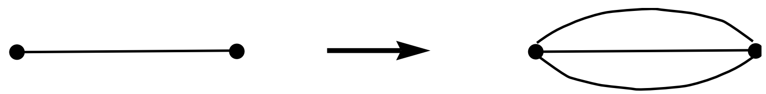
$$D(x,y) = (1+y)e^{S(x,y)} - 1,$$

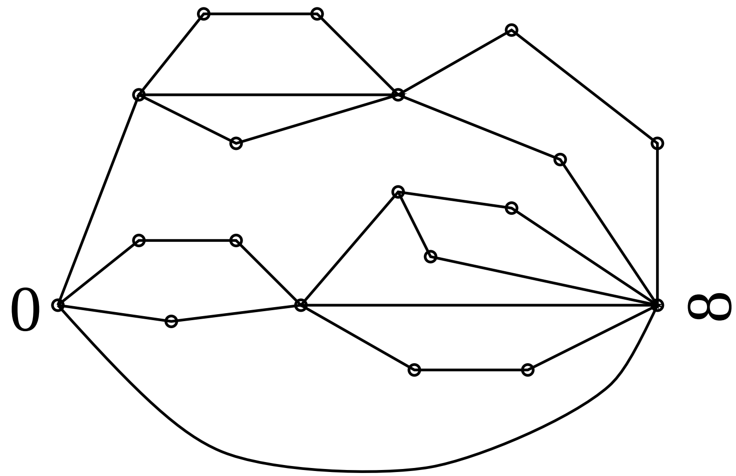
$$S(x,y) = (D(x,y) - S(x,y))xD(x,y).$$

Series-Parallel Graphs

Series-parallel networks: series-parallel extension of an edge

Series-extension: 

Parallel-extension: 

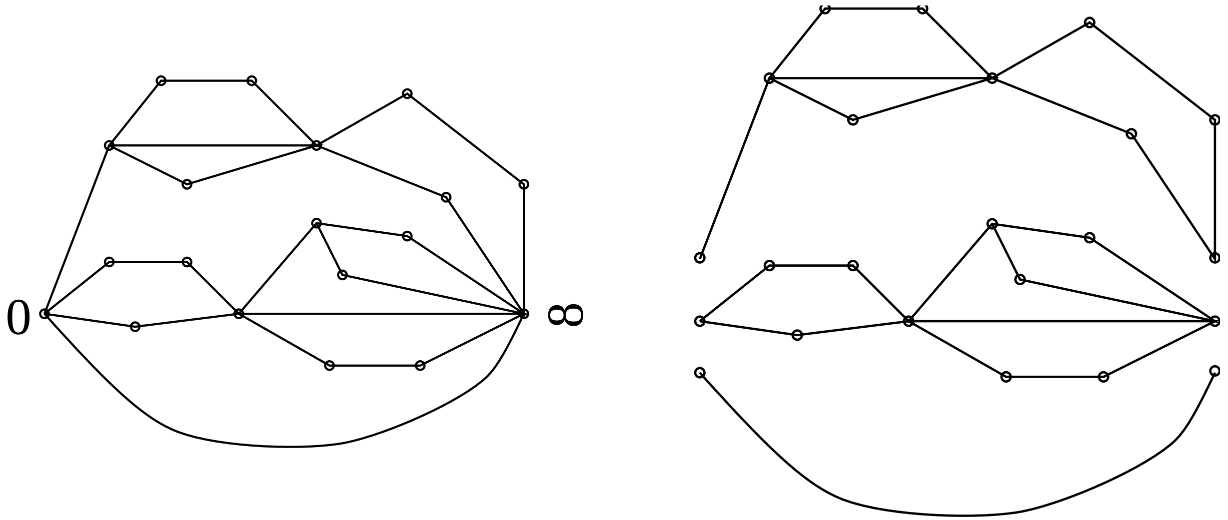


There are always two **poles** $(0, \infty)$ coming from the original two vertices.

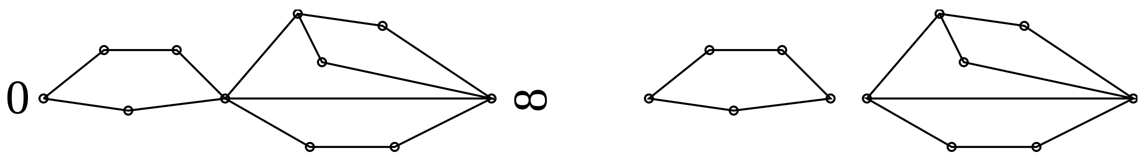
Series-Parallel Graphs

Series-parallel networks

Parallel decomposition of a Series-parallel network:



Series decomposition of a series-parallel network



Series-Parallel Graphs

Series-parallel networks

$d_{n,m}$ ... number of SP-networks with $n+2$ vertices and m edges

$s_{n,m}$ ... number of **series** SP-networks $n+2$ vertices and m edges

$$D(x,y) = \sum_{n,m} d_{n,m} \frac{x^n}{n!} y^m, \qquad S(x,y) = \sum_{n,m} s_{n,m} \frac{x^n}{n!} y^m,$$

$$\begin{aligned} D(x,y) &= e^{S(x,y)} - 1 + y e^{S(x,y)} \\ &= (1+y) e^{S(x,y)} - 1, \end{aligned}$$

$$S(x,y) = (D(x,y) - S(x,y)) x D(x,y)$$

Series-Parallel Graphs

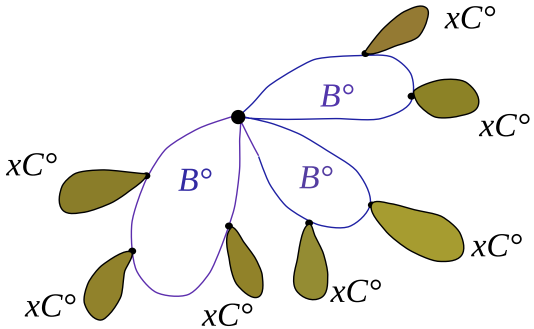
2-connected SP-graphs

A SP-network network with non-adjacent poles (which is counted by $e^{S(x,y)}$) is obtained by distinguishing, orienting and then deleting any edge of an arbitrary 2-connected series-parallel graph:

$$\frac{\partial B(x,y)}{\partial y} = \frac{x^2}{2} e^{S(x,y)}$$
$$= \frac{x^2}{2} \frac{1 + D(x,y)}{1 + y}$$

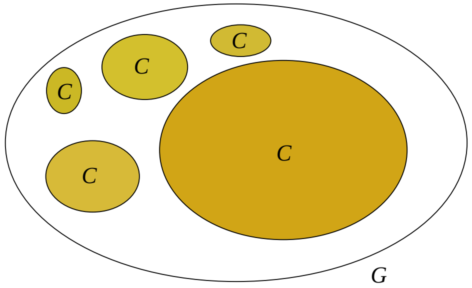
Series-Parallel Graphs

Connected SP-graphs



$$\frac{\partial C(x,y)}{\partial x} = \exp\left(\frac{\partial B}{\partial x}\left(x\frac{\partial C(x,y)}{\partial x}, y\right)\right)$$

All SP-graphs



$$G(x,y) = e^{C(x,y)}$$

Series-Parallel Graphs

Asymptotics

$$D(x,y) = (1+y)\exp\left(\frac{xD(x,y)^2}{1+xD(x,y)}\right) - 1$$

$$\Longrightarrow \quad D(x,y) = g(x,y) - h(x,y)\sqrt{1 - \frac{x}{\rho(y)}},$$

with $\rho(1) = \rho_1 = 0.12800...$.

Series-Parallel Graphs

Asymptotics

$$\Longrightarrow \quad \frac{\partial B(x,y)}{\partial y} = \frac{x^2}{2}\frac{1+D(x,y)}{1+y}D(x,y)$$

$$= g_2(x,y) - h_2(x,y)\sqrt{1-\frac{x}{\rho(y)}}$$

$$!!!! \quad \Longrightarrow \quad B(x,y) = g_3(x,y) + h_3(x,y)\left(1-\frac{x}{\rho(y)}\right)^{\frac{3}{2}}$$

$$\Longrightarrow \quad \boxed{b_n \sim b \cdot \rho(1)^{-n} n^{-\frac{5}{2}} n!}$$

Series-Parallel Graphs

Asymptotics $(C' := \frac{\partial}{\partial x} C)$

$$C'(x,y) = e^{B'(xC'(x,y),y)},\ v(x,y) = xC'(x,y),\ \Phi(x,y,v) = xe^{B'(v,y)}$$

$$\Longrightarrow \quad \boxed{v(x,y) = \Phi(x,y,v(x))}$$

$$\Longrightarrow \quad v(x,y) = xC'(x,y) = g_4(x,y) - h_4(x,y)\sqrt{1 - \frac{x}{\rho_2(y)}}$$

with $\rho_2(1) = 0.11021....$ (Note that $v(\rho) = 0.1279695... < \rho_1$!!!)

$$\Longrightarrow \quad C(x,y) = g_5(x,y) + h_5(x,y)\left(1 - \frac{x}{\rho_2(y)}\right)^{\frac{3}{2}}.$$

$$\Longrightarrow \quad \boxed{c_n \sim c\,\rho_2^{-n} n^{-\frac{5}{2}} n!}$$

Series-Parallel Graphs

Asymptotics

$$C(x,y) = g_5(x,y) + h_5(x,y)\left(1 - \frac{x}{\rho(y)}\right)^{\frac{3}{2}}$$

$$\Longrightarrow \quad G(x,y) = e^{C(x,y)} = g_6(x,y) + h_6(x,y)\left(1 - \frac{x}{\rho_2(y)}\right)^{\frac{3}{2}}.$$

$$\Longrightarrow \quad \boxed{g_n \sim g \cdot \rho_2^{-n} n^{-\frac{5}{2}} n!}$$

Series-Parallel Graphs

The number of edges

$$G(x,y) = g_6(x,y) + h_6(x,y)\left(1 - \frac{x}{\rho(y)}\right)^{\frac{3}{2}}.$$

Theorem

The number of edges X_n in an series-parallel graph of size n satisfies a **central limit theorem** with

$$\mathbb{E}\, X_n = \mu n + O(1) \quad \text{and} \quad \mathbb{V}\, X_n = \sigma^2 n + O(1),$$

where $\mu = 1.61673...$ and $\sigma^2 = 0.55347...$

Planar Graphs

Generating functions

$$G(x,y) = \exp\left(C(x,y)\right),$$

$$\frac{\partial C(x,y)}{\partial x} = \exp\left(\frac{\partial B}{\partial x}\left(x\frac{\partial C(x,y)}{\partial x}, y\right)\right),$$

$$\frac{\partial B(x,y)}{\partial y} = \frac{x^2}{2}\frac{1+D(x,y)}{1+y},$$

$$\frac{M(x,D)}{2x^2 D} = \log\left(\frac{1+D}{1+y}\right) - \frac{xD^2}{1+xD},$$

$$M(x,y) = x^2y^2\left(\frac{1}{1+xy} + \frac{1}{1+y} - 1 - \frac{(1+U)^2(1+V)^2}{(1+U+V)^3}\right),$$

$$U = xy(1+V)^2,$$

$$V = y(1+U)^2.$$

Planar Graphs

3-connected planar graphs

$M(x,y)$ ... generating function for the number of **3-connected edge-rooted planar maps** according to the number of vertices and edges

Whitney's theorem: every 3-connected planar graph has a unique embedding into the plane.

$\Longrightarrow \quad T^\bullet(x,y) = \frac{1}{2}M(x,y)$: ... generating function for the number of **3-connected labelled edge-rooted planar graphs**

Planar Graphs

Planar networks

A **network** N is a (multi-)graph with two distinguished vertices, called its poles (usually labelled 0 and ∞) such that the (multi-)graph $\hat{N}$ obtained from N by adding an edge between the poles of N is 2-connected.

Let M be a network and $X = (N_e, e \in E(M))$ a system of networks indexed by the edge-set $E(M)$ of M. Then $N = M(X)$ is called the **superposition** with core M and components N_e and is obtained by replacing all edges $e \in E(M)$ by the corresponding network N_e (and, of course, by identifying the poles of N_e with the end vertices of e accordingly).

A network N is called an h-**network** if it can be represented by $N = M(X)$, where the core M has the property that the graph $\hat{M}$ obtained by adding an edge joining the poles is 3-connected and has at least 4 vertices. Similarly $N = M(X)$ is called a p-**network** if M consists of 2 or more edges that connect the poles, and it is called an s-**network** if M consists of 2 or more edges that connect the poles in series.

Planar Graphs

Planar networks

Trakhtenbrot's canonical network decomposition theorem: any network with at least 2 edges belongs to exactly one of the 3 classes of h-, p- or s-networks. Furthermore, any h-network has a unique decomposition of the form $N = M(X)$, and a p-network (or any s-network) can be uniquely decomposed into components which are not themselves p-networks (or s-networks).

Planar Graphs

Planar networks

$K(x, y)$ ... generating function corresponding to all planar networks where the two poles are not connected by an edge.

$D(x, y)$ ... generating function corresponding to all planar networks with at least one edge

$S(x, y)$ ... generating function corresponding to all s-networks

$F(x, y) = D(x, y) - S(x, y)$ ... the generating function corresponding to all non-s-networks (with at least one edge)

$N(x, y)$ ... generating function corresponding to all non-p-networks.

Planar Graphs

Planar networks

$$
\begin{aligned}
\frac{\partial B(x,y)}{\partial y} &= \frac{x^2}{2} K(x,y), \\
D(x,y) &= (1+y)K(x,y) - 1, \\
K(x,y) &= e^{N(x,y)}, \\
S(x,y) &= xD(x,y)(D(x,y) - S(x,y)), \\
\frac{T^\bullet(x, D(x,y))}{x^2 D(x,y))} &= N(x,y) - S(x,y).
\end{aligned}
$$

$$
\Longrightarrow \quad
\begin{aligned}
\frac{M(x,D)}{2x^2 D} &= \log\left(\frac{1+D}{1+y}\right) - \frac{xD^2}{1+xD} \\
\frac{\partial B(x,y)}{\partial y} &= \frac{x^2}{2} \frac{1+D(x,y)}{1+y}
\end{aligned}
$$

Planar Graphs

Asymptotics

$$
\begin{aligned}
& U(x,y) = xy(1+V(x,y))^2, \\
& V(x,y) = y(1+U(x,y))^2 \\
\Longrightarrow \quad & U(x,y) = xy(1+y(1+U(x,y))^2)^2 \\
\Longrightarrow \quad & U(x,y) = g(x,y) - h(x,y)\sqrt{1-\frac{y}{\tau(x)}} \\
\Longrightarrow \quad & V(x,y) = g_2(x,y) - h_2(x,y)\sqrt{1-\frac{y}{\tau(x)}}
\end{aligned}
$$

$$
\begin{aligned}
& M(x,y) = x^2y^2\left(\frac{1}{1+xy} + \frac{1}{1+y} - 1 - \frac{(1+U)^2(1+V)^2}{(1+U+V)^3}\right) \\
!!! \quad \Longrightarrow \quad & M(x,y) = g_3(x,y) + h_3(x,y)\left(1-\frac{y}{\tau(x)}\right)^{\frac{3}{2}}
\end{aligned}
$$

due to cancellation of the $\sqrt{1-y/\tau(x)}$-term

Planar Graphs

Asymptotics

$$\frac{M(x,D)}{2x^2D} = \log\left(\frac{1+D}{1+y}\right) - \frac{xD^2}{1+xD}$$

$$!!! \quad \Longrightarrow \quad D(x,y) = g_4(x,y) + h_4(x,y)\left(1 - \frac{x}{R(y)}\right)^{\frac{3}{2}}$$

due to interaction of the singularities!!!

$$\frac{\partial B(x,y)}{\partial y} = \frac{x^2}{2}\frac{1+D(x,y)}{1+y},$$

$$!!! \quad \Longrightarrow \quad B(x,y) = g_5(x,y) + h_5(x,y)\left(1 - \frac{x}{R(y)}\right)^{\frac{5}{2}}$$

$$\Longrightarrow \quad \boxed{b_n \sim b \cdot R(1)^{-n} n^{-\frac{7}{2}} n!}$$

Planar Graphs

Asymptotics

$$B'(x,y) = g_6(x,y) + h_6(x,y)\left(1 - \frac{x}{R(y)}\right)^{\frac{3}{2}},$$

$$C'(x,y) = e^{B'(xC'(x,y),y)},$$

$$!!! \quad \Longrightarrow \quad C'(x,y) = g_7(x,y) + h_7(x,y)\left(1 - \frac{x}{r(y)}\right)^{\frac{3}{2}}$$

due to interaction of the singularities!!!

$$\Longrightarrow \quad C(x,y) = g_8(x,y) + h_8(x,y)\left(1 - \frac{x}{r(y)}\right)^{\frac{5}{2}}.$$

$$\Longrightarrow \quad \boxed{c_n \sim c\, r(1)^{-n} n^{-\frac{7}{2}} n!}$$

Planar Graphs

Asymptotics

$$C(x,y) = g_8(x,y) + h_8(x,y)\left(1 - \frac{x}{r(y)}\right)^{\frac{5}{2}}$$

$$\Longrightarrow \quad G(x,y) = e^{C(x,y)} = g_9(x,y) + h_9(x,y)\left(1 - \frac{x}{r(y)}\right)^{\frac{5}{2}}.$$

$$\Longrightarrow \quad \boxed{g_n \sim g \cdot r(1)^{-n} n^{-\frac{7}{2}} n!}$$

Planar Graphs

The number of edges

$$G(x,y) = e^{C(x,y)} = g_9(x,y) + h_9(x,y)\left(1 - \frac{x}{r(y)}\right)^{\frac{5}{2}}.$$

Theorem

The number of edges X_n in a planar graph of size n satisfies a **central limit theorem** with

$$\mathbb{E}\, X_n = \mu n + O(1) \quad \text{and} \quad \mathbb{V}\, X_n = \sigma^2 n + O(1),$$

where $\mu = 2.2132652...$ and $\sigma^2 = 0.4303471...$

Degree Distribution

Outerplanar graphs

Theorem

$X_n^{(k)}$ ... number of vertices of degree k in random 2-connected, connected or unrestricted **labelled outerplanar** graphs with n vertices.

$\Longrightarrow$ $X_n^{(k)}$ satisfies a **central limit theorem** with

$$\mathbb{E}\, X_n^{(k)} \sim \mu_k n \quad \text{and} \quad \mathbb{V}\, X_n^{(k)} \sim \sigma_k^2 n.$$

Degree Distribution

Outerplanar graphs $p(w) = \sum_{k \ge 1} \mu_k w^k$

- **2-connected**

$$p(w) = \frac{2(3-2\sqrt{2})w^2}{(1-(\sqrt{2}-1)w)^2}$$

- **connected** or **unrestricted**:

$$p(w) = \frac{c_1 w^2}{(1-c_2 w)^2} \exp\left(c_3 w + \frac{c_4 w^2}{(1-c_2 w)}\right)$$

(with certain constants $c_1, c_2, c_3, c_4 > 0$).

Degree Distribution

Outerplanar graphs

Theorem

Δ_n ... maximum degree of outerplanar graphs of size n

$$\Longrightarrow \quad \frac{\Delta_n}{\log n} \to c \quad \text{in probability}$$

$$\mathbb{E}\,\Delta_n \sim c \log n.$$

(Application of first and second moment method.)

Degree Distribution

Series-parallel graphs

Theorem

$X_n^{(k)}$ ... number of vertices of degree k in random 2-connected, connected or unrestricted **labelled series-parallel** graphs with n vertices.

$\Longrightarrow$ $X_n^{(k)}$ satisfies a **central limit theorem** with

$$\mathbb{E}\, X_n^{(k)} \sim \mu_k n \quad \text{and} \quad \mathbb{V}\, X_n^{(k)} \sim \sigma_k^2 n.$$

Degree Distribution

2-connected series-parallel graphs $p(w) = \sum_{k\geq 1} \mu_k w^k$:

$$\boxed{p(w) = \frac{B_1(1,w)}{B_1(1,1)}},$$

where $B_1(y,w)$ is given by the following procedure ...

Degree Distribution

$$\frac{E_0(y)^3}{E_0(y)-1} = \left(\log\frac{1+E_0(y)}{1+R(y)} - E_0(y)\right)^2,$$

$$R(y) = \frac{\sqrt{1-1/E_0(y)}-1}{E_0(y)},$$

$$E_1(y) = -\left(\frac{2R(y)E_0(y)^2(1+R(y)E_0(y))^2}{(2R(y)E_0(y)+R(y)^2E_0(y)^2)^2+2R(y)(1+R(y)E_0(y))}\right)^{\frac{1}{2}},$$

$$D_0(y,w) = (1+yw)e^{\frac{R(y)E_0(y)}{1+R(y)E_0(y)}D_0(y,w)} - 1,$$

$$D_1(y,w) = \frac{(1+D_0(y,w))^{\frac{R(y)E_1(y)D_0(y,w)}{1+R(y)E_0(y)}}}{1-(1+D_0(y,w))^{\frac{R(y)E_0(y)D_0(y,w)}{1+R(y)E_0(y)}}},$$

$$B_0(y,w) = \frac{R(y)D_0(y,w)}{1+R(y)E_0(y)} - \frac{R(y)^2E_0(y)D_0(y,w)^2}{2(1+R(y)E_0(y))},$$

$$B_1(y,w) = \frac{R(y)D_1(y,w)}{1+R(y)E_0(y)} - \frac{R(y)^2E_0(y)D_0(y,w)D_1(y,w)}{1+R(y)E_0(y)} - \frac{R(y)^2E_1(y)D_0(y,w)(1+D_0(y,w)/2)}{(1+R(y)E_0(y))^2}.$$

Degree Distribution

Series-parallel graphs

Theorem

Δ_n ... maximum degree of series-parallel graphs of size n

$$\Longrightarrow \quad \frac{\Delta_n}{\log n} \to c \quad \text{in probability}$$

$$\mathbb{E}\,\Delta_n \sim c \log n.$$

Degree Distribution

Planar graphs

Theorem

$X_n^{(k)}$ ... number of vertices of degree k in random 3-connected, 2-connected, connected or unrestricted **labelled planar** graphs with n vertices.

$\Longrightarrow \; \mathbb{E}\, X_n^{(k)} \sim p_k\, n$

For $k \le 2$, $X_n^{(k)}$ satisfies also a central limit theorem.

Degree Distribution

unrestricted planar graphs $p(w) = \sum_{k \geq 1} p_k w^k$:

p_1	p_2	p_3	p_4	p_5	p_6
0.0367284	0.1625794	0.2354360	0.1867737	0.1295023	0.0861805

$$p(w) = -e^{B_0(1,w)-B_0(1,1)} B_2(1,w) + e^{B_0(1,w)-B_0(1,1)} \frac{1 + B_2(1,1)}{B_3(1,1)} B_3(1,w)$$

where $B_j(y,w)$ are given by the following procedure ...

Degree Distribution

- Implicit equation for $D_0(y,w)$:

$$1 + D_0 = (1 + y\boxed{w})\exp\left(\frac{\sqrt{S}(D_0(t-1)+t)}{4(3t+1)(D_0+1)} - \right.$$
$$\left. - \frac{D_0^2(t^4-12t^2+20t-9) + D_0(2t^4+6t^3-6t^2+10t-12) + t^4+6t^3+9t^2}{4(t+3)(D_0+1)(3t+1)}\right),$$

where $t = t(y)$ satisfies $y+1 = \frac{1+2t}{(1+3t)(1-t)}\exp\left(-\frac{1}{2}\frac{t^2(1-t)(18+36t+5t^2)}{(3+t)(1+2t)(1+3t)^2}\right)$.
and $S = (D_0(t-1)+t)(D_0(t-1)^3 + t(t+3)^2)$.

- Explicit expressions in terms of $D_0(y,w)$ (**SEVERAL PAGES !!!!**):

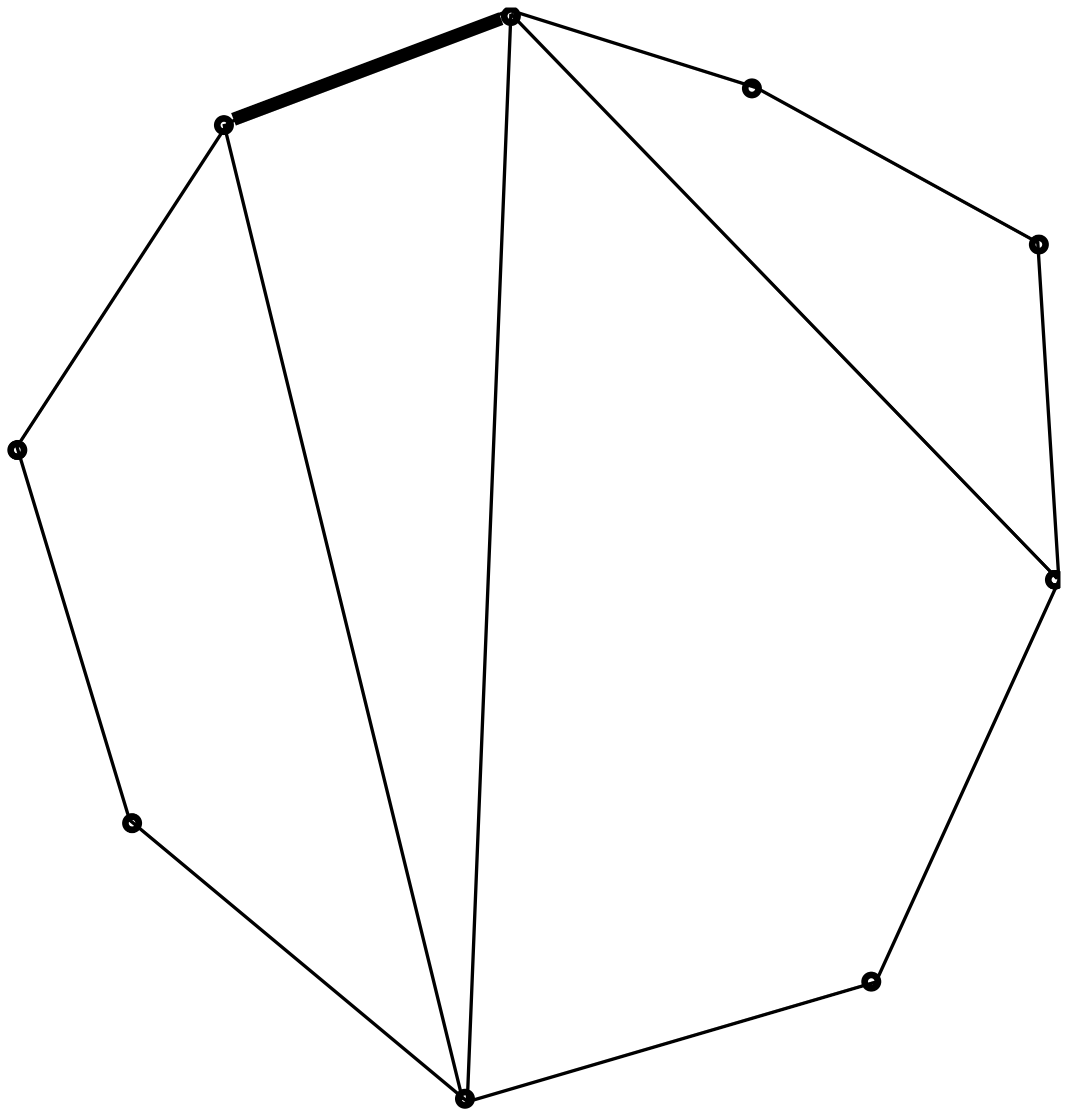
$$D_2(y,w),\ D_3(y,w),\ B_0(y,w),\ B_2(y,w),\ B_3(y,w)$$

- Explict expression for $p(w)$:

$$\boxed{p(w) = -e^{B_0(1,w)-B_0(1,1)}B_2(1,w) + e^{B_0(1,w)-B_0(1,1)}\frac{1+B_2(1,1)}{B_3(1,1)}B_3(1,w)}$$

Nodes of Given Degree

Dissections:



Nodes of Given Degree

- v_2 counts the number of nodes with degree 2,

- v_3 counts the number of nodes with degree 3,

- v counts the number of nodes with degree > 3, and

- in all cases the two **nodes of the rooted edge are are not taken into account.**

Nodes of Given Degree

- $A_{ij}(v_2, v_3, v)$ ... generating function of dissections with the properties that the left node of the rooted edge has degree i and right one has degree i, $2 \le i, j \le 3$

- $A_{i\infty}(v_2, v_3, v)$ ... generating function of dissections with the properties that the left node of the rooted edge has degree i and the right has degree > 3,

- $A_{\infty\infty}(v_2, v_3, v)$ ... generating function of dissections with the properties that both nodes of the rooted edge have degree > 3.

Nodes of Given Degree

The sum

$$A(v_2, v_3, v) = A_{22} + 2A_{23} + A_{33} + 2A_{2\infty} + 2A_{3\infty} + A_{\infty\infty}$$

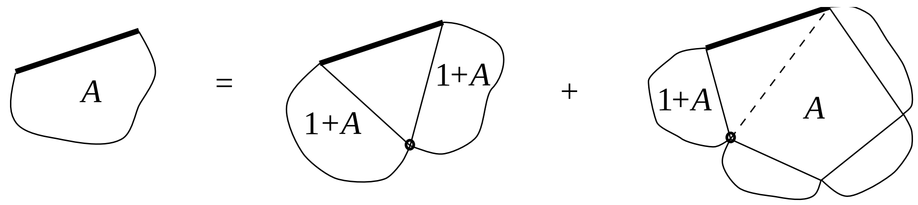
is the generating function of all dissections (with the corresponding counting in v_2, v_3, v).

In particular,

$$A(x) = A(x, x, x) = \frac{1 - 3x - \sqrt{1 - 6x + x^2}}{4x}.$$

Nodes of Given Degree

Lemma 3



$$
\begin{aligned}
A_{22} &= v_2 \\
&\quad + v_2 A_{22} + v_3 A_{23} + v A_{2\infty}, \\
A_{23} &= v_3 A_{22} + v(A_{23} + A_{2\infty}) \\
&= v_2 A_{23} + v_3 A_{33} + v A_{3\infty}, \\
A_{33} &= v(A_{22} + A_{23} + A_{2\infty})^2 \\
&\quad + v(A_{22} + A_{23} + A_{2\infty})(A_{23} + A_{33} + A_{3\infty}), \\
A_{2\infty} &= v_3 A_{23} + v(A_{33} + A_{3\infty}) + v(A_{2\infty} + A_{3\infty} + A_{\infty\infty}) \\
&\quad + v_2 A_{2\infty} + v_3 A_{3\infty} + v A_{\infty\infty}, \\
A_{3\infty} &= v(A_{23} + A_{33} + A_{3\infty})(A_{2\infty} + A_{3\infty} + A_{\infty\infty}) \\
&\quad + v(A_{22} + A_{23} + A_{2\infty})(A_{2\infty} + A_{3\infty} + A_{\infty\infty}), \\
A_{\infty\infty} &= v(A_{23} + A_{33} + A_{3\infty} + A_{2\infty} + A_{3\infty} + A_{\infty\infty})^2 \\
&\quad + v(A_{23} + A_{33} + A_{3\infty} + A_{2\infty} + A_{3\infty} + A_{\infty\infty})(A_{2\infty} + A_{3\infty} + A_{\infty\infty}).
\end{aligned}
$$

Nodes of Given Degree

Remark

All functions $A_{ij}(v_2, v_3, v)$ have a **squareroot singularity** due to the COMBINATORIAL CENTRAL LIMIT THEOREM II!!!

Nodes of Given Degree

- $B_i^\bullet(v_1, v_2, v_3, v)$ ... exponential genenerating functions of 2-connected labelled and marked outer planar graphs, where the root node (that is not counted) has degree i, $1 \leq i \leq 3$.

- $B_\infty^\bullet(v_1, v_2, v_3, v)$ ... exponential genenerating functions of 2-connected labelled and marked outer planar graphs, where the root node (that is not counted) has degree > 3.

Nodes of Given Degree

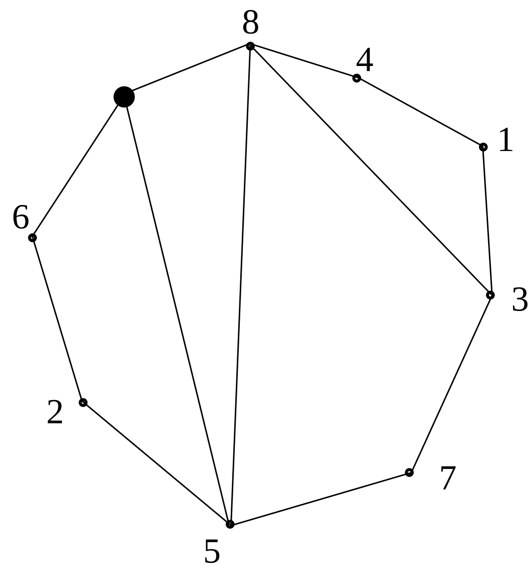
Lemma 4

$$B_1^{\bullet}(v_1, v_2, v_3, v) = v_1,$$

$$B_2^{\bullet}(v_1, v_2, v_3, v) = \frac{1}{2}\left(v_2 A_{22} + v_3 A_{23} + v A_{2\infty}\right),$$

$$B_3^{\bullet}(v_1, v_2, v_3, v) = \frac{1}{2}\left(v_2 A_{23} + v_3 A_{33} + v A_{3\infty}\right),$$

$$B_\infty^{\bullet}(v_1, v_2, v_3, v) = \frac{1}{2}\left(v_2 A_{2\infty} + v_3 A_{3\infty} + v A_{\infty\infty}\right).$$



Nodes of Given Degree

Remark

All functions $B_i^\bullet(v_1, v_2, v_3, v)$ have a **squareroot singularity** since all functions $A_{ij}(v_2, v_3, v)$ have squareroot singularities!!!

Nodes of Given Degree

- $C_i^\bullet(v_1, v_2, v_3, v)$ ... exponential generating functions of connected labelled and marked outer planar graphs, where the root node (that is not counted) has degree i, $0 \le i \le 3$.

- $C_\infty^\bullet(v_1, v_2, v_3, v)$ ... exponential generating functions of connected labelled and marked outer planar graphs, where the root node (that is not counted) has degree > 3.

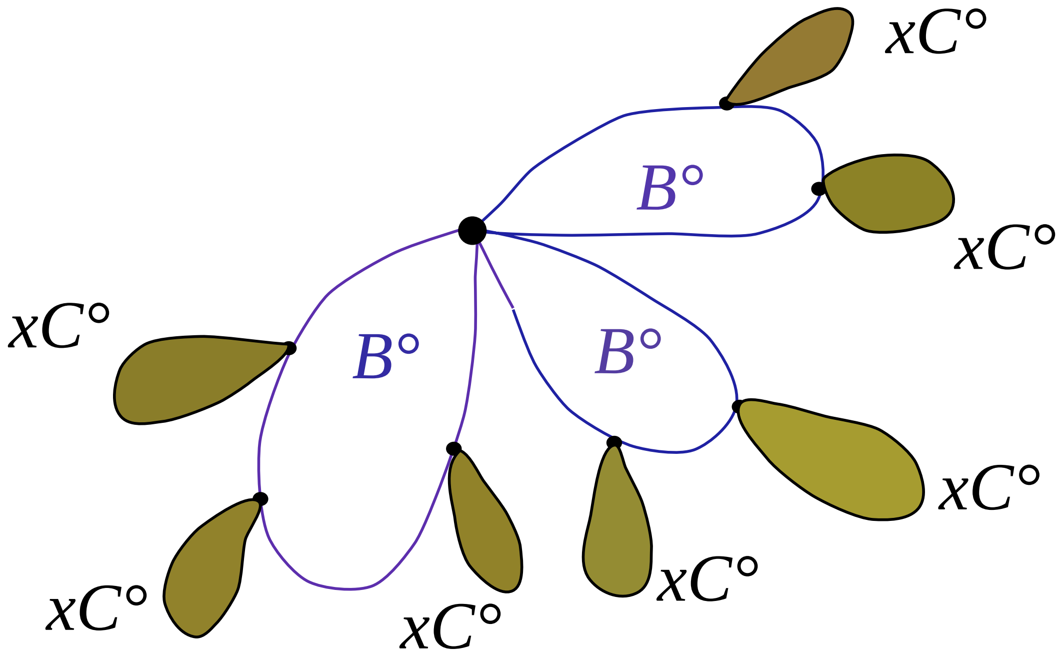
Nodes of Given Degree

Lemma 5

$$
\begin{aligned}
C_0^\bullet(v_1,v_2,v_3,v) &= 1,\\
C_1^\bullet(v_1,v_2,v_3,v) &= B_1^\bullet(W_1,W_2,W_3,W),\\
C_2^\bullet(v_1,v_2,v_3,v) &= \frac{1}{2!}(B_1^\bullet(W_1,W_2,W_3,W))^2 + B_2^\bullet(W_1,W_2,W_3,W),\\
C_3^\bullet(v_1,v_2,v_3,v) &= \frac{1}{3!}(B_1^\bullet(W_1,W_2,W_3,W))^3\\
&\quad + \frac{1}{1!1!}B_1^\bullet(W_1,W_2,W_3,W)B_2^\bullet(W_1,W_2,W_3,W)\\
&\quad + B_3^\bullet(W_1,W_2,W_3,W),\\
C_\infty^\bullet(v_1,v_2,v_3,v) &= e^{B_1^\bullet(W_1,W_2,W_3,W)+B_2^\bullet(\ldots)+B_3^\bullet(\ldots)+B_\infty^\bullet(W_1,W_2,W_3,W)}\\
&\quad - 1 - B_1^\bullet(W_1,W_2,W_3,W) - B_2^\bullet(\ldots) - B_3^\bullet(\ldots)\\
&\quad - \frac{1}{1!}(B_1^\bullet(W_1,W_2,W_3,W))^2 - \frac{1}{3!}(B_1^\bullet(W_1,W_2,W_3,W))^3\\
&\quad - \frac{1}{1!1!}B_1^\bullet(W_1,W_2,W_3,W)B_2^\bullet(W_1,W_2,W_3,W),
\end{aligned}
$$

where on the right hand side

$$W_1 = v_1 C_0^\bullet + v_2 C_1^\bullet + v_3 C_2^\bullet + v(C_3^\bullet + C_\infty^\bullet),$$
$$W_2 = v_2 C_0^\bullet + v_3 C_1^\bullet + v(C_2^\bullet + C_3^\bullet + C_\infty^\bullet),$$
$$W_3 = v_3 C_0^\bullet + v(C_1^\bullet + C_2^\bullet + C_3^\bullet + C_\infty^\bullet),$$
$$W = v(C_0^\bullet + C_1^\bullet + C_2^\bullet + C_3^\bullet + C_\infty^\bullet).$$



Nodes of Given Degree

Remark

All functions $C_i^\bullet(v_1, v_2, v_3, v)$ have a **squareroot singularity** due to the COMBINATORIAL CENTRAL LIMIT THEOREM II!!!

Nodes of Given Degree

Counting nodes of degree 3:

$C(v_1, v_2, v_3, v)$ ... exponential generating function of all connected labelled outer planar graphs

$C_{d=3}(x,u)$ ... exponential generating function that counts the number of nodes with x and the number of nodes of degree $d=3$ with u:

$$C_{d=3}(x,u) = C(x,x,xu,x).$$

Also:

$$\frac{\partial C_{d=3}(x,u)}{\partial x} = C_1^\bullet + C_2^\bullet + uC_3^\bullet + C_\infty^\bullet \quad \text{and} \quad \frac{\partial C_{d=3}(x,u)}{\partial u} = xC_3^\bullet$$

Nodes of Given Degree

Central limit theorem

$$\frac{\partial C_{d=3}(x,u)}{\partial x} = C_1^\bullet + C_2^\bullet + uC_3^\bullet + C_\infty^\bullet$$

$$\Longrightarrow \quad \frac{\partial C_{d=3}(x,u)}{\partial x} g(x,y) - h(x,y)\sqrt{1-\frac{x}{\rho(y)}}$$

$$\Longrightarrow \quad C_{d=3}(x,u) = g_2(x,y) + h_2(x,y)\left(1-\frac{x}{\rho(y)}\right)^{\frac{3}{2}}$$

$\Longrightarrow$ The number of nodes of degree 3 in outerplanar graphs satisfies a **central limit theorem**.

Degree Distribution of Planar Graphs

$C^\bullet = \frac{\partial C}{\partial x}$ ... GF, where one vertex is marked but not counted

w ... additional variable that *counts* the **degree of the marked vertex**

Generating functions:

$G^\bullet(x,y,w)$	**all rooted** planar graphs
$C^\bullet(x,y,w)$	**connected rooted** planar graphs
$B^\bullet(x,y,w)$	**2-connected rooted** planar graphs
$T^\bullet(x,y,w)$	**3-connected rooted** planar graphs

Note that $G^\bullet(x,y,1) = \dfrac{\partial G}{\partial x}(x,y)$ etc.

Degree Distribution of Planar Graphs

$$
\begin{aligned}
G^\bullet(x,y,w) &= \exp\left(C(x,y,1)\right) C^\bullet(x,y,w),\\
C^\bullet(x,y,w) &= \exp\left(B^\bullet\left(xC^\bullet(x,y,1),y,w\right)\right),\\
w\frac{\partial B^\bullet(x,y,w)}{\partial w} &= xyw\exp\left(S(x,y,w)+\frac{1}{x^2D(x,y,w)}T^\bullet\left(x,D(x,y,1),\frac{D(x,y,w)}{D(x,y,1)}\right)\right)\\
D(x,y,w) &= (1+yw)\exp\left(S(x,y,w)+\frac{1}{x^2D(x,y,w)}\times\right.\\
&\qquad\left.\times\, T^\bullet\left(x,D(x,y,1),\frac{D(x,y,w)}{D(x,y,1)}\right)\right)-1\\
S(x,y,w) &= xD(x,y,1)\left(D(x,y,w)-S(x,y,w)\right),\\
T^\bullet(x,y,w) &= \frac{x^2y^2w^2}{2}\left(\frac{1}{1+wy}+\frac{1}{1+xy}-1-\right.\\
&\qquad\left.-\frac{(u+1)^2\left(-w_1(u,v,w)+(u-w+1)\sqrt{w_2(u,v,w)}\right)}{2w(vw+u^2+2u+1)(1+u+v)^3}\right),\\
u(x,y) &= xy(1+v(x,y))^2,\quad v(x,y)=y(1+u(x,y))^2.
\end{aligned}
$$

Degree Distribution of Planar Graphs

3-connected planar graphs

$$\boxed{T^\bullet(x,y,w) = \tilde{T}_0(y,w) + \tilde{T}_2(y,w)\tilde{X}^2 + \tilde{T}_3(y,w)\tilde{X}^3 + O(\tilde{X}^4)}$$

with

$$\tilde{X} = \sqrt{1 - \frac{x}{r(y)}}$$

Degree Distribution of Planar Graphs

2-connected planar graphs

$$\Longrightarrow \quad \boxed{D(x,y,w) = D_0(y,w) + D_2(y,w)X^2 + D_3(y,w)X^3 + O(X^4)},$$

$$\Longrightarrow \quad \boxed{B^\bullet(x,y,w) = B_0(y,w) + B_2(y,w)X^2 + B_3(y,w)X^3 + O(X^4)}$$

with

$$X = \sqrt{1 - \frac{x}{R(y)}}$$

Degree Distribution of Planar Graphs

Lemma

$$f(x) = \sum_{n\ge 0} \boxed{a_n}\frac{x^n}{n!} = f_0 + f_2 X^2 + f_3 \boxed{X^3} + \mathcal{O}(X^4), \qquad X = \sqrt{1-\frac{x}{\rho}},$$

$$H(x,z,w) = h_0(x,w) + h_2(x,w)Z^2 + h_3(x,w)\boxed{Z^3} + \mathcal{O}(Z^4),$$

$$Z = \sqrt{1-\frac{z}{\boxed{f(\rho)}}},$$

$$f_H(x) = H(x,\boxed{f(x)},w) = \sum_{n\ge 0} \boxed{b_n(w)}\frac{x^n}{n!}$$

$$\Longrightarrow \quad \boxed{\lim_{n\to\infty} \frac{b_n(w)}{a_n} = -\frac{h_2(\rho,w)}{f_0} + \frac{h_3(\rho,w)}{f_3}\left(-\frac{f_2}{f_0}\right)^{3/2}}.$$

Degree Distribution of Planar Graphs

Connected planar graphs

$$C^\bullet(x,1,w) = \exp\left(B^\bullet\left(xC'(x),1,w\right)\right)$$

Application of the lemma with

$$f(x) = xC'(x)$$

and

$$H(x,z,w) = xe^{B^\bullet(z,1,w)}.$$

Contents 3

III. CONTINUOUS LIMITING OBJECTS

- Weak Convergence
- The Depth-First-Search of Rooted Trees
- The Continuum Random Tree
- The Profile of Galton-Watson trees
- The Schaeffer Bijection
- The ISE (Integrated SuperBrownian Excursion)

Asymptotics on Random Discrete Objects

Levels of complexity:

1. **Asymptotic enumeration**

2. **Distribution of (shape) parameters**

3. **Asymptotic shape (= continuous limiting object)**

Weak Convergence

X_n, X ... (real) random variables:

$$\boxed{X_n \xrightarrow{\text{d}} X} \quad :\Longleftrightarrow \quad \boxed{\lim_{n\to\infty} \mathbb{P}\{X_n \le x\} = \mathbb{P}\{X \le x\}}$$

for all points of continuity of $F_X(x) = \mathbb{P}\{X \le x\}$

$$\Longleftrightarrow \quad \boxed{\boxed{\lim_{n\to\infty} \mathbb{E}\, G(X_n) = \mathbb{E}\, G(X)}}$$

for all **bounded** continuous functionals $G : \mathbb{R} \to \mathbb{R}$

$$\Longleftrightarrow \quad \boxed{\lim_{n\to\infty} \mathbb{E}\, e^{itX_n} = \mathbb{E}\, e^{itX}}$$

for all real t
(Levy's criterion)

Weak Convergence

Polish space: (S, d) ... complete, separable, metric space

Examples: $\mathbb{R}$, $\mathbb{R}^k$, $C[0,1]$, $\mathcal{M}_0(X)$ (probability measures on X)

S-**valued random variable**: $\boxed{X : \Omega \to S}$ ... measurable function

$S = \mathbb{R}$: random variable

$S = \mathbb{R}^k$: k-dimensional random vector

$S = C[0,1]$: **stochastic process** $(X(t), 0 \leq t \leq 1)$

$S = \mathcal{M}_0(X)$: random measure

Weak Convergence

Definition

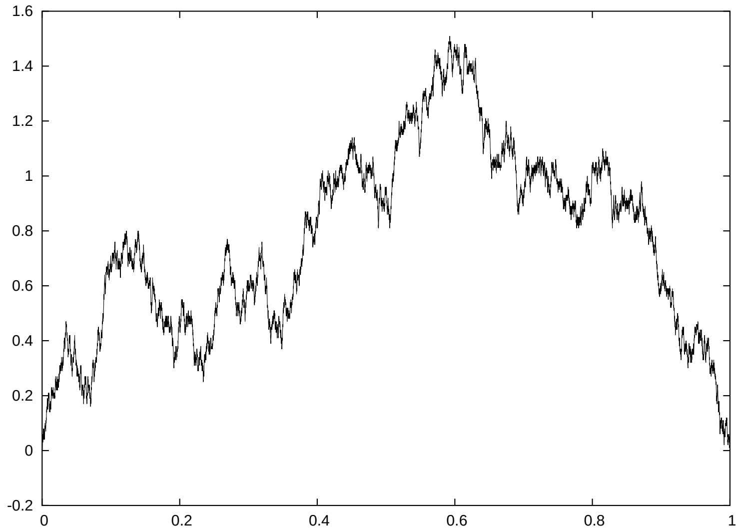
$X_n, X : \Omega \to S$ ... S-valued random variables ((S, d) ... Polish space)

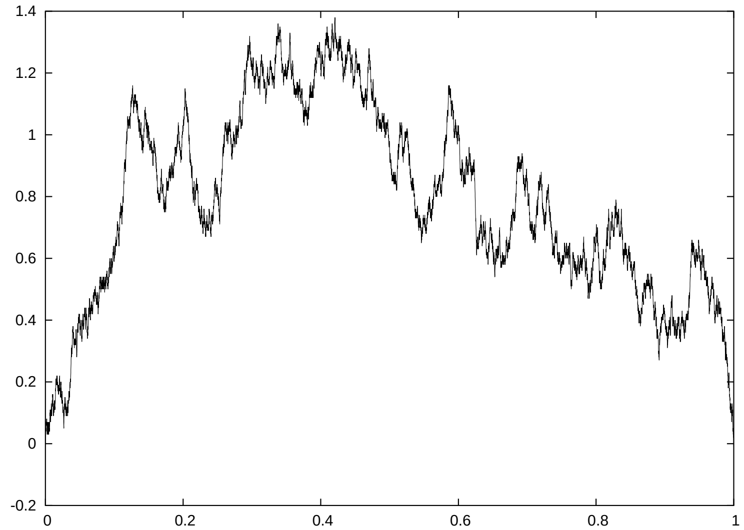
$$\boxed{X_n \xrightarrow{\text{d}} X} \quad :\Longleftrightarrow \quad \boxed{\boxed{\lim_{n\to\infty} \mathbb{E}\, G(X_n) = \mathbb{E}\, G(X)}}$$

for all **bounded** continuous functionals $G : S \to \mathbb{R}$

Weak Convergence

Stochastic process: random function





Weak Convergence

Stochastic process

$X_n : \Omega \to C[0,1]$ sequence of stochastic processes, $X : \Omega \to C[0,1]$

- $X_n \xrightarrow{\text{d}} X \quad \Longrightarrow \quad F(X_n) \xrightarrow{\text{d}} F(X)$ for all continuous $F : S \to S'$.

- $X_n \xrightarrow{\text{d}} X \quad \Longrightarrow \quad X_n(t_0) \xrightarrow{\text{d}} X(t_0)$ for all fixed $t_0 \in [0,1]$.

- $X_n \xrightarrow{\text{d}} X \quad \Longrightarrow \quad (X_n(t_1), \ldots, X_n(t_k)) \xrightarrow{\text{d}} (X(t_1), \ldots, X(t_k))$
  for all $k \geq 1$ and all fixed $t_1, \ldots, t_k \in [0,1]$.

The converse statement is not necessarily true, one needs **tightness**.

Weak Convergence

Stochastic process

$X_n : \Omega \to C[0,1]$ sequence of stochastic processes, $X : \Omega \to C[0,1]$

1. $(X_n(t_1), \ldots, X_n(t_k)) \xrightarrow{\text{d}} (X(t_1), \ldots, X(t_k))$
   for all $k \geq 1$ and all fixed $t_1, \ldots, t_k \in [0,1]$

2. $\mathbb{E}\left(|X_n(0)|^\beta\right) \leq C$
   for some constant $C > 0$ and an exponent $\beta > 0$

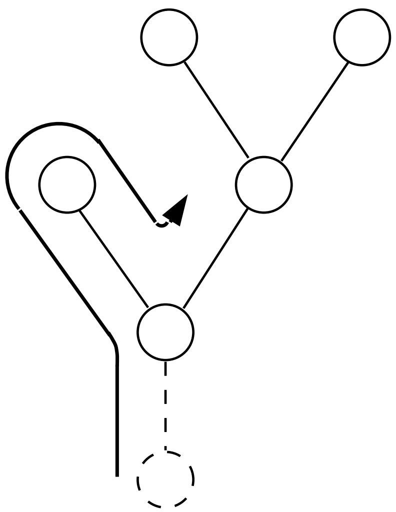
3. $\mathbb{E}\left(|X_n(t) - X_n(s)|^\beta\right) \leq C|t-s|^\alpha$ for all $s, t \in [0,1]$
   for some constant $C > 0$ and exponents $\alpha > 1$ and $\beta > 0$.

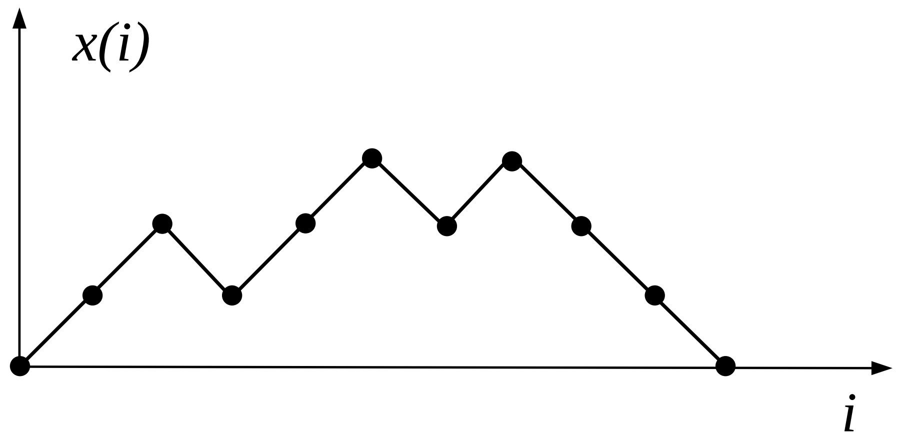
Then

$$\boxed{(X_n(t), 0 \leq t \leq 1) \xrightarrow{\text{d}} (X(t), 0 \leq t \leq 1)}.$$

Depth-First-Search

Rooted trees and discrete excursions





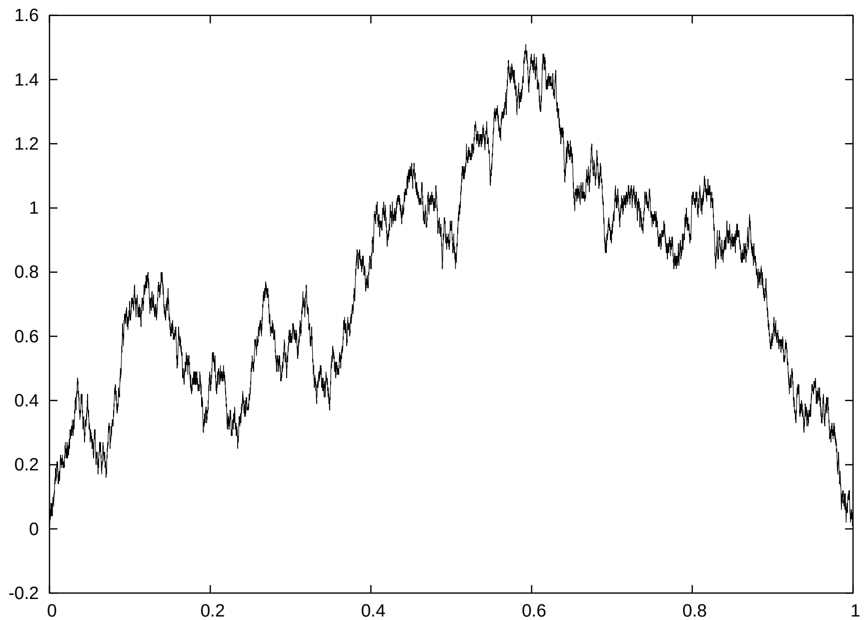
Bijection between

Catalan trees $\longleftrightarrow$ Dyck paths

random trees of size n $\longleftrightarrow$ **random** Dyck paths of length $2n$

Depth-First-Search

Brownian excursion $(e(t), 0 \leq t \leq 1)$



Rescaled Brownian motion between 2 zeros.

Random function in $C[0,1]$.

Depth-First-Search

Kaigh's Theorem

$(X_n(t), 0 \le t \le 2n)$ ... random Dyck path of length $2n$.

$$\Longrightarrow \quad \left(\frac{1}{\sqrt{2n}} X_n(2nt), 0 \le t \le 1\right) \xrightarrow{\mathrm{d}} (e(t), 0 \le t \le 1).$$

Remark. This theorem also holds for more general random walks with independent increments conditioned to be an excursion.

Real Trees

T ... tree, $\mathcal{T}$ ... embedding of T into the plane $\mathbb{R}^2$

$\Longrightarrow$ $\mathcal{T}$ is a metric space (and a **real tree** in the following sense):

Definition

A metric space $(\mathcal{T}, d)$ is a **real tree** if the following two properties hold for every $x, y \in \mathcal{T}$.

1. There is a unique isometric map $h_{x,y} : [0, d(x,y)] \to \mathcal{T}$ such that $h_{x,y}(0) = x$ and $h_{x,y}(d(x,y)) = y$.

2. If q is a continuous injective map from $[0,1]$ into $\mathcal{T}$ with $q(0) = x$ and $q(1) = y$ then

$$q([0,1]) = h_{x,y}([0, d(x,y)]).$$

A rooted real tree $(\mathcal{T}, d)$ is a real tree with a distinguished vertex $r = r(\mathcal{T})$ called the root.

Real Trees

Two real trees $(\mathcal{T}_1, d_1)$, $(\mathcal{T}_2, d_2)$ are **equivalent** if there is a root-preserving isometry that maps $\mathcal{T}_1$ onto $\mathcal{T}_2$.

$\mathbb{T}$ ... set of all equivalence classes of rooted compact real trees.

Gromov-Hausdorff Distance $d_{\mathsf{GH}}(\mathcal{T}_1, \mathcal{T}_2)$ of two real trees $\mathcal{T}_1, \mathcal{T}_2$ is the infimum of the Hausdorff distance of all isometric embeddings of $\mathcal{T}_1, \mathcal{T}_2$ into the same metric space.

Hausdorff distance: $\delta_{\mathsf{Haus}}(X, Y) = \max\left\{ \sup_{x \in X} \inf_{y \in Y} d(x, y), \sup_{y \in Y} \inf_{x \in X} d(x, y) \right\}$

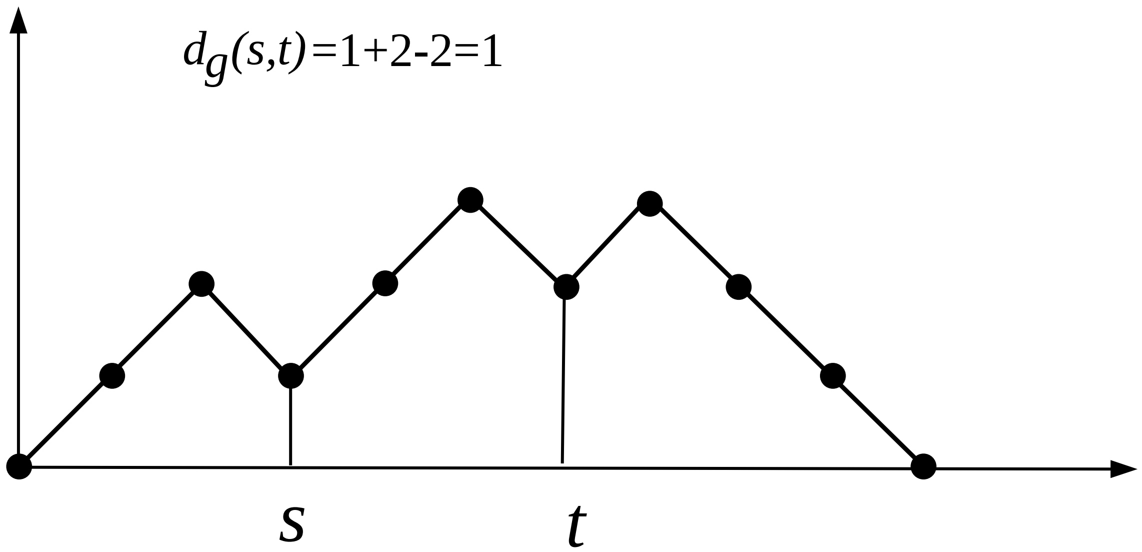
Theorem

The metric space $(\mathbb{T}, d_{\mathsf{GH}})$ is a Polish space.

Real Trees

$g : [0,1] \to [0,\infty)$ ... continuous, ≥ 0, $g(0) = g(1) = 0$

$$d_g(s,t) = g(s) + g(t) - 2 \inf_{\min\{s,t\} \leq u \leq \max\{s,t\}} g(u)$$

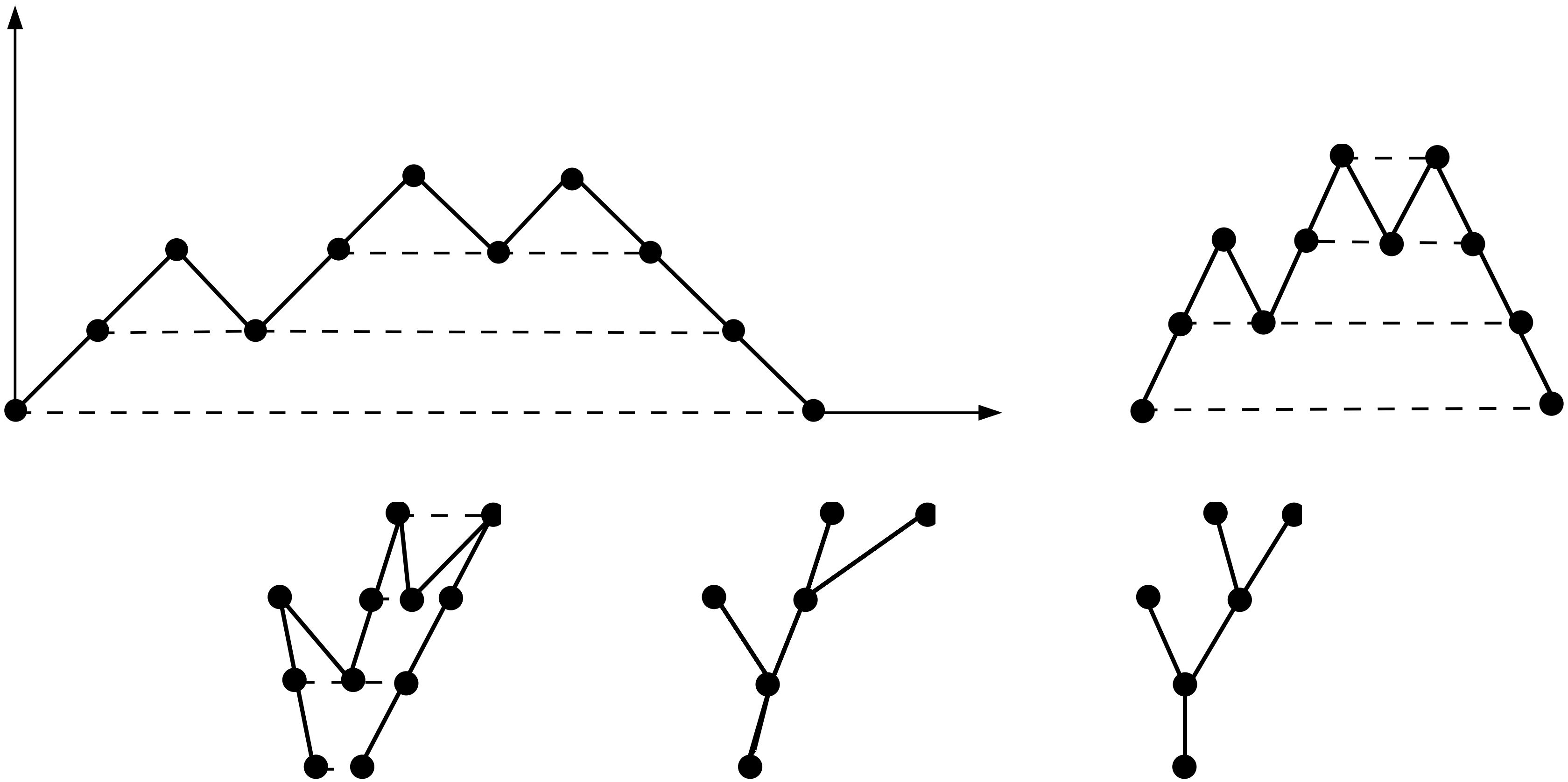


$$s \sim t \iff d_g(s,t) = 0 \qquad \mathcal{T}_g = [0,1]/\sim$$

$$\Longrightarrow \quad (\mathcal{T}_g, d_g) \quad \text{is a compact real tree.}$$

Real Trees

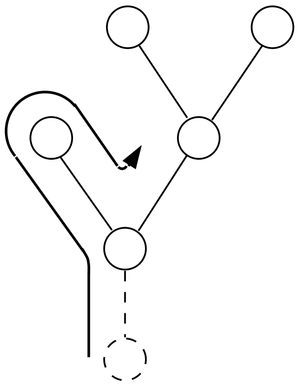
Construction of a real tree $\mathcal{T}_g$

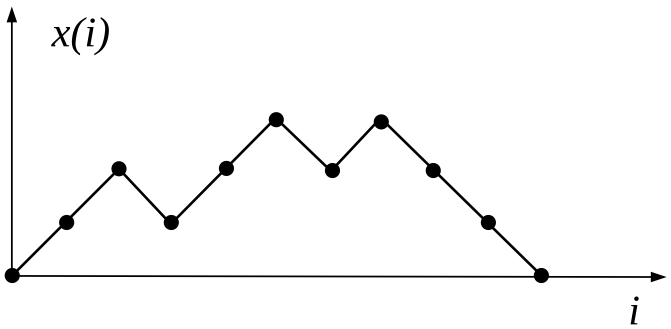


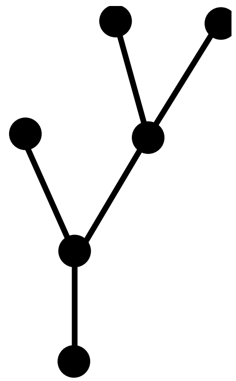
The mapping $C[0,1] \to \mathbb{T}$, $g \mapsto \mathcal{T}_g$ is **continuous**.

Real Trees

Catalan trees as real trees



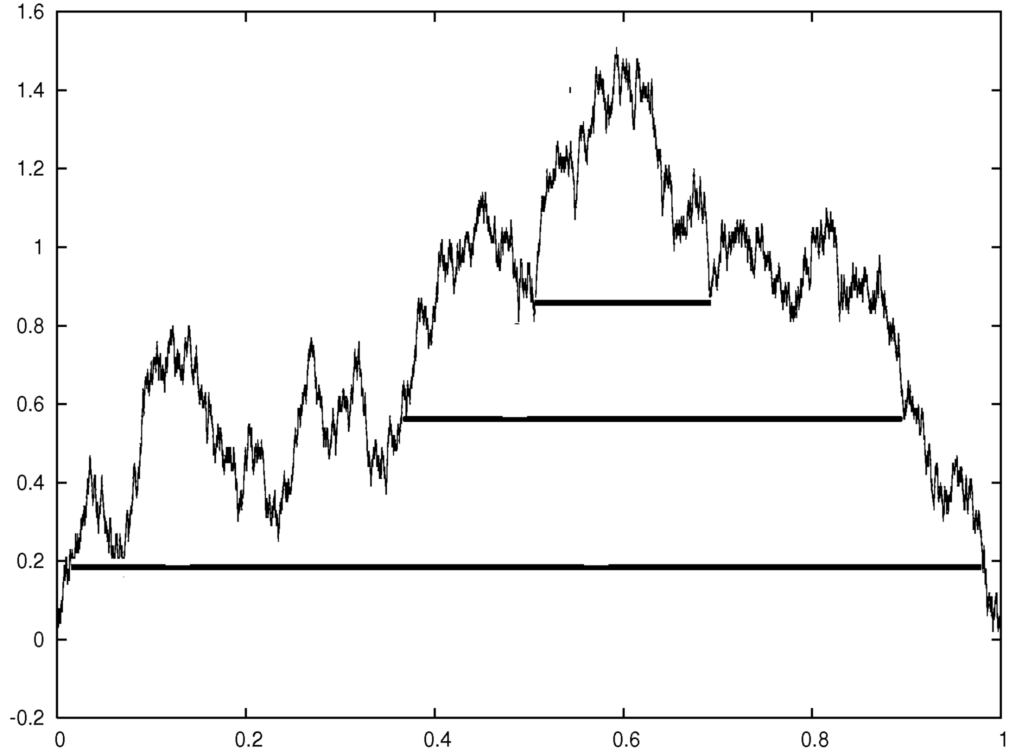




T_n $\qquad$ $X_n = X_{T_n}$ $\qquad$ $\mathcal{T}_{X_n}$

Real Trees

Continuum random tree $\mathcal{T}_{2e}$ (with Brownian excursion $e(t)$)



Real Trees

Theorem

$(X_n(t), 0 \le t \le 2n)$ ... random Dyck paths of length $2n$
or the depth-first-search process of Catalan trees of size n.

$$\Longrightarrow \quad \boxed{\frac{1}{\sqrt{2n}} \mathcal{T}_{X_n} \xrightarrow{\mathrm{d}} \mathcal{T}_{2e}}$$

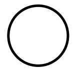
In other words...

Scaled Catalan trees (interpreted as "real trees") converge weakly to the continuum random tree.

Galton-Watson Trees

Galton-Watson branching process

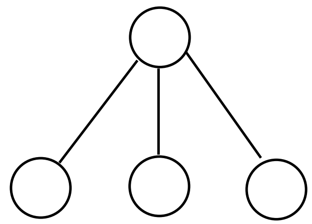
ξ ... offspring distribution, $\varphi_k = \mathbb{P}\{\xi = k\}$, $\varphi_0 > 0$



Galton-Watson Trees

Galton-Watson branching process

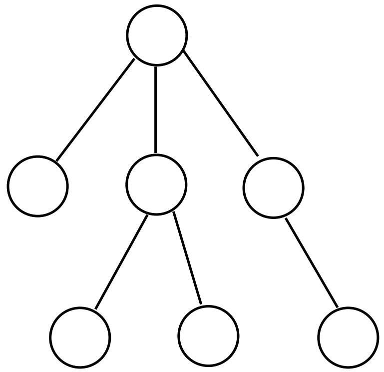
ξ ... offspring distribution, $\varphi_k = \mathbb{P}\{\xi = k\}$, $\varphi_0 > 0$



Galton-Watson Trees

Galton-Watson branching process

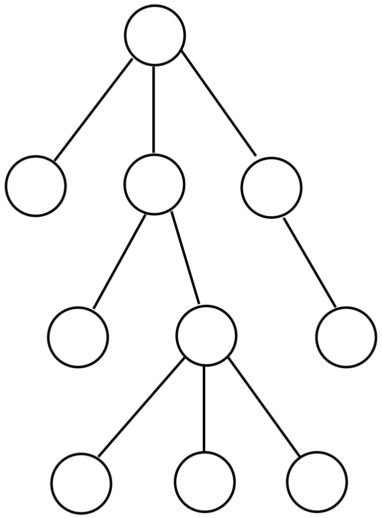
ξ ... offspring distribution, $\varphi_k = \mathbb{P}\{\xi = k\}$, $\varphi_0 > 0$



Galton-Watson Trees

Galton-Watson branching process

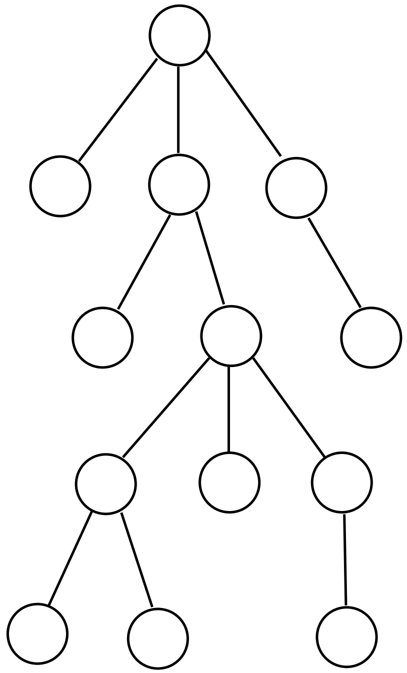
ξ ... offspring distribution, $\varphi_k = \mathbb{P}\{\xi = k\}$, $\varphi_0 > 0$



Galton-Watson Trees

Galton-Watson branching process

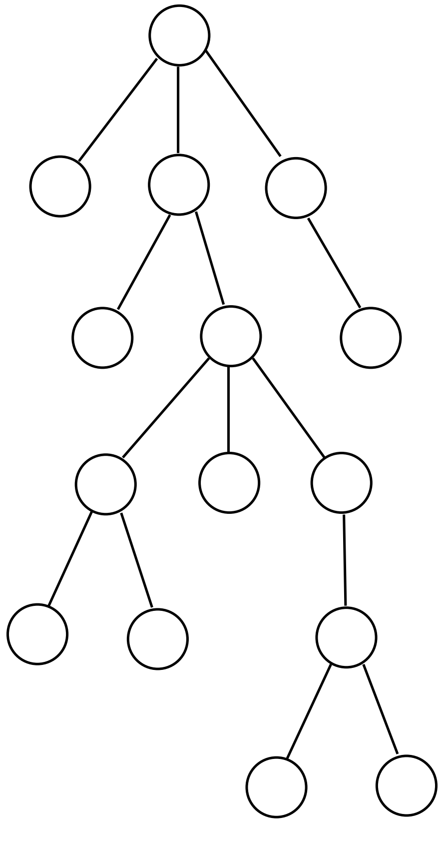
ξ ... offspring distribution, $\varphi_k = \mathbb{P}\{\xi = k\}$, $\varphi_0 > 0$



Galton-Watson Trees

Galton-Watson branching process

ξ ... offspring distribution, $\varphi_k = \mathbb{P}\{\xi = k\}$, $\varphi_0 > 0$



Galton-Watson Trees

Galton-Watson branching process. $(Z_k)_{k\geq 0}$

$Z_0 = 1$, and for $k \geq 1$

$$\boxed{Z_k = \sum_{j=1}^{Z_{k-1}} \xi_j^{(k)}},$$

where the $(\xi_j^{(k)})_{k,j}$ are iid random variables distributed as ξ.

Z_k ... number of nodes in k-th generation

$Z = Z_0 + Z_1 + Z_2 + \cdots$ ... total progeny

Galton-Watson Trees

Generating functions

$$y_n = \mathbb{P}\{Z = n\}, \qquad y(x) = \sum_{n\geq 1} y_n x^n$$

$$\Phi(w) = \mathbb{E}\, w^{\xi} = \sum_{k\geq 0} \varphi_k w^k$$

$$\Longrightarrow \quad \boxed{y(x) = x\,\Phi(y(x))}$$

Conditioned Galton-Watson tree

GW-branching process conditioned on the total progeny $Z = n$.

Galton-Watson Trees

Example. $\mathbb{P}\{\xi = k\} = 2^{-k-1}$, $\Phi(w) = 1/(2-w)$

$\Longrightarrow$ all trees of size n have the same probability

$\Longrightarrow$ conditioned GW-tree of size n is the same model as the **Catalan tree model** (with the uniform distribution on trees of size n)

Example. $\Phi(w) = \frac{1}{2}(1+w)^2$: **binary trees** with n internal nodes.

Example. $\Phi(w) = \frac{1}{3}(1+w+w^2)$: **Motzkin trees**

Example. $\Phi(w) = e^{w-1}$: **Cayley trees**

Galton-Watson Trees

General assumption: $\boxed{\mathbb{E}\,\xi = 1}$, $\boxed{0 < \mathbb{V}\mathrm{ar}\,\xi = \sigma^2 < \infty}$

Theorem (Aldous)

$X_n(t)$ ... depth-first-search of conditioned GW-trees of size n

$$\Longrightarrow \quad \boxed{\left(\frac{\sigma}{2\sqrt{n}} X_n(2nt), 0 \le t \le 1\right) \xrightarrow{\mathrm{d}} (e(t), 0 \le t \le 1)}.$$

Corollary

$$\boxed{\frac{\sigma}{\sqrt{n}} \mathcal{T}_{X_n} \xrightarrow{\mathrm{d}} \mathcal{T}_{2e}}$$

Galton-Watson Trees

Corollary H_n ... height of conditioned GW-trees of size n:

$$\Longrightarrow \quad \boxed{\frac{1}{\sqrt{n}} H_n \xrightarrow{\mathrm{d}} \frac{2}{\sigma} \max_{0 \le t \le 1} e(t)}$$

Remark. Distribution function of $\max\limits_{0 \le t \le 1} e(t)$:

$$\mathbb{P}\{\max_{0 \le t \le 1} e(t) \le x\} = 1 - 2 \sum_{k=1}^{\infty} (4x^2k^2 - 1)e^{-2x^2k^2}$$

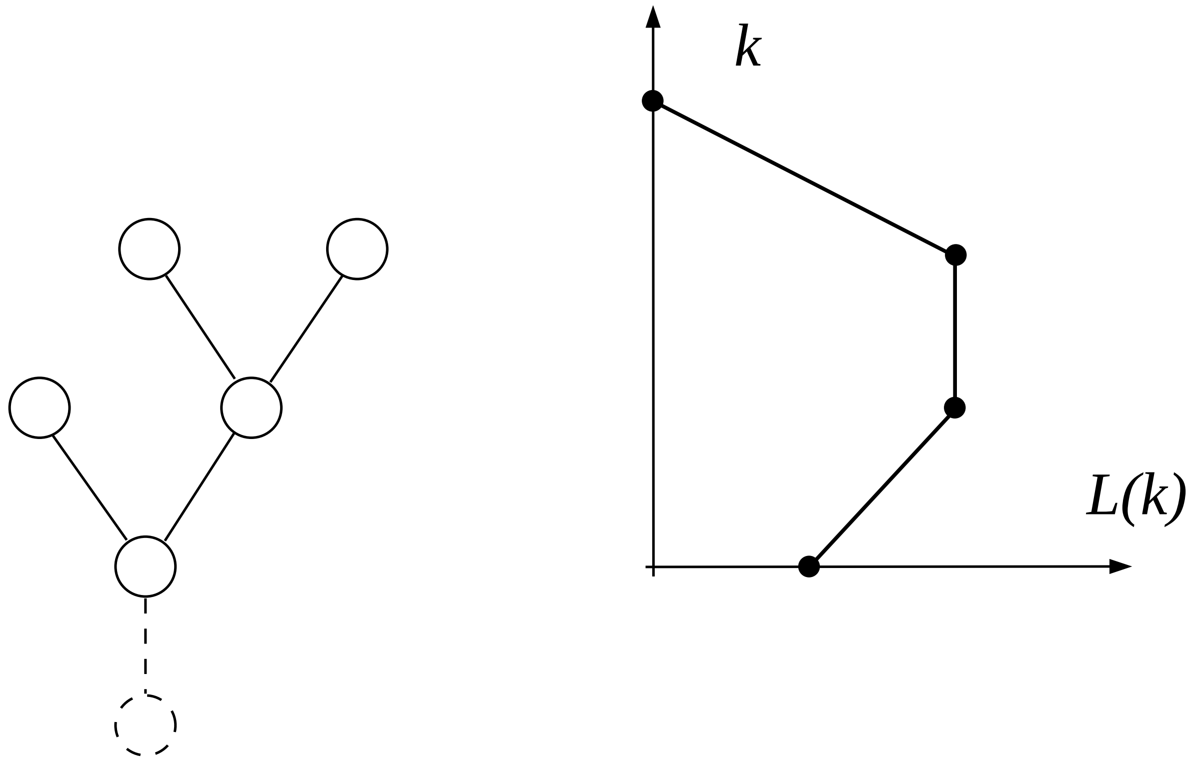
Galton-Watson Trees

Profile

$L_T(k)$ ... number of nodes at distance k from the root

$(L_T(k))_{k\geq 0}$ ... profile of T

$(L_T(s), s \geq 0)$ ... linearly interpolated profile of T



Galton-Watson Trees

Value distribution

$$\mu_T = \frac{1}{|T|} \sum_{k \geq 0} L_T(k)\, \delta_k$$

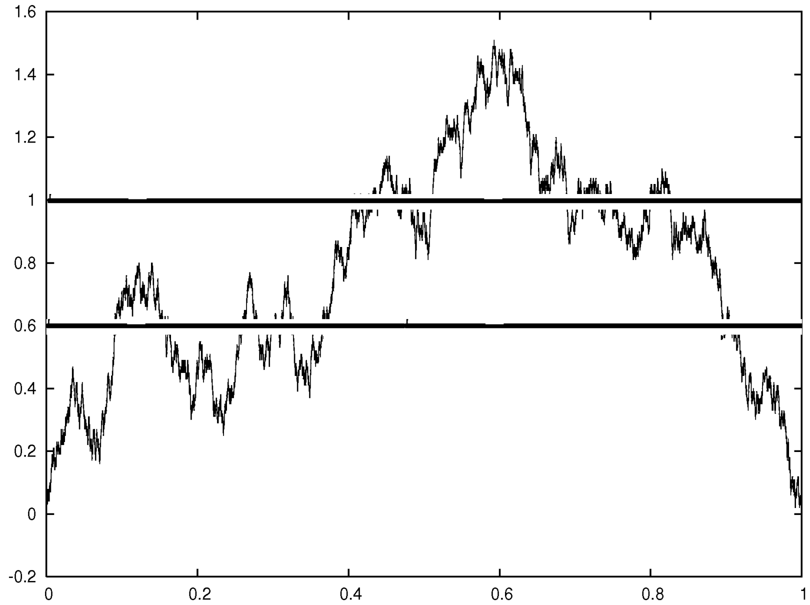
δ_x ... δ-distribution concentrated at x

Galton-Watson Trees

Occupation measure: random measure on $\mathbb{R}$

$$\mu(A) = \int_0^1 \mathbf{1}_A(e(t)\, dt$$

measure how long $e(t)$ stays in set A



Galton-Watson Trees

Theorem (Aldous)

$(L_n(k), k \ge 0)$ ... random profile of conditioned GW-trees of size n

$$\Longrightarrow \quad \boxed{\frac{1}{n} \sum_{k \ge 0} L_n(k)\, \delta_{(\sigma/2)k/\sqrt{n}} \xrightarrow{\mathrm{d}} \mu}$$

Galton-Watson Trees

Local time of the Brownian excursion: random density of μ

$$\boxed{l(s) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^1 \mathbf{1}_{[s,s+\varepsilon]}(e(t))\, dt}$$

Theorem (D.+Gittenberger)

$(L_n(s), s \ge 0)$ ... random profile of conditioned GW-trees of size n

$$\Longrightarrow \quad \boxed{\left(\frac{1}{\sqrt{n}} L_n(s\sqrt{n}),\ s \ge 0\right) \xrightarrow{\text{d}} \left(\frac{\sigma}{2} l\left(\frac{\sigma}{2} s\right),\ s \ge 0\right)}$$

Proof with asymptotics on generating functions (very involved)!!!

Galton-Watson Trees

Width

$$W = \max_{k \ge 0} L(k) = \max_{t \ge 0} L(t),$$

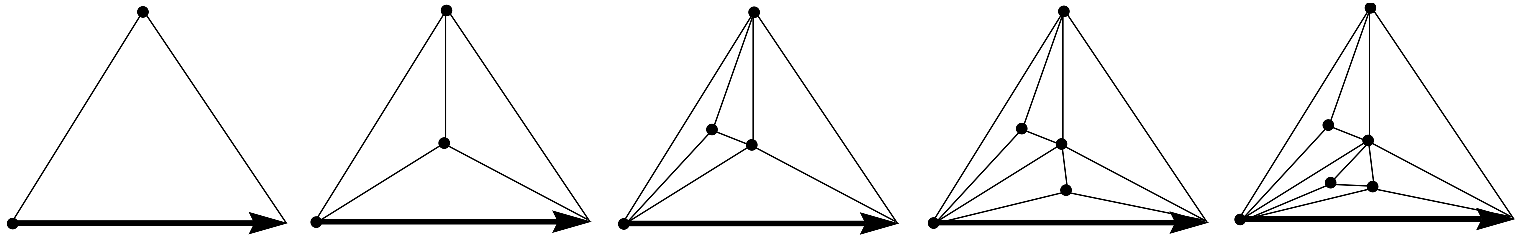
maximal number of nodes in a level.

Corollary

$$\boxed{\frac{1}{\sqrt{n}} W_n \xrightarrow{\mathrm{d}} \frac{\sigma}{2} \sup_{0 \le t \le 1} l(t)}$$

Remark. $\sup_{t \ge 0} l(t) = 2 \sup_{0 \le t \le 1} e(t)$ (in distribution)

Stacked Triangulations



Theorem (Albenque+Marckert)

M_n ... uniform stacked triangulations with $2n$ faces with graph distance as metric:

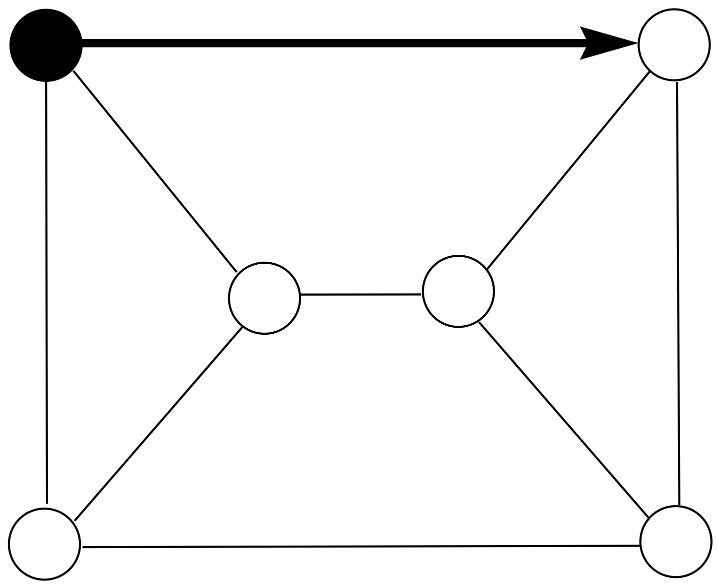
$$\Longrightarrow \quad \boxed{\frac{11}{\sqrt{6n}} M_n \xrightarrow{\text{d}} \mathcal{T}_{2e}}$$

in the Gromov-Hausdorff topology.

Remark. The continuum random tree $\mathcal{T}_{2e}$ seems to be a universal continuous limiting object.

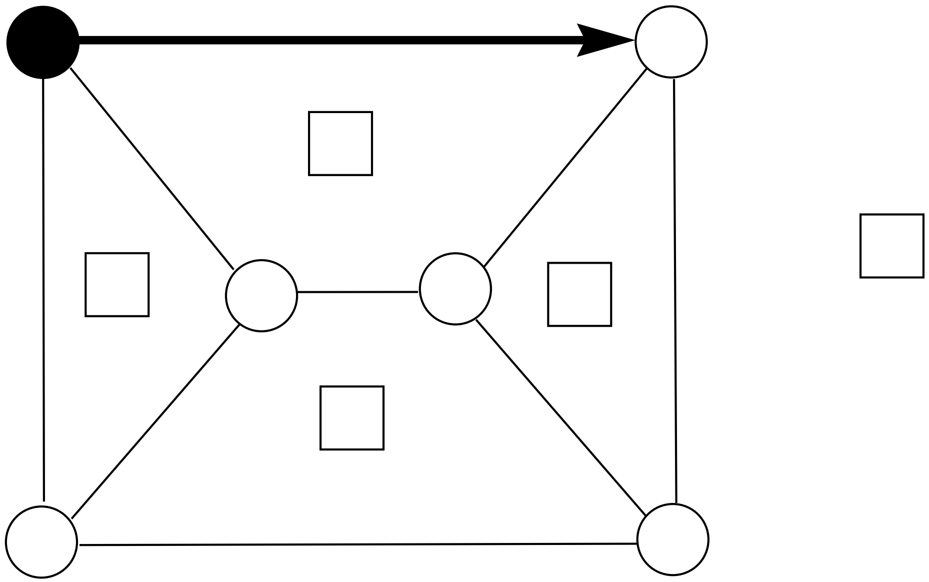
Quadrangulations

Bijection between 2-connected maps and quadrangulations



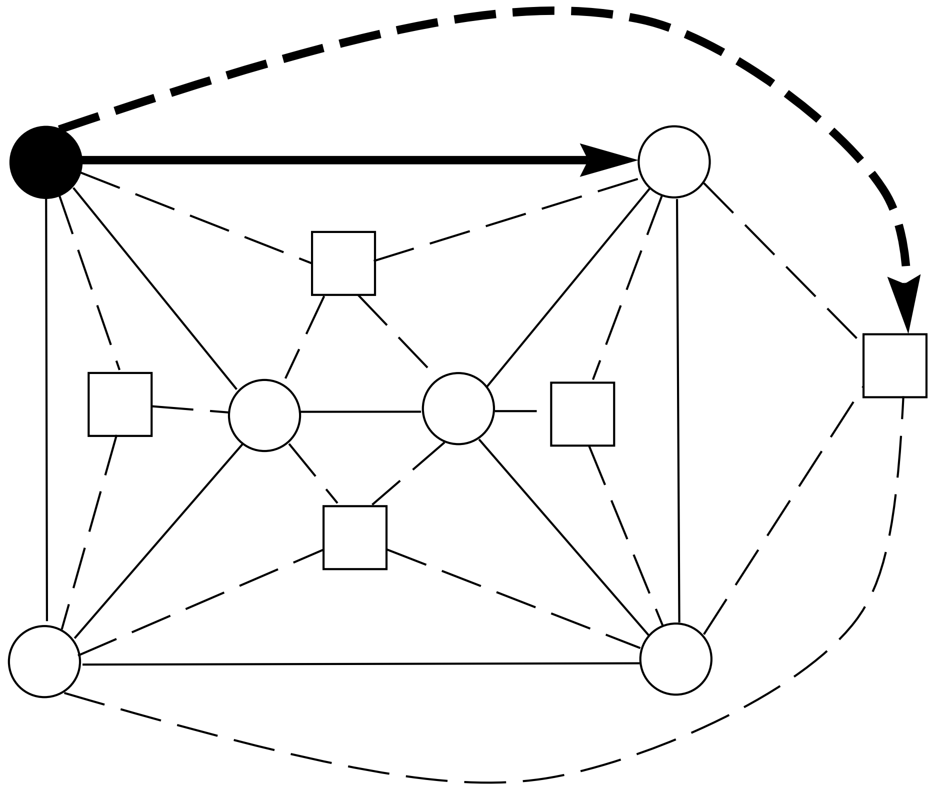
Quadrangulations

Bijection between 2-connected maps and quadrangulations



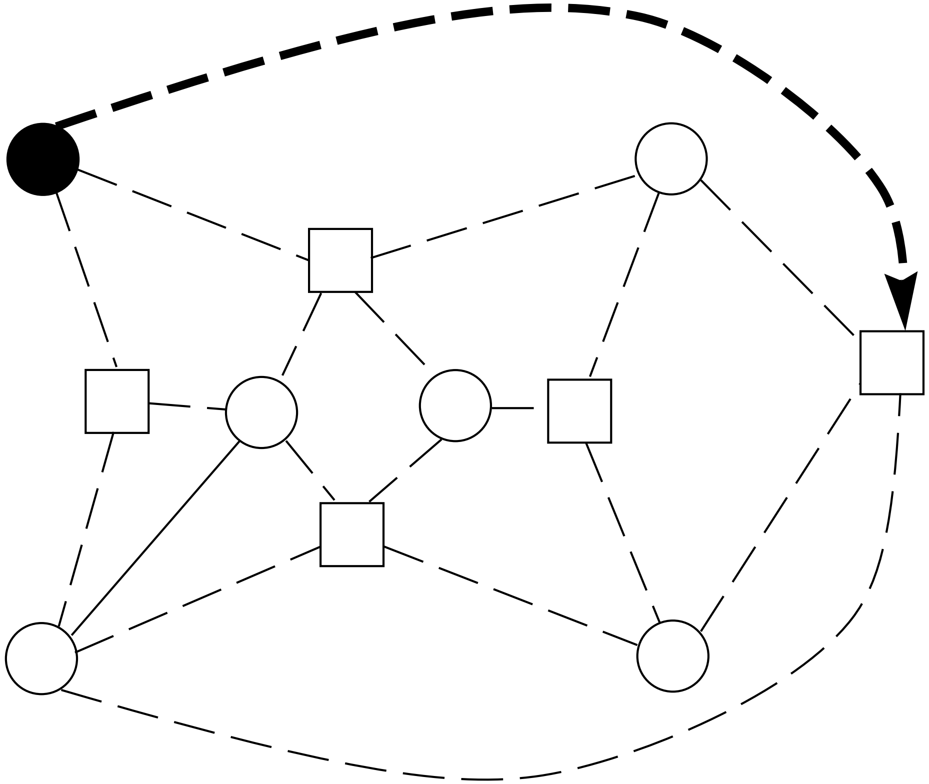
Quadrangulations

Bijection between 2-connected maps and quadrangulations



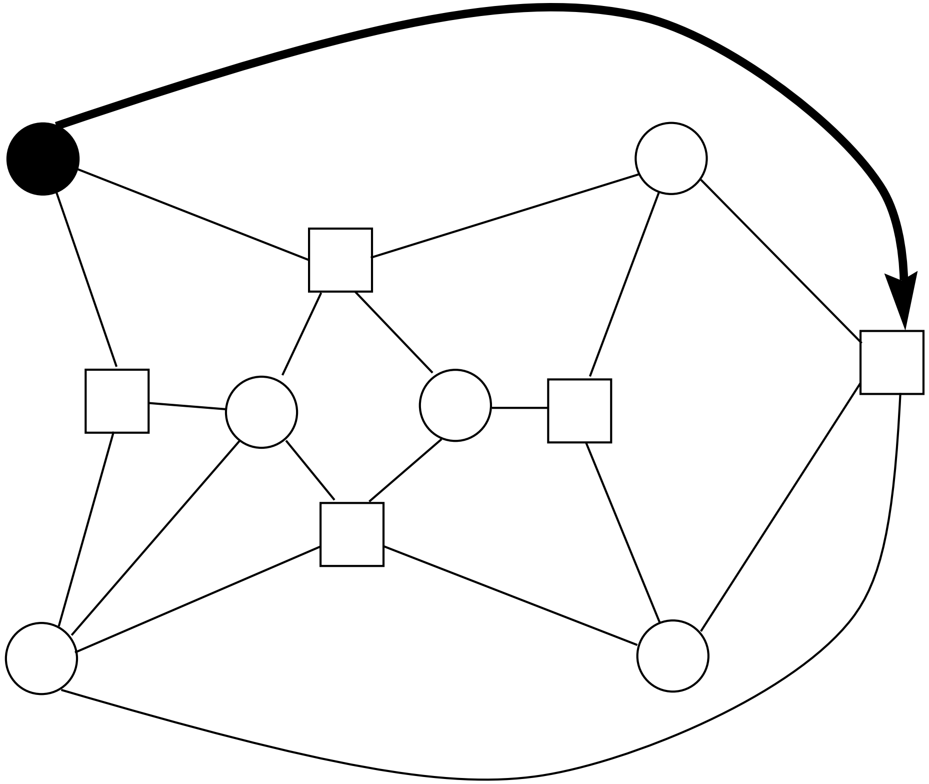
Quadrangulations

Bijection between 2-connected maps and quadrangulations



Quadrangulations

Bijection between 2-connected maps and quadrangulations



Quadrangulations

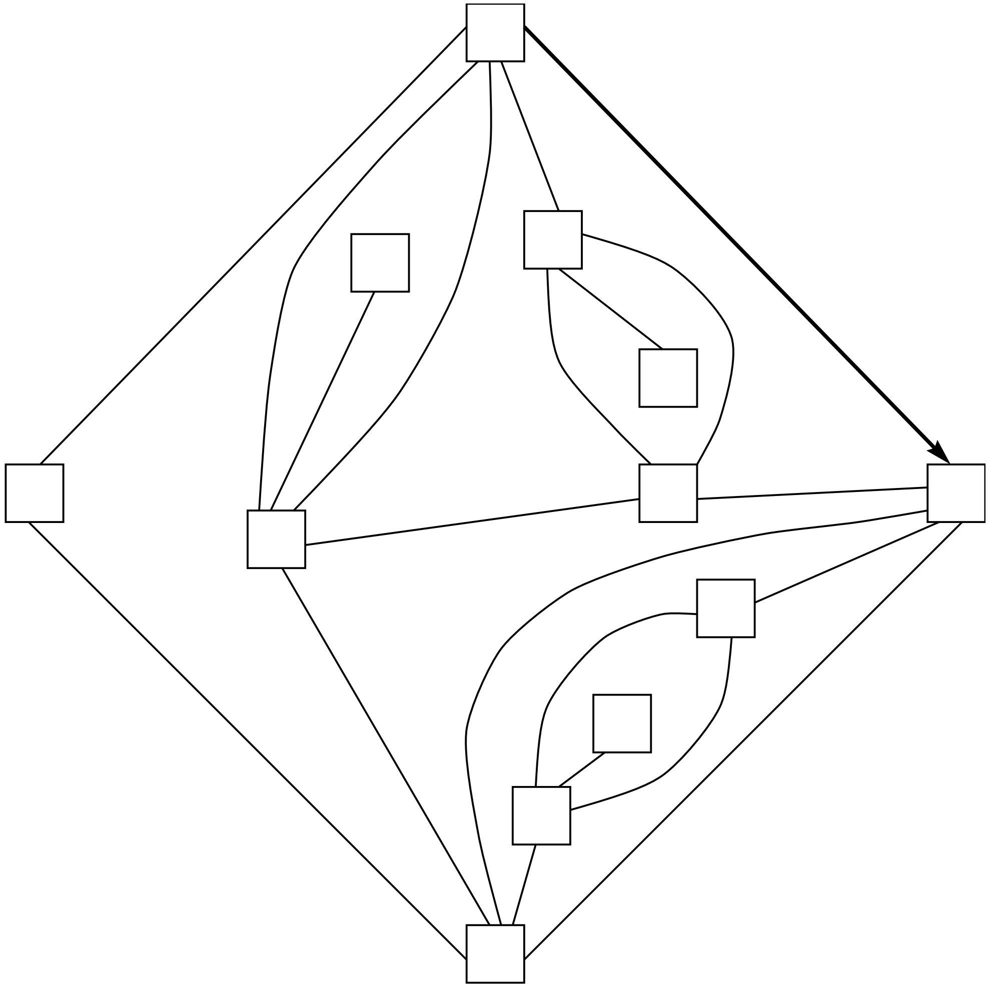
3-connected maps

In this bijection **3-connected maps** correspond so **simple quadrangulations** (every circle different from the outer cirlce of length 4 determines a face).

This correspondence is important for the counting procedure of planar graphs.

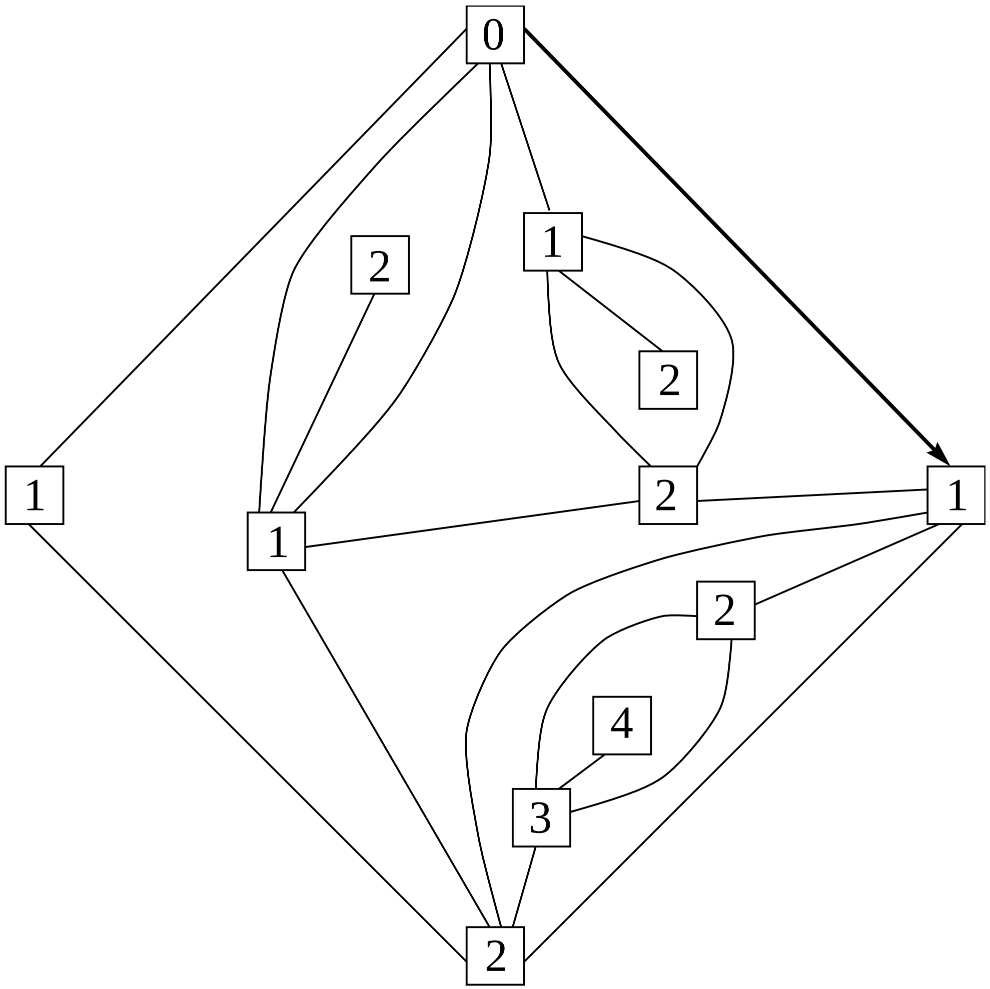
Quadrangulations

Schaeffer bijection: start with a quadrangulation



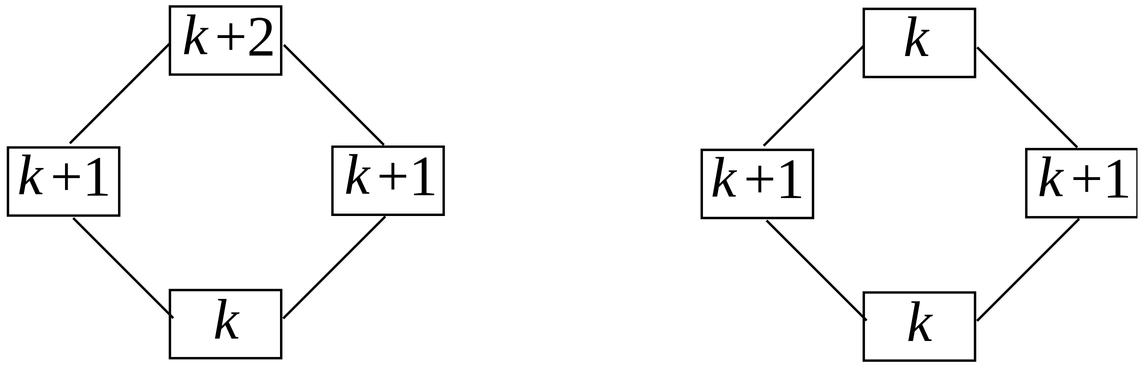
Quadrangulations

Schaeffer bijection: calulate the distance to the root vertex



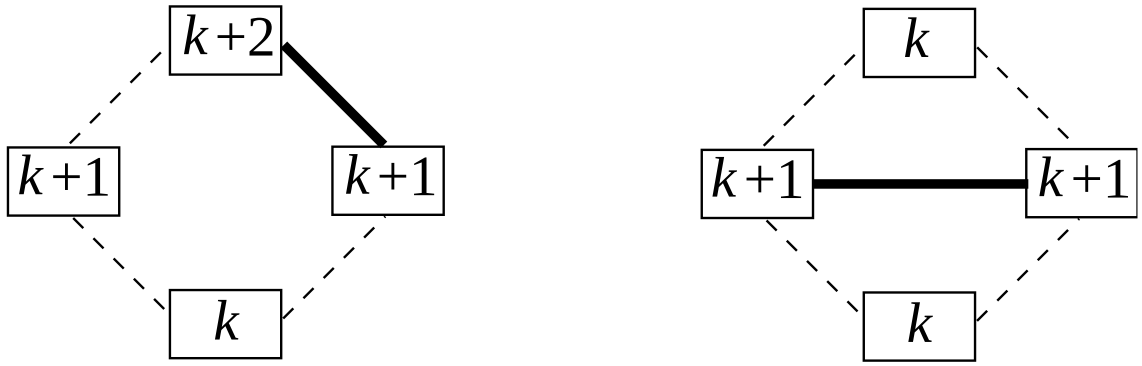
Quadrangulations

Schaeffer bijection: there are only two possible constellations



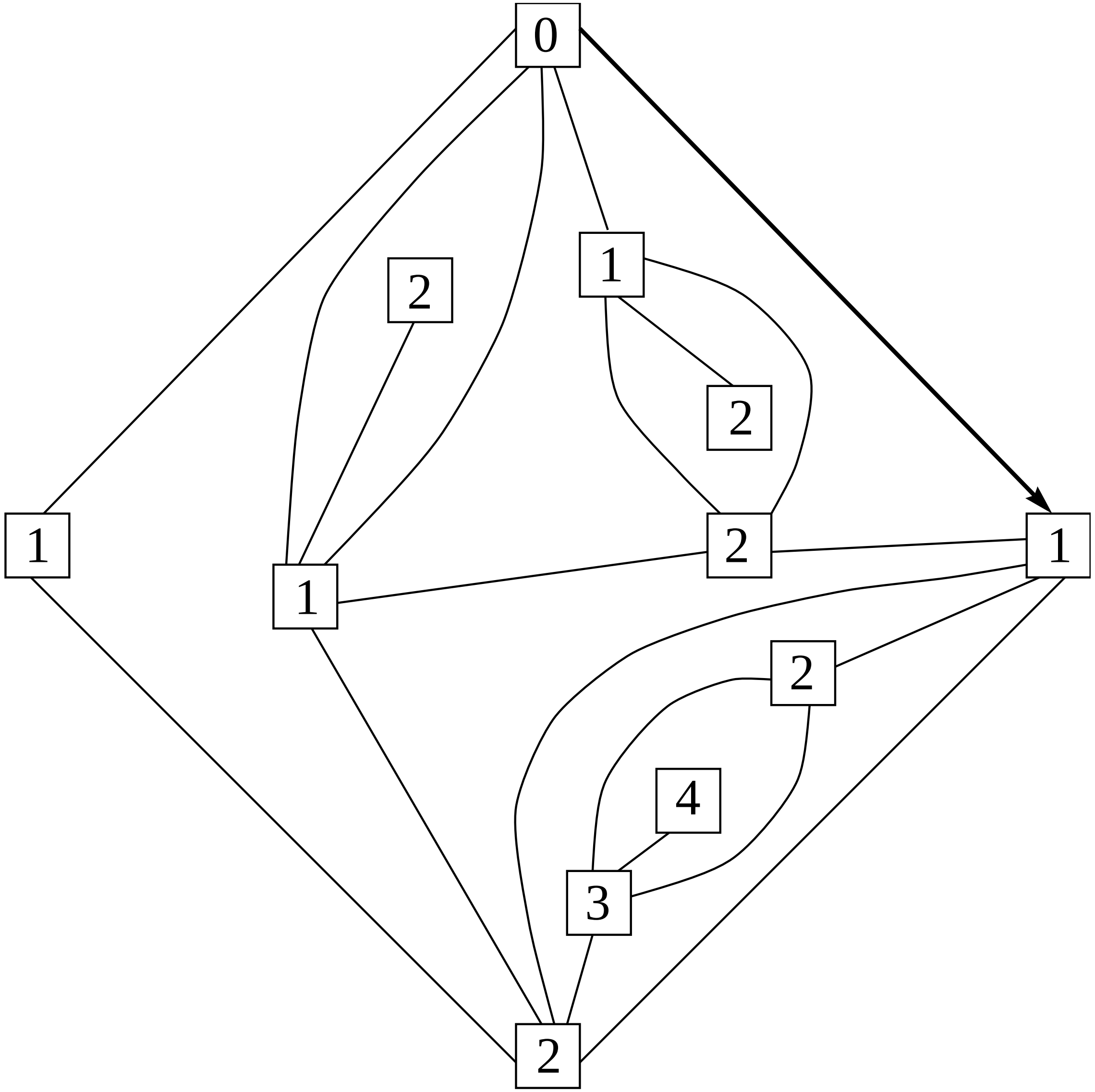
Quadrangulations

Schaeffer bijection: include **fat** edges



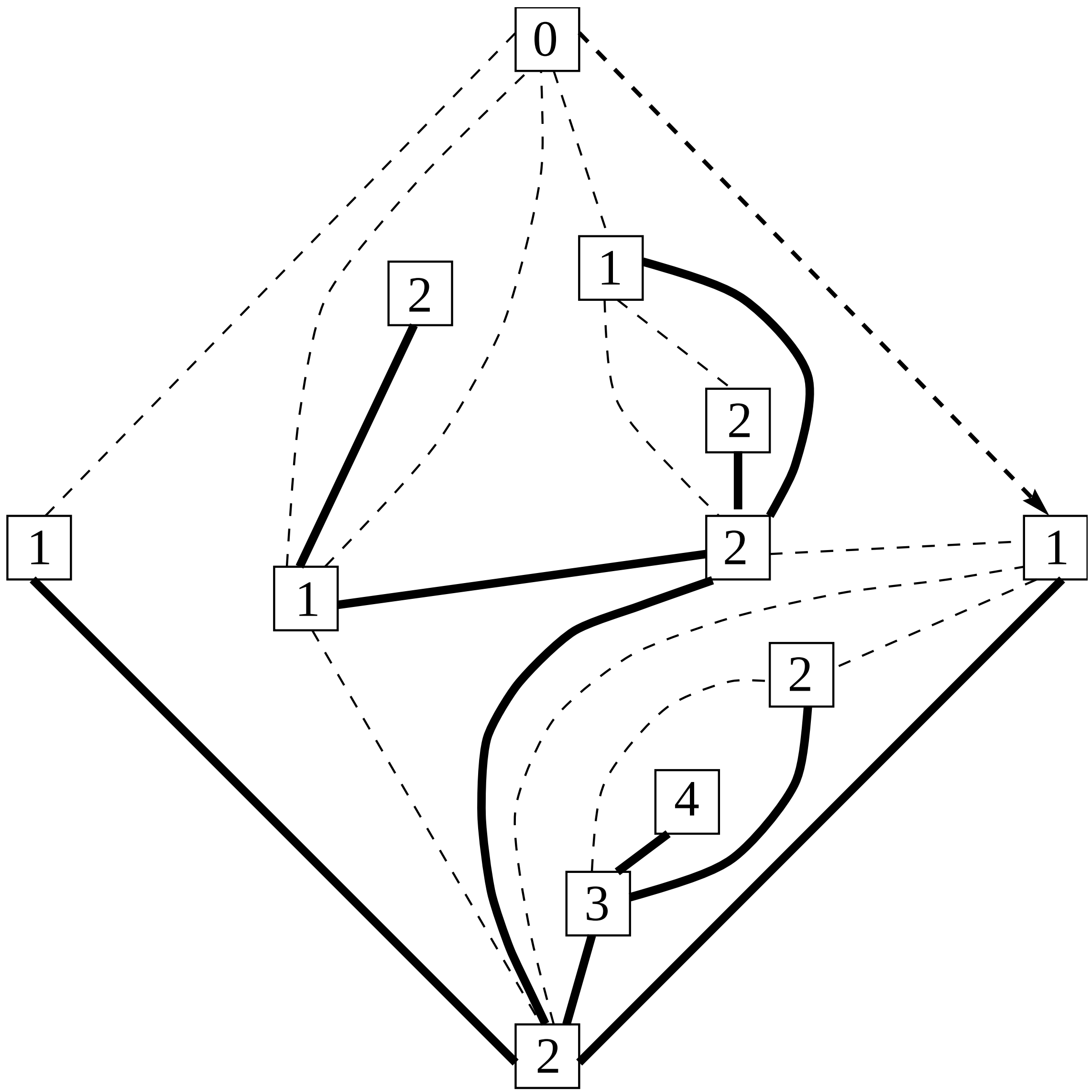
Quadrangulations

Schaeffer bijection: include **fat** edges



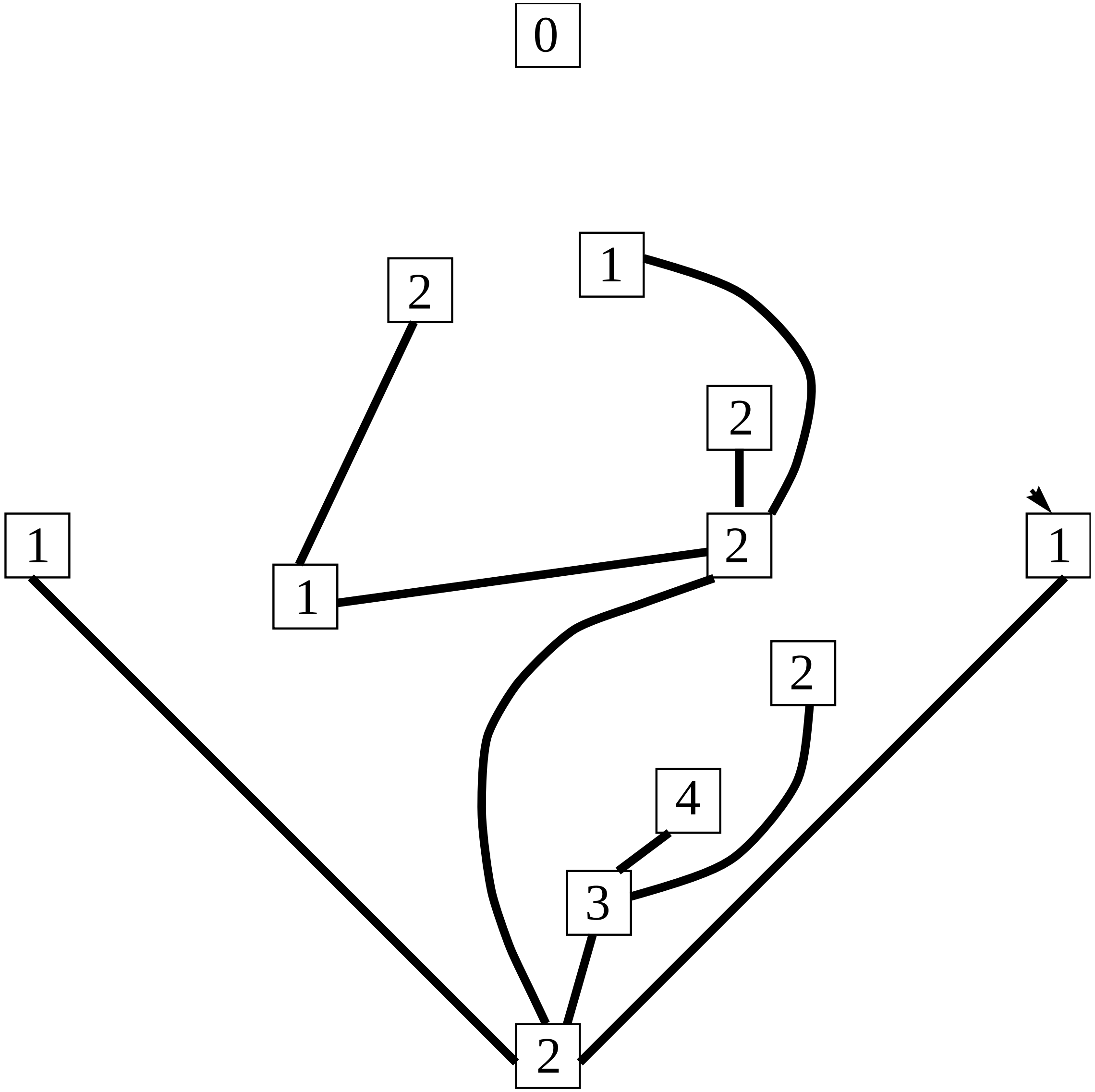
Quadrangulations

Schaeffer bijection: include **fat** edges



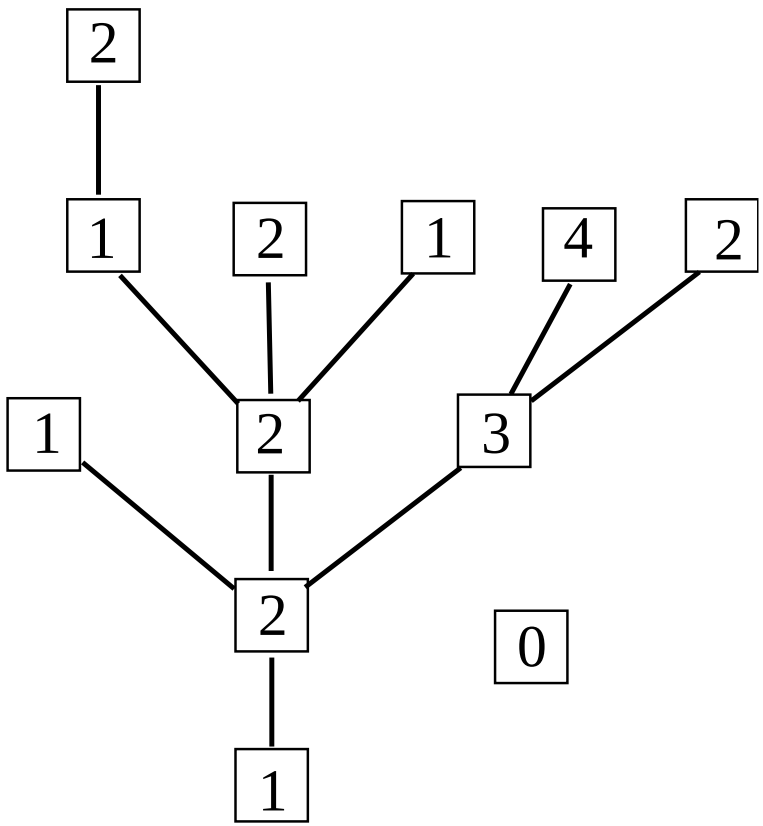
Quadrangulations

Schaeffer bijection: delete the dotted edges



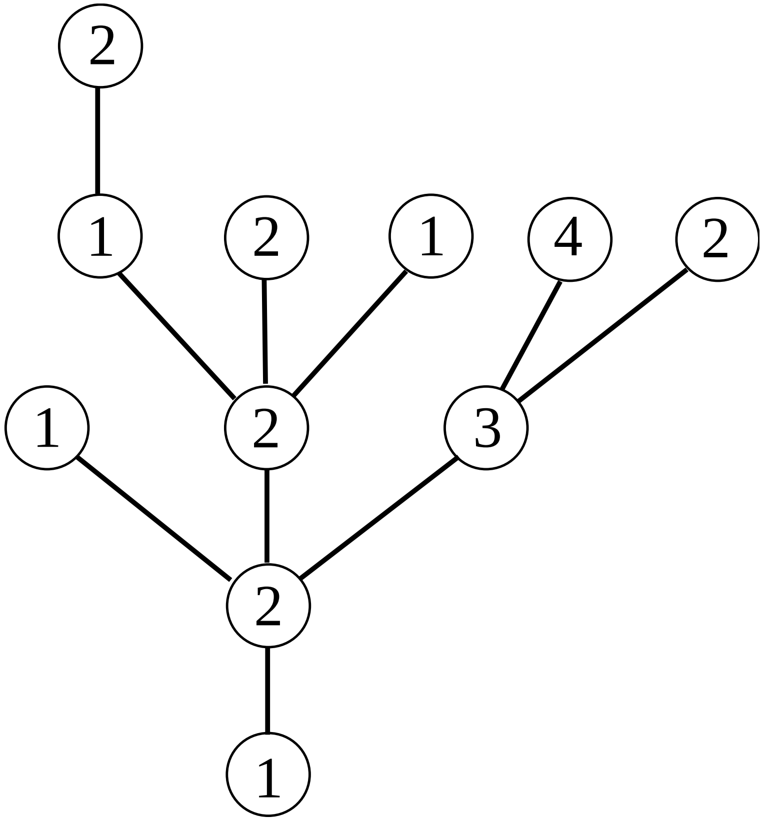
Quadrangulations

Schaeffer bijection: a labelled tree occurs



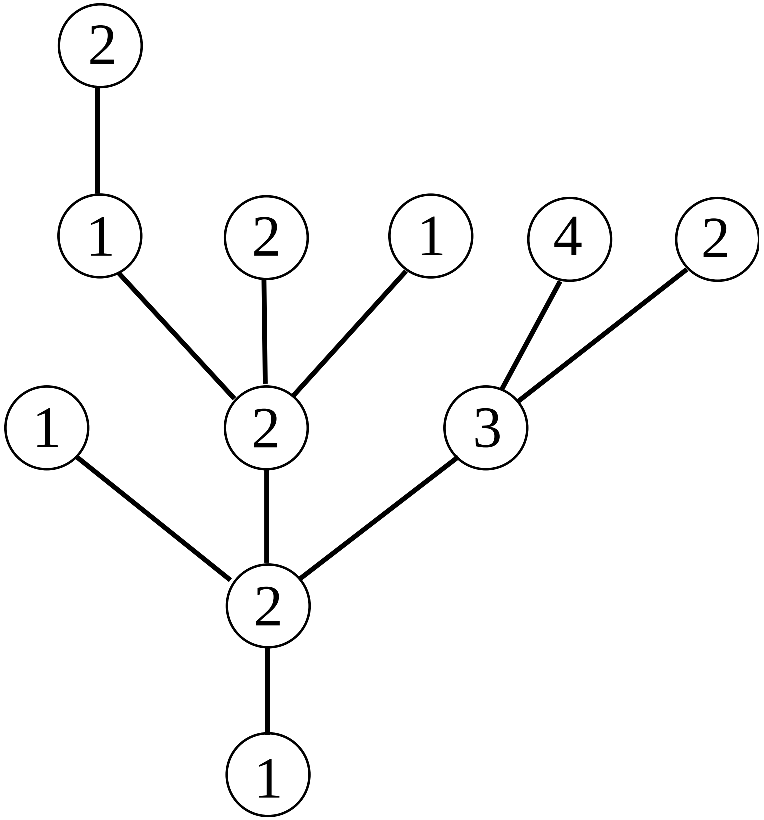
Quadrangulations

Schaeffer bijection: a labelled tree occurs



Well-Labelled Trees

Positive labels, root has label 1, adjacent labels differ at most by 1:



Well-Labelled Trees

$H_{n,k}$ ... number of vertices of distance k from the root vertex in a quadrangulation of size n

$\lambda_{n,k}$ ... number of vertices with label k from in a well-labelled tree with n edges

Theorem (Schaeffer)

There exists a **bijection** between **edge-rooted quadrangulations** with n faces and **well-labelled trees** with n edges, such that the **distance profile** $(H_{n,k})_{k\geq 1}$ of a quadrangulation is mapped onto the **label distribution** $(\lambda_{n,k})_{k\geq 1}$ of the corresponding well-labelled tree.

Well-Labelled Trees

Counting

q_n ... number of well-labelled trees of size n:

$$\boxed{q_n = \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}}$$

$T_j(y)$ ... generating function of those generalised well-labelled trees where the root has label j and where the exponent of y counts the number of edges:

$$\boxed{T_j(y) = \frac{1}{1 - y(T_{j-1}(y) + T_j(y) + T_{j+1}(y))}} \quad (j \geq 1).$$

with the convention $T_0(y) = 0$

Well-Labelled Trees

Theorem

$$T(y) = \frac{1}{1 - 3yT(y)} = \frac{1 - \sqrt{1 - 12y}}{6y},$$

$$Z(y) + \frac{1}{Z(y)} + 1 = \frac{1}{yT(y)^2}.$$

$$\Longrightarrow \quad \boxed{T_j(y) = T(y)\frac{(1 - Z(y)^j)(1 - Z(y)^{j+3})}{(1 - Z(y)^{j+1})(1 - Z(y)^{j+2})}}$$

Well-Labelled Trees

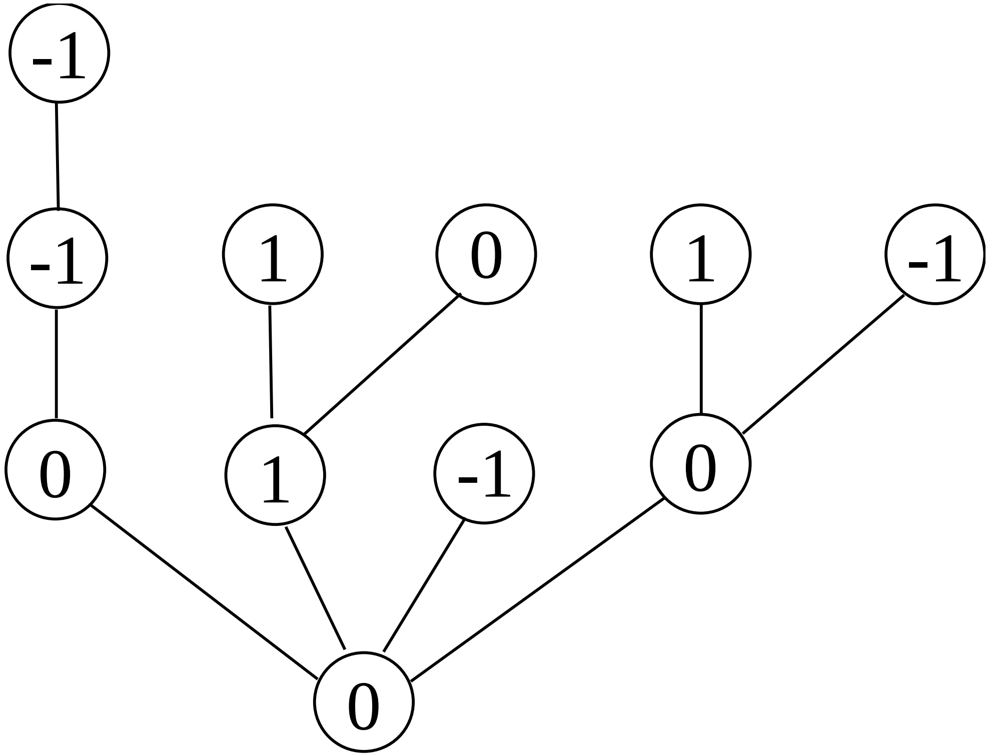
Counting

$$\begin{aligned} T_1(y) &= T(y)\frac{(1-Z(y))(1-Z(y)^4)}{(1-Z(y)^2)(1-Z(y)^3)} \\ &= T(y)\frac{1+Z(y)^2}{1+Z(y)+Z(y)^2} \\ &= T(y)(1-tT(y)^2), \end{aligned}$$

$$\Longrightarrow \quad q_n = [y^n]T_1(y) = \frac{2\cdot 3^n}{(n+1)(n+2)}\binom{2n}{n}.$$

Embedded Trees

Integer labels, root has label 0, adjacent labels differ at most by 1:

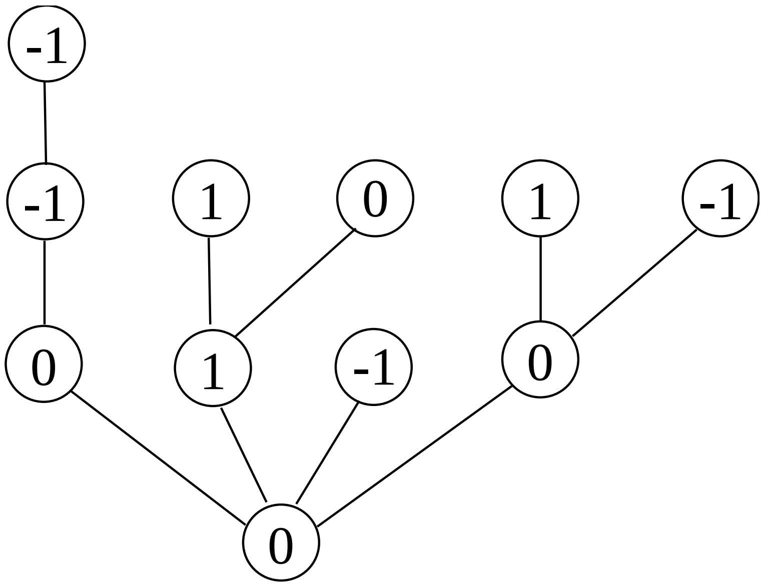


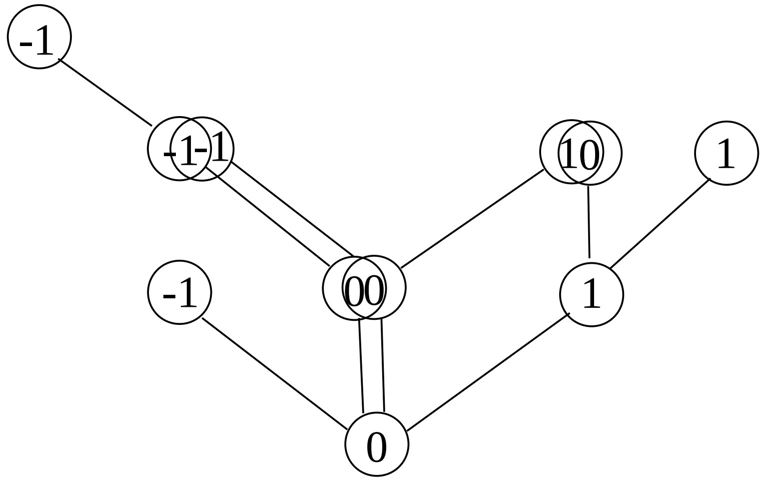
$$u_n = 3^n p_{n+1} = \frac{3^n}{n+1}\binom{2n}{n},$$

(the number of embedded trees with n edges)

Embedded Trees

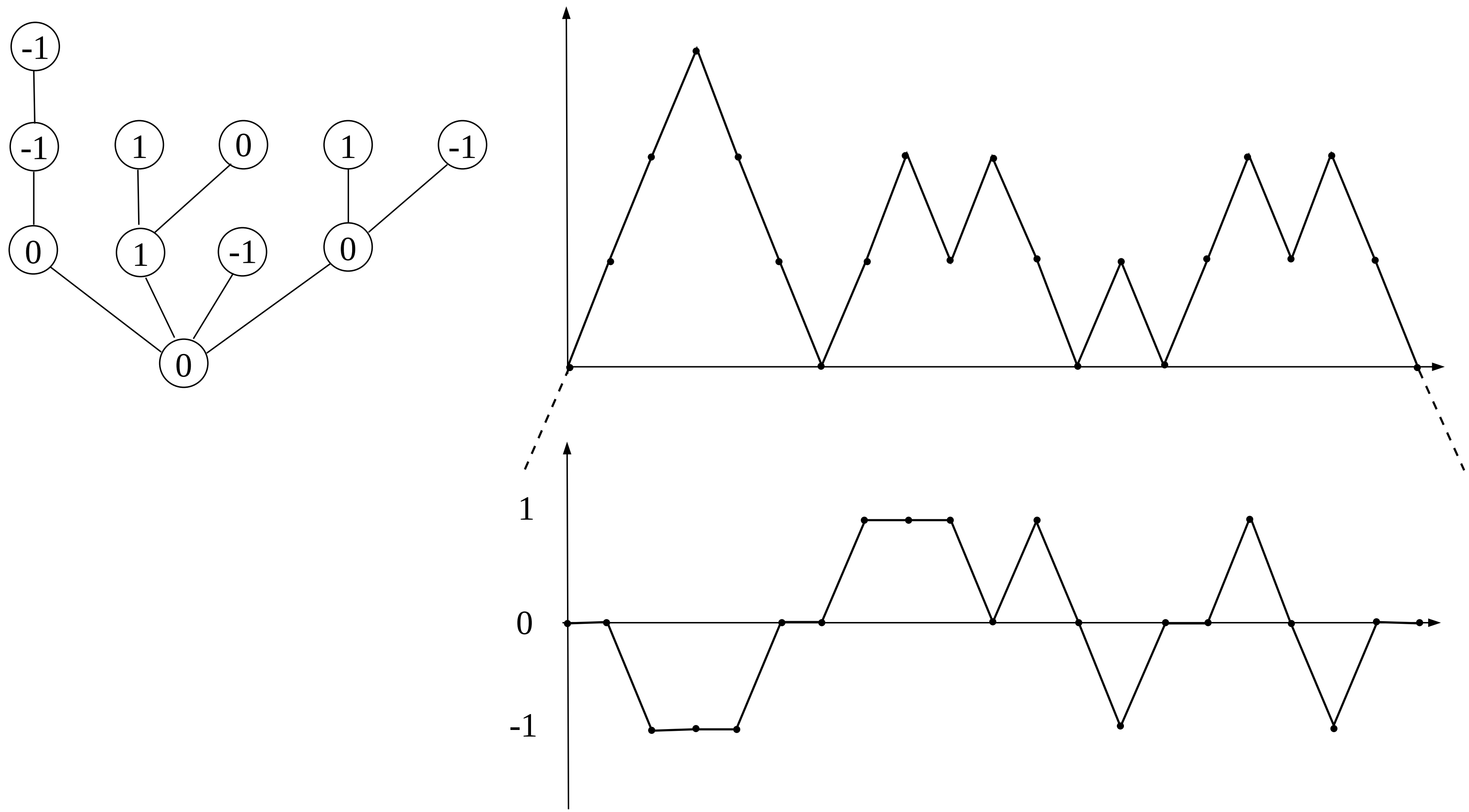
Interpretation as embedding





Brownian Snake

Discrete Brownian snake



Brownian Snake

$g : [0,1] \to [0,\infty)$ ... continuous, ≥ 0, $g(0) = g(1) = 0$

$$d_g(s,t) = g(s) + g(t) - 2 \inf_{\min\{s,t\} \le u \le \max\{s,t\}} g(u) = 0.$$

Gaussian process

$\boxed{(W_g(t),\, t \ge 0)}$: $W_g(0) = 0$,

$$\mathbb{E}(W_g(t)) = 0, \quad \mathbb{C}\mathrm{ov}(W_g(s), W_g(t)) = \inf_{\min\{s,t\} \le u \le \max\{s,t\}} g(u).$$

A Gaussian process $(X(t), t \in I)$ (with zero mean) is completely determined by a **positive definite covariance function** $B(s,t)$. All finite dimensional random vectors $(X(t_1), \ldots, X(t_k))$ are **normally distributed** with covariance matrix $(B(t_i, t_j))_{1 \le i,j \le k}$.

Brownian snake: $\boxed{W(t) = W_{2e}(t)}$.

Brownian Snake

Theorem (Chassaing+Marckert)

Consider a conditioned GW-trees with offspring distribution ξ and labels given by independent increments following a distribution η with $\mathbb{E}\,\eta = 0$.

$W_n(s)$ ... discrete Brownian snake corresponding to these trees and labels

$$\Longrightarrow \quad \boxed{\left(\frac{\gamma}{n^{1/4}} W_n(2nt), 0 \le t \le 1\right) \xrightarrow{\mathrm{d}} (W(t), 0 \le t \le 1)}$$

with $\gamma = (\mathbb{V}\mathrm{ar}\,\eta)^{-\frac{1}{2}} (\mathbb{V}\mathrm{ar}\,\xi)^{\frac{1}{4}}$.

Integrated SuperBrownian Excursion (ISE)

Occupation measure of the Brownian snake: random measure

$$\mu_{\text{ISE}}(A) = \int_0^1 \mathbf{1}_A(W(t))\,dt$$

Density of the ISE: random density

$$f_{\text{ISE}}(s) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^1 \mathbf{1}_{[s,s+\varepsilon]}(W(t))\,dt$$

Remark. The ISE has **finite support** $[L_{\text{ISE}}, R_{\text{ISM}}]$
(Its length $R_{\text{ISE}} - L_{\text{ISE}}$ is a random variable.)

Continuous Limits

Theorem (Aldous)

Consider a conditioned GW-trees with offspring distribution ξ and labels given by independent increments following a distribution η with $\mathbb{E}\,\eta = 0$.

$\ell(v)$ ... label of vertex v

$$\Longrightarrow \quad \boxed{\frac{1}{n}\sum_{v\in V(T_n)} \delta_{\gamma n^{-1/4}\ell(v)} \xrightarrow{\text{d}} \mu_{\text{ISE}}}$$

with $\gamma = (\mathbb{V}\text{ar}\,\eta)^{-\frac{1}{2}}(\mathbb{V}\text{ar}\,\xi)^{\frac{1}{4}}$.

Continuous Limits

Theorem (Devroye+Janson)

Suppose additionally that η is integer valued and aperiodic.

$(X_n(j))_{j\in\mathbb{Z}}$ ... profile corresponding to η
$(X_n(j)$ ... number of nodes with label $j)$
$(X_n(t), -\infty < t < \infty)$ ... the linearly interpolated process:

$$\Longrightarrow \quad \boxed{\left(n^{-3/4}X_n(n^{1/4}t),\ -\infty < t < \infty\right) \xrightarrow{\mathrm{d}} (\gamma f_{\mathrm{ISE}}(\gamma t),\ -\infty < t < \infty)}$$

Continuous Limits

Theorem (Chassaing+Marckert)

Let $(\lambda_{n,k})$ denote the **height profile** and r_n the **maximum distance** from the root vertex in **random quadrangulations** with n vertices:

$$\Longrightarrow \quad \boxed{\frac{1}{n}\sum_{k\geq 0}\lambda_{n,k}\,\delta_{\gamma n^{-1/4}k} \xrightarrow{\text{d}} \widehat{\mu}_{\text{ISE}}}$$

and

$$\Longrightarrow \quad \boxed{\gamma n^{-1/4} r_n \xrightarrow{\text{d}} R_{\text{ISE}} - L_{\text{ISE}}},$$

where $\gamma = 2^{-1/4}$.

Thank You!