ENZ für Algebra

On Plethysms of type $s_\mu[t\, s_\nu]$ for non-integral t and a few Applications

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62th Sèminaire Lotharingien de Combinatoire

joint work with P.D. Jarvis, Hobart, TAS & R.C. King, Southampton

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Integer partitions, Young diagrams / tableaux

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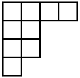
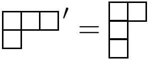
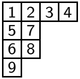
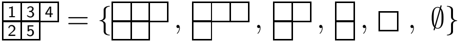
$\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_l$

Let $\lambda \vdash n$ be a partition of the integer n

- $|\lambda| = n$ *weight* of the partition
- $\ell(\lambda) = l$ number of parts, *length* of the partition
- $\lambda = (\lambda_1, \ldots, \lambda_l) = [1^{m_1}, 2^{m_2}, \ldots, n^{m_n}]$
  additive and multiplicative representation
-  *shape* or *(Ferrers) Young diagram* of a partition
-  $\lambda \mapsto \lambda'$ *conjugation* of partitions
- 

  (semi) standard *Young tableau*, box $x = (i, j) \in SYT$
  (weakly) increasing rows, strictly increasing columns
- 

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$\mathcal{H}(G)$ Hopf algebra (groups)

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Group

- $f, g, h, e \in G$ group elements
- $f(gh) = (fg)h$
- $fg = h, \;\; ge = g = eg, \;\; g^{-1}$ exists with $g^{-1}g = e$

Representative functions

- $\phi, \psi, \mathbf{1} \in \mathcal{H}(G) = \{G \to \mathbb{C}\}$ functions
- $(\psi\phi)(g) = \psi(g)\phi(g)$ pointwise product (assoc.)
- $\mathbf{1}(g) = 1$ unit element

Maps:

$\Delta : \mathcal{H}(G) \to \mathcal{H}(G) \otimes \mathcal{H}(G)$; $\epsilon : \mathcal{H}(G) \to \mathbb{C}$; $\mathbf{S} : \mathcal{H}(G) \to \mathcal{H}(G)$

- Coproduct: $\Delta(\psi)(g, h) = \psi(g)\psi(h)$
- Counit: $\epsilon(\psi) = \psi(e)$
- Antipode: $\mathrm{S}(\psi)(g) = \psi(g^{-1})$

$\mathcal{H}(G)$ Hopf algebra, cont.

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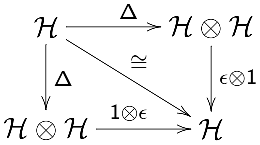
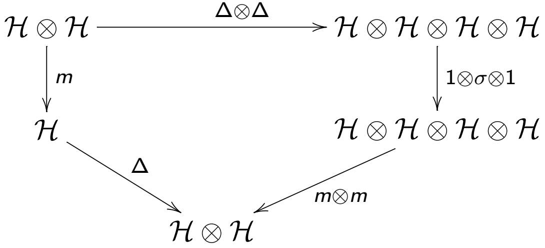
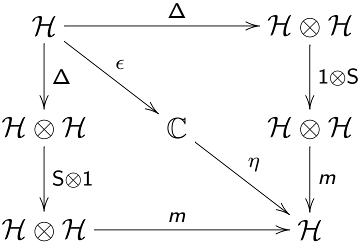
Hopf algebra; CD version (counit; pentagon; antipode)







$$\sigma(\psi \otimes \phi) = \phi \otimes \psi$$

$$\eta : \mathbb{C} \to \mathcal{H}$$

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Symm(X) Hopf algebra (outer HA)

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Symm(X)

- $Symm(X)$ is the sub-Hopf algebra $K_0[Rep(GL(\infty))]$ (i.e. Grothendieck ring) of the repr. category $Rep(GL(\infty))$) (in formally infinite many variables to avoid syzygies).

  Let A, B, C $(A[X], B[X], C[X])$ be in Symm(X):

- product: $AB = C$; $\quad s_\mu s_\nu = \sum c^\lambda_{\mu,\nu} s_\lambda$ $\quad$ ($c^\lambda_{\mu,\nu}$ LR-coeff.)
- unit: $A1 = A = 1A$; $\quad s_\mu s_{(0)} = s_\mu = s_{(0)} s_\mu$
- coproduct: (Sweedler index notation)
  $\Delta(A[X]) = A[X+Y] = A_{(1)}[X]A_{(2)}[Y] \cong A_{(1)} \otimes A_{(2)}$
  $\Delta s_\lambda = \sum_{(\lambda)} s_{\lambda_{(1)}} \otimes s_{\lambda_{(2)}} = \sum_{\mu,\nu} c^\lambda_{\mu,\nu} s_\mu \otimes s_\nu$ $\quad$ ($c^\lambda_{\mu,\nu}$ LR-coeff.)
- counit: $\epsilon(A) = \delta_{A,1}$; $\quad \epsilon(s_\mu) = \delta_{\mu,(0)}$

Needs of course a proof to be a HA (Geissiger 77, Zelevinsky 81)
Symm(X) is a PSH (positive selfadjoint HA)

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Symm(X) additional structure

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inner product (S_n tensor product of characters)

- tensor product in $R(S_n)$ yields via Frobenius characteristics a second product on Symm(X)
- $s_\mu \star s_\nu = \sum_\lambda g^\lambda_{\mu,\nu} s_\lambda$
- $g^\lambda_{\mu,\nu}$ is symmetric in all three partitions, computable via the Murnaghan-Nakayama rule
- $\delta(s_\lambda[X]) = s_\lambda[X \otimes Y] = \sum_{[\lambda]} s_{\lambda_{[1]}}[X] s_{\lambda_{[2]}}[Y] \cong \sum_{[\lambda]} s_{\lambda_{[1]}} \otimes s_{\lambda_{[2]}}$

Scalar product (self duality)

$\{s_\lambda\}_{\lambda \vdash n, n \in \mathbb{N}}$ basis of irreducibles $\Rightarrow$ orthonormal (Schur's lemma)

$$\langle . \mid . \rangle : \Lambda \otimes \Lambda \to \mathbb{Z} \qquad \langle s_\mu \mid s_\nu \rangle = \delta_{\mu,\nu}$$

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Symm(X) additional structure, cont.


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Schur functors (on $R(G) = Rep(G)$)

Let V, W be G-modules of $R(G)$. Consider a the Schur functor
$\mathbb{S}_\lambda : R(G) \to R(G)$; $\quad \mathbb{S}_\lambda(V) = W =: V^\lambda \quad$ (Weyl module)

$\mathbb{S}_{(n)}(V) = \mathrm{Sym}(V^{\otimes^n}) = V^{(n)}$
$\mathbb{S}_{(1^n)}(V) = V^{\wedge^n} = V^{(1^n)}$

$$\mathbb{S}_\mu \circ \mathbb{S}_\nu(V) = \mathbb{S}_\mu(V^\nu) = (V^\nu)^\mu \qquad U \xrightarrow{\mathbb{S}_\nu} V \xrightarrow{\mathbb{S}_\mu} W, \quad U \xrightarrow[\mathbb{S}_\mu \circ \mathbb{S}_\nu]{} W$$

Plethysm (composition on Symm(X) $\cong K_0[Rep(G)]$)

$$s_\mu[s_\nu] = \sum_\lambda a^\lambda_{\mu,\nu} s_\lambda$$

$s_\nu[X] = Y = y_1[X] + y_2[X] + \ldots + y_m[X]$
with m number of monomials (incl. multiplicities)
$\mathbb{S}_\nu(V) = V^\nu = W \quad$ and $\quad \mathbb{S}_\mu(\mathbb{S}_\nu(V)) = \mathbb{S}_\mu(W) = W^\mu = (V^\nu)^\mu$

$s_\mu[s_\nu][X] = s_\mu[s_\nu[X]] = s_\mu[Y[X]]$

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Symm(X) : summary of costructures

σ (succ. map), HA (outer HA), BI (bialgebra), HG (Hopf gebra)

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$$\sigma : \quad \sigma(s_\lambda)[X] = s_\lambda[x_0 + X] = \sum_{(n),\nu} c^{\lambda}_{(n),\nu} s_{(n)}[x_0] s_\nu[X]$$

$$= \sum_{\lambda \subset \nu} x_0^n \, s_\nu \text{ with } \nu \setminus \lambda \text{ horizontal strip with } n \text{ boxes}$$

$$\text{HA:} \quad \Delta(s_\lambda)[X] = s_\lambda[X + Y] = s_{\lambda_{(1)}}[X] s_{\lambda_{(2)}}[Y] = \sum_{\mu,\nu} c^{\lambda}_{\mu,\nu} s_\mu[X] s_\nu[Y]$$

$$\text{BI:} \quad \delta(s_\lambda)[X] = s_\lambda[XY] = s_{\lambda_{[1]}}[X] s_{\lambda_{[2]}}[Y] = \sum_{\mu,\nu} g^{\lambda}_{\mu,\nu} s_\mu[X] s_\nu[Y]$$

$$\text{HG:} \quad \hat{\Delta}(s_\lambda)[X] = s_\lambda[Y^X] = s_{\lambda_{\langle 1\rangle}}[X] s_{\lambda_{\langle 2\rangle}}[Y] = \sum_{\mu,\nu} p^{\lambda}_{\mu,\nu} s_\mu[X] s_\nu[Y]$$

I call this **Arithmetic Complexity**

Plethysm properties

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Let $a, b, c, .. \in R$ and $A, B, C, .. \in Symm[X]$

$\varepsilon^n(s_\lambda(x_1, x_2, \ldots)) = s_\lambda(1, \ldots, 1, 0, \ldots)$ with n ones

defining rules :

$(aA + bB)[C] = aA[C] + bB[C]$	left distributivity
$(AB)[C] = A[C]B[C]$	left multiplicativity
$A[B + C] = A_{(1)}[B]A_{(2)}[C]$	right additivity ($'+'$ convolution on $Symm(X)$)
$A[-B] = (\mathrm{S}(A))[B]$	antipode (negative alphabets)
$A[BC] = A_{[1]}[B]A_{[2]}[C]$	right multiplicativity ($'\cdot'$ convolution on $Symm(X)$)
$A[B * C] = A[\sum_{i=1}^{N} g_{BC}^{D_i} D_i]$ $= \prod_{i=1}^{N} \varepsilon^{g_{BC}^{D_i}}(A_{(i)[1]}) A_{(i)[2]}[D_i]$	right inner product expansion
$(A[B])[C] = A[B[C]]$	associativity

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Relating outer and inner coproducts $'+'$

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We give two ways to compute the plethysm $s_\mu[n\, s_\nu]$:

- Lemma [FJK] : $(A \otimes B)[C] = A[C] \otimes B[C]$
- Lemma : $A[B + B] = A_{(1)}[B] A_{(2)}[B] = \big(m^1 \circ \Delta^1)(A)\big)[B]$
  : $A[n\, B] = \prod_{i=1}^n A_{(i)}[B] = \big(m^{(n-1)} \circ \Delta^{(n-1)})(A)\big)[B]$

Proof: additive version

$$
\begin{aligned}
s_\mu[n\, s_\nu] &= s_\mu[s_\nu + (n-1)\, s_\nu] && n > 1 \\
&= s_{\mu(1)}[s_\nu]\, s_{\mu(2)}[(n-1)\, s_\nu] \\
&= \ldots \\
&= \prod_{i=1}^{n} s_{\mu(i)}[s_\nu] \\
&= \Big(m^{(n-1)} \circ \Delta^{(n-1)}\Big)(s_\mu)[s_\nu]
\end{aligned}
$$

To stop the induction process we have to require: $n \in \mathbb{N}$

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Relating outer and inner coproducts '.'

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- $\varepsilon^n(s_{(1)}) = 1 + \ldots + 1 = \dim_{GL(n)}(s_{(1)})$ is a Hom
- Lemma [FJKW] : $\Delta(A[B]) = A[\Delta(B)]$
- with $m(A \otimes B) = AB$ and the HA axioms we get further Lemma [FJK] : $A[B \otimes C] = A_{[1]}[B] \otimes A_{[2]}[C]$

2nd proof: multiplicative version

$$
\begin{aligned}
s_\mu[n\, s_\nu] &= m(s_\mu[n \otimes s_\nu]) \\
&= m(s_\mu[\varepsilon^n(s_{(1)}) \otimes s_\nu]) \\
&= m \circ (\varepsilon^n \otimes \mathrm{Id})(s_\mu[s_{(1)} \otimes s_\nu]) \\
&= m(\varepsilon^n(s_{\mu[1]}) \otimes s_{\mu[2]}[s_\nu]) \\
&= \varepsilon^n(s_{\mu[1]})\, s_{\mu[2]}[s_\nu] \\
&= \dim_{GL(n)}(s_{\mu[1]})\, s_{\mu[2]}[s_\nu]
\end{aligned}
$$

Relating outer and inner coproducts Δ, δ

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- $A = A[s_{(1)}]$ for all A
- $[1+n](A) = (m^n\Delta^n)(A) = A[(1+n)\, s_{(1)}]$
  $[1+n](A[B]) = (m^n\Delta^n)(A[B]) = A[(1+n)\, B]$
  $[1+n]$ is a **scaling operator**
- $[1+n](A[B]) = A[(1+n)\, B] = \dim_{GL(n+1)}(A_{[1]})A_{[2]}[B]$
  $[1+n] = (m^n\Delta^n) = (\varepsilon^{(n+1)} \otimes \mathrm{Id})\delta$

Result : t-generalization

Since we have that ε^n is a binomial expression in n it can be generalized to arbitrary $t \in \mathbb{C}$, that is $\varepsilon^n \Rightarrow \varepsilon^t$:

$$[1+t] := (\varepsilon^{1+t} \otimes \mathrm{Id})\delta \qquad (= \left(m^t\Delta^t\right) \quad t \in \mathbb{N})$$

$$[1+t](A[B]) = A[(1+t)\, B] = \dim_{GL(t+1)}(A_{[1]})A_{[2]}[B]$$

For $t \notin \mathbb{N}$ generically non-recursive.

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Other Homs and specializations

Specializations (eg. q, t-case : no time ... $\to$ [sketch])

- Let $\Omega[X] = M[X] = \sum h_n(X)$ complete symmetric functions
  $[1+t]\Omega[X] = \Omega[(1+t)X]$ well known plethystic formula
- $\langle A \mid B \rangle = \varepsilon^1(A * B)$: counts 1-dim. GL irreps
  $\varepsilon^1(s_\lambda) = \delta_{\lambda,(n)}$: shows that the series $M = \Omega$ is defined by ε^1

  duality : $\Lambda \underset{M}{\overset{\varepsilon^1}{\rightleftarrows}} \mathbb{Z} \qquad \Lambda \underset{M^{1+t}}{\overset{\varepsilon^{1+t}}{\rightleftarrows}} \mathbb{Z}$
- $\delta(M[X]) = M[XY] = \sum_\lambda s_\lambda[X]\, s_\lambda[Y]$: Cauchy kernel
  $\delta(M[-X]) = L[XY] = \sum_\lambda (-1)^{|\lambda|} s_\lambda[X]\, s_{\lambda'}[Y]$: Cauchy-Binet
- There are abundantly many further specializations and homomorphisms available to generalize the above given result:
  $M[X] \to M[\frac{1}{1-q}X] = M[\sum_{n\geq 0} q^n X] = \prod_{n\geq 0} M[q^n X] =$
  $\prod_{n\geq 0} \left(\sum_\lambda \dim_{GL(q^n)}(s_\lambda)\, s_\lambda \right)$
  $A[B] \to A[\frac{1}{1-q}B] = \prod_{i\geq 1} \left(\dim_{GL(q^n)}(A_{(i)[1]})\, A_{(i)[2]}[B] \right)$
  for a general A and a general "alphabet" B.

Applications:

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Computational benefits

Timing of the iterated and directly evaluated plethysms $s_{(3)}[n\, s_{(1,1)}]$:

multiplicity	recursive	direct
n=1	0.01	0.02
n=10	0.08	0.02
n=100	0.89	0.01
n=1000	7.32	0.01
n=10000	—	0.01

SchurFkt Maple package [BF.,Rafal Ablamowicz]: Figures obscured by Maple's garbage collection.

SCHUR cannot do the $n = 100, 1000$ cases in reasonable time (Maple algorithm uses remember hash tables)

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Applications:

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Jack symmetric function (eg. [Brenti98])

Define the scalar product

$$\langle A \mid B\rangle_{\alpha\beta} := \langle A[\alpha s_{(1)}] \mid B[\beta s_{(1)}]\rangle = \varepsilon^{\alpha\beta}(A{*}B) = (A{*}B)[1,\ldots,1,0,\ldots]$$

Schur functions are no longer an orthonormal basis for $\langle . \mid . \rangle_a$

$$\left(\begin{array}{c|cc} \langle|\rangle_{q^2} & s_{(2)} & s_{(11)} \\ \hline s_{(2)} & \frac{1}{2}q^2(q^2+1) & \frac{1}{2}q^2(q^2-1) \\ s_{(11)} & \frac{1}{2}q^2(q^2-1) & \frac{1}{2}q^2(q^2+1) \end{array}\right)$$

Jack symmetric functions can be defined as:

$$J_\lambda(x;\alpha) = \prod_{i=1}^{\ell(\lambda)} J_{\lambda_i}(x;\alpha) \qquad J_{\lambda_i}(x;\alpha) = \sum_{\mu\vdash n} \alpha^{n-\ell(\mu)} n! \frac{p_\mu}{z_\mu}$$

which results in: $h_\lambda[\frac{1}{\alpha}x] = \frac{1}{\alpha^{|\lambda|}\lambda!} J_\lambda(x;\alpha)$

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Applications:

Powers of Schur function series

let G be a group like Schur function series (M,L,...) i.e. $\Delta G = G \otimes G$. For such series we have:

$$G[n\, s_{(1)}] = G^n = GG \cdots G$$

and of course with $G^+ := G - s_{(0)}$

$$\log(1 + G^+) = -\sum_{n \geq 1} \frac{(-1)^n}{n} (G^+)^n = -\sum_{n \geq 1} \sum_{k=0}^{n} \frac{(-1)^k}{n} \binom{n}{k} (G^+)^k$$

$$\log(1 + M^+) = p_{(1)} + \frac{1}{2} p_2 + \frac{1}{3} p_3 + \frac{1}{4} p_4 + \ldots$$

which can be generalized

Applications:

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Powers of Schur function series (cont.)

We look at $D = M[s_{(2)}]$, $C = M[-s_{(2)}]$, $M_3 = M[s_{(3)}]$ etc.

$$\log(1+D^+) = \frac{1}{2}(p_2 + p_{(11)}) + \frac{1}{4}(p_4 + p_{(22)}) \ldots + \frac{1}{2n}(p_{2n} + p_{(nn)}) \ldots$$

$$\log(1+C^+) = -\frac{1}{2}(p_2 + p_{(11)}) - \frac{1}{4}(p_4 + p_{(22)}) \ldots - \frac{1}{2n}(p_{2n} + p_{(nn)}) \ldots$$

$$\log(1+M_3^+) = \frac{1}{3}(p_3 + p_{(111)}) + \frac{1}{6}(p_6 + p_{(222)}) \ldots + \frac{1}{3n}(p_{3n} + p_{(nnn)}) \ldots$$

Of course : $\log(1+C^+) + \log(1+D^+) = 0 \Leftrightarrow CD = 1$

Furthermore we have with $\Theta := \sum_\lambda s_\lambda s_\lambda$ and $\Theta\tilde{\Theta} = 1$ that

$$D^n = D[n\, s_{(1)}]\tilde{\Theta}^{n-1}$$

$$[1+n]D = (m^n \Delta^n)D = D[(1+n)\, s_{(1)}]\Theta^n$$

a relation important for string functions and QFT.

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Thank you!

Thanks to the organizers !

Peano axioms (NNO : natural number objects)

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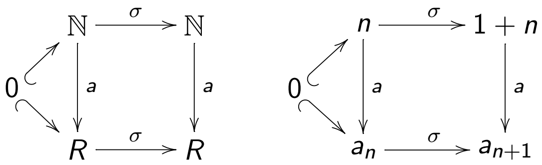
Mathematical induction (adding ‘+1‘)

$\sigma : \mathbb{N} \to \mathbb{N}$ successor map, R target ring of series coeff,
$a : \mathbb{N} \to R$ series (actual a ring, dualizing gives co-structures):



Arithmetic complexity (of ordinary arithmetics)

- $\sigma(n) = 1 + n$ successor map
- $\sigma^m(n) = 1 + (1 + \ldots (1 + n) \ldots) =: \mathrm{add}(m, n) = m + n$
  iteration of σ : **addition**
- $\mathrm{add}(m, 0)^n = (\sigma^m(0))^n = \sigma^{mn}(0) =: \mathrm{mul}(m, n) = mn$
  iteration of $\mathrm{add(n, \ )}$: **multiplication**

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Peano axioms (NNO : natural number objects)

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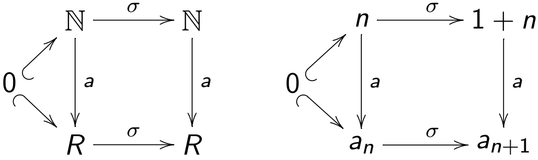

Arithmetic complexity (of ordinary arithmetics)

- $\mathrm{mul}(m,1)^n = \mathrm{mul}(m,\sigma(0))^n = (\mathrm{add}(m,0)^{\sigma(0)})^n$

$$= \prod_{i=1}^{n} \mathrm{add}(m,0)^{\sigma(0)} = \prod_{i=1}^{n} (\sigma^m(0))^{\sigma(0)}$$

$$= \prod_{i=1}^{n} (\sigma^m(0)) = \sigma^{m^n}(0)$$

Peano axioms (NNO : natural number objects)

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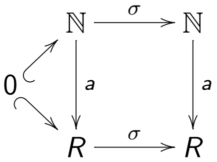

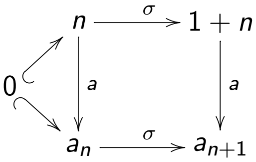

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  iteration of σ : **addition**
- $\mathrm{add}(m, 0)^n = (\sigma^m(0))^n = \sigma^{mn}(0) =: \mathrm{mul}(m, n) = mn$
  iteration of $\mathrm{add(n,)}$: **multiplication**
- $\mathrm{mul}(m, 1)^n = \mathrm{mul}(m, 1)^n =: \exp(m, n) = m^n$
  iteration of $\mathrm{mul(n,)}$: **exponentiation**

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