Superization and (q,t)-specialization in combinatorial Hopf algebras

SLC 62, February 2009

Jean-Christophe Novelli

Institut Gaspard-Monge / Université Paris-Est Marne-la-Vallée

Joint papers with F. Hivert and J.-Y. Thibon

Introduction

Our theme: applications of combinatorial Hopf algebras to (q, t) enumeration of signed permutations with a given (tree) shape.

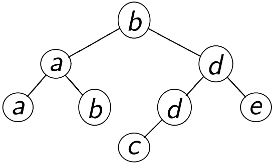
Our way: use the *numerous* constraints of Hopf morphisms to

- find the right (intermediate) questions,
- solve the algebraic part,
- fix the remaining (un)easy combinatorial lemmas.

Words and trees: the binary search tree

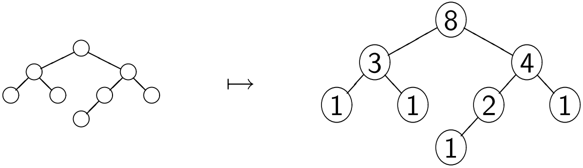
Warning: we start by inserting the *rightmost* letter of the word !

cadbaedb $\longmapsto$



Hook-length formula for trees (Knuth)

Given a tree shape, count the number of permutations having this tree as binary search tree. Put in each node the size of its corresponding subtree. Can be computed by integrating at each node the product of its children (1 in the leaves).



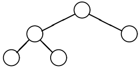
Theorem

The number of permutations having T as binary search tree is

$$\frac{n!}{\prod_{\bullet \in T} h(\bullet)}$$

A natural q-analog

Same question with the q-enumeration of permutations by their major index (sum of positions of descents):

$T =$ 

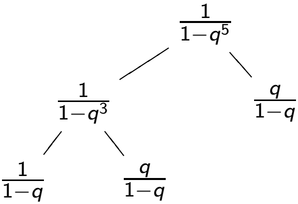
Eight $(5!/5/3)$ permutations:

$$53124, 35124, 51324, 31524, 15324, 31254, 13524, 13254.$$

Hence the polynomial

$$q^2 + 2\,q^3 + 2\,q^4 + 2\,q^5 + q^6.$$

The q-hook-length formula (Björner-Wachs)



And

$$q^2 + 2\,q^3 + 2\,q^4 + 2\,q^5 + q^6 = \frac{\prod_{i=1}^{5}(1-q^i)}{(1-q^5)(1-q^3)(1-q)^3/q^2}.$$

Can be computed by means of the q-integration.

The l.h.s within Hopf algebras

Through the specialization of alphabets $A = \frac{1}{1-q}$ in the Hopf algebras of permutations **FQSym** and of trees **PBT**:

$$\mathbf{F}_\sigma(A) := \sum_{w;\mathrm{Std}(w)=\sigma^{-1}} w,$$

so that

$$\mathbf{F}_\sigma\left(\frac{1}{1-q}\right) = \frac{q^{\mathrm{maj}\,(\sigma)}}{(q)_n}.$$

and

$$\mathbf{P}_T = \sum_{BST(\sigma)=T} \mathbf{F}_\sigma.$$

The r.h.s.

Find a way to get the r.h.s.: the natural induction from the algebraic point of view!

By definition of the dendriform operations:

$$\mathbf{P}_T = \mathbf{P}_{T_1} \succ \mathbf{P}_1 \prec \mathbf{P}_{T_2}.$$

where, on **FQSym**:

$$⧢ = \prec + \succ ,$$

$$\sigma \prec \tau = (\sigma' ⧢ \tau)\sigma_n \quad \text{and} \quad \sigma \succ \tau = (\sigma ⧢ \tau')\tau_p \, .$$

Both dendriform operations have a nice behaviour with the specialization of alphabet.

Petit aparté

Thanks to the specializations of alphabet, for free: a refinement of a well-known theorem of Foata-Schützenberger.

Theorem (Björner-Wachs)

The inverse major index and the number of inversions are equidistributed over sets of permutations having a given binary search tree.

How to find (q, t) analogs?

Hook-content formula (Littlewood-Stanley, originally for tableaux).

$$s_\lambda\left(\frac{1-t}{1-q}\right) = q^{n(\lambda)} \prod_{x\in\lambda} \frac{1-tq^{c(x)}}{1-q^{h(x)}} \,.$$

Specialization of $s_\lambda(X-Y)$ defined by

$$p_n(X-Y) := p_n(X) - p_n(Y)$$

or *supersymmetric functions*

$$p_n(X|Y) := p_n(X) - (-1)^n p_n(Y) := p_n^{\#}$$

Towards (q, t) analogs: Hopf algebras

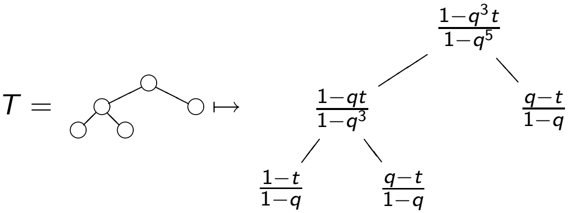
We need *noncommutative supersymmetric functions* and generalizations, for permutations, binary trees, ...

Commutative case: $(1-t)/(1-q) = 1/(1-q) - t/(1-q)$.

Noncommutative case: different versions.

In combinatorial terms: same question as before but count *signed* permutations by their major index *and* their number of negative signs.

(q, t) analogs for trees (N.-Thibon)



Each of the eight possible unsigned permutations give rise to 32 signed permutations and to polynomials that simplify to

$$(1+t)(q+t)^2(1+qt)(1+q^3t)(1+q)^2(1+q^2).$$

A key combinatorial ingredient

Theorem

The list of the descent sets of $u \sqcup\!\sqcup v$ if u and v are on disjoint alphabets only depends on the descent sets of u and v. The same holds for half-products.

For example,

$$\begin{aligned} 634 \prec 125 =& 631254 + 613254 + 612354 + 612534 + 163254 \\ &+ 162354 + 162534 + 126354 + 126534 + 125634, \end{aligned}$$

$$\begin{aligned} 312 \prec 456 =& 314562 + 341562 + 345162 + 345612 + 431562 \\ &+ 435162 + 435612 + 453162 + 453612 + 456312, \end{aligned}$$

and one can check that both sums have descent compositions

$$132,\ 141,\ 1131,\ 1221,\ 222,\ 231,\ 2121,\ 312,\ 321,\ 42.$$

Other enumeration results

In combinatorial Hopf algebras, a natural analog of permutations is the set of *packed words*.

Definition

w is packed if ($k > 1$ appears in $w \Rightarrow k - 1$ also appears).

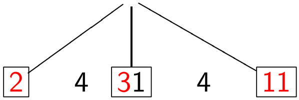
It is a way to see set compositions as a subset of all words.

In this setting, binary trees transform into plane trees and usual (q-)integrals into discrete integrals.

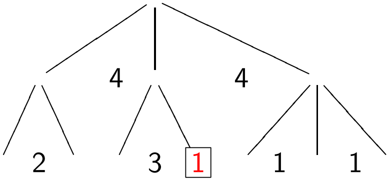
Plane tree of 2431411

2431411

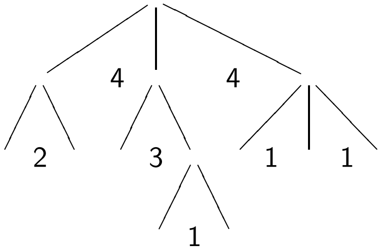
Plane tree of 2431411



Plane tree of 2431411



Plane tree of 2431411



"Hook-length" type formula for plane trees

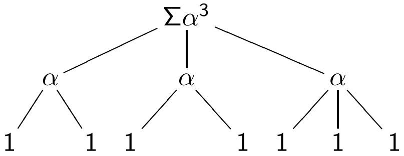
Theorem (Hivert-N.-Thibon)

The coefficients c_k of the expansion of the discrete integral $F(R)$ of the root of the plane tree as

$$F(R) = \sum_k c_k \binom{\alpha}{k}$$

are the number of packed words u with maximal letter k such that $\mathcal{T}(u) = T$.

Example

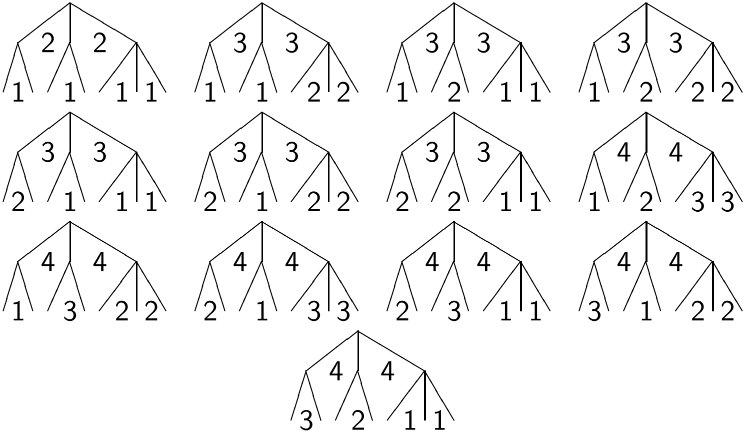


gives

$$\sum_{i=0}^{\alpha-1} i^3 = 6\binom{\alpha}{4} + 6\binom{\alpha}{3} + \binom{\alpha}{2}$$

so that there are $6+6+1=13$ packed words whose plane trees have this shape.

The 13 packed words



Conclusion

- Other combinatorial Hopf algebras ? (e.g., can one explain the hook-content formula for Schur functions in a similar way ?)
- Other specializations ?
- How to connect this to other applications of tree expanded series?