Trace Generating Functions of Plane Partitions

Soichi OKADA
(Nagoya University)

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We use an approach of Okounkov and Reshetikhin to derive trace generating functions of

- Reverse plane partitions (Gansner's Hook Formula),
- Diagonally strict reverse plane partitions,
- and more ...

Diagrams and Shifted Diagrams

For a partition λ, we denote its diagram by $D(\lambda)$:

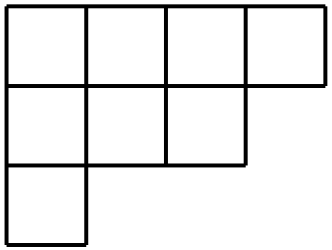
$$D(\lambda) = \{(i,j) \in \mathbb{P}^2 : 1 \leq j \leq \lambda_i\}.$$

For a strict partition μ, we denote its shifted diagram by $S(\mu)$:

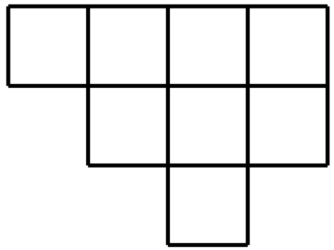
$$S(\mu) = \{(i,j) \in \mathbb{P}^2 : i \leq j \leq \mu_i + i - 1\}.$$

Example :

$D((4,3,1))$



$S((4,3,1))$



Reverse Plane Partitions

A (weak) reverse plane partition of shape λ is an array of non-negative integers

$$\pi = \begin{array}{llllll} \pi_{1,1} & \pi_{1,2} & & \cdots\cdots & & \pi_{1,\lambda_1} \\ \pi_{2,1} & \pi_{2,2} & \cdots & & \pi_{2,\lambda_2} & \\ \vdots & \vdots & & & & \\ \pi_{r,1} & \pi_{r,2} & \cdots & \pi_{r,\lambda_r} & & \end{array}$$

(i.e., a map $D(\lambda) \longrightarrow \mathbb{N}$) satisfying

$$\pi_{i,j} \le \pi_{i,j+1}, \quad \pi_{i,j} \le \pi_{i+1,j}.$$

Let $\mathcal{R}(D(\lambda))$ be the set of reverse plane partitions of shape λ :

$$\mathcal{R}(D(\lambda)) = \{\pi : \text{reverse plane partition of shape } \lambda\}.$$

A shifted (weak) reverse plane partition of shifted shape μ is an array of non-negative integers

$$\sigma = \begin{array}{cccccccc} \sigma_{1,1} & \sigma_{1,2} & \sigma_{1,3} & & \cdots\cdots & & & \sigma_{1,\mu_1} \\ & \sigma_{2,2} & \sigma_{2,3} & & \cdots & & \sigma_{2,\mu_2+1} & \\ & & \ddots & & & & & \\ & & & \sigma_{r,r} & \cdots & \sigma_{r,\mu_r+r-1} & & \end{array}$$

(i.e., a map $S(\mu) \longrightarrow \mathbb{N}$) satisfying

$$\sigma_{i,j} \leq \sigma_{i,j+1}, \quad \sigma_{i,j} \leq \sigma_{i+1,j}.$$

Let $\mathcal{R}(S(\mu))$ be the set of shifted reverse plane partitions of shape μ :

$$\mathcal{R}(S(\mu)) = \{\sigma : \text{shifted reverse plane partition of shape } \mu\}.$$

Trace and Trace Generating Function

Given a reverse (shifted) plane partition $\pi = (\pi_{i,j})$, we define the k-th trace $t_k(\pi)$ by

$$t_k(\pi) = \sum_i \pi_{i,i+k}.$$

We are interested in the trace generating function

$$\sum_\pi \prod_k q_k^{t_k(\pi)}.$$

Example :

$$\pi = \begin{matrix} 0 & 1 & 3 & 3 \\ 1 & 1 & 3 & \\ 2 & 4 & & \end{matrix} \quad \text{has the traces} \quad \begin{matrix} \cdots, & t_{-3}(\pi) = 0, \\ t_{-2}(\pi) = 2, & t_{-1}(\pi) = 5, \\ t_0(\pi) = 1, & t_1(\pi) = 4, \\ t_2(\pi) = 3, & t_3(\pi) = 3, \\ t_4(\pi) = 0, & \cdots. \end{matrix}$$

Hook and Shifted Hook

For a partition λ and a cell $(i,j) \in D(\lambda)$ of the Ferrers diagram, the hook at (i,j) in $D(\lambda)$ is defined by

$$\begin{aligned} H_{D(\lambda)}(i,j) = \{(i,j)\} &\cup \{(i,l) \in D(\lambda) : l > j\} \\ &\cup \{(k,j) \in D(\lambda) : k > i\}. \end{aligned}$$

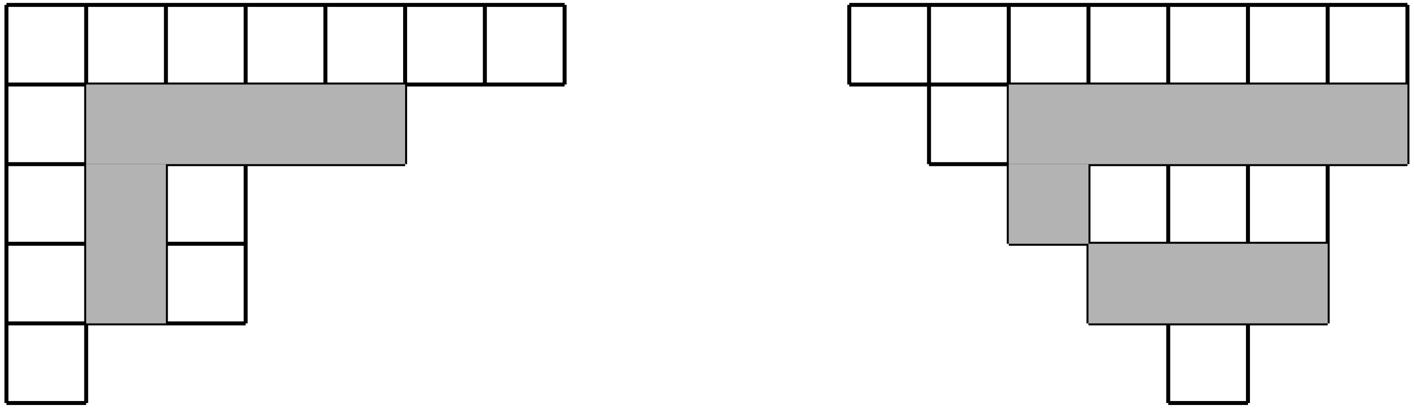
For a strict partition μ and a cell $(i,j) \in S(\mu)$ of the shifted diagram, the shifted hook at (i,j) in $S(\mu)$ is defined by

$$\begin{aligned} H_{S(\mu)}(i,j) = \{(i,j)\} &\cup \{(i,l) \in S(\mu) : l > j\} \\ &\cup \{(k,j) \in S(\mu) : k > i\} \\ &\cup \{(j+1,l) \in S(\mu) : l > j\}. \end{aligned}$$

Example :

The hook at $(2,2)$ in $D((7,5,3,3,1))$

The shifted hook at $(2,3)$ in $S((7,6,4,3,1))$



Theorem 1 (Gansner)
(1) For a partition λ, the trace generating function of $\mathcal{R}(D(\lambda))$ is given by

$$\sum_{\pi \in \mathcal{R}(D(\lambda))} \prod_{k} q_k^{t_k(\pi)} = \prod_{x \in D(\lambda)} \frac{1}{1 - q[H_{D(\lambda)}(x)]},$$

where we put

$$q[H] = \prod_{(k,l) \in H} q_{l-k}$$

for a finite subset $H \subset \mathbb{P}^2$.
(2) For a strict partition μ, the trace generating function of $\mathcal{R}(S(\mu))$ is given by

$$\sum_{\sigma \in \mathcal{R}(S(\mu))} \prod_{k \geq 0} q_k^{t_k(\sigma)} = \prod_{x \in S(\mu)} \frac{1}{1 - q[H_{S(\mu)}(x)]}.$$

Proof of Theorem 1

A reverse plane partition π is decomposed into two shifted reverse plane partitions

$$\pi^{+} = (\pi_{i,j})_{1 \le i \le j}, \quad \text{and} \quad \pi^{-} = (\pi_{j,i})_{1 \le i \le j}$$

with the same profile. So we compute the trace generating function of shifted reverse plane partitions with a given profile.

Let ρ and ν be two strict partitions such that $S(\rho) \supset S(\nu)$. A shifted skew plane partition of shifted skew shape ρ/ν is a map $\sigma : S(\rho) - S(\nu) \longrightarrow \mathbb{N}$ satisfying

$$\sigma_{i,j} \ge \sigma_{i,j+1}, \quad \sigma_{i,j} \ge \sigma_{i+1,j}.$$

Then

a reverse shifted plane partition of shifted shape μ

can be viewed as

a shifted skew plane partition of shifted skew shape δ_N/ν

where $N \geq \mu_1$, $\delta_N = (N, N-1, \cdots, 2, 1)$ and ν is given by

$$\{\mu_1, \cdots, \mu_p\} \sqcup \{\nu_1, \cdots, \nu_q\} = \{1, 2, \cdots, N\}.$$

Example :

$$\begin{array}{cccccc} 0 & 0 & 1 & 2 & 3 & 3 \\ & 1 & 2 & 3 & 3 & 3 \\ & & 2 & 4 & & \end{array}$$

shifted reverse plane partition of shifted shape $(6, 5, 2)$

$$\begin{array}{cccccc} * & * & * & * & 3 & 3 \\ & * & * & * & 3 & 3 \\ & & * & 4 & 3 & 2 \\ & & & 2 & 2 & 1 \\ & & & & 1 & 0 \\ & & & & & 0 \end{array}$$

shifted skew plane partition of shifted skew shape $\delta_6/(4, 3, 1)$.

Hence it is enough to compute the trace generating function of shifted skew plane partitions with a given shifted skew shape δ_N/ν and a given profile τ.

Given a shifted skew array of non-negative integers σ of shifted skew shape δ_N/ν, we define

$$\sigma[t] = (\sigma_{i,i+t})_i \quad (t = 0, 1, 2 \cdots).$$

Example :

If $\sigma = \begin{array}{cccccc} * & * & * & * & 3 & 3 \\ & * & * & * & 3 & 3 \\ & & * & 4 & 3 & 2 \\ & & & 2 & 2 & 1 \\ & & & & 1 & 0 \\ & & & & & 0 \end{array}$, then we have $\begin{array}{ll} \sigma[0] = (2,1,0), & \sigma[1] = (4,2,0), \\ \sigma[2] = (3,1), & \sigma[3] = (3,2), \\ \sigma[4] = (3,3), & \sigma[5] = (3). \end{array}$

A key is the following observation.

Lemma The following are equivalent:

(i) σ is a shifted skew plane partition.

(ii) Each $\sigma[t]$ is a partition and

$$\begin{cases} \sigma[t-1] \prec \sigma[t] & \text{if } t \in \{\nu_1, \cdots, \nu_q\}, \\ \sigma[t-1] \succ \sigma[t] & \text{if } t \notin \{\nu_1, \cdots, \nu_q\}. \end{cases}$$

where we write $\alpha \succ \beta$ if

$$\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \cdots ,$$

i.e., the skew diagram α/β is a horizontal strip.

Let h_k and $h_k^\perp$ be the multiplication and skewing operators on the ring of symmetric functions Λ associated to the complete symmetric function h_k. Consider the vertex operators

$$H^+(t) = \sum_{k\geq 0} h_k t^k, \quad H^-(t) = \sum_{k\geq 0} h_k^\perp t^k.$$

and the operator $D(q) : \Lambda \to \Lambda$ defined by

$$D(q)s_\lambda = q^{|\lambda|} s_\lambda.$$

First we apply the Pieri rule

$$H^+(t)s_\lambda = \sum_{\kappa\succ\lambda} t^{|\kappa|-|\lambda|} s_\kappa \quad H^-(t)s_\lambda = \sum_{\kappa\prec\lambda} t^{|\lambda|-|\kappa|} s_\kappa,$$

and Lemma above to obtain

Lemma If we define ε_k by

$$\varepsilon_k = \begin{cases} + & \text{if } k \notin \{\nu_1, \cdots, \nu_q\}, \\ - & \text{if } k \in \{\nu_1, \cdots, \nu_q\}, \end{cases}$$

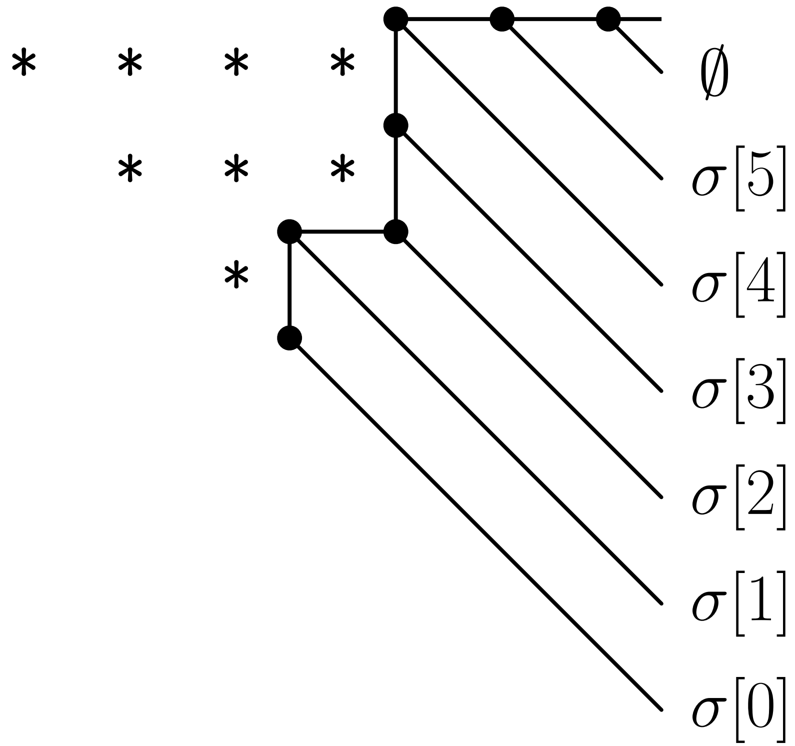
then we have

$$D(q_0)H^{\varepsilon_1}(1)D(q_1)H^{\varepsilon_2}(1)D(q_2)H^{\varepsilon_2}(1)\cdots H^{\varepsilon_{N-1}}(1)D(q_{N-1})H^{\varepsilon_N}(1)1$$
$$= \sum_{\tau} \left(\sum_{\sigma} \prod_{k \geq 0} q_k^{t_k(\sigma)} \right) s_\tau,$$

where the inner summation is taken over all shifted skew plan partitions with skew shifted shape δ_N/ν and profile $\sigma[0] = \tau$.

Example : If $\nu = (4,3,1)$ and $N = 6$, then we compute

$$D(q_0)H^-(1)D(q_1)H^+(1)D(q_2)H^-(1)D(q_3)H^-(1)$$
$$D(q_4)H^+(1)D(q_5)H^+(1)1.$$



$$\sigma[0] \prec \sigma[1] \succ \sigma[2] \prec \sigma[3] \prec \sigma[4] \succ \sigma[5] \succ \emptyset.$$

Next, we use the commutation relations

$$D(q) \circ H^{+}(t) = H^{+}(qt) \circ D(q),$$
$$D(q) \circ H^{-}(t) = H^{-}(q^{-1}t) \circ D(q),$$

to obtain

$$\begin{aligned} D(q_0)H^{\varepsilon_1}(1)D(q_1)H^{\varepsilon_2}(1)D(q_2)H^{\varepsilon_2}(1) \cdots H^{\varepsilon_{N-1}}(1)D(q_{N-1})H^{\varepsilon_N}(1) \\ = H^{\varepsilon_1}(z_1^{\varepsilon_1})H^{\varepsilon_2}(z_2^{\varepsilon_2}) \cdots H^{\varepsilon_N}(z_N^{\varepsilon_N})D(z_N), \end{aligned}$$

where we put

$$z_k = q_0 q_1 \cdots q_{k-1}.$$

Further, by using the commutation relation

$$H^{-}(s) \circ H^{+}(t) = \frac{1}{1-st} H^{+}(t) \circ H^{-}(s),$$

we can derive

$$H^{\varepsilon_1}(z_1^{\varepsilon_1}) H^{\varepsilon_2}(z_2^{\varepsilon_2}) \cdots H^{\varepsilon_N}(z_N^{\varepsilon_N}) \\ = \prod_{\nu_i < \mu_j} \frac{1}{1 - z_{\nu_i}^{-1} z_{\mu_j}} \prod_{j=1}^{p} H^{+}(z_{\mu_j}) \prod_{i=1}^{q} H^{-}(z_{\nu_i}).$$

Recall that $\mu = (\mu_1, \cdots, \mu_p)$, $\nu = (\nu_1, \cdots, \nu_q)$ and

$$\{\mu_1, \cdots, \mu_p\} \sqcup \{\nu_1, \cdots, \nu_q\} = \{1, 2, \cdots, N\}.$$

Finally, by using the Cauchy identity

$$\prod_{i=1}^{n} H^{+}(t_i) \cdot 1 = \sum_{\tau} s_{\tau}(t_1, \cdots, t_n) s_{\tau},$$

we have

Proposition The trace generating function of shifted skew plane partitions of shape δ_N/ν with profile τ is given by

$$\sum_{\sigma} \prod_{k \geq 0} q_k^{t_k(\sigma)} = \prod_{\nu_i < \mu_j} \frac{1}{1 - z_{\nu_i}^{-1} z_{\mu_j}} \cdot s_{\tau}(z_{\mu_1}, \cdots, z_{\mu_p}),$$

where $\{\mu_1, \cdots, \mu_p\} \sqcup \{\nu_1, \cdots, \nu_q\} = \{1, 2, \cdots, N\}$, and

$$z_k = q_0 q_1 \cdots q_{k-1}.$$

Now Theorem 1 (1) follows from the Cauchy identity

$$\sum_{\tau} s_{\tau}(x_1, \cdots, x_m) s_{\tau}(y_1, \cdots, y_n) = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1}{1 - x_i y_j},$$

and (2) follows from the Schur–Littlewood identity

$$\sum_{\tau} s_{\tau}(x_1, \cdots, x_m) = \prod_{i=1}^{m} \frac{1}{1 - x_i} \prod_{1 \leq i < j \leq m} \frac{1}{1 - x_i x_j}.$$

This completes the proof of Theorem 1.

Generalization

We can play the same game for

(1) Schur's P functions,

(2) Hall–Littlewood functions, or Macdonald functions

instead of Schur functions to obtain

(1) trace generating functions for diagonally strict reverse plane partitions,

(2) weighted trace generating functions for reverse plane partitions,

respectively (see also Foda–Wheeler-Zuparic, Vuletić).

Diagonally Strict Reverse Plane Partitions

We say that a reverse (shifted) plane partition π is diagonally strict if

$$\pi_{i,j} < \pi_{i+1,j+1} \quad \text{if the both sides are positive.}$$

If a reverse plane partition π is diagonally strict, then we see that, for each k, the skew diagram

$$\pi^{-1}(k) = \{(i,j) \in \mathbb{P}^2 : \pi_{ij} = k\}$$

contains no $\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline\end{array}$, so it is a disjoint union of rim hooks. We put

$$p_k(\pi) = \text{the number of connected components of } \pi^{-1}(k),$$
$$p(\pi) = \sum_k p_k(\pi).$$

Example : If

$$\pi = \begin{array}{cccc} 0 & 1 & 3 & 3 \\ 1 & 1 & 3 & \\ 2 & 3 & & \end{array},$$

then we have

$$p_1(\pi) = 1, \quad p_2(\pi) = 1, \quad p_3(\pi) = 2, \quad p(\pi) = 4.$$

Theorem 2 For a partition λ, the trace generating function of diagonally strict reverse plane partitions of shape λ is given by

$$\sum_{\pi} 2^{p(\pi)} \prod_{k} q_k^{t_k(\pi)} = \prod_{x \in D(\lambda)} \frac{1 + q[H_{D(\lambda)}(x)]}{1 - q[H_{D(\lambda)}(x)]}.$$

Weighted Trace Generating Function (HL case)

Theorem 3 For a partition λ, the weighted trace generating function of reverse plane partitions of shape λ is given by

$$\sum_{\pi} A_\pi(t) \prod_k q_k^{t_k(\pi)} = \prod_{x \in D(\lambda)} \frac{1 - t \cdot q[H_{D(\lambda)}(x)]}{1 - q[H_{D(\lambda)}(x)]}.$$

Remark : We have

$$A_\pi(0) = 1,$$

$$A_\pi(-1) = \begin{cases} 2^{p(\pi)} & \text{if } \pi \text{ is diagonally strict,} \\ 0 & \text{otherwise.} \end{cases}$$

So Theorem 3 reduces to Theorem 1 (1) and Theorem 2 if $t = 0$ and $t = -1$ respectively.

The weight $A_\pi(t)$ is defined as follows.

Given a skew diagram θ and a cell $(i,j) \in \theta$, the level $h_\theta(i,j)$ is defined by

$$h_\theta(i,j) = \begin{array}{l}\text{the smallest positive integer } k\\ \text{satisfying } (i-k, j-k) \notin \theta.\end{array}$$

Then, for each l,

$$\{(i,j) \in \theta : h_\theta(i,j) = l\}$$

is a disjoint union of rim hooks. Each connected component is called a border component of level l. We set

$$P_\theta(t) = \prod_{l \geq 1} (1 - t^l)^{\#\text{ border components of level } l}.$$

Example : If $\theta = (6,5,5,2)/(2,1)$, then

		1	1	1	1
	1	1	2	2	
1	1	2	2	3	
2	2				

and

$$P_\theta(t) = (1-t)(1-t^2)^2(1-t^3).$$

Then the weight $A_\pi(t)$ of a reverse plane partition π is defined by

$$A_\pi(t) = \prod_{k \geq 1} P_{\pi^{-1}(k)}(t).$$